RIMS Kôkyûroku Bessatsu
B61 (2017), 163–180

On parametric Stokes phenomena of higher order linear ODEs

Dedicated to Professor Takahiro Kawai on the occasion of his 70th birthday.

By

Shinji Sasaki*

Abstract

We study an example of parametric Stokes phenomena of higher order linear ODEs with a large parameter, taking (1,4)-hypergeometric equation as a concrete example. Parametric Stokes phenomena occur when degenerate configuration of Stokes curves appears and degenerate configuration of Stokes curves of higher order equations is described by bidirectional binary trees (which we call "Stokes trees" for short) introduced by Honda. In this paper we investigate the situation where a Stokes tree with three edges appears and present an explicit formula describing the parametric Stokes phenomenon in this situation.

§ 1. Introduction and result

In this paper, we study parametric Stokes phenomena of higher order linear ODEs with a large parameter, taking $(1,4)$-hypergeometric equation as a concrete example.

The equation we consider is a linear ODE with a large parameter η of the following form

$$\left[\frac{d^n}{dx^n} + a_1(x)\eta\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x)\eta^n\right]\psi = 0. \tag{1.1}$$

This equation has a WKB solution

$$\psi = \exp\left(\int^x S\,dx\right), \tag{1.2}$$

Received July 31, 2016. Revised October 16, 2016. Accepted October 17, 2016.
2010 Mathematics Subject Classification(s): 34M40, 34M60, 34E05.
Key Words: *exact WKB analysis, parametric Stokes phenomenon, new Stokes curve.*
*Kindai University, Higashi-Osaka, Osaka 577-8502, Japan.
e-mail: `sasaki@kurims.kyoto-u.ac.jp`

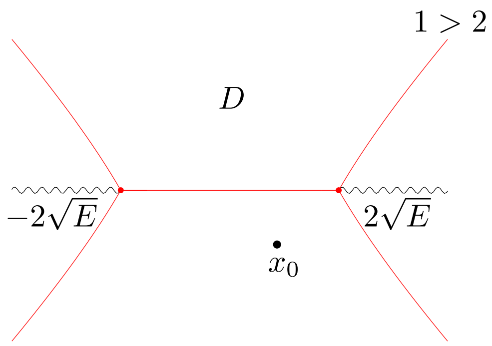


Figure 1. Stokes geometry of the Weber equation with $E > 0$.

with $S = S(x, \eta) = S_{-1}(x)\eta + S_0(x) + S_1(x)\eta^{-1} + \cdots$, where S_{-1} is a characteristic root $\xi_j(x)$ $(j = 1, 2, \dots, n)$ satisfying the characteristic equation

$$\xi^n + a_1\xi^{n-1} + \cdots + a_n = 0. \tag{1.3}$$

Once we fix S_{-1}, then S_0, $S_1, \dots$ are determined uniquely. We denote S with $S_{-1} = \xi_j(x)$ by $S^{(j)}$. An (ordinary) turning point of type (k, l) is a point t at which $\xi_k(t) = \xi_l(t)$ holds, and an (ordinary) Stokes curve of type (k, l) is a curve defined by $\operatorname{Im} \int_t^x (\xi_k - \xi_l)\, dx = 0$, where t is a turning point of type (k, l). Further we say that the Stokes curve is of type $k > l$ if $\operatorname{Re} \int_t^x (\xi_k - \xi_l)\, dx > 0$ holds on it.

In the case $n = 2$, if the Stokes geometry is degenerate, i.e., if there is a Stokes curve connecting a pair of turning points (such a Stokes curve is sometimes called a Stokes segment), a Stokes phenomenon occurs with WKB solutions according as a parameter moves.

Example 1.1 (Weber equation)**.** The Weber equation with a large parameter η

$$\left[\frac{d^2}{dx^2} - \eta^2\left(E - \frac{x^2}{4}\right)\right]\psi = 0 \tag{1.4}$$

has a Stokes segment if $E > 0$ (Figure 1). This degeneration is resolved by adding a small imaginary part to the parameter E. If $\operatorname{Im} E < 0$, we have Figure 2, and if $\operatorname{Im} E > 0$, Figure 3. Related to this change of geometry, a Stokes phenomenon occurs with WKB solutions; Let a point x_0 and a domain D be as in Figure 1. We place branch cuts (wavy lines) and fix the branch of ξ_j's so that the dominance relations are given as in Figure 1. Define $\psi_j := \exp \int_{x_0}^x S^{(j)}\, dx$ and denote by $\Psi_j^{(\pm)}$ the Borel sum of ψ_j in D for $\pm \operatorname{Im} E > 0$ $(j = 1, 2)$. Then, after the analytic continuation to a neighborhood of $\operatorname{Im} E = 0$, they satisfy the following relation:

$$\begin{cases} \Psi_1^{(-)} &= \left[1 - \exp\left(\int_{\gamma_0} S\, dx\right)\right] \Psi_1^{(+)} \\ \Psi_2^{(-)} &= \left[1 - \exp\left(\int_{\gamma_0} S\, dx\right)\right]^{-1} \Psi_2^{(+)} \end{cases} \tag{1.5}$$

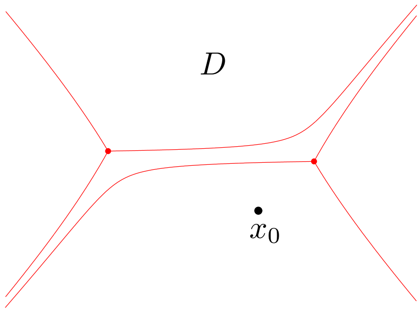


Figure 2. Stokes geometry of the Weber equation with $\mathrm{Re}\,E > 0,\ \mathrm{Im}\,E < 0$.

