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Parametric Borel summability for semilinear partial differential equation

By

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Abstract

In [1], Balser and Kostov studied the parametric Borel summability for a system of ordinary differential equations of Fuchsian type. They noted that a certain Diophantine phenomenon enters in the summability. In fact, they showed, by a counter example, that a Diophantine condition is necessary in general for the parametric Borel summability. In this paper we shall show the parametric Borel summability for a first order semilinear partial differential equation as well as an ordinary differential equation which does not necessarily satisfy Balser-Kostov's Diophantine-type condition.

§ 1. Introduction

In 2002, Balser and Kostov showed the parametric Borel summability for a first order system of ordinary differential equations under a certain Diophantine type condition for eigenvalues of the linear part. (cf. [1]). They also showed that such a condition is necessary in general. On the other hand, in [2] it was shown that the Diophantine phenomena do not appear in the case of an irregular singular ordinary differential equations.

In our preceding paper [3] we proved the parametric Borel summability for a first order semilinear system of partial differential equations of Fuchsian type under the condition similar to Balser-Kostov's one. In this paper, we shall study the case where the Diophantine type condition for the eigenvalues of the linear part is not satisfied. (cf. [1]). Because the Borel summability without Balser-Kostov's condition does not hold in

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general, we need to restrict the class of functions which we use in proving the summability. In fact, we introduce the class of functions whose Borel transforms are holomorphic and of exponential growth in some direction as well as in its antipodal direction. Then we show the parametric Borel summability for a Fuchsian semilinear partial differential equation which does not satisfy Balser-Kostov's condition. We remark that the summability in such a restricted class is new even in the case of ordinary differential equation not satisfying Balser-Kostov's condition.

This paper is organized as follows. In Section 2, we state the main theorem, Theorem 2.1. In Section 3, we prove Gevrey estimate for the formal power series. In Section 4, we prove elementary properties of the convolution needed for the proof of Theorem 2.1. In Section 5, we prove Theorem 2.1, after having prepared technical lemmas.

§ 2. Statement of results

Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $n \geq 1$ be the variable in $\mathbb{C}^n$. For $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$ $(j = 1, 2, \ldots, n)$ we define

$$\mathcal{L} := \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}. \tag{2.1}$$

Let $N \geq 1$ be an integer and let $f(x, u, \eta) = (f^{(1)}(x, u, \eta), \ldots, f^{(N)}(x, u, \eta))$ be a holomorphic vector function in some neighborhood of the origin of $x \in \mathbb{C}^n$, $\eta \in \mathbb{C}$ and $u = (u^{(1)}, \ldots, u^{(N)}) \in \mathbb{C}^N$. We assume that $f(x, u, \eta)$ is an entire function of u and η. We consider the semilinear system of equations of u

$$\eta \mathcal{L} u = f(x, u, \eta), \tag{2.2}$$

where $\eta \in \mathbb{C}$ is a complex parameter. We assume

$$f(0, 0, \eta) \equiv 0 \text{ for all } \eta, \quad \det(\nabla_u f(0, 0, 0)) \neq 0 \tag{2.3}$$

where $\nabla_u f(0, 0, 0)$ denotes the Jacobi matrix of $f(x, u, \eta)$ with respect to u at the point $x = 0, u = 0, \eta = 0$. We assume

$$\nabla_u f(0, 0, \eta) \equiv \nabla_u f(0, 0, 0), \quad \text{for all } \eta, \tag{2.4}$$

and

$$\nabla_u f(x, 0, 0) \text{ is a diagonal matrix.} \tag{2.5}$$

We set

$$\nabla_u f(0, 0, 0) = \operatorname{diag}(\mu_1, \ldots, \mu_N), \tag{2.6}$$

where $\operatorname{diag}(\mu_1, \dots, \mu_N)$ is the diagonal matrix with diagonal components given by $\mu_1, \dots, \mu_N$ in this order. By multiplying the equation with some nonzero constant one may assume that $\lambda_n = 1$. In the following we always assume the condition. We assume

$$\Re\lambda_j > 0, \qquad j = 1, \dots, n, \tag{2.7}$$

and

$$\Re\mu_k \neq 0, \qquad k = 1, \dots, N. \tag{2.8}$$

If $\Re\mu_k > 0$ for all k or if $\Re\mu_k < 0$ for all k, then we are in the situation studied in [3]. Hence we assume that there exist k and ℓ such that $\Re\mu_k > 0$ and $\Re\mu_\ell < 0$ in the rest of this paper.

