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# Transformations of KZ type equations

*Dedicated to Professor Hikosaburo Komatsu and Professor Takahiro Kawai*

By

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## Abstract

The middle convolution introduced by Katz is extended to an operation on a regular holonomic system by Haraoka. We study this operation on a KZ type equation and we clarify how the conjugacy classes of resulting reside matrices under this operation are determined in terms of the original residue matrices and examine the relation with other related transformations.

## § 1. Introduction

Katz [4] introduces the middle convolution and Dettweiler-Reiter [2] interprets this and an addition as operations on a Fuchsian system of Schlesinger canonical form. The author [5] defines these operations on linear ordinary differential equations of arbitrary order and studies the analytic properties of their solutions.

Since the rigid Fuchsian ordinary differential equation is obtained by a successive application of these operations on the trivial equation $u' = 0$, we obtain many global analytic properties of its solution by the analysis of these operations (cf. [5]). In this analysis it is a key that the structure of the transformation by these operations on the space of Fuchsian differential equations is understood as an action of the Weyl group of a Kac-Moody root space by Crawley-Boevey [CB] and then the equations are classified by their local monodromies of its solutions or their spectral types which are identified with roots of the root space.

Haraoka [3] extends the middle convolution to an operation on regular holonomic systems. As a consequence, any rigid Fuchsian system of Schlesinger canonical form

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can be naturally extended to a KZ type equation. If the rigid Fuchsian system has more than three singular points, the solution of the corresponding KZ type equation can be regarded as a hypergeometric function with several variables and hence we have a plenty of examples as is shown in §5. Appell's hypergeometric function is the simplest example. The irreducibility of the monodromy group of the solution space of the KZ type equation is studied by [7] in this point of view.

For our further study of the KZ type equation we need to analyze the transformation of local monodromies or the spectral type under these operations, which we will study in this paper. The integrability condition of the equation plays an essential role in analyzing the transformation. We will also examine the symmetry of the KZ type equation related to these operations, which also give other transformations of the equation.

## § 2. KZ type equation and integrability condition

A Pfaffian system

$$\mathcal{M}\ :\ \mathrm{d}u = \sum_{0\le i<j\le n} A_{i,j}\frac{\mathrm{d}(x_i - x_j)}{x_i - x_j}u \tag{2.1}$$

with an unknown $N$ vector $u$ and constant square matrices $A_{i,j}$ of size $N$ is called a KZ (Knizhnik-Zamolodchikov) type equation of rank $N$, which equals the system of the equations

$$\frac{\partial u}{\partial x_i} = \sum_{\substack{0\le\nu\le n\\ \nu\ne i}} \frac{A_{i,\nu}}{x_i - x_\nu}u \qquad (i = 0,\dots,n) \tag{2.2}$$

with denoting $A_{j,i} = A_{i,j}$. The matrix $A_{i,j}$ is called the *residue matrix* of $\mathcal{M}$ at $x_i = x_j$. Here we always assume the *integrability condition*

$$\begin{cases} [A_{i,j}, A_{k,\ell}] = 0 & (\forall\{i,j,k,\ell\}\subset\{0,\dots,n\}),\\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0 & (\forall\{i,j,k\}\subset\{0,\dots,n\}), \end{cases} \tag{2.3}$$

which follows from the condition dd$u = 0$. Here $i$, $j$, $k$, $\ell$ are mutually different indices.

*Remark* 1. We can also study the KZ type equation (2.1) such that $u$ is a function of $(x_0,\dots,x_q)$ and $x_\nu = a_\nu \in \mathbb{C}$ for $\nu = q+1,\dots,n$. In this case $A_{i,j}$ have no meaning for $\{i,j\}\subset q+1,\dots,n$ and in the integrability condition (2.3) the relations containing such $A_{i,j}$ are not necessary. Most of the results in this paper are naturally extended to this case (cf. Theorem 4.1 ii)).

**Definition 2.1.** We use the following notation.

$$\begin{aligned}
A_{i,i} &= A_\emptyset = A_i = 0, \quad A_{i,j} = A_{j,i} \quad (i,\, j \in \{0,1,\ldots,n+1\}),\\
A_{i,n+1} &:= -\sum_{\nu=0}^{n} A_{i,\nu},\\
A_{i_1,i_2,\ldots,i_k} &:= \sum_{1\le p<q\le k} A_{i_p,i_q} \quad (\{i_1,\ldots,i_k\} \subset \{0,\ldots,n+1\}).
\end{aligned}$$

The matrix $A_{i,n+1}$ is called the residue matrix of $\mathcal{M}$ at $x_i = \infty$.

We have

$$\begin{aligned}
[A_{1,2}, A_{0,n+1}] &= -[A_{1,2}, \sum_{\nu=1}^{n} A_{0,\nu}] = -[A_{1,2}, A_{0,1} + A_{0,2}] = 0,\\
[A_{0,n+1}, A_{0,1} + A_{1,n+1}] &= [\sum_{\nu=0}^{n} A_{0,\nu}, \sum_{k=2}^{n} A_{1,k}] = \sum_{k=2}^{n} [A_{0,1} + A_{0,k}, A_{1,k}] = 0,\\
[A_{0,1}, A_{0,1} + A_{0,n+1} + A_{1,n+1}] &= [A_{0,1} + A_{0,n+1} + A_{1,n+1}, A_{0,1} + A_{0,n+1} + A_{1,n+1}]\\
&\quad - [A_{0,n+1}, A_{0,1} + A_{0,n+1} + A_{1,n+1}]\\
&\quad - [A_{1,n+1}, A_{0,1} + A_{0,n+1} + A_{1,n+1}] = 0
\end{aligned}$$

and therefore in general, $A_{i,j}$ satisfy

$$[A_{i,j}, A_{k,\ell}] = 0 \quad (\forall\{i,j,k,\ell\} \subset \{0,\ldots,n+1\}), \tag{2.4}$$

$$[A_{i,j}, A_{i,k} + A_{j,k}] = 0 \quad (\forall\{i,j,k\} \subset \{0,\ldots,n+1\}). \tag{2.5}$$

Conversely the assumption (2.4) implies (2.3) and moreover (2.5), which follows from

$$[A_{i,j}, A_{i,k} + A_{j,k}] = -[A_{i,j}, \sum_{\ell\in\{0,\ldots,n+1\}\setminus\{i,j\}} A_{\ell,k}]$$

for $\{i,j,k\} \subset \{0,\ldots,n\}$.

Hereafter we assume (2.4) and (2.5). If $1 \le k \le n+1$, we have

$$[A_{0,1}, A_{0,\ldots,k}] = [A_{0,1}, \sum_{0\le i<j\le k} A_{i,j}] = [A_{0,1}, A_{0,1} + \sum_{j=2}^{k} A_{0,j} + \sum_{j=2}^{k} A_{1,j}] = 0.$$

Then by the symmetry of indices we have

$$[A_{i,j}, A_J] = 0 \ \text{ if } \{i,j\} \subset J \text{ or } \{i,j\} \cap J = \emptyset \tag{2.6}$$

for $J \subset \{0,\ldots,n+1\}$ and therefore we have the following lemma.

**Lemma 2.2.** *If $I$, $J \subset \{0, \dots, n+1\}$ satisfy $I \cap J = \emptyset$ or $I \subset J$, then*

$$[A_I, A_J] = 0.$$

Note that

$$\begin{aligned}\sum_{i=0}^{n} A_{i,n+1} &= \sum_{i=0}^{n}\Bigl(-\sum_{j=0}^{n} A_{i,j}\Bigr) = -2A_{0,\dots,n},\\ A_{k,\dots,n+1} &= A_{k,\dots,n} + \sum_{i=k}^{n} A_{i,n+1} = \frac{1}{2}\sum_{i=k}^{n}\sum_{j=k}^{n} A_{i,j} - \sum_{i=k}^{n}\sum_{j=0}^{n} A_{i,j}\\ &= -\sum_{i=k}^{n}\sum_{j=0}^{k-1} A_{i,j} - \frac{1}{2}\sum_{i=k}^{n}\sum_{j=k}^{n} A_{i,j},\\ A_{0,\dots,k-1} - A_{k,\dots,n+1} &= \frac{1}{2}\Bigl(\sum_{i=0}^{k-1}\sum_{j=0}^{k-1} A_{i,j} + 2\sum_{i=k}^{n}\sum_{j=0}^{k-1} A_{i,j} + \sum_{i=k}^{n}\sum_{j=k}^{n} A_{i,j}\Bigr) = A_{0,1,\dots,n}.\end{aligned}$$

We assume that the system $\{A_{i,j}\}$ is irreducible, namely, there exists no proper invariant subspace $V$ of $\mathbb{C}^N$ with $A_{i,j}V \subset V$ for all $\{i, j\}$. Since any $A_{i,j}$ with $\{i, j\} \subset \{0, \dots, n\}$ satisfies $[A_{i,j}, A_{0,\dots,n}] = 0$, $A_{0,\dots,n}$ commutes with any $A_{i,j}$. Hence $A_{0,\dots,n}$ is a scalar matrix and there is a complex number $\kappa$ satisfying

$$A_{0,\dots,n} = \kappa I_N \tag{2.7}$$

and therefore

$$\sum_{i=0}^{n} A_{i,n+1} = -2\kappa I_N, \tag{2.8}$$

$$A_I - A_{\{0,\dots,n+1\}\setminus I} = \kappa I_N \quad (\forall I \subset \{0, \dots, n\}). \tag{2.9}$$

## § 3. Addition and Middle convolution of KZ type equation

In this section we review the middle convolution of the KZ type equation defined by [3]. The *addition* $\mathcal{M}' = \mathrm{Ad}\,((x_k - x_\ell)^\lambda)\mathcal{M}$ with $1 \le k < j \le n$ and $\lambda \in \mathbb{C}$ is a transformation of (2.1) defined by

$$\mathcal{M}' : \mathrm{d}u' = \left(\sum_{0\le i<j\le n} (A_{i,j} + \lambda\delta_{i,k}\delta_{j,\ell})\frac{\mathrm{d}(x_i - x_j)}{x_i - x_j}\right) u'.$$