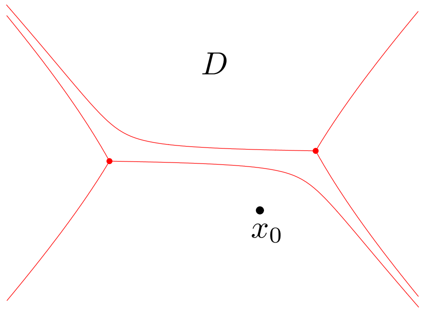


Figure 3. Stokes geometry of the Weber equation with $\mathrm{Re}\,E > 0,\ \mathrm{Im}\,E > 0$.

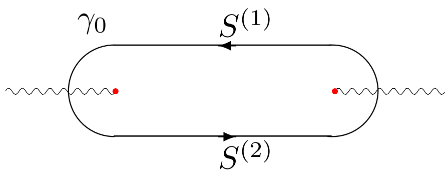


Figure 4. Path γ_0 and the branch of S on it.

(cf. [14], [5], [13]), where the path γ_0 and the branch of S on it are given in Figure 4. The branch of S is $S^{(1)}$ on the upper-half part and $S^{(2)}$ on the lower-half part. One important point here is that such a parametric Stokes phenomenon occurs with a WKB solution whose path of integration crosses a Stokes segment in question. As a matter of fact, in the case of (1.5) a path of integration for ψ_j is a path connecting x_0 and x in the region D, which crosses a Stokes segment $[-2\sqrt{E}, 2\sqrt{E}]$. On the other hand, if a path of integration does not cross a Stokes segment, no parametric Stokes phenomenon occurs. For example, $\exp \int_{\gamma_0} S dx$, which appears in Formula (1.5), is known to be Borel summable near the positive real axis $\{E \in \mathbb{C} \mid \mathrm{Im}\,E = 0, \mathrm{Re}\,E > 0\}$ and no Stokes phenomenon occurs with it.

Remark. In many cases, the formula (1.5) is represented in terms of $S_{\mathrm{odd}} := (S^{(1)} - S^{(2)})/2$:

$$
\begin{cases}
\Psi_1^{(-)} = \left[1 + \exp\left(-2\int_{-2\sqrt{E}}^{2\sqrt{E}} S_{\mathrm{odd}}\, dx\right)\right] \Psi_1^{(+)} \\
\Psi_2^{(-)} = \left[1 + \exp\left(-2\int_{-2\sqrt{E}}^{2\sqrt{E}} S_{\mathrm{odd}}\, dx\right)\right]^{-1} \Psi_2^{(+)}
\end{cases}. \tag{1.6}
$$

Here, $2\int_{-2\sqrt{E}}^{2\sqrt{E}} S_{\mathrm{odd}}\, dx$ is interpreted as the integral along γ_0 with the integrand being $(S^{(1)} - S^{(2)})/2$ on the lower-half part and $(S^{(2)} - S^{(1)})/2$ on the upper-half part. The relation between (1.6) and (1.5) is discussed in A.1.

When discussing higher order ($n \geq 3$) equations, we need the so-called new Stokes curves ([4]) which emanate from virtual turning points ([2], [10]). Due to this, we encounter more complicated degeneration.

Example 1.2.

$$\left(\frac{d^3}{dx^3} + \frac{2}{3}c\eta\frac{d^2}{dx^2} + \frac{1}{3}x\eta^2\frac{d}{dx} - \frac{\alpha}{3}\eta^3\right)\psi = 0, \tag{1.7}$$

where c and α are complex constants. This is the restriction of the $(1,4)$-hypergeometric system ([12])

$$\begin{cases} \left(\dfrac{\partial^3}{\partial x_1^3} + \dfrac{2}{3}x_2\eta\dfrac{\partial^2}{\partial x_1^2} + \dfrac{1}{3}x_1\eta^2\dfrac{\partial}{\partial x_1} - \dfrac{\alpha}{3}\eta^3\right)\psi = 0 \\ \left(\eta\dfrac{\partial}{\partial x_2} - \dfrac{\partial^2}{\partial x_1^2}\right)\psi = 0 \end{cases} \tag{1.8}$$

to the first variable $x_1 (=: x)$. As is discussed in [7], we need several new Stokes curves emanating from virtual turning points to obtain its complete Stokes geometry. For example, when $c = 3$ and $\alpha = (100/9) - 6i$, the Stokes geometry of (1.7) is given in Figure 5, where two virtual turning points and new Stokes curves emanating from them appear. (An inert portion of a new Stokes curve is drawn by a dotted line there.) All WKB solutions of (1.7) are expected to be Borel summable except on these Stokes curves including (active portions of) new Stokes curves.

Now, if we change the parameter α, we encounter the following degenerate configuration of Stokes curves peculiar to higher order equations: Let $c = 0$ and $\alpha \in i\mathbb{R}_{<0}$. Then the Stokes geometry of (1.7) become the one given in Figure 6. We see that three ordinary simple turning points are connected with three virtual turning points by Stokes curves simultaneously. These three Stokes curves connecting simple turning points and virtual turning points cross at one point, and the three solid portions of Stokes curves between simple turning points and the unique crossing point form a tree consisting of three edges. This is an example of "bidirectional binary trees" introduced by Honda in his study of linear differential equations associated with Noumi-Yamada systems ([8], cf. [10, Chapter 2] also). In this paper we call it a "Stokes tree" for short. The appearance of Stokes trees can be considered as degeneration of Stokes curves peculiar to higher order equations.