We construct the solution $u = v(x, \eta)$ of (2.2) in the formal power series of η

$$v(x, \eta) = \sum_{\nu=0}^{\infty} \eta^\nu v_\nu(x) = v_0(x) + \eta v_1(x) + \cdots , \tag{2.9}$$

where the coefficient $v_\nu(x)$ is a holomorphic vector function of x in some open set independent of ν. We set $v_\nu(x) \equiv v_\nu = (v_\nu^{(1)}, \dots, v_\nu^{(N)})$. We denote by Ω_0 the neighborhood of the origin on which every coefficient $v_\nu(x)$ is defined. For the details we refer §3.

In order to state our results we recall some definitions from the summability theory. The formal Borel transform of $v(x, \eta)$ is defined by

$$B(v)(x, y) := \sum_{\nu=0}^{\infty} v_\nu(x) \frac{y^\nu}{\Gamma(\nu + 1)}, \tag{2.10}$$

where $\Gamma(z)$ is the Gamma function. For $\xi \in \mathbb{R}$ and $\theta > 0$ we define

$$E_+(\xi, \theta) := \left\{ z \in \mathbb{C} \,\middle|\, dist(z, \mathbb{R}_+ e^{i\xi}) < \frac{\theta}{2} \right\}, \tag{2.11}$$

where $\mathbb{R}_+ = \{t, t \geq 0\}$, and $dist(z, \mathbb{R}_+ e^{i\xi})$ denotes the distance from z to the set $\mathbb{R}_+ e^{i\xi}$. We say that $v(x, \eta)$ is 1-summable in the direction ξ with respect to η if $B(v)(x, y)$ converges in some neighborhood of $(x, y) = (0, 0)$, and there exist a neighborhood U of $x = 0$ and a $\theta > 0$ such that $B(v)(x, y)$ can be analytically continued to the set $\{(x, y) \in U \times E_+(\xi, \theta)\}$ and of exponential growth of order 1 with respect to y in $E_+(\xi, \theta)$ when $x \in U$. For the sake of simplicity we denote the analytic continuation with the same notation $B(v)(x, y)$. The Borel sum $V(x, \eta)$ of $v(x, \eta)$ is, then, given by the Laplace transform

$$V(x, \eta) := \eta^{-1} \int_0^{\infty e^{i\xi}} e^{-y\eta^{-1}} B(v)(x, y) dy. \tag{2.12}$$

Let C_0^+ be the convex closed positive cone with vertex at the origin containing μ_k such that $\Re\mu_k > 0$ and $1 \le k \le N$. C_0^- is defined similarly by replacing the condition $\Re\mu_k > 0$ with $\Re\mu_k < 0$. Write

$$C_0^+ = \{z \in \mathbb{C} | -\theta_2^+ \le \arg z \le \theta_1^+\}, \quad C_0^- = \{z \in \mathbb{C} | -\theta_2^- \le \arg z - \pi \le \theta_1^-\}. \tag{2.13}$$

for some $-\pi/2 < \theta_1^\pm < \pi/2$ and $-\pi/2 < \theta_2^\pm < \pi/2$ with $-\theta_2^\pm \le \theta_1^\pm$. Define $\theta_1 := \max\{\theta_1^+, \theta_1^-\}$, $\theta_2 := \max\{\theta_2^+, \theta_2^-\}$. Then we have

Theorem 2.1. *Suppose (2.3), (2.4), (2.5) and (2.7). Then there exists a neighborhood U of $x = 0$ such that $v(x, \eta)$ is 1-summable in the direction $\arg\eta$ with $\pi/2+\theta_1 < \arg\eta < 3\pi/2 - \theta_2$ when $x \in U$. Moreover, there exists a neighborhood W of $\eta = 0$ such that $V(x, \eta)$ is holomorphic and satisfies (2.2) when $x \in U$, $\theta_1 < \arg\eta < 2\pi - \theta_2$, $\eta \in W$.*

§ 3. Formal series solutions and Gevrey estimate

Formal series expansion in a parameter. We substitute the expansion (2.9) into (2.2) with $u = v$. The left-hand side is given by

$$\eta\mathcal{L}v = \sum_{\nu=0}^{\infty} \mathcal{L}v_\nu(x)\eta^{\nu+1}. \tag{3.1}$$