The convolution $\tilde{\mathcal{M}} = \widetilde{\mathrm{mc}}_{x_0,\mu}\,\mathcal{M}$ of $\mathcal{M}$ with $\mu \in \mathbb{C}$ is given by

$$\tilde{\mathcal{M}} : \mathrm{d}\tilde{u} = \left(\sum_{0\le i<j\le n} \tilde{A}_{i,j}\frac{\mathrm{d}(x_i - x_j)}{x_i - x_j}\right) \tilde{u}$$

with

$$\begin{aligned}\tilde{A}_{0,k} &= \begin{array}{cc} & \begin{matrix} \phantom{A_{0,1}} & \phantom{\cdots} & k & \phantom{\cdots} & \phantom{A_{0,n}} \end{matrix} \\ \begin{matrix} \\ \\ k \\ \\ \\ \end{matrix} & \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ A_{0,1} & \cdots & A_{0,k}+\mu & \cdots & A_{0,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \end{array} \in M(nN,\mathbb{C}) \qquad (1 \le k \le n) \\ &= \Big((A_{0,q}+\mu\delta_{p,q})\delta_{p,k}\Big)_{\substack{1\le p\le n\\ 1\le q\le n}},\end{aligned} \tag{3.1}$$

$$\begin{aligned}\tilde{A}_{i,j} &= \begin{array}{cc} & \begin{matrix} \phantom{A_{i,j}} & \phantom{\ddots} & i & \phantom{\ddots} & j & \phantom{\ddots} & \phantom{A_{i,j}} \end{matrix} \\ \begin{matrix} \\ \\ i \\ \\ j \\ \\ \\ \end{matrix} & \begin{pmatrix} A_{i,j} & & & & & & \\ & \ddots & & & & & \\ & & A_{i,j}+A_{0,j} & & -A_{0,j} & & \\ & & & \ddots & & & \\ & & -A_{0,i} & & A_{i,j}+A_{0,i} & & \\ & & & & & \ddots & \\ & & & & & & A_{i,j} \end{pmatrix} \end{array} \in M(nN,\mathbb{C}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \le i < j \le n) \\ &= \Big(A_{i,j}\delta_{p,q} + A_{0,j}\delta_{p,i}(\delta_{q,i}-\delta_{q.j}) + A_{0,i}\delta_{p,j}(\delta_{q,j}-\delta_{q,i})\Big)_{\substack{1\le p\le n\\ 1\le q\le n}}.\end{aligned} \tag{3.3}$$

Here $\tilde{A}_{0,k}$ and $\tilde{A}_{i,j}$ are block matrices of size $n$ whose entries are square matrices of size $N$ and $M(r,\mathbb{C})$ denotes the space of square matrices of size $r$ whose entries are complex numbers.

**Definition 3.1.** The above matrices $\tilde{A}_{0,k}$ and $\tilde{A}_{i,j}$ define linear transformations of $\mathbb{C}^{nN}$ and then we introduce the following notation.

$$\begin{aligned} V &:= \mathbb{C}^{nN} = \Big\{ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \;\Big|\; v_j \in \mathbb{C}^N \quad (j = 1, \ldots, n) \Big\}, \\ p_j(\mathbf{v}) &:= v_j \qquad \Big(\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^{nN},\ j = 1, \ldots, n\Big), \\ \iota_j(v) &:= (v)_j := \begin{pmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \end{pmatrix} \qquad (p_i \circ \iota_j(v) = \delta_{i,j} v,\ v \in \mathbb{C}^N), \\ V_j &:= \iota_j(\mathbb{C}^N) \simeq \mathbb{C}^N, \\ V_{\{j_1,\ldots,j_k\}} &:= V_{j_1} \oplus \cdots \oplus V_{j_k}, \\ \iota_{j_1,\ldots,j_k}(v) &:= (v)_{j_1,\ldots,j_k} := \iota_{j_1}(v) + \cdots + \iota_{j_k}(v) \qquad (v \in \mathbb{C}^N), \end{aligned}$$

$$V_{j_1,\dots,j_k} := \iota_{j_1,\dots,j_k}(\mathbb{C}^N) \simeq \mathbb{C}^N.$$

Under the above notation we put

$$\begin{aligned}
\mathcal{K}_j &= \iota_j(\operatorname{Ker} A_{0,j}) = \begin{array}{c} \\ \\ j \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ \operatorname{Ker} A_{0,j} \\ 0 \\ \vdots \end{pmatrix} \simeq \operatorname{Ker} A_{0,j},\\
\mathcal{K}_\infty &= \left\{ \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} \mid A_{0,\infty} v = \mu v \right\} = \iota_{1,\dots,n}(\operatorname{Ker}(A_{0,\infty} - \mu)) \simeq \operatorname{Ker}(A_{0,\infty} - \mu),\\
\mathcal{K}' &= \bigoplus_{j=1}^{n} \mathcal{K}_j, \quad \mathcal{K} = \mathcal{K}_\infty + \mathcal{K}'.
\end{aligned}$$

Then $\mathcal{K}_j$ and $\mathcal{K}_\infty$ are stable under $\tilde{A}_{i,j}$ and $\tilde{A}_{i,j}$ induce linear transformations on $\mathbb{C}^{nN}/\mathcal{K}$ and then the corresponding matrices with respect to a base of $\mathbb{C}^{nN}/\mathcal{K}$ are denoted by $\bar{A}_{i,j}$, respectively. Note that $\mathcal{K}_\infty \cap \mathcal{K}' = \{0\}$ if $\mu \neq 0$. Then $\bar{A}_{i,j}$ are square matrices of size $nN - \dim \mathcal{K}$ and the *middle convolution* $\bar{\mathcal{M}} = \mathrm{mc}_{x_0,\mu}\, \mathcal{M}$ is the KZ type equation

$$\bar{\mathcal{M}} \,:\, \mathrm{d}\bar{u} = \left( \sum_{0 \le i < j \le n} \bar{A}_{i,j} \frac{\mathrm{d}(x - x_j)}{x_i - x_i} \right) \bar{u}. \tag{3.4}$$

Here we note that $\mathrm{mc}_{x_0,-\mu} \circ \mathrm{mc}_{x_0,\mu} = id$.

When $x_1, \dots, x_n$ are fixed in (2.1), $u$ is a solution of the ordinary differential equation

$$\frac{du}{dx} = \sum_{j=1}^{n} \frac{A_{0,j}}{x - x_j} u \tag{3.5}$$

with $x = x_0$. Dettweiler-Reiter [2] studied its middle convolution

$$\frac{d\bar{u}}{dx} = \sum_{j=1}^{n} \frac{\bar{A}_{0,j}}{x - x_j} \bar{u} \tag{3.6}$$

and analyzed the conjugacy classes of $\bar{A}_{0,1}, \dots, \bar{A}_{0,n+1}$ in terms of those of $A_{0,1}, \dots, A_{0,n+1}$. The main purpose of this paper is to determine the conjugacy classes of other $\bar{A}_{i,j}$.

## § 4. Conjugacy classes of residue matrices

Let $A$ be a square matrix of size $N$. Then there exist positive integers $r, m_1, \dots, m_r$ and complex numbers $\lambda_1, \dots, \lambda_r$ such that

$$\begin{gathered}
N = m_1 + \dots + m_r,\\
m_\nu \ge m_{\nu'} \ \text{ if } \lambda_\nu = \lambda_{\nu'} \text{ and } \nu > \nu'
\end{gathered}$$

and moreover

$$\operatorname{rank}\prod_{j=1}^{k}(A-\lambda_j)=N-(m_1+\cdots+m_k)\qquad(k=1,\dots,r).\tag{4.1}$$

Then the set

$$\bigl\{[\lambda_1]_{m_1},\dots,[\lambda_r]_{m_r}\bigr\}\tag{4.2}$$

determines the conjugacy class of $A$ and we call this set the *eigenvalue class* of $A$ and denote it by $[A]$. If $m_j=1$ in (4.2), the element $[\lambda_j]_{m_j}$ in (4.2) may be simply expressed by $\lambda_j$. Moreover we put

$$[A]_k=\bigl\{[\lambda_1]_{km_1},\dots,[\lambda_r]_{km_r}\bigr\}\quad\text{and}\quad[[\lambda]_m]_k=[\lambda]_{km}$$

for a positive integer $k$.

If a square matrix $B$ belongs to the closure of the conjugacy class in $M(N,\mathbb{C})$, $B$ is said to be in the eigenvalue class (4.2) in the *weak sense*. If $\lambda_i\neq\lambda_j$ for $1\le i<j\le r$, then $A$ is semisimple and the conjugacy class is closed and the eigenvalue class of the matrix $B$ equals (4.2) in the above.

For the KZ type equations which we want to study, the reside matrices $A_{i,j}$ have complex parameters and in most cases they are semisimple for generic values of the parameters. For some special values of the parameters $A_{i,j}$ may not be semisimple, but the conjugacy classes of $A_{i,j}$ may be obtained by the analytic continuation of the parameters and therefore we assume $A_{i,j}$ are semisimple (cf. [5, Theorem 12.10] and [7, Lemma 2.1]). Even without this assumption most of our results in this paper are valid.

If $A$, $B\in M(N,\mathbb{C})$ satisfy $AB=BA$, then $A$ and $B$ have a simultaneous (generalized) eigenspace decomposition and we can define a *simultaneous eigenvalue class* denoted by $[A:B]$. For example $A=\begin{pmatrix}0&&&\\&0&&\\&&0&\\&&&-1\end{pmatrix}$ and $B=\begin{pmatrix}1&&&\\&2&&\\&&2&\\&&&3\end{pmatrix}$, we put $[A:B]=\bigl\{[0:1]_1,[0:2]_2,[-1:3]_1\bigr\}$.