Remark. A Stokes segment can be regarded as a special case of Stokes trees. If we regard a Stokes segment as the simplest example of Stokes trees, the Stokes tree

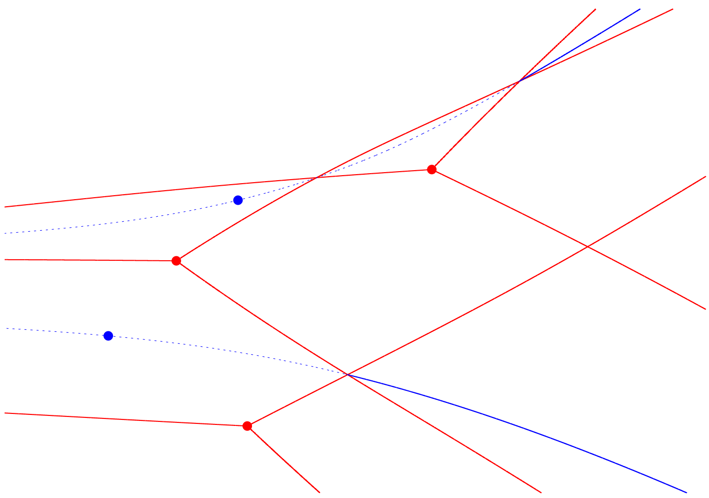


Figure 5. Stokes geometry of (1.7) with $c = 3$, $\alpha = (100/9) - 6i$. This is equivalent to that of [2, Example 2.5] by change of variables $\psi = e^{-2cx\eta/9}\varphi$, $x = -18z - 14$.

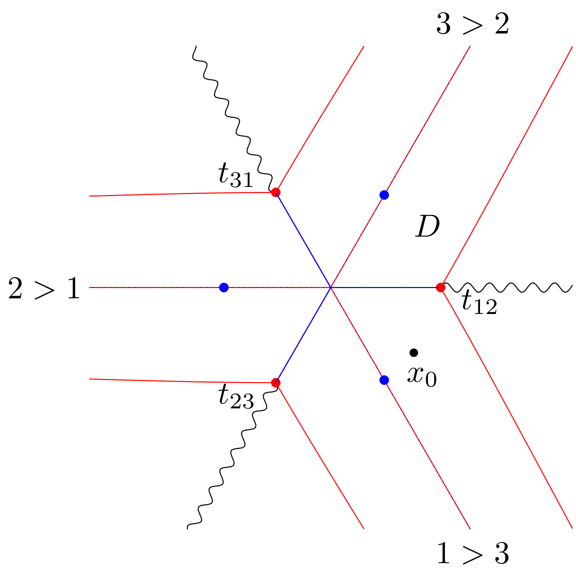


Figure 6. Stokes geometry of Equation (1.7) with $\alpha \in i\mathbb{R}_{<0}$.

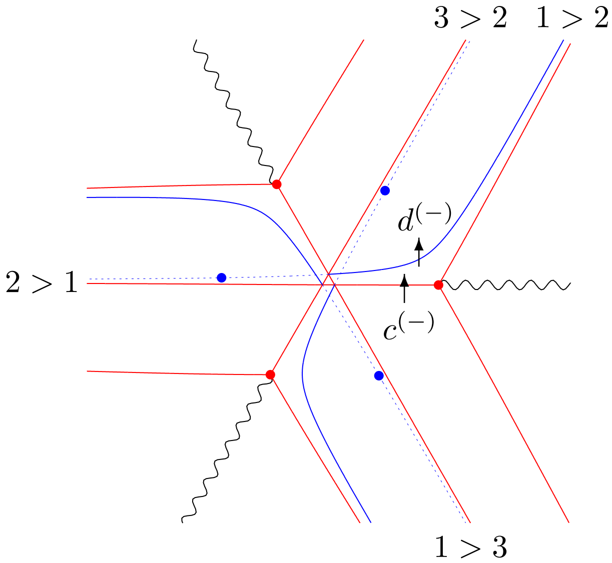


Figure 7. Stokes geometry of Equation (1.7) with $\operatorname{Re}\alpha < 0,\ \operatorname{Im}\alpha < 0$.

appearing in Figure 6 is the second simplest one. There are more complicated Stokes trees involving more than three edges. See [9] and [8] for more details.

The Stokes tree in Figure 6 is resolved by adding a small real part to the parameter α. The Stokes geometry for $\operatorname{Re}\alpha < 0$ is given in Figure 7, and that for $\operatorname{Re}\alpha > 0$ in Figure 8. Comparing Figures 7 and 8, we find the configuration of solid portions of new Stokes curves abruptly changes. In parallel to the case of Stokes segments discussed in Example 1.1, due to this abrupt change of the Stokes geometry, we can expect that a parametric Stokes phenomenon should occur with a WKB solution of (1.7) whose path of integration crosses the Stokes tree in question. The aim of this paper is to analyze explicitly such a parametric Stokes phenomenon relevant to the Stokes tree. To be more specific, let a point x_0 and a domain D be as in Figure 6, and consider WKB solutions $\psi_j := \exp \int_{x_0}^{x} S^{(j)}\,dx$ $(j = 1, 2, 3)$ of (1.7). Letting $\Psi_j^{(\pm)}$ denote the Borel sum of ψ_j in D for $\pm\operatorname{Re}\alpha > 0$, we then find that the following formula similar to (1.5) holds in this case:

$$(1.9)\qquad \begin{cases} \Psi_1^{(-)} &= \left[1 - \exp\left(\int_\gamma S\,dx\right)\right] \Psi_1^{(+)} \\ \Psi_2^{(-)} &= \left[1 - \exp\left(\int_\gamma S\,dx\right)\right]^{-1} \Psi_2^{(+)} \\ \Psi_3^{(-)} &= \Psi_3^{(+)} \end{cases}$$

holds, where the path γ and the branch of S on it are as given in Figure 9. Note that the path of integration for ψ_j crosses one of the edges of the Stokes tree in question, that is, the edge of type $(1, 2)$. As the path of integration for ψ_j crosses the edge of type $(1, 2)$, a parametric Stokes phenomenon occurs only with ψ_1 and ψ_2, not with ψ_3.