By the partial Taylor expansion of f with respect to η we have $f(x, u, \eta) = \sum_{\ell=0}^{\infty} f_\ell(x, u)\eta^\ell$. Hence the right-hand side of (2.2) is written as

$$\begin{aligned} f(x, v, \eta) &= \sum_{\ell=0}^{\infty} \eta^\ell f_\ell(x, v_0 + v_1\eta + v_2\eta^2 + \cdots) \\ &= f_0(x, v_0) + \eta(\nabla_u f_0)(x, v_0)v_1 + \eta f_1(x, v_0) + O(\eta^2). \end{aligned} \tag{3.2}$$

By comparing the coefficients of η, we obtain for $\eta^0 = 1$

$$f_0(x, v_0(x)) = 0 \tag{3.3}$$

and for η

$$\mathcal{L}v_0 - f_1(x, v_0) = (\nabla_u f_0)(x, v_0)v_1. \tag{3.4}$$

We solve (3.3) with the condition $v_0(0) = 0$ by means of an implicit function theorem on some Ω_0 in view of the assumption (2.3) and $f_0(0, 0) = f(0, 0, 0)$. Next we solve v_1 from (3.4) on Ω_0 where we may assume $\det(\nabla_u f_0(x, v_0(x))) \ne 0$ on Ω_0 since $\det(\nabla_u f(0, 0, 0)) \ne 0$.

In order to determine $v_\nu(x)$ $(\nu \geq 2)$ we compare the coefficients of η^ν in the both sides of (2.2). Indeed, we differentiate (3.2) $(\nu-1)$-times with respect to η and we put $\eta = 0$. Then we obtain

$$\mathcal{L}v_{\nu-1} = (\nabla_u f_0)(x, v_0)v_\nu + (\text{terms consisting of } v_k,\ k < \nu). \tag{3.5}$$

We observe that the second term in the right-hand side appear from the terms $\eta^\ell f_\ell(x, v_0 + v_1\eta + v_2\eta^2 + \cdots)$ for $\ell \geq 0$ in (3.2), which are products of terms of the form $v_{i_j}\eta^{i_j}$ and η^ℓ such that

$$i_1 + i_2 + \cdots + i_k + \ell = \nu,\ i_1 \geq 0, \ldots, i_k \geq 0, i_j \neq 0$$

for some $k \geq 2$ and $j \leq k$. It follows that all terms v_k's in the second term satisfy $k < \nu$. Therefore one can write (3.5) in the following way

$$\nabla_u f_0(x, v_0)v_\nu = H_\nu(x, v_0, v_1, \ldots, v_{\nu-1}) \quad \text{for all } \nu \geq 2.$$

Since $\det(\nabla_u f(x, v_0(x), 0)) \neq 0$ on Ω_0, one can inductively determine v_ν.

The next theorem gives the existence of a formal solution.

Proposition 3.1. *Suppose that $v_0(x)$ is holomorphic and satisfy (3.3) with $v_0(0) = 0$ on Ω_0 for some neighborhood of the origin Ω_0. Assume that* $\det(\nabla_u f_0(x, v_0)) \neq 0$ *on Ω_0. Then every coefficient v_ν, $\nu \geq 1$ of (2.9) is uniquely determined as a holomorphic function on Ω_0.*

Proof. Suppose that $v_k(x)$ is determined up to some $\ell - 1$ in the neighborhood of the origin for some $\ell \geq 1$. Then, by an implicit function theorem one can determine $v_\ell(x)$ uniquely in some neighborhood of the origin depending on ℓ. Because $v_k(x)$ are determined recursively by differentiations and algebraic calculations, the recurrence formula for $v_\ell(x)$ implies that $v_\ell(x)$ is holomorphic on Ω_0. □

By the same argument as that of Proposition 3.2 in [3] we have the Gevrey estimate of order 1 with respect to the parameter η. Namely we have

Proposition 3.2. *Assume that $f(x, u, \eta)$ be analytic with respect to x in some neighborhood of the origin $0 \in \mathbb{C}^n$ and an entire function of $u \in \mathbb{C}^N$ and $\eta \in \mathbb{C}$. Let v in (2.9) be the formal series solution given by Proposition 3.1. Then there exist a neighborhood U of $x = 0$ and a neighborhood W of $y = 0$ in $\mathbb{C}$ such that $B(v)(x, y)$ converges in $U \times W$.*