Putting

$$[A_{0,j}]=\bigl\{[\lambda_{j,1}]_{m_{j,1}},\dots,[\lambda_{j,r_j}]_{m_{j,r_j}}\bigr\}\quad(j=1,\dots,n+1),$$

we define the *generalized Riemann scheme* of the equation (3.5) by

$$\left\{\begin{matrix}x=x_1&x=x_2&\cdots&x=x_k&\cdots&x=x_{n+1}=\infty\\ [\lambda_{1,1}]_{m_{1,1}}&[\lambda_{2,1}]_{m_{1,1}}&\cdots&[\lambda_{k,1}]_{m_{k,1}}&\cdots&[\lambda_{n+1,1}]_{m_{n+1,1}}\\ \vdots&\vdots&\cdots&\vdots&\cdots&\vdots\\ [\lambda_{1,r_1}]_{m_{1,r_1}}&[\lambda_{2,r_2}]_{m_{2,r_2}}&\cdots&[\lambda_{k,r_k}]_{m_{k,r_k}}&\cdots&[\lambda_{n+1,r_{n+1}}]_{m_{n+1,r_{n+1}}}\end{matrix}\right\}.\tag{4.3}$$

Here and hereafter in this section we use the notation

$$A_{0,\infty}=A_{0,n+1},\ \lambda_{\infty,\nu}=\lambda_{n+1,\nu},\ r_\infty=r_{n+1}\ \text{ and }\ m_{\infty,\nu}=m_{n+1,\nu}. \tag{4.4}$$

For $\tilde{A}_{0,k}$ given by (3.1) we have

$$\begin{aligned}\tilde{A}_{0,k}(v)_k&=\bigl((A_{0,k}+\mu)v\bigr)_k,\\ \tilde{A}_{0,k}(v)_\nu&\equiv 0\mod V_k\qquad(\nu\in\{1,\dots,n\}\setminus\{k\}),\\ \tilde{A}_{0,k}(\mathbf{v})&=\mu\mathbf{v}\qquad(\mathbf{v}\in\mathcal{K}_k),\\ \tilde{A}_{0,k}(\mathbf{v})&=0\qquad(\mathbf{v}\in\mathcal{K}_\nu,\ \nu\in\{1,\dots,n,\infty\}\setminus\{k\}).\end{aligned}$$

Hence we have

$$[\tilde{A}_{0,k}]=[A_{0,k}+\mu]\cup[0]_{(n-1)N}$$

and

$$[\tilde{A}_{0,k}]|_{\mathcal{K}_\nu}=\begin{cases}[\mu|_{\operatorname{Ker}A_{0,k}}] & (\nu=k),\\ [0|_{\operatorname{Ker}A_{0,\nu}}] & (\nu\in\{1,\dots,n\}\setminus\{k\}),\\ [0|_{\operatorname{Ker}(A_{0,\infty}-\mu)}] & (\nu=\infty),\end{cases}$$

from which we obtain

$$[\bar{A}_{0,k}]=\bigl([A_{0,k}+\mu]\cup[0]_{(n-1)N-\dim\mathcal{K}+\dim\mathcal{K}_k}\bigr)\setminus[\mu]_{\dim\mathcal{K}_k}. \tag{4.5}$$

Here for $A$, $B\in M(N,\mathbb{C})$ and a subspace $U\in\mathbb{C}^N$ satisfying $AU\subset U$ and $BU\subset U$, put $[A]|_U=[A|_U]$ and $[A:B]|_U=[A|_U:B|_U]$ for simplicity.

For the residue matrix at $x_0=\infty$ we have

$$A_{0,\infty}=-\sum_{\nu=1}^n A_{0,\nu},\quad \tilde{A}_{0,\infty}=-\sum_{\nu=1}^n\tilde{A}_{0,\nu},$$

and therefore

$$\tilde{A}_{0,\infty}=-\begin{pmatrix}A_{0,1}+\mu\cdots & A_{0,k} & \cdots & A_{0,n}\\ \vdots\quad\ddots & \vdots & \cdots & \vdots\\ A_{0,1}\quad\cdots & A_{0,k}+\mu & \cdots & A_{0,n}\\ \vdots\quad\cdots & \vdots & \ddots & \vdots\\ A_{0,1}\quad\cdots & A_{0,k} & \cdots & A_{0,n}+\mu\end{pmatrix}=-\Bigl(A_{0,q}+\mu\delta_{p,q}I_N\Bigr)_{\substack{1\le p\le n,\\ 1\le q\le n}}$$

$$\begin{aligned}\tilde{A}_{0,\infty}(v)_{1,\dots,n} &= \bigl((A_{0,\infty}-\mu)v\bigr)_{1,\dots,n},\\ \tilde{A}_{0,\infty}(v)_{\nu} &\equiv (-\mu v)_{\nu} \mod V_{1,\dots,n},\\ [\tilde{A}_{0,\infty}] &= [A_{0,\infty}-\mu]\cup[-\mu]_{(n-1)N},\\ [\tilde{A}_{0,\infty}]|_{\mathcal{K}_\nu} &= \begin{cases}[-\mu|_{\mathrm{Ker}\,A_{0,\nu}}] & (1\le\nu\le n),\\ [0|_{\mathrm{Ker}\,(A_{0,\infty}-\mu)}] & (\nu=\infty),\end{cases}\\ [\bar{A}_{0,\infty}] &= \bigl([A_{0,\infty}-\mu]\cup[-\mu]_{(n-1)N-\dim\mathcal{K}'}\bigr)\setminus[0]_{\dim\mathcal{K}_\infty}. \qquad (4.6)\end{aligned}$$

Suppose $\lambda_{k,1}=0$ for $k=1,\dots,n$ and $\lambda_{n+1,1}=\mu$ and $m_{i,1}\ge m_{i,\nu}$ if $\lambda_{i,1}=\lambda_{i,\nu}$, which can be assumed by allowing $m_{i,1}=0$ for some $i$. Moreover we assume $\mu\ne0$ and $\lambda_{i,j}$ are generic under the Fuchs condition

$$\sum_{j=1}^{n+1}\sum_{\nu=1}^{r_j} m_{j,\nu}\lambda_{j,\nu}=0, \qquad (4.7)$$

which follows from $A_{1,1}+\cdots+A_{0,n+1}=0$. Then the above calculation shows that the generalized Riemann scheme of (3.6) equals

$$\left\{\begin{matrix} x=x_1 & \cdots & x=x_k & \cdots & x=x_{n+1}=\infty\\ [0]_{m_{1,1}-d} & \cdots & [0]_{m_{k,1}-d} & \cdots & [-\mu]_{m_{n+1,1}-d}\\ [\lambda_{1,2}+\mu]_{m_{1,2}} & \cdots & [\lambda_{k,2}+\mu]_{m_{k,2}} & \cdots & [\lambda_{n+1,2}-\mu]_{m_{n+1,2}}\\ \vdots & \cdots & \vdots & \cdots & \vdots\\ [\lambda_{1,r_1}+\mu]_{m_{1,r_1}} & \cdots & [\lambda_{k,r_k}+\mu]_{m_{k,r_k}} & \cdots & [\lambda_{n+1,r_{n+1}}-\mu]_{m_{n+1,r_{n+1}}}\end{matrix}\right\} \qquad (4.8)$$

with the integer

$$d=\sum_{j=1}^{n+1} m_{j,1}-(n-1)N.$$

This result is given in [2].

Note that $\operatorname{rank}\bar{\mathcal{M}}=\operatorname{rank}\mathcal{M}-d$ and therefore the rank of the KZ type equation may be changed by the middle convolution. We will calculate $[\bar{A}_{i,j}]$ for $1\le i<j\le n+1$. By the definition (3.2) we have

$$\begin{aligned}\tilde{A}_{i,j}(v)_{i,j} &= (A_{i,j}v)_{i,j},\\ \tilde{A}_{i,j}(v)_{i} &\equiv (A_{0,i,j}v)_{i} \mod V_{i,j},\\ \tilde{A}_{i,j}(v)_{\nu} &= A_{i,j}(v)_{\nu} \quad (\nu\ne i,\, j)\end{aligned}$$

and therefore

$$[\tilde{A}_{i,j}]=[A_{i,j}]_{n-1}\cup[A_{0,i,j}], \qquad (4.9)$$

$$[\tilde{A}_{i,j}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{i,j}|_{\operatorname{Ker} A_{0,\nu}}] & (\nu \in \{1,\dots,n\} \setminus \{i,j\}), \\ [A_{0,i,j}|_{\operatorname{Ker} A_{0,\nu}}] & (\nu \in \{i,j\}), \\ [A_{i,j}|_{\operatorname{Ker}(A_{0,\infty}-\mu)}] & (\nu = \infty). \end{cases} \tag{4.10}$$