In the subsequent section, assuming the Borel summability of WKB solutions whose

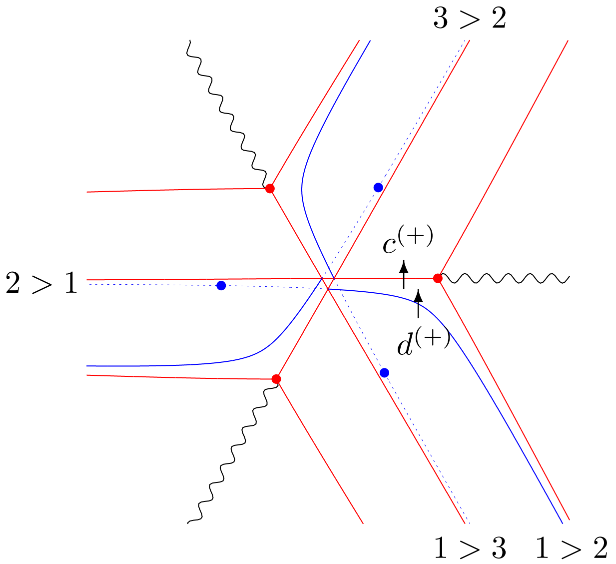


Figure 8. Stokes geometry of Equation (1.7) with $\mathrm{Re}\,\alpha > 0$, $\mathrm{Im}\,\alpha < 0$.

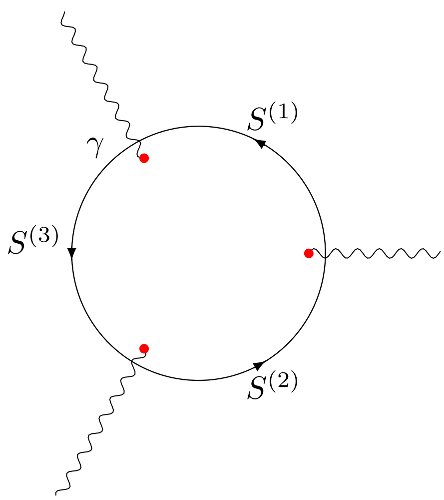


Figure 9. Path γ and the branch of S on it.

path of integration does not cross the Stokes tree in question, we explain the derivation of Formula (1.9). A key point of discussion is the abrupt change of the Stokes geometry between Figures 7 and 8. In particular, the computation of Stokes coefficients on new Stokes curves plays an important role in the derivation.

Remark. Although in this paper we deal with Equation (1.7) only for the sake of definiteness, the derivation explained in the subsequent section is valid for more general equations where a Stokes tree of the same type appears.

Remark. It is reasonable to assume the Borel summability of WKB solutions whose path of integration does not cross the Stokes tree, since it is verified for second order equations (with replacing "Stokes tree" by "Stokes segment"; cf. [11]) and further Figures 6–8 are considered to give a complete Stokes geometry of Equation (1.7). However, to verify the validity of this assumption for general equations is one of the most important open problems in the exact WKB analysis of higher order equations.

§ 2. Derivation of Formula (1.9)

Here we verify the derivation of the formula (1.9) for ψ_1 only. To this aim, we start with the Borel sums $\hat{\Psi}_1^{(\pm)}$ of a WKB solution ψ_1 near $x = x_0$ and $\alpha = \pm\epsilon - i$ with ϵ being a small positive number, and calculate their analytic continuations to the domain D.

In Figures 7 and 8 Stokes coefficients attached to each Stokes curve are denoted by $c^{(\pm)}$ and $d^{(\pm)}$. For example, when x moves across the Stokes curve to which $d^{(-)}$ is attached in Figure 7, ψ_1 changes as $\psi_1 \mapsto \psi_1 + d^{(-)}\psi_2$. Using such a connection formula repeatedly, we obtain the analytic continuations of $\hat{\Psi}_1^{(\pm)}$ to D. For example, taking Figure 7 into account, we obtain

$$\hat{\Psi}_1^{(-)} = \Psi_1^{(-)} + d^{(-)}\Psi_2^{(-)}, \tag{2.1}$$

and similarly, in view of Figure 8, we obtain

$$\hat{\Psi}_1^{(+)} = (1 + c^{(+)}d^{(+)})\Psi_1^{(+)} + d^{(+)}\Psi_2^{(+)}. \tag{2.2}$$

Now we try to make (2.1) and (2.2) more concrete by using the explicit form of the connection formula for WKB solutions of a higher order equation (1.1) when one crosses an ordinary Stokes curve emanating from a simple turning point. Let t be a simple turning point of type (k, l) and C be an ordinary Stokes curve of type $k > l$ emanating from t (Figure 10). For WKB solutions $\phi_j = \exp \int_p^x S^{(j)}\,dx$ $(j = 1, 2, 3)$

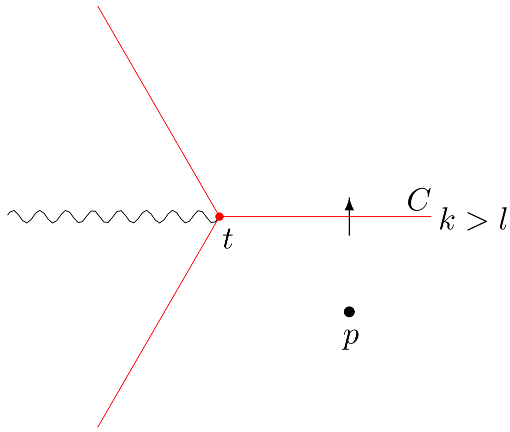


Figure 10. A simple turning point t and an ordinary Stokes curve C of type $k > l$.

