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On gravity water waves

By

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Abstract

Considering a 2D gravity water wave, we study the propagation of the singularity of the surface profile. We announce a conjecture that this singularity propagates at the fluid velocity, which means that a stagnation occurs there.

§ 1. Introduction

We consider the free surface of a 2D incompressible inviscid fluid without surface tension. There are many problems about the motion of the free surface governed by the Euler equation. For example, the relation between the fluid bed and the fluid surface, and the relation between the pressure and the fluid surface are studied. We only refer to an introductory explanation in chapter 8 of [2] for such problems. In this article we discuss about the singularity propagation of the free surface profile. However, the author has not given a complete proof of the result, and this is an interim report of such an investigation.

Let $x = (x_0, x') = (x'', x_2) = (x_0, x_1, x_2) \in \mathbf{R}^3$. We denote the fluid surface by $H = \{x \in \mathbf{R}^3;\ x_2 = h(x'')\}$, and $H(x_0) = \{x' \in \mathbf{R}^2;\ (x_0, x') \in H\}$. We assume that $\omega = \{x \in \mathbf{R}^3;\ x_2 < h(x'')\}$ is occupied by a incompressible inviscid fluid without surface tension. We assume that H has an isolated singularity at a moving point $A(x_0) = (A_1(x_0), A_2(x_0)) \in H(x_0)$.

Let $u(x) = (u_1(x), u_2(x))$ be the fluid velocity, and $p(x)$ be the pressure. We

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consider an initial-boundary value problem for the Euler equation:

$$
(1.1)\qquad \begin{cases}
\partial_{x_0} u_1 + u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 = -\dfrac{\partial_{x_1} p}{\rho}, \\
\partial_{x_0} u_2 + u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 = -\dfrac{\partial_{x_2} p}{\rho} - g, \\
\operatorname{div} u = 0, \\
\partial_{x_0} h + u_1 \partial_{x_1} h = u_2 & \text{on } H, \\
u_j(0, x_1, x_2) = u_j^0(x_1, x_2), & 1 \le j \le 2, \\
\partial_n^j p(x_0, x_1, h(x_0, x_1)) = p^j(x_0, x_1), & 0 \le j \le 1, \\
h(0, x_1) = h^0(x_1).
\end{cases}
$$

Here ρ is the density of the fluid, which is a constant by assumption, g is the gravitation constant, and n denotes the unit normal vector to H, pointing upwards. We assume that H has sufficient regularity and n is well-defined. Furthermore, $u_j^0(x_1, x_2)$, $p^j(x_0, x_1)$ and $h^0(x_1)$ are some given functions. We must assume

$$
(1.2)\qquad \operatorname{div} u^0 = 0.
$$

Note that we impose initial conditions for $u(x)$ and $h(x'')$, but boundary conditions for $p(x)$. It is natural to assume that $p^0(x_0, x_1)$ is equal to one atmospheric pressure. For technical reason we impose not only the Dirichlet condition but also the Neumann condition for $p(x)$. We need to determine u_1, u_2, p, h.

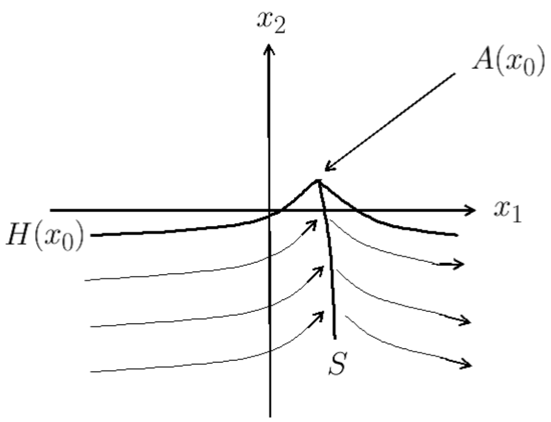

Figure 1. Singularities of the solution

We consider the following situation. We assume that $u_1(x), u_2(x), p(x)$ have singularities along a hypersurface $S = \{x \in \omega;\ x_1 = s(x_0, x_2)\}$, and that $(\{x_0\} \times H(x_0)) \cap S =$

$\{(x_0, A(x_0))\}$ as is illustrated in Figure 1. In this figure, we describe as if the velocity is discontinuous along S, but in fact the sungularity is not so outstanding. As we shall assume later, it has sufficient continuity along S, and analytic outside of S.