Hence we should know $[A_{i,j} : A_{0,k}]$, $[A_{0,i,j} : A_{0,i}]$ and $[A_{i,j} : A_{0,\infty}]$ for $\{i,j,k\} \subset \{1,\dots,n\}$. Then

$$[\tilde{A}_{i,j} : \tilde{A}_{0,k}] = [A_{i,j} : A_{0,k} + \mu] \cup [A_{i,j} : 0]_{n-2} \cup [A_{0,i,j} : 0], \tag{4.11}$$

$$[\tilde{A}_{i,j} : \tilde{A}_{0,k}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{i,j}|_{\operatorname{Ker} A_{0,\nu}} : 0] & (\nu \in \{1,\dots,n\} \setminus \{i,j,k\}), \\ [A_{i,j}|_{\operatorname{Ker} A_{0,k}} : \mu] & (\nu = k), \\ [A_{0,i,j}|_{\operatorname{Ker} A_{0,\nu}} : 0] & (\nu \in \{i,j\}), \\ [A_{i,j}|_{\operatorname{Ker}(A_{0,\infty}-\mu)} : 0] & (\nu = \infty) \end{cases} \tag{4.12}$$

and

$$[\tilde{A}_{i,j} : \tilde{A}_{0,\infty}] = [A_{i,j} : A_{0,\infty} - \mu] \cup [A_{i,j} : -\mu]_{n-2} \cup [A_{0,i,j} : -\mu], \tag{4.13}$$

$$[\tilde{A}_{i,j} : \tilde{A}_{0,\infty}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{i,j}|_{\operatorname{Ker} A_{0,\nu}} : -\mu] & (\nu \in \{1,\dots,n\} \setminus \{i,j\}), \\ [A_{0,i,j}|_{\operatorname{Ker} A_{0,\nu}} : -\mu] & (\nu \in \{i,j\}), \\ [A_{i,j}|_{\operatorname{Ker}(A_{0,\infty}-\mu)} : 0] & (\nu = \infty). \end{cases} \tag{4.14}$$

Since

$$A_{i,\infty} = -\sum_{\substack{0 \le \nu \le n \\ \nu \ne i}} A_{i,\nu} \quad \text{and} \quad \tilde{A}_{i,\infty} = -\sum_{\substack{0 \le \nu \le n \\ \nu \ne i}} \tilde{A}_{i,\nu}, \tag{4.15}$$

we have

$$\tilde{A}_{i,\infty} = \begin{array}{cc} & \begin{array}{cccccc} \phantom{A_{i,\infty}} & \phantom{A_{i,\infty}} & & i & & \phantom{A_{i,\infty}} \end{array} \\ \begin{array}{c} \\ \\ \\ i \\ \\ \\ \end{array} & \left( \begin{array}{cccccc} A_{i,\infty} & & & A_{0,i} & & \\ & A_{i,\infty} & & A_{0,i} & & \\ & & \ddots & \vdots & & \\ & & & A_{i,\infty} + A_{0,\infty} + A_{0,i} - \mu & & \\ & & & \vdots & \ddots & \\ & & & A_{0,i} & & A_{i,\infty} \end{array} \right) \end{array}$$

and therefore

$$\begin{aligned} \tilde{A}_{i,\infty}(v)_\nu &= (A_{i,\infty} v)_\nu \qquad (\nu \ne i), \\ \tilde{A}_{i,\infty}(v)_i &\equiv \big((A_{0,i,\infty} - \mu)v\big)_i \mod V_{1,\dots,i-1,i+1,\dots,n}, \\ \tilde{A}_{i,\infty}(v)_{1,\dots,n} &= ((A_{0,i,\infty} - \mu)v)_{1,\dots,n} \quad ((v)_{1,\dots,n} \in \mathcal{K}_\infty), \end{aligned}$$

$$[\tilde{A}_{i,\infty}] = [A_{i,\infty}]_{n-1} \cup [A_{0,i,\infty} - \mu], \tag{4.16}$$

$$[\tilde{A}_{i,\infty}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{i,\infty}|_{\operatorname{Ker} A_{0,\nu}}] & (\nu \in \{1,\dots,n\} \setminus \{i\}), \\ [(A_{0,i,\infty} - \mu)|_{\operatorname{Ker} A_{0,i}}] & (\nu = i), \\ [(A_{0,i,\infty} - \mu)|_{\operatorname{Ker}(A_{0,\infty} - \mu)}] & (\nu = \infty). \end{cases} \tag{4.17}$$

Thus we have

$$[\tilde{A}_{i,\infty} : \tilde{A}_{0,k}] = [A_{i,\infty} : 0]_{n-2} \cup [A_{i,\infty} : A_{0,k} + \mu] \cup [A_{0,i,\infty} - \mu : 0], \tag{4.18}$$

$$[\tilde{A}_{i,\infty} : \tilde{A}_{0,k}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{i,\infty}|_{\operatorname{Ker} A_{0,\nu}} : 0] & (\nu \in \{1,\dots,n\} \setminus \{i,k\}), \\ [A_{i,\infty}|_{\operatorname{Ker} A_{0,k}} : \mu] & (\nu = k), \\ [(A_{0,i,\infty} - \mu)|_{\operatorname{Ker} A_{0,i}} : 0] & (\nu = i), \\ [(A_{0,i,\infty} - \mu)|_{\operatorname{Ker}(A_{0,\infty} - \mu)} : 0] & (\nu = \infty). \end{cases} \tag{4.19}$$

We will study a successive application of additions and middle convolutions to KZ type equations. Hence to complete our calculation it is sufficient to know $[A_{0,i,j} : A_{0,i}]$, $[A_{0,i,\infty} : A_{0,i}]$, $[A_{0,i,\infty} : A_{0,\infty}]$ and their transformations under the middle convolution. The calculation is symmetric for the indices in $\{1,\dots,n\}$ and hence we have only to examine $[A_{0,1,2} : A_{0,1}]$, $[A_{0,1,\infty} : A_{0,1}]$ and $[A_{0,1,\infty} : A_{0,\infty}]$.

Since

$$A_{0,1,2} = A_{0,1} + A_{0,2} + A_{1,2} \quad \text{and} \quad \tilde{A}_{0,1,2} = \tilde{A}_{0,1} + \tilde{A}_{0,2} + \tilde{A}_{1,2},$$

we have

$$\tilde{A}_{0,1,2} = \begin{pmatrix} A_{0,1,2} + \mu & & A_{0,3} & \cdots & A_{0,n} \\ & A_{0,1,2} + \mu & A_{0,3} & \cdots & A_{0,n} \\ & & A_{1,2} & & \\ & & & \ddots & \\ & & & & A_{1,2} \end{pmatrix},$$

$$\begin{aligned} \tilde{A}_{0,1,2}(v)_\nu &= \big((A_{0,1,2} + \mu)v\big)_\nu \qquad (\nu = 1, 2), \\ \tilde{A}_{0,1,2}(v)_\nu &\equiv \big(A_{1,2}v\big)_\nu \mod V_{\{1,2\}} \qquad (\nu = 3,\dots,n), \\ \tilde{A}_{0,1,2}(v)_\nu &= \big(A_{1,2}v\big)_\nu \quad (v \in \operatorname{Ker} A_{0,\nu},\ \nu = 3,\dots,n), \\ \tilde{A}_{0,1,2}(v)_{1,\dots,n} &= (A_{1,2}v)_{1,\dots,n} \quad (v \in \operatorname{Ker}(A_{0,\infty} - \mu)) \end{aligned}$$

and therefore

$$\begin{aligned} [\tilde{A}_{0,1,2}] &= [A_{0,1,2} + \mu]_2 \cup [A_{1,2}]_{n-2}, \\ [\tilde{A}_{0,1,2} : \tilde{A}_{0,1}] &= [A_{0,1,2} + \mu : A_{0,1} + \mu] \cup [A_{0,1,2} + \mu : 0] \cup [A_{1,2} : 0]_{n-2}, \end{aligned}$$

$$[\tilde{A}_{0,1,2} : \tilde{A}_{0,1}]|_{\mathcal{K}_\nu} = \begin{cases} [(A_{0,1,2}+\mu)|_{\mathrm{Ker}\,A_{0,\nu}} : \mu] & (\nu = 1), \\ [(A_{0,1,2}+\mu)|_{\mathrm{Ker}\,A_{0,\nu}} : 0] & (\nu = 2), \\ [A_{1,2}|_{\mathrm{Ker}\,A_{0,\nu}} : 0] & (\nu = 3,\dots,n), \\ [A_{1,2}|_{\mathrm{Ker}\,(A_{0,\infty}-\mu)} : 0] & (\nu = \infty). \end{cases}$$

The indices 1 and 2 in the above may be changed into arbitrary different numbers in $\{1,\dots,n\}$. Similarly we have

$$[\tilde{A}_{0,1,2} : \tilde{A}_{1,2}] = [A_{0,1,2}+\mu : A_{0,1,2}] \cup [A_{0,1,2}+\mu : A_{1,2}] \cup [A_{1,2} : A_{1,2}]_{n-2}, \tag{4.20}$$

$$[\tilde{A}_{0,1,2} : \tilde{A}_{1,2}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{0,1,2}+\mu : A_{0,1,2}]|_{\mathrm{Ker}\,A_{0,\nu}} & (\nu = 1,\ 2), \\ [A_{1,2} : A_{1,2}]|_{\mathrm{Ker}\,A_{0,\nu}} & (\nu = 3,\dots,n), \\ [A_{1,2} : A_{1,2}]|_{\mathrm{Ker}\,(A_{0,\infty}-\mu)} & (\nu = \infty), \end{cases} \tag{4.21}$$

which will be used in §7.