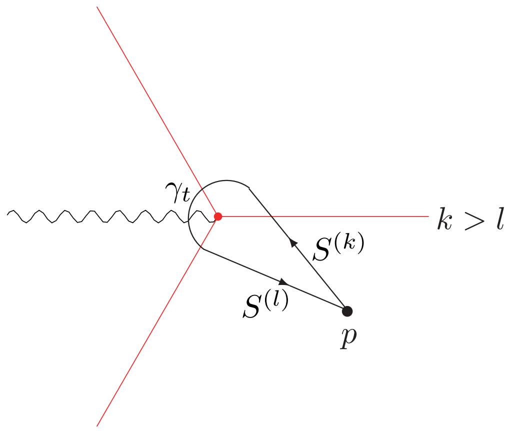


Figure 11. Path γ_t for the connection formula and the branch of S on it.

which are normalized at a point p, the following connection formula holds as x crosses C (Figure 10):

$$\begin{cases} \phi_k \mapsto \phi_k - e^{\int_{\gamma_t} S\,dx} \phi_l \\ \phi_j \mapsto \phi_j \quad (j \neq k) \end{cases}, \tag{2.3}$$

where γ_t is a path which starts and ends at p with rounding t counterclockwise once. Here the branch of S on γ_t is taken so that $S = S^{(k)}$ at the beginning and $S = S^{(l)}$ at the end (Figure 10). For the reader's convenience, we outline how this formula is deduced in A.2. By using this formula, we have

$$c^{(\pm)} = \exp\left(\int_{\gamma_c} S\,dx \right), \tag{2.4}$$

where γ_c is a path rounding t_{12} and the branch of S on it begins with $S^{(2)}$ and ends

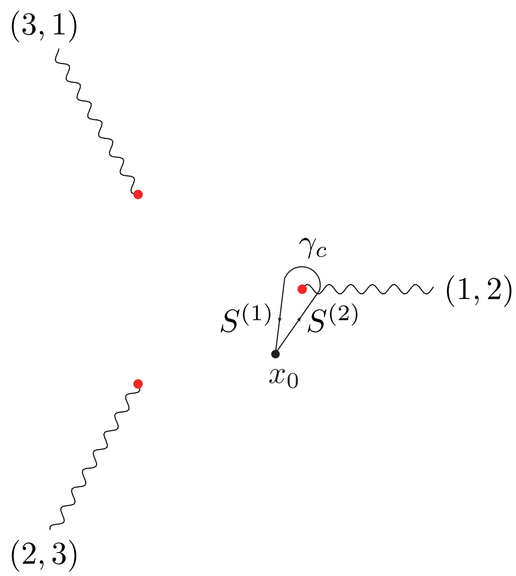


Figure 12. Path γ_c for the Stokes coefficients $c^{(\pm)}$.

with $S^{(1)}$. (Cf. Figure 12. Here, as is shown in Figure 6, t_{12} denotes the turning point where the branch cut of type $(1, 2)$ emanates.)

To determine $d^{(\pm)}$, we recall the determination of the Stokes coefficients on new Stokes curves. The new Stokes curve to which $d^{(\pm)}$ is attached appears from a crossing point of two ordinary Stokes curves specified in Figure 13. Therefore, if we write the Stokes coefficients for the ordinary Stokes curves by $a^{(\pm)}$ and $b^{(\pm)}$, we have

$$d^{(\pm)} = -a^{(\pm)}b^{(\pm)}. \tag{2.5}$$

This formula (2.5) is derived by the single-valuedness argument around the crossing point in question ([2],[10]). Due to the connection formula (2.3), $a^{(\pm)}$ and $b^{(\pm)}$ are described as

$$a^{(\pm)} = \exp\left(\int_{\gamma_a} S\,dx\right), \tag{2.6}$$

$$b^{(\pm)} = \exp\left(\int_{\gamma_b} S\,dx\right), \tag{2.7}$$

where γ_a and γ_b are as given in Figure 14. (The branch of S on γ_a and γ_b are also specified there.) Therefore we obtain

$$d^{(\pm)} = -\exp\left(\int_{\gamma_d} S\,dx\right). \tag{2.8}$$

Here γ_d is a path rounding t_{31} and t_{23} where the branch cuts of type $(3, 1)$ and $(2, 3)$ respectively emanate (cf. Figure 15 and Figure 6).

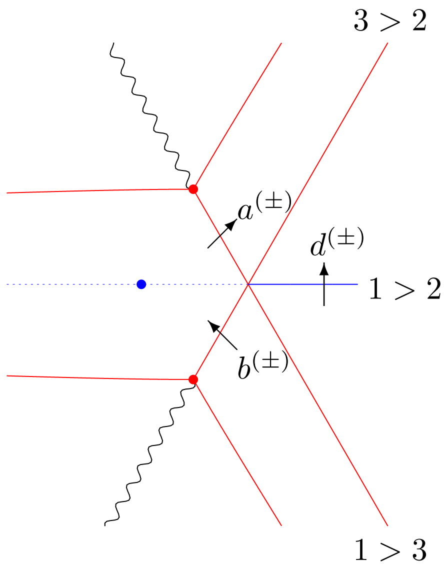


Figure 13. A new Stokes curve appearing from a crossing point of two ordinary Stokes curves.

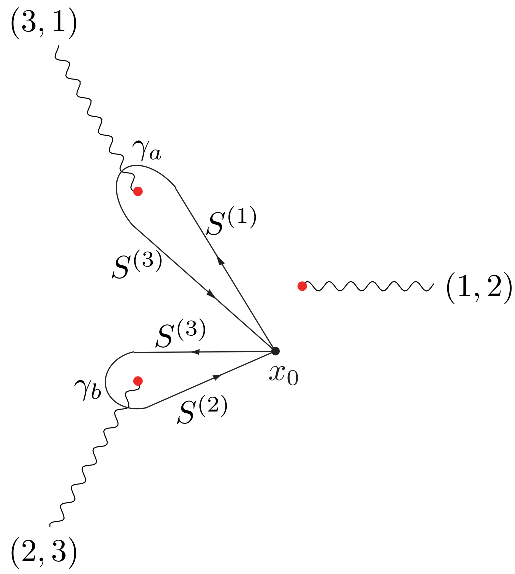


Figure 14. Path γ_a and γ_b for the Stokes coefficients $a^{(\pm)}$ and $b^{(\pm)}$.