§ 4. Convolution estimate

For $\theta > 0$ let Ω be an open set containing $E_+(\pi, \theta)$ such that if $z \in \Omega$, then $z - t \in \Omega$ for every $t \geq 0$ and $z\sigma \in \Omega$ for every $0 \leq \sigma \leq 1$. Let $H(\Omega)$ be the set of holomorphic functions in Ω. For $c > 0$, we define the space $\mathcal{H}_c(\Omega)$ as the set of those $h \in H(\Omega)$ such that there exists $K \geq 0$ for which

$$|h(z)| \leq K e^{c|\Re z|}(1+|z|)^{-2} \quad \text{for all } z \in \Omega. \tag{4.1}$$

Obviously, $\mathcal{H}_c(\Omega)$ is the Banach space with the norm

$$\|h\|_{\Omega,c} := \sup_{z\in\Omega} |h(z)|(1+|z|)^2 e^{-c|\Re z|}. \tag{4.2}$$

The convolution $f * g$ ($f, g \in \mathcal{H}_c(\Omega)$) is defined by

$$(f*g)(z) := \frac{d}{dz}\int_0^z f(z-t)g(t)dt = \frac{d}{dz}\int_0^z f(t)g(z-t)dt. \tag{4.3}$$

Write $f'(z) = (df/dz)(z)$. Then we have

Proposition 4.1. *For every $f, g \in \mathcal{H}_c(\Omega)$ such that $f(0) = g(0) = 0$ and $f', g' \in \mathcal{H}_c(\Omega)$ we have $f * g \in \mathcal{H}_c(\Omega)$ with the estimate*

$$\|f*g\|_{\Omega,c} \leq 8\|f'\|_{\Omega,c}\|g\|_{\Omega,c}, \quad \|f*g\|_{\Omega,c} \leq 8\|f\|_{\Omega,c}\|g'\|_{\Omega,c}. \tag{4.4}$$

Proposition 4.1 can be proved by the similar argument as in the proof of Proposition 4.2 in [3]. Because the domain of the integration of the convolution in our case is different from the one in [3], we give the key point of the proof. Consider

$$\left|\int_0^z f'(z-t)g(t)dt\right| \leq \|f'\|_{\Omega,c}\|g\|_{\Omega,c}\left|\int_0^z e^{c|\Re t|+c|\Re(z-t)|}(1+|z-t|)^{-2}(1+|t|)^{-2}|dt|\right|.$$

Because $\Re t$ and $\Re(z-t)$ have the same sign, by the definition of Ω, we have

$$e^{c|\Re t|+c|\Re(z-t)|} = e^{c|\Re t+\Re(z-t)|} = e^{c|\Re z|}.$$

The estimate of the right-hand side integral and the remaining argument are idential with those given in the proof of Proposition 4.2 in [3].

For $f \in \mathcal{H}_c(\Omega)$ we define $D_z^{-1}f(z) := \int_0^z f(t)dt$, where the integration is taken on the straight line connecting the origin and z. Then we have

Lemma 4.2. *D_z^{-1} is a continuous linear operator defined on $\mathcal{H}_c(\Omega)$ into $\mathcal{H}_c(\Omega)$.*

Proof. Let $f \in \mathcal{H}_c(\Omega)$. By definition we have

$$\begin{aligned} e^{-c|\Re z|}(1+|z|)^2|D_z^{-1}f(z)| &\leq \int_0^z |f(t)|e^{-c|\Re z|}(1+|z|)^2|dt| \\ &\leq \|f\|_c \int_0^z e^{c|\Re t|-c|\Re z|}\frac{(1+|z|)^2}{(1+|t|)^2}|dt|. \end{aligned} \tag{4.5}$$

If $|\Re t| \le (1-\epsilon)|\Re z|$, then we have $e^{c|\Re t|-c|\Re z|}(1+|z|)^2 \le e^{-c\epsilon|\Re z|}(1+|z|)^2$. We see that the right-hand side is bounded by some constant K_0 independent of z and f. Moreover, the integral $\int_{|\Re t|\le(1-\epsilon)|\Re z|}(1+|t|)^{-2}|dt|$ is bounded by $\int_{-\infty}^{\infty}(1+|t|)^{-2}|dt|$. Hence it is bounded by some constant K_1. It follows that

$$\int_{|\Re t|\le(1-\epsilon)|\Re z|} e^{c|\Re t|-c|\Re z|}\frac{(1+|z|)^2}{(1+|t|)^2}|dt| \le K_0K_1. \tag{4.6}$$