Since

$$A_{0,1,\infty} = A_{0,1} + A_{0,\infty} + A_{1,\infty} \ \text{ and } \ \tilde{A}_{0,1} = \tilde{A}_{0,1} + \tilde{A}_{0,\infty} + \tilde{A}_{1,\infty},$$

we have

$$\tilde{A}_{0,1,\infty} = \begin{pmatrix} A_{0,1,\infty}-\mu & & & & \\ & A_{1,\infty}-\mu-A_{0,2} & -A_{0,3} & \cdots & -A_{0,n} \\ & -A_{0,2} & A_{1,\infty}-\mu-A_{0,3} & \cdots & -A_{0,n} \\ & \vdots & \vdots & \ddots & \vdots \\ & -A_{0,2} & -A_{0,3} & \cdots & A_{1,\infty}-\mu-A_{0,n} \end{pmatrix},$$

$$\begin{aligned} \tilde{A}_{0,1,\infty}(v)_1 &= \big((A_{0,1,\infty}-\mu)v\big)_1, \\ \tilde{A}_{0,1,\infty}(v)_{2,\dots,n} &= \big((A_{0,1,\infty}-\mu)v\big)_{2,\dots,n}, \\ \tilde{A}_{0,1,\infty}(v)_\nu &\equiv ((A_{1,\infty}-\mu)v)_\nu \mod V_{2,\dots,n} \qquad (2 \le \nu \le n), \\ \tilde{A}_{0,1,\infty}(v)_\nu &= \big((A_{1,\infty}-\mu)v\big)_\nu \qquad (v \in \mathrm{Ker}\,A_{0,\nu},\ \nu = 2,\dots,n), \end{aligned}$$

and therefore

$$[\tilde{A}_{0,1,\infty}] = [A_{0,1,\infty}-\mu]_2 \cup [A_{1,\infty}-\mu]_{n-2},$$

$$[\tilde{A}_{0,1,\infty}|_{\mathcal{K}_\nu}] = \begin{cases} [(A_{0,1,\infty}-\mu)|_{\mathrm{Ker}\,A_{0,1}}] & (\nu = 1), \\ [(A_{1,\infty}-\mu)|_{\mathrm{Ker}\,A_{0,\nu}}] & (\nu = 2,\dots,n), \\ [(A_{0,1,\infty}-\mu)|_{\mathrm{Ker}\,(A_{0,\infty}-\mu)}] & (\nu = \infty), \end{cases}$$

$$[\tilde{A}_{0,1,\infty} : \tilde{A}_{0,1}] = [A_{0,1,\infty}-\mu : A_{0,1}+\mu] \cup [A_{0,1,\infty}-\mu : 0] \cup [A_{1,\infty}-\mu : 0]_{n-2},$$

$$[\tilde{A}_{0,1,\infty} : \tilde{A}_{0,1}]|_{\mathcal{K}_\nu} = \begin{cases} [(A_{0,1,\infty}-\mu)|_{\mathrm{Ker}\,A_{0,1}} : \mu] & (\nu = 1), \\ [(A_{1,\infty}-\mu)|_{\mathrm{Ker}\,A_{0,\nu}} : 0] & (\nu = 2,\dots,n), \\ [(A_{0,1,\infty}-\mu)|_{\mathrm{Ker}\,(A_{0,\infty}-\mu)} : 0] & (\nu = \infty), \end{cases}$$

$$[\tilde{A}_{0,1,\infty} : \tilde{A}_{0,\infty}] = [A_{0,1,\infty} - \mu : A_{0,\infty} - \mu] \cup [A_{0,1,\infty} - \mu : -\mu] \cup [A_{1,\infty} - \mu : -\mu]_{n-2},$$

$$[\tilde{A}_{0,1,\infty} : \tilde{A}_{0,\infty}]|_{\mathcal{K}_\nu} = \begin{cases} [(A_{0,1,\infty} - \mu)|_{\mathrm{Ker}\,A_{0,1}} : -\mu] & (\nu = 1), \\ [(A_{1,\infty} - \mu)|_{\mathrm{Ker}\,A_{0,\nu}} : -\mu] & (\nu = 2, \ldots, n), \\ [(A_{0,1,\infty} - \mu)|_{\mathrm{Ker}\,(A_{0,\infty}-\mu)} : 0] & (\nu = \infty). \end{cases}$$

The index 1 in the above may be changed into arbitrary number in $\{1, \ldots, n\}$.

Thus we have the following theorem.

**Theorem 4.1.** i) *If* $[A_{i,j} : A_{0,k}]$ *and* $[A_{0,i,j} : A_{0,i}]$ *of the KZ type equation* (2.1) *are known for all mutually different indices* $i$, $j$ *and* $k$ *in* $\{1, \ldots, n, \infty\}$, *their transformations under the middle convolution* $\mathrm{mc}_{x_0,\mu}$ *can be obtained by the calculation in this section if the eigenvalues of the reside matrices are generic.*

ii) *Choose* $i_0 \in \{1, \ldots, n, \infty\}$. *Then if* $[A_{i_0,j} : A_{0,k}]$, $[A_{0,i_0,j} : A_{0,i_0}]$ *and* $[A_{0,i_0,j} : A_{0,j}]$ *of the KZ type equation* (2.1) *are known for all mutually different indices* $j$ *and* $k$ *in* $\{1, \ldots, n, \infty\} \setminus \{i_0\}$, *their transformations under the middle convolution* $\mathrm{mc}_{x_0,\mu}$ *can be obtained by the calculation in this section if the eigenvalues of the reside matrices are generic.*

*Remark* 2. i) If $[A_{0,k}]$ of the KZ type equation (2.1) are known for $1 \le k \le n+1$, then their transformations under the middle convolution $\mathrm{mc}_{x_0,\mu}$ can be obtained by the calculation in this section if the eigenvalues are generic, which was given by [2].

ii) By our calculation we have

$$\tilde{A}_{0,\infty} + \sum_{i=1}^{n} \tilde{A}_{i,\infty} = \begin{pmatrix} \sum_{j=0}^{n} A_{i,\infty} - 2\mu & & \\ & \ddots & \\ & & \sum_{j=0}^{n} A_{j,\infty} - 2\mu \end{pmatrix}$$

and therefore if $A_{0,\ldots,n} = \kappa$ under identifying a scalar matrix with the scalar, then

$$\bar{\kappa} := \bar{A}_{0,\ldots,n} = \kappa + \mu. \tag{4.22}$$

## § 5. Rigid Fuchsian ordinary differential equations and KZ type equations

Consider a Fuchsian ordinary differential equation of Schlesinger canonical form. Suppose the equation is of rank $N$ and has $q+3$ singular points. By a fractional linear transformation we may assume that the singular points are $0, 1, \infty$ and $y_1, \ldots, y_q$. Then the equation is

$$\mathcal{N} : \frac{du}{dx} = \left( \frac{A_{x=0}}{x} + \frac{A_{x=1}}{x-1} + \frac{A_{x=y_1}}{x-y_1} + \cdots + \frac{A_{x=y_q}}{x-y_q} \right) u \tag{5.1}$$

with constant square matrices $A_{x=0}$, $A_{x=1}$, $A_{x=y_1}, \dots, A_{x=y_q}$ of size $N$. Here $u$ is a column vector of $N$ unknown functions of $x$. The matrix $A_{x=c}$ in the above is called the residue matrix of the equation at the singular point $x = c$ $(c \in \{0, 1, y_1, \dots, y_q, \infty\})$ and

$$A_{x=\infty} = -\bigl(A_{x=0} + A_{x=1} + A_{x=y_1} + \cdots + A_{x=y_q}\bigr). \tag{5.2}$$

We define that the set of *generalized exponents* of the equation at the singular point $x = c$ by the eigenvalue class $\{[\lambda_{c,1}]_{m_{c,1}}, \dots, [\lambda_{c,r_c}]_{m_{c,r_c}}\}$ of the matrix $A_{x=c}$ defined in §4. Note that this set of generalized exponents determines the conjugacy class of the matrix $A_{x=c}$. The number

$$\operatorname{idx}\mathcal{N} = \Biggl(\sum_{c\in\{0,1,y_1,\dots,y_m,\infty\}} \sum_{\nu=1}^{r_c} m_{c,\nu}^2\Biggr) - (m+1)N^2$$

is called the *index of rigidity* of $\mathcal{N}$, which is defined by Katz [4] and the $q+3$ partitions

$$N = m_{c,1} + \cdots + m_{c,r_c} \qquad (c \in \{0, 1, y_1, \dots, y_q, \infty\})$$

of $N$ is called the spectral type of $\mathcal{N}$, which can be expressed by

$$m_{0,1} \cdots m_{0,r_0}, \cdots, m_{\infty,1} \cdots m_{\infty,r_\infty}$$

if there is no confusion. Then the generalized Riemann scheme of $\mathcal{N}$ is

$$\begin{Bmatrix} x=0 & x=1 & \cdots & x=y_k & \cdots & x=\infty \\ [\lambda_{0,1}]_{m_{0,1}} & [\lambda_{1,1}]_{m_{1,1}} & \cdots & [\lambda_{y_k,1}]_{m_{k,1}} & \cdots & [\lambda_{\infty,1}]_{m_{\infty,1}} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ [\lambda_{0,r_0}]_{m_{0,r_0}} & [\lambda_{1,r_1}]_{m_{1,r_1}} & \cdots & [\lambda_{y_k,r_k}]_{m_{k,r_k}} & \cdots & [\lambda_{\infty,r_\infty}]_{m_{\infty,r_\infty}} \end{Bmatrix}. \tag{5.3}$$

When the generalized Riemann scheme is given, the dimension of the moduli space of the corresponding equations equals $2 - \operatorname{idx}\mathcal{N}$. Then if $\operatorname{idx}\mathcal{N} = 2$ and the equation is irreducible, namely, the space of its solutions has an irreducible monodromy group, the equation and its spectral type is called *rigid*. In this case, the equation and the monodromy group is uniquely determined by the generalized Riemann scheme.