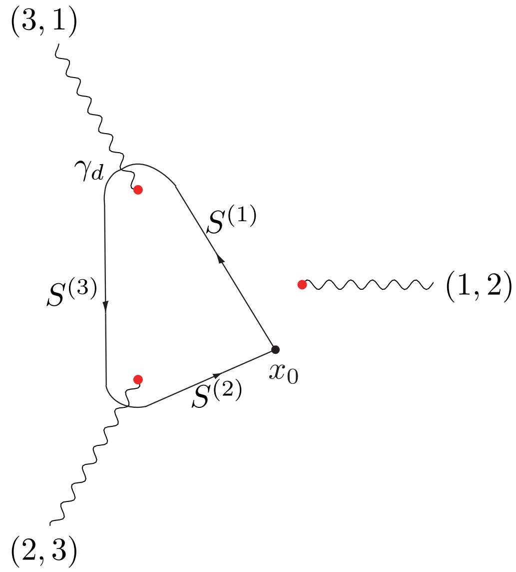


Figure 15. Path γ_d for the Stokes coefficients $d^{(\pm)}$ and the branch of S on it.

Thus the results of analytic continuation become

$$\hat{\Psi}_1^{(-)} = \Psi_1^{(-)} - e^{\int_{\gamma_d} S\,dx}\Psi_2^{(-)}, \tag{2.9}$$

$$\hat{\Psi}_1^{(+)} = \left(1 - e^{\int_{\gamma} S\,dx}\right)\Psi_1^{(+)} - e^{\int_{\gamma_d} S\,dx}\Psi_2^{(+)}. \tag{2.10}$$

Now we compare these two equalities. Since ψ_1 is normalized at $x = x_0$ and its path of integration does not cross the Stokes tree in question (or any Stokes curve) near $x = x_0$, we may assume that ψ_1 is Borel summable near $x = x_0$ and $\alpha = -i$, i.e.,

$$\hat{\Psi}_1^{(-)} = \hat{\Psi}_1^{(+)}. \tag{2.11}$$

Therefore the left-hand sides of (2.9) and (2.10) are equal to each other. Next we consider $e^{\int_{\gamma_d} S\,dx}\psi_2$. We note that

$$e^{\int_{\gamma_d} S\,dx}\psi_2 = \exp\left(\int_{\gamma_{d,2,x}} S\,dx\right) \tag{2.12}$$

holds, where $\gamma_{d,2,x}$ is a path given in Figure 16.

Then we use the fact that the integral of S along a path rounding a simple turning point counterclockwise twice is $\int S\,dx = -\pi i$ (cf. Lemma A.1.2), and consider $-e^{\int_{\gamma_d} S\,dx}\psi_2$ instead (minus sign is added). This is given by integral of S along a path given in Figure 17(a). Deforming the path rounding t_{12} (Figure 17(b)), we have five segments connecting x_0 and x ($l_1, l_2, \ldots, l_5$, in order from the left). After several cancellations such as $\int_{l_1} S^{(2)}\,dx + \int_{l_2} S^{(2)}\,dx = 0$, we obtain Figure 17(c). Now the path does not cross the Stokes tree, and so we may assume that $e^{\int_{\gamma_d} S\,dx}\psi_2$ is Borel summable on D and near $\alpha = -i$, i.e.,

$$d^{(-)}\Psi_2^{(-)} = d^{(+)}\Psi_2^{(+)}. \tag{2.13}$$

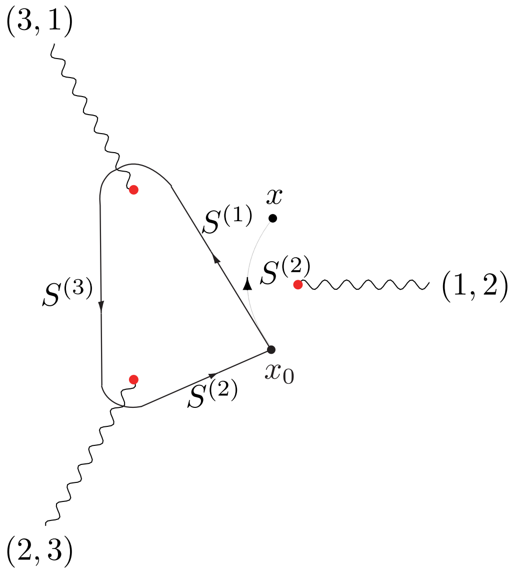


Figure 16. Path $\gamma_{d,2,x}$ and the branch of S on it.

Therefore the second term in right-hand side of (2.9) is equal to that of (2.10). Combining this with (2.11), we find that the first term of the right-hand side of (2.9) coincides with that of (2.10);

$$\Psi_1^{(-)} = \left(1 - e^{\int_\gamma S\,dx}\right) \Psi_1^{(+)}. \tag{2.14}$$

Thus we have obtained the formula (1.9) for ψ_1.

§ A.1. Some integral formulas around a simple turning point

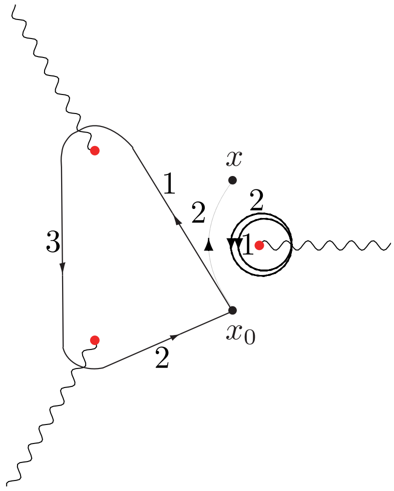
We consider a second order equation

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0. \tag{A.1.1}$$

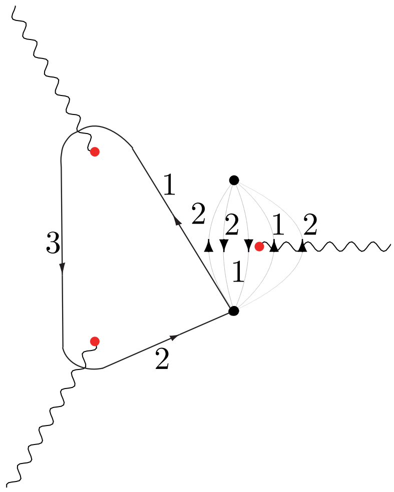
Let t be a simple turning point of (A.1.1) and consider the contour integral $\oint S_{\text{even}}\,dx$ around t, where $S_{\text{even}} := (S^{(1)} + S^{(2)})/2$.