We consider the following additional conditions.

$$\begin{cases} \partial_{x_0} s + u_2 \partial_{x_2} s = u_1 & \text{on } S, \\ \partial_{x_0} A_j = u_j & 1 \leq j \leq 2, \\ s(0, x_2) = s^0(x_2), & \\ A_j(0) = 0, & 1 \leq j \leq 2. \end{cases} \tag{1.3}$$

We must assume

$$h^0(0) = s^0(0) = 0. \tag{1.4}$$

Let us remark the following facts.

(a) By definition, the velocity of $A(x_0)$ is called the wave velocity, i.e., the shape of the surface looks like moving with velocity dA/dx_0.

(b) It is known that the singularity set S in the fluid moves at the fluid velocity (See [3, 4]).

The wave velocity and the fluid velocity are different notions. If they coincide at a point, then such a point is called a stagnant point (In fluid dynamics, a stagnant point is a point in a flow field where the velocity of the fluid is zero. But here we use stagnation in the former sense). In many studies about the fluid surface, it is assumed that there are no stagnant points, but perhaps the following is true:

(c) Usually, the wave velocity and the fluid velocity take different values. In this case, singularities never appear on the fluid surface.

(d) If the wave velocity and the fluid velocity take the same value at a point on the fluid surface, then the surface may have a singularity there.

Many people study case (c), and we consider case (d).

Let us state the problem more precisely. Considering x as a coordinate system in $\mathbf{C}^3$, we assume that the functions are holomorphic on some complex domains. Let us denote by $\mathcal{R}(X)$ the universal covering space of a complex domain X, and by $\mathcal{O}(Y)$ the set of single-valued holomorphic functions on Y. Let $0 < q < 1$. We define

$$
\begin{aligned}
\Omega &= \{x \in \mathbf{C}^3;\ |x_j| < r\ (0 \le j \le 2)\},\\
\Omega' &= \{(x_1, x_2) \in \mathbf{C}^2;\ |x_j| < r\ (1 \le j \le 2),\ x_1^2 + x_2^2 \ne 0\},\\
\Omega'' &= \{(x_0, x_1) \in \mathbf{C}^2;\ |x_j| < r\ (0 \le j \le 1),\ x_1 \ne A_1(x_0)\},\\
B(r) &= \{z \in \mathbf{C};\ |z| < r\},\\
\mathcal{O}^{k-q}(\mathcal{R}(\Omega')) &= \{f(x') \in \mathcal{O}(\mathcal{R}(\Omega'));\\
&\quad |\partial_{x'}^{\alpha'} f(x)| \le \exists M(\min_{l=\pm 1} |x_1 + \sqrt{-1} l x_2|)^{(|\alpha'|-k+q)_+},\ |\alpha'| \le k\},\\
\mathcal{O}^{k-q}(\mathcal{R}(\Omega'')) &= \{f(x'') \in \mathcal{O}(\mathcal{R}(\Omega''));\\
&\quad |\partial_{x_1}^{l} f(x'')| \le \exists M|x_1 - A_1(x_0)|^{(l-k+q)_+},\ l \le k\},\\
\mathcal{O}^{k-q}(\mathcal{R}(B(r) \setminus \{0\})) &= \{f(z) \in \mathcal{O}(\mathcal{R}(B(r) \setminus \{0\}));\\
&\quad |\partial_z^{l} f(z)| \le \exists M|z|^{(l-k+q)_+},\ l \le k\}.
\end{aligned}
$$

for $k \in \{1, 2, 3, \cdots\}$. Here we have denoted $t_+ = \max(t, 0)$. We assume the following conditions.

$$
\begin{aligned}
&u_1^0(x'), u_2^0(x') \in \mathcal{O}^{3-q}(\mathcal{R}(\Omega')),\\
&h^0(x_1), s^0(x_2) \in \mathcal{O}^{3-q}(\mathcal{R}(B(r) \setminus \{0\})),\\
&|u_j^0(x')|, |\partial_{x_1} h^0(x_1)|, |\partial_{x_2} s^0(x_2)| \ll 1, \quad 1 \le j \le 2.
\end{aligned}
$$

Note also that the last assumption means that the surface is very calm. It may be possible to weaken this assumption. Furthermore, we require that $p^0(x'')$ and $p^1(x'')$ are given in such a way that we have

$$p(x'', h(x'')), \partial_{x_1} p(x'', h(x'')), \partial_{x_2} p(x'', h(x'')) \in \mathcal{O}^{3-q}(\mathcal{R}(\Omega'')).$$

Here we do not discuss about this condition any more. Under these assumptions, the author is trying to show the following results. For the sake of convenience, we only consider solution for real variables on a small neighborhood of the origin.