Next we shall estimate

$$\int_{|\Re z|\ge|\Re t|\ge(1-\epsilon)|\Re z|} e^{c|\Re t|-c|\Re z|}\frac{(1+|z|)^2}{(1+|t|)^2}|dt|. \tag{4.7}$$

Because (4.7) is bounded if z moves in a bounded set in Ω, we may assume that $|z|$ is sufficiently large. It follows that there exists $0 < c_0 < 1$ such that $|\Re z| \ge c_0|z|$ for such z. Because $|\Re t| \ge (1-\epsilon)|\Re z|$, we have

$$(1+|t|)^2 \ge (1+|\Re t|)^2 \ge (1+(1-\epsilon)c_0|z|)^2 \ge (1-\epsilon)^2c_0^2(1+|z|)^2.$$

On the other hand, since the integral is taken on the straight line connecting 0 and z, it follows that $\Re z$ and $\Re t$ has the same sign. Hence we have $|\Re t| - |\Re z| = -|\Re(z-t)|$. Therefore (4.7) is bounded by

$$(1-\epsilon)^{-2}c_0^{-2}\int_{|\Re z|\ge|\Re t|\ge(1-\epsilon)|\Re z|} e^{-c|\Re(t-z)|}|dt|.$$

By setting $s = t - z$ we see that the integral is equal to

$$(1-\epsilon)^{-2}c_0^{-2}\int_{0\le|\Re s|\le\epsilon|\Re z|} e^{-c|\Re s|}|ds|.$$

Clearly, it is bounded by some constant independent of z. Therefore, by (4.6) there exists $K_3 > 0$ independent of f such that the right-hand side of (4.5) is bounded by $K_3\|f\|_c$. □

§ 5. Proof of Theorem 2.1

First we define a function space. Let D be an open connected set in the neighborhood of the origin of $\mathbb{C}^n$ and define $\Omega := \{z \in \mathbb{C} \mid |\Im z| < \theta/2\}$. Let $H(D,\Omega)$ be the set of holomorphic functions in $(x,y) \in D \times \Omega$. Then we define $\mathcal{H}_c(D,\Omega)$ as the set of those $h \equiv h(x,y) \in H(D,\Omega)$ such that there exists $K_0 \ge 0$ for which

$$\sup_{x\in D}|h(x,y)| \le K_0e^{c|\Re y|}(1+|y|)^{-2} \quad \text{for all } y \in \Omega. \tag{5.1}$$

The space $\mathcal{H}_c(D,\Omega)$ is a Banach space with the norm $\|h\|_c = \inf K_0$ where K_0 is given in (5.1).

Proof of Theorem 2.1. We first show the summability of $v(x,\eta)$ in the direction $\arg\eta = \pi$ when $x \in U$, where U is given in Proposition 3.2. In terms of (2.2) with u replaced by $v_0 + u$, $f_0(x,v_0) = 0$ and

$$f_\ell(x,v) = f_\ell(x,v_0) + \nabla_u f_\ell(x,v_0)\cdot u + \sum_{|\beta|\ge 2} r_{\beta,\ell}(x,v_0)u^\beta, \quad \ell = 0,1,2,\ldots$$

we obtain

$$\mathcal{L}u = -\mathcal{L}v_0 + \eta^{-1}\nabla_u f_0(x,v_0)u + \eta^{-1}\sum_{|\beta|\ge 2} r_{\beta,0}(x,v_0)u^\beta \\ + \sum_{\ell\ge 1}\eta^{\ell-1}\left(f_\ell(x,v_0) + \nabla_u f_\ell(x,v_0)\cdot u + \sum_{|\beta|\ge 2} r_{\beta,\ell}(x,v_0)u^\beta\right). \tag{5.2}$$

Let $\hat{u}(y) := \mathcal{B}(u)$ be the formal Borel transform of u with respect to η, where y is the dual variable of η. By the Borel transform of (5.2) and by recalling that η^{-1} corresponds to $\partial/\partial y$, we obtain

$$\mathcal{L}\hat{u} = -\mathcal{L}v_0 + \nabla_u f_0(x,v_0)\frac{\partial \hat{u}}{\partial y} + \frac{\partial}{\partial y}\sum_{|\beta|\ge 2} r_{\beta,0}(x,v_0)(\hat{u})^{*\beta} \\ + \sum_{\ell\ge 1} D_y^{1-\ell}\left(f_\ell(x,v_0) + \nabla_u f_\ell(x,v_0)\cdot \hat{u} + \sum_{|\beta|\ge 2} r_{\beta,\ell}(x,v_0)(\hat{u})^{*\beta}\right). \tag{5.3}$$