Lemma A.1.1.

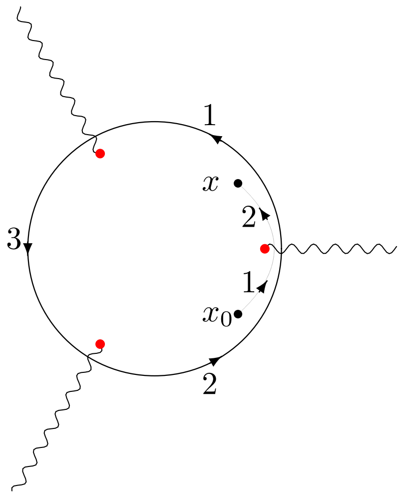
$$\oint S_{\text{even}}\,dx = -\pi i/2. \tag{A.1.2}$$



(a)



(b)



(c)

Figure 17. Deformation of path $\gamma_{d,2,x}$.

Proof. First we note that $S_{\text{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}}$ holds. Therefore

$$S_{\text{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}} \tag{A.1.3}$$

$$= -\frac{1}{2}\frac{d}{dx}\log\left[\eta\sqrt{Q}\left(1+\frac{S_{\text{odd}}-\eta\sqrt{Q}}{\eta\sqrt{Q}}\right)\right] \tag{A.1.4}$$

$$= -\frac{1}{2}\frac{d}{dx}\log\sqrt{Q} - \frac{1}{2}\frac{d}{dx}\log\left(1+\frac{S_{\text{odd}}-\eta\sqrt{Q}}{\eta\sqrt{Q}}\right). \tag{A.1.5}$$

Here $(S_{\text{odd}}-\eta\sqrt{Q})/\eta\sqrt{Q}$ is single-valued around t, and hence the second term vanishes after the integration. Since t is a simple zero of Q, we obtain the result. □

Lemma A.1.1 leads to

$$\exp\left(\int_{\gamma_0} S_{\text{even}}\,dx\right) = -1 \tag{A.1.6}$$

for the Weber equation, and hence we find the relations (1.5) and (1.6) are equivalent.

Lemma A.1.1 is applicable also to the Voros connection formula ([14]): as x crosses counterclockwise a Stokes curve of type $k > l$ emanating from t, the following connection formula holds.

$$\psi_k \mapsto \psi_k + i\exp\left(\int_{\gamma_t} S_{\text{odd}}\right)\psi_l, \tag{A.1.7}$$

where ψ_j's are WKB solutions normalized at a point p and γ_t is a path given in Figure 11. In view of Lemma A.1.1, this formula can be described also as

$$\psi_k \mapsto \psi_k - \exp\left(\int_{\gamma_t} S\right)\psi_l, \tag{A.1.8}$$

where the branch of S is $S^{(k)}$ at the beginning of γ_t and $S^{(l)}$ at the end.

Next we consider an integral of S around a simple turning point.

Lemma A.1.2.

$$\int S\,dx = -\pi i, \tag{A.1.9}$$

where the integral is done along a path rounding t counterclockwise twice (Figure 18).

Proof. Since $S_{\text{odd}} := (S^{(1)} - S^{(2)})/2$ is of the form

$$S_{\text{odd}} = (x-t)^{1/2} f(x,\eta) \tag{A.1.10}$$

with a function f single-valued around t, S_{odd} vanishes after the integration. Therefore only S_{even} contributes the integral. In view of Lemma A.1.1, we obtain $\int S\,dx = -\pi i$. □

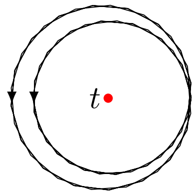


Figure 18. A path rounding a simple turning point counterclockwise twice.

§ A.2. Outline for derivation of the connection formula (2.3)

The connection formula (2.3) is explicitly calculated for a third order equation (actually a holonomic system) in [6]. Here, for the reader's convenience, we outline how this is deduced from the results of [2], [1] and [3].

We consider a third order equation

$$H\psi := \left[\frac{d^3}{dx^3} + a(x)\eta^2\frac{d}{dx} + b(x)\eta^3\right]\psi = 0. \tag{A.2.1}$$

Let $\xi_1(x), \xi_2(x), \xi_3(x)$ be characteristic roots of (A.2.1) and assume that (A.2.1) has a simple turning point at $x = t$ of type (k, l). For this equation, the following holds:

Theorem A.2.1 ([2, Theorem 1.4]). *Near the turning point t, there exist Borel transformable series $L(x,\eta) = \sum_{n=0}^{\infty} L_n(x)\eta^{-n}$, $M(x,\eta) = \sum_{n=0}^{\infty} M_n(x)\eta^{-n}$ and $N(x,\eta) = \sum_{n=0}^{\infty} N_n(x)\eta^{-n}$ with holomorphic coefficients $L_n(x)$, $M_n(x)$ and $N_n(x)$ $(n = 0, 1, 2, \dots)$ such that the following decomposition holds:*

$$H = \left(\frac{d}{dx} - L(x,\eta)\eta\right)\left(\frac{d^2}{dx^2} + M(x,\eta)\eta\frac{d}{dx} + N(x,\eta)\eta^2\right). \tag{A.2.2}$$

Here the second order operator $H_2 := (d^2/dx^2) + M(x,\eta)\eta(d/dx) + N(x,\eta)\eta^2$ has ξ_k and ξ_l as its characteristic roots.