1. $A_1(x_0)$ and $A_2(x_0)$ are analytic.

2. $h(x_0, x_1)$ is analytic for $x_1 \ne A_1(x_0)$.

3. $s(x_0, x_2)$ is analytic for $x_2 \ne A_2(x_0)$.

4. $u_1(x), u_2(x), p(x)$ are analytic for $x_1 \ne s(x_0, x_2)$ (This result is different in a complex domain, as we shall later see).

5. The derivatives of these functions of order at most two have Hölder continuity of exponent $1 - q$ containing their singularity sets.

The fluid surface has a singularity at $A(x_0)$, and this point is moving with the fluid velocity. This means that $A(x_0)$ is a stagnant point.

§ 2. Sketch of the main idea

We briefly see only the main idea of the investigation. We use two fundamental methods in fluid dynamics.

First, let us rewrite the divergence zero condition $\operatorname{div} u = 0$ in (1.1) following a common method in this field (See [1] for example). We assume that all other equations in (1.1), but for the divergence zero condition, and (1.2) are true. Then we have

$$\partial_{x_1}(\partial_{x_0} u_1 + u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1) = -\frac{\partial_{x_1}^2 p}{\rho},$$

and thus

$$(\partial_{x_0} + u_1 \partial_{x_1} + u_2 \partial_{x_2})\partial_{x_1} u_1 + \partial_{x_1} u_1 \cdot \partial_{x_1} u_1 + \partial_{x_1} u_2 \cdot \partial_{x_2} u_1 = -\frac{\partial_{x_1}^2 p}{\rho}.$$

Similarly we have

$$(\partial_{x_0} + u_1 \partial_{x_1} + u_2 \partial_{x_2})\partial_{x_2} u_2 + \partial_{x_2} u_1 \cdot \partial_{x_1} u_2 + \partial_{x_2} u_2 \cdot \partial_{x_2} u_2 = -\frac{\partial_{x_2}^2 p}{\rho}.$$

It follows that

$$(\partial_{x_0} + u_1 \partial_{x_1} + u_2 \partial_{x_2})(\operatorname{div} u) + \sum_{1 \le j,k \le 2} \partial_{x_j} u_k \cdot \partial_{x_k} u_j = -\frac{\Delta_x p}{\rho},$$

where $\Delta_x p = \partial_{x_1}^2 p + \partial_{x_2}^2 p$. Under the condition (1.2) we have

$$\begin{aligned} \operatorname{div} u = 0 &\Longleftrightarrow (\partial_{x_0} + u_1 \partial_{x_1} + u_2 \partial_{x_2})(\operatorname{div} u) = 0 \\ &\Longleftrightarrow \Delta_x p = -\rho \textstyle\sum_{1 \le j,k \le 2} \partial_{x_j} u_k \cdot \partial_{x_k} u_j. \end{aligned}$$

Therefore in (1.1) we may replace $\operatorname{div} u = 0$ by $\Delta_x p = -\rho \sum_{1 \le j,k \le 2} \partial_{x_j} u_k \cdot \partial_{x_k} u_j$. Again we write the important parts of (1.1).

$$\begin{cases} \partial_{x_0} u_1 + u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 = -\dfrac{\partial_{x_1} p}{\rho}, \\ \partial_{x_0} u_2 + u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 = -\dfrac{\partial_{x_2} p}{\rho} - g, \\ \Delta_x p = -\rho \sum_{1 \le j,k \le 2} \partial_{x_j} u_k \cdot \partial_{x_k} u_j. \end{cases}$$

The singularity propagation is governed by the left hand sides. This means that there are three singularity sets of the solution in a complex domain. One of them is S, and the others are the characteristic hypersurfaces of Δ_x through $(A_1(x_0), A_2(x_0))$.