where $(\hat{u})^{*\beta} = (\hat{u}_1)^{*\beta_1}\cdots(\hat{u}_N)^{*\beta_N}$, $\beta = (\beta_1,\ldots,\beta_N)$, and $(\hat{u}_j)^{*\beta_j}$ is the β_j-convolution product, $(\hat{u}_j)^{*\beta_j} = \hat{u}_j * \cdots * \hat{u}_j$. Here $D_y^{1-\ell} = (D_y^{-1})^{\ell-1}$ and D_y^{-1} is the integration, $D_y^{-1}g(y) = \int_0^y g(t)dt$.

Let v be the formal solution given by Proposition 3.1 and let $B(v)$ be the formal Borel transform of v. Define $\hat{u}(x,y) := B(v) - v_0$. Then $\hat{u}(x,y)$, being analytic on $(x,y) \in U \times W$ and satisfying $\hat{u}(x,0) \equiv 0$ in x, is the solution of (5.3) in a neighborhood of $y = 0$. We shall show that every solution of (5.3) being analytic at $y = 0$ and satisfying $\hat{u}(x,0) \equiv 0$ is uniquely determined. Indeed, by definition the convolution product of $y^i/i!$ and $y^j/j!$ is equal to $y^{i+j}/(i+j)!$. Hence, if we expand $\hat{u}$ in the power series of y and we insert it into (5.3), then every coefficient of the expansion can be uniquely determined from the recurrence relation because $\nabla_u f_0(x,v_0)$ is invertible. Therefore, if we can show the existence of the solution of (5.3) being analytic in $(x,y) \in U \times \Omega$ and of exponential growth with respect to y in Ω, then we have the analytic continuation of the formal Borel transform of v which is of exponential growth in $y \in \Omega$. Hence we have the summability of v. Therefore, it is sufficient to prove the following theorem.

Theorem 5.1. *There exist $c > 0$, a neighborhood of $x = 0$, D and Ω as in (5.1) such that (5.3) has a solution $\hat{u}$ in $\mathcal{H}_c(D, \Omega)$.*

For the proof of Theorem 5.1 we prepare six lemmas. Let $c > 0$, D and Ω be given as in the above. We may assume that D is contained in an open ball centered at the origin. In order to prove the solvability of (5.3) when x is in some neighborhood of the origin and $y \in \Omega$ we first consider

$$\mathcal{L}w - (\nabla_u f_0)(x, 0)\frac{\partial w}{\partial y} = g(x, y), \tag{5.4}$$

where $w = (w_1, \ldots, w_N)$ and $g = g(x, y) = (g_1, \ldots, g_N)$, $g_j \in \mathcal{H}_c(D, \Omega)$ is a given vector function and $f_0(x, u) = f(x, u, 0)$.

By the assumption (2.5), for every $j, 1 \le j \le N$ we denote the j-th diagonal component of $(\nabla_u f_0)(x, 0)$ by $(\nabla_u f_0)_j(x, 0)$. We use the method of characteristics in order to solve (5.4). Namely, we consider

$$\frac{d\zeta}{\zeta} = \frac{dx_k}{\lambda_k x_k} = -\frac{dy}{(\nabla_u f_0)_j(x, 0)}, \quad k = 1, 2, \ldots, n-1. \tag{5.5}$$

Let $b \in \mathbb{C}$, $b \neq 0$ be sufficiently small and $y_0 \in \Omega$ be given. By integrating (5.5) we have

$$x_k = x_k(\zeta) = c_k \zeta^{\lambda_k} \ (k = 1, \ldots, n-1), \ y = y_0 - \Phi_j(\zeta, b), \tag{5.6}$$

where

$$\Phi_j(\zeta, b) = \int_b^\zeta (\nabla_u f_0)_j(x_1(s), \cdots, x_{n-1}(s), s, 0)s^{-1}ds, \tag{5.7}$$

and the integral is taken along the non self-intersecting curve which does not encircle the origin. Then we make analytic continuation around the origin. Here $y_0 := y(b) \in \Omega$ is the initial value of $y = y(\zeta)$ at $\zeta = b$ and c_k's are chosen so that the initial point $x^{(0)} := (x_1(b), \ldots, x_{n-1}(b), b)$ lies in D. Define $\Phi(\zeta, b) := (\Phi_1(\zeta, b), \ldots, \Phi_N(\zeta, b))$. Then we have