By the correspondence of the spectral type and an element of a space spanned by a star-shaped Kac-Moody root system, which is introduced by [1], the existence of an irreducible equation with a given spectral type is characterized that the spectral type corresponds to a positive root of the system. In particular, the rigid spectral type correspond to the real root of the root system. Moreover for a given rigid spectral type and generic numbers $\lambda_{c,\nu} \in \mathbb{C}$ under the Fuchs condition

$$\sum_{c\in\{0,1,y_1,\dots,y_q,\infty\}} \sum_{\nu=1}^{r_c} m_{c,\nu}\lambda_{c,\nu} = 0, \tag{5.4}$$

there uniquely exists an irreducible equation $\mathcal{N}$ with the corresponding generalized Riemann scheme (5.3). Here we note that the Fuchs condition follows from (5.2). Moreover any rigid irreducible equation $\mathcal{N}$ can be obtained by a successive application of middle convolutions and additions from the trivial equation $\mathrm{d}u = 0$ with rank 1 and therefore $\mathcal{N}$ can be extended to a KZ type equation

(5.5)
$$\tilde{\mathcal{N}} : \begin{cases} \dfrac{\partial u}{\partial x} = \Big(\dfrac{A_{x=0}}{x} + \dfrac{A_{x=1}}{x-1} + \displaystyle\sum_{\nu=1}^{q} \dfrac{A_{x=y_\nu}}{x-y_\nu}\Big)u, \\ \dfrac{\partial u}{\partial y_i} = \Big(\dfrac{A_{i,q+1}}{y_i} + \dfrac{A_{i,q+2}}{y_i-1} + \dfrac{A_{x=y_i}}{y_i-x} + \displaystyle\sum_{\nu\in\{1,\dots,n\}\setminus\{i\}} \dfrac{A_{i,\nu}}{y_i-y_\nu}\Big)u \quad (i=1,\dots,q). \end{cases}$$

Putting

$$x = \frac{x_0 - x_{q+2}}{x_{q+1}-x_{q+2}} \text{ and } y_i = \frac{x_i - x_{q+2}}{x_{q+1}-x_{q+2}} \quad (i=1,\dots,q), \tag{5.6}$$

this equation is transformed into our equation (2.1) with

$$\begin{gathered} A_{0,i} = A_{x=y_i},\ A_{0,q+1} = A_{x=1},\ A_{0,q+2} = A_{x=0},\ n = q+2, \\ A_{q+1,q+2} = -\sum_{0\le i<j\le q} A_{i,j} - \sum_{i=1}^{q} \left(A_{i,q+1} + A_{i,q+2}\right). \end{gathered} \tag{5.7}$$

Note that $\kappa$ given by (2.7) equals 0 in this case. The solution of this KZ type equation has an integral representation since the middle convolution corresponds to an integral transformation given by a Riemann-Liouville integral and it is a hypergeometric function with (essentially) $q+1$ variables.

Thus we have a plenty of KZ type equations corresponding to rigid spectral types. The numbers of rigid spectral types with rank at most 15 are as follows.

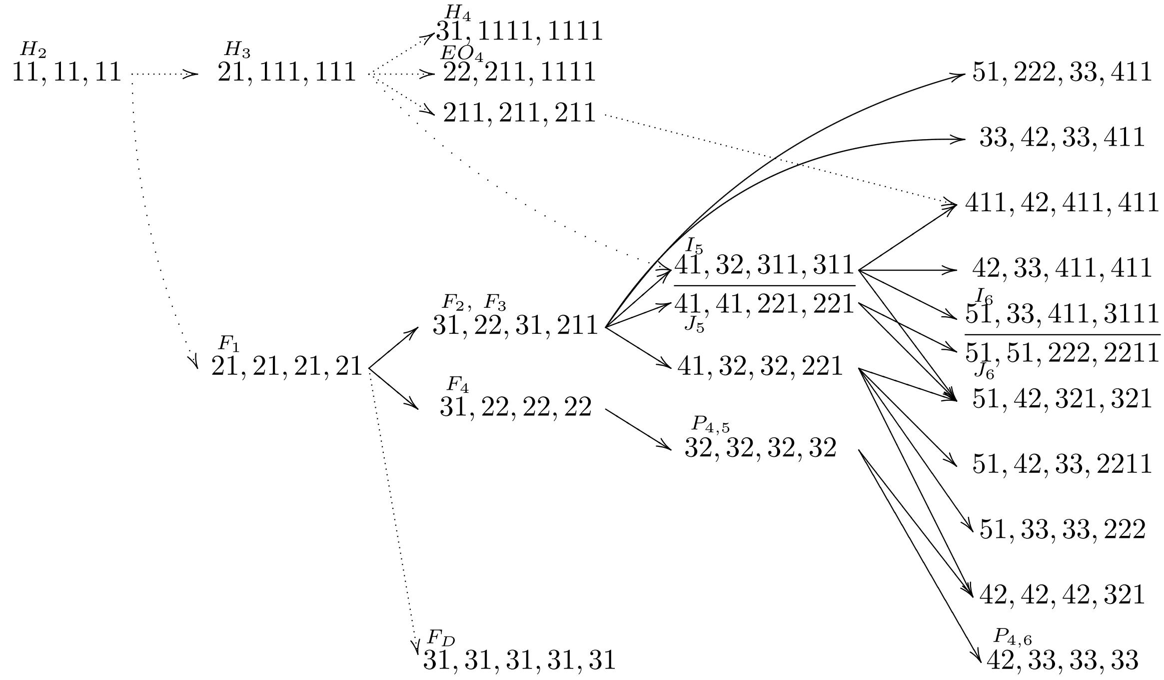
**The numbers of rigid spectral types with rank** $\le 15$

| $q+1$\rank | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 3 | 5 | 13 | 20 | 45 | 74 | 142 | 212 | 421 | 588 | 1004 | 1481 |
| 2 | | 1 | 2 | 4 | 11 | 16 | 35 | 58 | 109 | 156 | 299 | 402 | 685 | 924 |
| 3 | | | 1 | 1 | 3 | 5 | 12 | 17 | 43 | 52 | 104 | 135 | 263 | 327 |
| 4 | | | | 1 | 0 | 2 | 3 | 5 | 8 | 14 | 24 | 39 | 60 | 79 |
| 5 | | | | | 1 | 0 | 0 | 2 | 3 | 4 | 6 | 6 | 14 | 20 |

There are 9 rigid spectral types whose order are smaller than 5 and larger than 1. The corresponding hypergeometric functions are Gauss hypergeometric function, generalized hypergeometric functions expressed by ${}_3F_2$ and ${}_4F_3$, a function belonging to the even family, another function corresponding to $211, 211, 211$ with a single variable and Appell's hypergeometric functions $F_j$ $(j = 1, 2, 3, 4)$ with two variables and Lauricella's $F_D$ with three variables.

Some of the spectral types are in the following figure.

**Hierarchy of rigid quartets** (cf. [5])



## § 6. A symmetry of KZ type equations

We define a KZ type equation is *homogeneous* if $\kappa$ defined by (2.7) equals 0. By the addition $\mathrm{Ad}\big((x_{n-1}-x_n)^{-\kappa}\big)$ which corresponds to the transformation of $u$ into $(x_{n-1}-x_n)^{\kappa}u$, the KZ equation is transformed to homogeneous.

If $\sigma$ is an element of transposition of the set of indices $\{0,1,\dots,n\}$, then the transformations of $A_{i,j}$ into $A_{\sigma(i),\sigma(j)}$ for $0 \le i < j \le n+1$ defines a transformation in homogeneous KZ equations. This group of transformations is isomorphic to the symmetric group $S_{n+2}$ whose number of elements is $(n+2)!$. The group is generated by $\sigma_0, \dots, \sigma_n$, where $\sigma_i = \sigma_{i,i+1}$ and $\sigma_{i,j}$ corresponds to the transposition of indices $i$ and $j$.

We realize these transformations in the equation (5.5) which is obtained from (2.1) by putting $x_n = 0$ and $x_{n-1} = 1$. Then the transposition of $y_i$ and $y_{i+1}$ corresponds to $\sigma_i$ for $1 \le i \le n-3$, the transposition of $x$ and $y$ corresponds to $\sigma_0$ and the map $(x, y_1, \dots, y_q) \mapsto (1-x, 1-y_1, \dots, 1-y_q)$ and the map $(x, y_1, \dots, y_q) \mapsto (\frac{1}{x}, \frac{1}{y_1}, \dots, \frac{1}{y_q})$ correspond to $\sigma_{n-1}$ and $\sigma_n$, respectively. Note that $q = n-2$ and $\{\sigma_i, \sigma_{0,n-1} \mid i \in \{0, \dots, n-3, n-1, n\}\}$ is a set of generators of the group and the following involution corresponds to $\sigma_{0,n-1}$. Then the element of the group corresponds to a coordinate transformation of the variable $(x, y_1, \dots, y_q)$.

Under the involution $\sigma : (x, y_1, \dots, y_q) \mapsto (X, Y_1, \dots, Y_q) = (\frac{1}{x}, \frac{y_1}{x}, \dots, \frac{y_q}{x})$, we have

$$\begin{aligned}
\frac{dX}{X} &= \frac{d(\frac{1}{x})}{\frac{1}{x}} = x(-\frac{1}{x^2}dx) = -\frac{dx}{x},\\
\frac{d(X-1)}{X-1} &= \frac{d(\frac{1}{x}-1)}{\frac{1}{x}-1} = \frac{-\frac{dx}{x}}{1-x} = \frac{dx}{x(x-1)} = \frac{d(x-1)}{x-1} - \frac{dx}{x},\\
\frac{dY_i}{Y_i} &= \frac{d(\frac{y_i}{x})}{\frac{y_i}{x}} = \frac{\frac{dy_i}{x} - \frac{y_i dx}{x^2}}{\frac{x}{y_i}} = \frac{dy_i}{y_i} - \frac{dx}{x},\\
\frac{d(Y_i-1)}{Y_i-1} &= \frac{\frac{dy_i}{x} - \frac{y_i dx}{x^2}}{\frac{y_i-x}{x}} = \frac{dy_i - \frac{y_i dx}{x}}{y_i - x} = \frac{d(y_i - x) - \frac{(y_i-x)dx}{x}}{y_i - x} = \frac{d(x-y_i)}{x-y_i} - \frac{dx}{x},\\
\frac{d(X-Y_i)}{X-Y_i} &= \frac{-\frac{dx}{x^2} - \frac{dy_i}{x} + \frac{y_i dx}{x^2}}{\frac{1}{x} - \frac{y_i}{x}} = \frac{(y_i-1)dx}{x(1-y_i)} + \frac{dy_i}{y_i-1} = -\frac{dx}{x} + \frac{d(y_i-1)}{y_i-1}.
\end{aligned}$$