Secondly, we employ the method of characteristic functions. This means to consider two functions $y_1(x)$ and $y_2(x)$ defined as follows:

$$
(2.1)\qquad \begin{cases} \partial_{x_0} y_1 + u_1 \partial_{x_1} y_1 + u_2 \partial_{x_2} y_1 = 0, \\ \partial_{x_0} y_2 + u_1 \partial_{x_1} y_2 + u_2 \partial_{x_2} y_2 = 0, \\ y_1(0, x_1, x_2) = x_1 - s^0(x_2), \\ y_2(0, x_1, x_2) = x_2 - h^0(x_1). \end{cases}
$$

These are characteristic functions of the flow. The singularity set S inside of the fluid is given by $y_1(x) = 0$, and the surface profile H is given by $y_2(x) = 0$. We define $y_0(x) = x_0$ and $y(x) = (y_0(x), y_1(x), y_2(x))$. We assume that we can define its inverse function $x(y) = (x_0(y), x_1(y), x_2(y))$. Then we have

$$
\partial_{x_0} + u_1 \partial_{x_1} + u_2 \partial_{x_2} = \sum_{0 \le j \le 2} (\partial_{x_0} y_j + u_1 \partial_{x_1} y_j + u_2 \partial_{x_2} y_j) \partial_{y_j}.
$$

Here we have

$$
\partial_{x_0} y_j + u_1 \partial_{x_1} y_j + u_2 \partial_{x_2} y_j = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0, \end{cases}
$$

and thus $\partial_{x_0} + u_1 \partial_{x_1} + u_2 \partial_{x_2} = \partial_{y_0}$. Furthermore, it follows that $u_j = (\partial_{x_0} + u_1 \partial_{x_1} + u_2 \partial_{x_2}) x_j = \partial_{y_0} x_j$ for $1 \le j \le 2$. We can rewrite (1.1) as follows.

$$
(2.2)\quad \begin{cases} \partial_{y_0}^2 x_1 (= \partial_{y_0} u_1) = -\dfrac{\partial_{x_1} p}{\rho}, & \\ \partial_{y_0}^2 x_2 (= \partial_{y_0} u_2) = -\dfrac{\partial_{x_2} p}{\rho} - g, & \\ \Delta_x p \qquad\qquad\quad = -\rho \sum_{1 \le j,k \le 2} \partial_{x_j} (\partial_{y_j} x_k) \cdot \partial_{x_k} (\partial_{y_0} x_j), & \\ \partial_{y_0}^j x_k (0, y_1, y_2) = x_k^j (y_1, y_2), & 0 \le j \le 1,\ 1 \le k \le 2, \\ \partial_{y_2}^j p(y_0, y_1, 0) \quad = p^j (y_0, y_1), & 0 \le j \le 1. \end{cases}
$$

Here the derivatives in x must be rewritten in terms of derivatives in y.

By assumption we have $y \sim x$ and thus $\Delta_x \sim \Delta_y = \partial_{y_1}^2 + \partial_{y_2}^2$. Using the new coordinate system, we have simplified the problem. In fact, we have $S = \{y_1 = 0\}$ and $H = \{y_2 = 0\}$. We can solve (2.2) outside of the following sets:

$$
\begin{aligned} K_0 &= \{y \in \mathbf{C}^3;\ y_1 \neq 0\}, \\ K_- &= \{y \in \mathbf{C}^3;\ |y_1 - \sqrt{-1} y_2| < |y_1 + \sqrt{-1} y_2|/2\}, \\ K_+ &= \{y \in \mathbf{C}^3;\ |y_1 + \sqrt{-1} y_2| < |y_1 - \sqrt{-1} y_2|/2\}, \end{aligned}
$$

Here $K_\pm$ is a conical neighborhood of $\{y_1 \mp \sqrt{-1} y_2\}$, which we do not calculate in precision. This is because they do not appear in the real domain. Considering the coordinate transformation from y to x, we obtain the solution of (1.1). We can define $h(x_0, x_1)$ and $s(x_0, x_2)$ by

$$x_2 = h(x_0, x_1) \Longleftrightarrow y_2 = 0,$$
$$x_1 = s(x_0, x_2) \Longleftrightarrow y_1 = 0.$$

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