Lemma 5.2. *Assume (2.7) and (2.8). Let $\zeta_0 \in D \setminus \{0\}$. Then, for every j, $1 \le j \le N$ there exists a curve $\gamma_{\zeta_0, j}$ passing ζ_0 and tending to the origin such that $\Im \Phi_j(\zeta, b) = \Im \Phi_j(\zeta_0, b)$ for every $\zeta \in \gamma_{\zeta_0, j}$, where $\Im \Phi_j$ denotes the imaginary part of Φ_j.*

Proof. The condition $\Im \Phi_j(\zeta, b) = \Im \Phi_j(\zeta_0, b)$ is equivalent to $\Im \Phi_j(\zeta, \zeta_0) = 0$. We shall look for the curve $\gamma_{\zeta_0, j}$ satisfying the latter condition. By the assumption (2.6) there exists a neighborhood Ω_0 of $x = 0$ such that $(\nabla_u f_0)_j(x, 0) = \mu_j + O(|x|)$ when $x \in \Omega_0$. We shall show that there exist $\tilde{\mu}_j$, $\Re \tilde{\mu}_j \neq 0$ $(j = 1, \ldots, n)$ such that

$$\Phi_j(\zeta, \zeta_0) = \int_{\zeta_0}^\zeta s^{-1}(\nabla_u f_0)_j(x, 0)ds = \tilde{\mu}_j \log\left(\frac{\zeta}{\zeta_0}\right) + R(\zeta), \tag{5.8}$$

where $x = x(s)$ is given by (5.6) with c_k sufficiently small so that the integrand $(\nabla_u f_0)_j(x(s), 0)$ is well-defined. First we observe that $\int_{\zeta_0}^{\zeta} s^{-1}\mu_j ds = \mu_j \log(\zeta/\zeta_0)$.

Consider the term containing x^α, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ in the Taylor expansion of $(\nabla_u f_0)_j(x, 0)$ at $x = 0$. In view of (5.6) with $\zeta = s$, the integrand is given by the (infinite) sum with respect to α of constant times of the following integral

$$\int_{\zeta_0}^{\zeta} s^{-1} \prod_{j=1}^{n} c_j^{\alpha_j} s^{\alpha_j \lambda_j} ds \tag{5.9}$$

where $\lambda_n = 1$ and $\alpha \neq 0$.

If α satisfies that $\sum_{j=1}^n \alpha_j \lambda_j = 0$, then by integration we get, from (5.9), $(\prod_{j=1}^n c_j^{\alpha_j}) \log(\zeta/\zeta_0)$. Because we may assume that $\prod c_j^{\alpha_j}$ is sufficiently small, there exists $\tilde{\mu}_j$ such that the first term of the right-hand side of (5.8) appears. We remark that such terms may appear if there is an alpha such that $\sum_{j=1}^n \alpha_j \lambda_j = 0$.

Next we consider the case $\lambda \cdot \alpha \equiv \sum_{j=1}^n \alpha_j \lambda_j \neq 0$. We shall show that the terms corresponding to this case yield $R(\zeta)$. By simple integration in (5.9) we have

$$(\prod_{j=1}^{n} c_j^{\alpha_j})(\zeta^{\lambda\cdot\alpha} - \zeta_0^{\lambda\cdot\alpha})/\lambda \cdot \alpha. \tag{5.10}$$

If ζ approaches the origin from a sector, then $\zeta^{\lambda\cdot\alpha}$ is bounded by $K_0^{|\alpha|+1}$ for some constant K_0 independent of α because $\Re\lambda_j > 0$ for every j. Hence, if we take c_j sufficiently small, then (5.10) is bounded in ζ as ζ approaches the origin from a sector. Therefore we have proved the boundedness of $R(\zeta)$.