We first note that a WKB solution $\phi_j = \exp(\int_p^x T^{(j)}\,dx)$ to $H_2\phi = 0$ in which $T^{(j)}(x,\eta) = T_{-1}(x)\eta + T_0 + T_1(x)\eta^{-1} + \cdots$ starts with ξ_j (i.e. $T_{-1} = \xi_j$) is a WKB solution to the third order equation $H\psi = 0$ $(j = k, l)$. For the equation $H_2\phi = 0$, we can apply the transformation theory to the Airy equation (cf. [1],[3]): Let $T_{\mathrm{odd}}(x,\eta)$ be the odd part of the solution for the Riccati equation associated with $H_2\phi = 0$. Then

$$\varphi_j := \frac{1}{\sqrt{T_{\mathrm{odd}}}}\exp\left[\int_t^x \left(\pm T_{\mathrm{odd}} - \frac{1}{2}M\eta\right)dx\right] \quad (j = k, l) \tag{A.2.3}$$

are WKB solutions to $H_2\phi = 0$. (cf. [2, (1.24)]) Here plus sign is for $j = k$ and minus for $j = l$. For the Borel transform $\varphi_{j,B}(x,y)$ of φ_j, the following holds.

Theorem A.2.2 ([2, Theorem 1.8]). *On a small neighborhood of* $(t,0) \in \mathbb{C}_x \times \mathbb{C}_y$, $\varphi_{j,B}(x,y)$ $(j = k,l)$ *have singularities only along* $R_k \cup R_l$, *where* $R_j = \{(x,y) \mid y = -\int_t^x \xi_j\, dx\}$ $(j = k,l)$. *Furthermore the singular part of* $\varphi_{k,B}(x,y)$ *(resp.,* $\varphi_{l,B}(x,y)$*) along* R_l *(resp.,* R_k*) coincides with* $i\varphi_{l,B}(x,y)$ *(resp.,* $-i\varphi_{k,B}(x,y)$*).*

If we assume Borel summability of the transformation, this leads to the connection formula for φ_j $(j = k,l)$: When we go across counterclockwise a Stokes curve of type $k > l$ emanating from t,

$$\begin{cases} \varphi_k \mapsto \varphi_k + i\varphi_l \\ \varphi_j \mapsto \varphi_j \quad (j \neq k) \end{cases} . \tag{A.2.4}$$

Then we compare φ and ϕ at $x = p$ to obtain connection formula for ϕ. By substituting $x = p$, we have

$$\varphi_j(p,\eta) = \frac{1}{\sqrt{T_{\mathrm{odd}}(p,\eta)}} \exp\left[\int_t^p \left(\pm T_{\mathrm{odd}} - \frac{1}{2}M\eta\right) dx\right], \tag{A.2.5}$$

$$\phi_j(p,\eta) = 1. \tag{A.2.6}$$

Therefore we have

$$\begin{cases} \phi_k \mapsto \phi_k + i\exp\left(\int_{\gamma_t} T_{\mathrm{odd}}\, dx\right)\phi_l \\ \phi_j \mapsto \phi_j \quad (j \neq k) \end{cases} , \tag{A.2.7}$$

where γ_t is a path starting from p, rounding t counterclockwise and ending at p (Figure 11). Finally we take into account the integral of T_{even} along γ_t (cf. Lemma A.1.1) to obtain

$$\begin{cases} \phi_k \mapsto \phi_k - \exp\left(\int_{\gamma_t} T\, dx\right)\phi_l \\ \phi_j \mapsto \phi_j \quad (j \neq k) \end{cases} . \tag{A.2.8}$$

Since $\phi^{(j)}$'s are solutions to the third order equation $H\psi = 0$ and $T^{(j)}$'s are solutions to the nonlinear equation associated with $H\psi = 0$, (A.2.8) means the connection formula (2.3).

References

[1] T. Aoki, T. Kawai, and Y. Takei. The Bender-Wu analysis and the Voros theory. In *ICM-90 Satellite Conference Proceedings*, pp. 1–29. Springer, 1991.

[2] T. Aoki, T. Kawai, and Y. Takei. New turning points in the exact WKB analysis for higher-order ordinary differential equations. In *Analyse algébrique des perturbations singulières: Méthodes résurgentes*, Vol. 1, pp. 69–84. Hermann, 1994.

[3] T. Aoki and J. Yoshida. Microlocal reduction of ordinary differential operators with a large parameter. *Publications of the Research Institute for Mathematical Sciences*, 29(6):959–975, 1993.

[4] H. L. Berk, W. M. Nevins, and K. V. Roberts. New Stokes' line in WKB theory. *Journal of Mathematical Physics*, 23(6):988–1002, 1982.

[5] E. Delabaere, H. Dillinger, and F. Pham. Résurgence de Voros et périodes des courbes hyperelliptiques. *Annales de l'institut Fourier*, 43(1):163–199, 1993.

[6] S. Hirose. Exact WKB analysis of the holonomic system satisfied by the Pearcey integral. Master's thesis, Kyoto University, 2010. (In Japanese.).

[7] S. Hirose. On the Stokes geometry for the Pearcey system and the $(1, 4)$ hypergeometric system. *RIMS Kôkyûroku Bessatsu*, B40:243–292, 2013.

[8] N. Honda. Some examples of the Stokes geometry for Noumi-Yamada systems. *RIMS Kôkyûroku*, 1516:21–167, 2006. (In Japanese.).

[9] N. Honda. On the Stokes geometry of the Noumi-Yamada system. *RIMS Kôkyûroku Bessatsu*, B2:45–72, 2007.

[10] N. Honda, T. Kawai, and Y. Takei. *Virtual Turning Points*, Vol. 4 of *SpringerBriefs in Mathematical Physics*. Springer, 2015.

[11] T. Koike and R. Schäfke. In preparation.

[12] K. Okamoto and H. Kimura. On particular solutions of the Garnier systems and the hypergeometric functions of several variables. *The Quarterly Journal of Mathematics*, 37(1):61–80, 1986.

[13] Y. Takei. Sato's conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points. *RIMS Kôkyûroku Bessatsu*, B10:205–224, 2008.

[14] A. Voros. The return of the quartic oscillator—the complex WKB method. *Annales de l'institut Henri Poincaré (A) Physique théorique*, 39(3):211–338, 1983.