Putting

$$\begin{aligned}
&A_{X=0}\frac{dX}{X} + A_{X=1}\frac{d(X-1)}{X-1} + \sum_{i=1}^{q}\left(A_{X=Y_i}\frac{d(X-Y_i)}{X-Y_i} + A_{Y_i=0}\frac{dY_i}{Y} + A_{Y_i=1}\frac{d(Y_i-1)}{Y_i-1}\right)\\
&= A_{x=0}\frac{dx}{x} + A_{x=1}\frac{d(x-1)}{x-1} + \sum_{i=1}^{q}\left(A_{x=y_i}\frac{d(x-y_i)}{x-y_i} + A_{y_i=0}\frac{dy_i}{y_i} + A_{y_i=1}\frac{d(y_i-1)}{y_i-1}\right),
\end{aligned}$$

we have

$$\begin{aligned}
&A_{X=0} = -A_{x=0} - A_{x=1} - \sum_{i=1}^{q}\left(A_{x=y_i} + A_{y_i=0} + A_{y_i=1}\right),\\
&A_{X=1} = A_{x=1}, \quad A_{Y_i=0} = A_{y_i=0},\\
&A_{Y_i=1} = A_{x=y_i}, \quad A_{X=Y_i} = A_{y_i=1}.
\end{aligned}$$

Hence it follows from (5.7) that this involution corresponds to the transposition of $x_0$ and $x_{n-1}$ of the KZ type equation (2.1).

## § 7. Systems of hypergeometric equations with two variables

In this section we examine the KZ equations corresponding to the hypergeometric functions with two variables, such as Appell's hypergeometric functions. Then the equation is (2.1) with $n = 3$ or (5.5) with $q = 1$. Then the group of symmetry defined by coordinate transformations is of order 120.

To get the conjugacy classes of the residue matrices under the middle convolution $\mathrm{mc}_{x_0,\mu}$ it is sufficient to know $[A_{i,j} : A_{0,k}]$, $[A_{i,j} : A_{0,4}]$, $[A_{0,i,j} : A_{0,i}]$, $[A_{0,i,4} : A_{0,i}]$ and

$[A_{0,i,4} : A_{0,4}]$ for any $(i,j,k)$ satisfying $\{i,j,k\} = \{1,2,3\}$ (cf. Theorem 4.1). By the relation (2.9) we have $A_{0,i,j} = A_{k,4} + \kappa$ and $A_{0,i,4} = A_{j,k} - \kappa$ and therefore

$$\begin{aligned}&[A_{0,i,4} : A_{0,i}] = [A_{j,k} - \kappa : A_{0,i}], \quad [A_{0,i,4} : A_{0,4}] = [A_{j,k} - \kappa : A_{0,4}],\\ &[A_{0,i,j} : A_{0,i}] = [A_{k,4} + \kappa : A_{0,i}], \quad [A_{0,i,j} : A_{i,j}] = [A_{k,4} + \kappa : A_{i,j}].\end{aligned}$$

Hence the calculation in §3 gives the transformation of simultaneous eigenvalue classes $[A_I : A_J]$ for all $I$, $J \subset \{0,1,2,3,4\}$ satisfying $\#I = \#J = 2$ and $I \cap J = \emptyset$ and therefore we can also get the transformation of them under the middle convolution $\mathrm{mc}_{x_j,\mu}$ for $0 \le j \le 3$. Here we note that there are 15 simultaneous eigenvalue classes and the calculation in §3 shows that their transformation under $\mathrm{mc}_{x_0,\mu}$ are obtained by the following:

**Theorem 7.1.** *Retain the above notation. For $\{i,j,k\} = \{1,2,3\}$ we have*

$$[\tilde{A}_{i,j} : \tilde{A}_{0,k}] = [A_{i,j} : A_{0,k} + \mu] \cup [A_{i,j} : 0] \cup [A_{k,4} + \kappa : 0],$$

$$[\tilde{A}_{i,j} : \tilde{A}_{0,k}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{k,4} + \kappa : 0]|_{\mathrm{Ker}\, A_{0,\nu}} & (\nu = i,\, j),\\ [A_{i,j} : \mu]|_{\mathrm{Ker}\, A_{0,k}} & (\nu = k),\\ [A_{i,j} : 0]|_{\mathrm{Ker}\, (A_{0,4} - \mu)} & (\nu = 4),\end{cases}$$

$$[\tilde{A}_{i,4} : \tilde{A}_{0,k}] = [A_{i,4} : A_{0,k} + \mu] \cup [A_{i,4} : 0] \cup [A_{j,k} - \kappa - \mu : 0],$$

$$[\tilde{A}_{i,4} : \tilde{A}_{0,k}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{j,k} - \kappa - \mu : 0]_{\mathrm{Ker}\, A_{0,i}} & (\nu = i),\\ [A_{i,4} : 0]|_{\mathrm{Ker}\, A_{0,j}} & (\nu = j),\\ [A_{i,4} : \mu]|_{\mathrm{Ker}\, A_{0,k}} & (\nu = k),\\ [A_{j,k} - \kappa - \mu : 0]|_{\mathrm{Ker}\, (A_{0,4} - \mu)} & (\nu = 4),\end{cases}$$

$$[\tilde{A}_{i,j} : \tilde{A}_{0,4}] = [A_{i,j} : A_{0,4} - \mu] \cup [A_{i,j} : -\mu] \cup [A_{k,4} + \kappa : -\mu],$$

$$[\tilde{A}_{i,j} : \tilde{A}_{0,4}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{k,4} + \kappa : -\mu]|_{\mathrm{Ker}\, A_{0,\nu}} & (\nu = i,\, j),\\ [A_{i,j} : -\mu]|_{\mathrm{Ker}\, A_{0,k}} & (\nu = k),\\ [A_{i,j} : 0]|_{\mathrm{Ker}\, (A_{0,4} - \mu)} & (\nu = 4),\end{cases}$$

$$[\tilde{A}_{i,j} : \tilde{A}_{k,4}] = [A_{i,j} : A_{i,j} - \kappa - \mu] \cup [A_{i,j} : A_{k,4}] \cup [A_{k,4} + \kappa : A_{k,4}],$$

$$[\tilde{A}_{i,j} : \tilde{A}_{k,4}]|_{\mathcal{K}_\nu} = \begin{cases} [A_{k,4} + \kappa : A_{k,4}]|_{\mathrm{Ker}\, A_{0,\nu}} & (\nu = i,\, j),\\ [A_{i,j} : A_{i,j} - \kappa - \mu]|_{\mathrm{Ker}\, A_{0,k}} & (\nu = k),\\ [A_{i,j} : A_{i,j} - \kappa - \mu]|_{\mathrm{Ker}\, (A_{0,4} - \mu)} & (\nu = 4).\end{cases}$$

*Remark* 3. i) When $n = 3$, the above theorem describes the transformation of equivalence classes of local monodromy of the KZ equation under the successive operation of additions, middle convolutions and transformations by its symmetry in §6.

ii) When $n > 3$, we require more simultaneous conjugacy classes, which we will study in another paper. When $n = 4$, we require simultaneous eigenvalue classes

$$[A_{I_2^{(1)}} : A_{I_2^{(2)}} : A_{I_3^{(3)}}] \quad \text{and} \quad [A_{I_2} : A_{I_3} : A_{I_4}] \tag{7.1}$$

for $\#I_2^{(\nu)} = 2,\ \ \#I_j = j,\ \ I_2^{(1)} \cup I_2^{(2)} \cup I_2^{(3)} = \{0, \dots, 5\}$ and $I_2 \subset I_3 \subset I_4 \subset \{0, \dots, 5\}$.

## § 8. Examples

Applying $\mathrm{Ad}\big(\prod_{\nu=1}^{3}(x_0 - x_\nu)^{\lambda_\nu}\big)$ to the trivial equation, we get a KZ type equation (2.1) of rank 1 with

$$A_{0,i} = \lambda_i, \quad A_{0,4} = -\lambda_{123}, \quad A_{i,j} = 0, \quad A_{i,4} = -\lambda_i, \quad \kappa = \lambda_{123}$$

for $\{i, j, k\} = \{1, 2, 3\}$. Here $\lambda_{ij} = \lambda_i + \lambda_j$ and $\lambda_{123} = \lambda_1 + \lambda_2 + \lambda_3$.

Then we apply $\mathrm{mc}_{x_0,\mu}$ with $\mu \in \mathbb{C}$ to this equation and we get a KZ type equation corresponding to Appell's $F_1$. In this case $\mathcal{K} = 0$ and the residue matrices satisfy

$$\begin{aligned} &[A_{0,i}] = \big\{\lambda_i + \mu,\ [0]_2\big\}, && [A_{0,4}] = \big\{-\lambda_{123} - \mu,\ [-\mu]_2\big\},\\ &[A_{i,j}] = \big\{\lambda_{ij},\ [0]_2\big\}, && [A_{i,4}] = \big\{-\lambda_{123} - \mu,\ [-\lambda_i]_2\big\}. \end{aligned} \tag{8.1}$$

The spectral type of the ordinary differential equation equation with the variable $x_0$ is $21, 21, 21, 21$ and the simultaneous conjugacy classes obtained by Theorem 4.1 are

$$\begin{aligned} [A_{i,j} : A_{0,k}] &= \big\{[0 : \lambda_k + \mu],\ [0 : 0],\ [-\lambda_{ij} : 0]\big\},\\ [A_{i,4} : A_{0,k}] &= \big\{[-\lambda_i : \lambda_k + \mu],\ [-\lambda_i : 0],\ [-\lambda_{123} - \mu : 0]\big\},\\ [A_{i,j} : A_{0,4}] &= \big\{[0 : -\lambda_{123} - \mu],\ [0 : -\mu],\ [-\lambda_{ij} : -\mu]\big\},\\ [A_{i,j} : A_{k,4}] &= \big\{[0 : -\lambda_{123} - \mu],\ [0 : -\lambda_k],\ [-\lambda_{ij} : -\lambda_k]\big\}. \end{aligned} \tag{8.2}$$

Putting

$$a_0 = -\mu,\ a_j = \lambda_j + \mu \ \text{ and } \ a_{ij} = a_i + a_j \quad (0 \le i < j \le 3),$$

we have

$$A_{0,1} = \begin{pmatrix} a_1 & a_{02} & a_{03} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_{0,2} = \begin{pmatrix} 0 & 0 & 0 \\ a_{01} & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_{0,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{01} & a_{02} & a_3 \end{pmatrix},$$

$$A_{1,2} = \begin{pmatrix} a_{02} & -a_{02} & 0 \\ -a_{01} & a_{01} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{03} & -a_{03} \\ 0 & -a_{02} & a_{02} \end{pmatrix}, \qquad A_{1,3} = \begin{pmatrix} a_{03} & 0 & -a_{03} \\ 0 & 0 & 0 \\ -a_{01} & 0 & a_{01} \end{pmatrix}.$$

Next apply $\mathrm{Ad}\big((x_0-x_1)^{-\lambda_1-\mu}\big)$ to the above equation. Then for $i=2,\,3$ and $\{p,q\}\subset\{1,2,3\}$, the residue matrices of the resulting KZ type equation satisfy

$$\begin{aligned}
&[A_{0,1}] = \{0,\ [-\lambda_1-\mu]_2\}, \quad &&[A_{0,i}] = \{\lambda_i+\mu,\ [0]_2\}, \quad &&[A_{0,4}] = \{-\lambda_{23},\ [\lambda_1]_2\},\\
&[A_{p,q}] = \{\lambda_{pq},\ [0]_2\}, &&[A_{1,4}] = \{-\lambda_{23},\ [\mu]_2\}, &&[A_{i,4}] = \{-\lambda_{123}-\mu,\ [-\lambda_i]_2\}
\end{aligned}$$

and

$$\begin{aligned}
[A_{i,j}:A_{0,1}] &= \big\{[0:0],\ [0:-\lambda_1-\mu],\ [-\lambda_{ij}:-\lambda_1-\mu]\big\},\\
[A_{i,4}:A_{0,1}] &= \big\{[-\lambda_i:0],\ [-\lambda_i:-\lambda_1-\mu],\ [-\lambda_{123}-\mu:-\lambda_1-\mu]\big\},\\
[A_{i,j}:A_{0,4}] &= \big\{[0:-\lambda_{23}],\ [0:\lambda_1],\ [-\lambda_{ij}:\lambda_1]\big\},\\
[A_{i,j}:A_{1,4}] &= \big\{[0:-\lambda_{23}],\ [0:\mu],\ [-\lambda_{ij}:\mu]\big\}.
\end{aligned}$$

and other simultaneous conjugacy classes are same as in (8.2).

Applying $\widetilde{\mathrm{mc}}_{x_0,\tau}$ and $\mathrm{mc}_{x_0,\tau}$ to the above, we have the resulting residue matrices with

$$\begin{aligned}
&[\tilde A_{0,1}] = \big\{\tau,\ [-\lambda_1-\mu+\tau]_2,\ [0]_6\big\}, &&[\tilde A_{0,i}] = \big\{\lambda_i+\mu+\tau,\ [\tau]_2,\ [0]_6\big\},\\
&[\tilde A_{0,4}] = \big\{\lambda_{23}-\tau,\ [\lambda_1-\tau]_2,\ [-\tau]_6\big\}, &&[\tilde A_{1,i}] = \big\{[\lambda_{1i}]_2, [0]_4, -\lambda_1-\mu, [\lambda_i]_2\big\},\\
&[\tilde A_{2,3}] = \big\{[\lambda_{23}]_2,\ [0]_5,\ [\lambda_{23}+\mu]_2\big\}, &&[\tilde A_{1,4}] = \big\{[-\lambda_{23}]_2,\ [\mu]_4,\ -\tau,\ [-\lambda_{23}-\tau]_2\big\},\\
&[\tilde A_{i,4}] = \big\{[-\lambda_{123}-\mu]_2,\ [-\lambda_i]_4,\ \lambda_1-\lambda_i-\tau,\ [-\lambda_{23}-\tau]_2\big\}, \quad &&\bar\kappa = \lambda_{23}+\tau
\end{aligned}$$

and

$$\begin{aligned}
&\dim\mathcal{K}_1 = 1,\quad \dim\mathcal{K}_2 = \dim\mathcal{K}_3 = 2,\quad \dim\mathcal{K}_4 = 0,\\
&\quad[\bar A_{0,1}] = \big\{[-\lambda_1-\mu+\tau]_2,\ [0]_2\big\},\quad [\bar A_{0,i}] = \big\{\lambda_i+\mu+\tau,\ [0]_3\big\}\quad (i=2,\,3),\\
&\quad[\bar A_{0,4}] = \big\{-\lambda_{23}-\tau,\ [\lambda_1-\tau]_2,\ -\tau\big\}.
\end{aligned}\tag{8.3}$$

Hence under the transformation $\mathrm{mc}_{x_0,\tau}$, the spectral type of the resulting ordinary differential equation with the variable $x_0$ is $22, 31, 31, 211$. Moreover owing to Theorem 4.1, we have

$$\begin{aligned}
&\tilde A_{1,i}|_{\mathcal{K}_1} = \lambda_i,\quad \tilde A_{2,3}|_{\mathcal{K}_1} = 0,\quad \tilde A_{1,4}|_{\mathcal{K}_1} = -\lambda_{23}-\tau,\quad \tilde A_{i,4}|_{\mathcal{K}_1} = -\lambda_i,\\
&[\tilde A_{1,i}|_{\mathcal{K}_i}] = \big\{-\lambda_1-\mu,\ \lambda_i\big\}, &&[\tilde A_{1,i}|_{\mathcal{K}_j}] = \big\{\lambda_{1i},\ 0\big\},\\
&[\tilde A_{2,3}|_{\mathcal{K}_i}] = \big\{0,\ \lambda_{23}+\mu\big\}, &&[\tilde A_{1,4}|_{\mathcal{K}_i}] = \big\{-\lambda_{23},\ \mu\big\},\\
&[\tilde A_{i,4}|_{\mathcal{K}_i}] = \big\{\lambda_1-\lambda_i-\tau,\ -\lambda_{23}-\tau\big\}, &&[\tilde A_{i,4}|_{\mathcal{K}_j}] = \big\{-\lambda_{123}-\mu,\ -\lambda_i\big\}
\end{aligned}$$

for $\{i,\,j\} = \{2,\,3\}$ and therefore we get

$$\begin{aligned}
&[\bar A_{1,i}] = \big\{\lambda_{1i},\ [0]_3\big\}, &&[\bar A_{2,3}] = \big\{[\lambda_{23}]_2,\ [0]_2\big\},\\
&[\bar A_{1,4}] = \big\{[\mu]_2,\ -\tau,\ -\lambda_{23}-\tau\big\}, \quad &&[\bar A_{i,4}] = \big\{-\lambda_{123}-\mu,\ [-\lambda_i]_2,\ -\lambda_{23}-\tau\big\}.
\end{aligned}\tag{8.4}$$

In fact, for example,

$$\begin{aligned}[\bar{A}_{1,2}] &= [\tilde{A}_{1,2}] \setminus \bigcup_{i=1}^{4} [\tilde{A}_{1,2}|_{\mathcal{K}_i}] \\ &= \big\{[\lambda_{12}]_2,\, [0]_4,\, -\lambda_1-\mu,\, [\lambda_1]_2\big\} \setminus \big(\{\lambda_1\} \cup \{-\lambda_1-\mu,\, \lambda_1\} \cup \{\lambda_{12},\, 0\}\big) \\ &= \big\{\lambda_{12},\, [0]_3\big\}.\end{aligned}$$

Putting

$$a_1 = \lambda_1 - \mu + \tau,\ a_2 = \lambda_2 + \mu + \tau,\ a_3 = \lambda_3 + \mu + \tau,\ a_4 = -\tau,\ a_5 = -\mu - \tau$$

and $a_{ij} = a_i + a_j$, we have

$$\bar{A}_{0,1} = \begin{pmatrix} a_1 & 0 & a_{14} & 0 \\ 0 & a_1 & 0 & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{A}_{1,2} = \begin{pmatrix} a_{24} & a_{25} & -a_{14} & 0 \\ 0 & 0 & 0 & 0 \\ -a_{24} & -a_{25} & a_{14} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{A}_{1,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{35} & a_{34} & 0 & -a_{14} \\ 0 & 0 & 0 & 0 \\ -a_{35} & -a_{34} & 0 & a_{14} \end{pmatrix},$$

$$\bar{A}_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{24} & a_{25} & a_2 & a_{25} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{A}_{0,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{35} & a_{34} & a_{35} & a_3 \end{pmatrix}, \quad \bar{A}_{2,3} = \begin{pmatrix} a_{35} & -a_{25} & 0 & 0 \\ -a_{35} & a_{25} & 0 & 0 \\ 0 & 0 & a_{35} & -a_{25} \\ 0 & 0 & -a_{35} & a_{25} \end{pmatrix}.$$

In this case the resulting KZ type equation corresponds to Appell's $F_2$ and $F_3$.

*Remark* 4. The calculation based on Theorem 4.1 and Theorem 7.1 is implemented in a library [8] of a computer algebra `Risa/Asir`.

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