We shall show the existence of the curve $\gamma_{\zeta_0,j}$ which tends to the origin. By (5.8) the relation $\Im\,\Phi_j(\zeta, \zeta_0) = 0$ is written as

$$\Im\,(\tilde{\mu}_j \log \zeta) + \Im\, R(\zeta) - \Im\,(\tilde{\mu}_j \log \zeta_0) = 0. \tag{5.11}$$

By setting

$$\tilde{\zeta} := \tilde{\mu}_j \log \zeta, \quad \tilde{\zeta}_0 := \tilde{\mu}_j \log \zeta_0, \tag{5.12}$$

and $\tilde{R}(\tilde{\zeta}) := R(e^{\tilde{\zeta}/\tilde{\mu}_j})$ (5.11) is written as

$$\Im\,\tilde{\zeta} + \Im\,\tilde{R}(\tilde{\zeta}) - \Im\,\tilde{\zeta}_0 = 0. \tag{5.13}$$

Set $\tilde{\zeta} = \tilde{x} + i\tilde{y}$ and $\tilde{\zeta}_0 = \tilde{x}_0 + i\tilde{y}_0$. We shall determine $\tilde{y} = \tilde{y}(\tilde{x})$ from (5.13) such that $\tilde{y}(\tilde{x}_0) = \tilde{y}_0$. We remark that (5.13) holds if $\tilde{\zeta} = \tilde{\zeta}_0$. On the other hand, we observe that $\tilde{R}(\tilde{\zeta})$ is bounded when ζ is in some neighborhood of the origin. Moreover, the derivative of $\tilde{R}(\tilde{\zeta})$ with respect to y is also bounded. Thus, if $|\tilde{x}_0|$ is sufficiently large and $|\tilde{x}| \geq |\tilde{x}_0|$, then the derivative with respect to $\tilde{y}$ of (5.13) does not vanish. By the implicit function theorem one can determine $\tilde{y} = \tilde{y}(\tilde{x})$ as an analytic function of $\tilde{x}$, if $|\tilde{x}_0|$ is sufficiently large and $|\tilde{x}| \geq |\tilde{x}_0|$. We denote the curve by $\tilde{\gamma}_{\tilde{\zeta}_0,j}$.

We transform $\tilde{\gamma}_{\tilde{\zeta}_0,j}$ in the $\tilde{\zeta}$ space to $\gamma_{\zeta_0,j}$ in the ζ space by the change of variables, (5.12). Set $\tilde{\mu}_j = \tilde{\mu}_{j,0} + i\tilde{\mu}_{j,1}$ with $\tilde{\mu}_{j,0} \neq 0$. Then we have

$$\frac{\tilde{\zeta}}{\tilde{\mu}_j} = \frac{1}{|\tilde{\mu}_j|^2} \left(\tilde{x}\tilde{\mu}_{j,0} - \tilde{y}\tilde{\mu}_{j,1} + i(\tilde{x}\tilde{\mu}_{j,1} + \tilde{y}\tilde{\mu}_{j,0}) \right). \tag{5.14}$$

We recall that $\tilde{y}(\tilde{x})$ is a bounded function of $\tilde{x}$ and its derivative tends to zero as $|\tilde{x}| \to \infty$. Because we have $\tilde{\mu}_{j,0} \neq 0$ by assumption, we have that

$$\begin{aligned} \tilde{x}\tilde{\mu}_{j,0} &= \Re\tilde{\mu}_j \Re\tilde{\zeta} = \Re\tilde{\mu}_j \Re(\tilde{\mu}_j \log\zeta) = \Re\left((\Re\tilde{\mu}_j)\tilde{\mu}_j \log\zeta \right) \\ &= (\Re\tilde{\mu}_j)^2 \log|\tilde{\zeta}| - (\Re\tilde{\mu}_j)(\Im\tilde{\mu}_j)(\arg\tilde{\zeta}). \end{aligned}$$

In view of the definition of $\tilde{\gamma}_{\tilde{\zeta}_0,j}$, $\arg\tilde{\zeta}$ is bounded. Hence we have $\tilde{x}\tilde{\mu}_{j,0} \to -\infty$. It follows that $\zeta = \exp(\tilde{\zeta}/\tilde{\mu}_j) \to 0$. □

Lemma 5.3. *Assume (2.7) and (2.8). Let $c \neq 0$ and $\zeta_0 \neq 0$ be given complex constants. Then, for every j, $1 \leq j \leq N$, $\Re\,\Phi_j(\zeta, c)$ is monotone when ζ approaches to the origin along the curve $\gamma_{\zeta_0,j}$.*

Proof. Because the curves $\gamma_{\zeta_0,j}$ and $\tilde{\gamma}_{\tilde{\zeta}_0,j}$ are diffeomorphic by the relation $\zeta = \exp(\tilde{\zeta}/\mu_j)$ it is sufficient to show that $\Re\,\Phi_j(\zeta,\zeta_0)$ is a monotone function when $\tilde{\zeta}$ moves along $\tilde{\gamma}_{\tilde{\zeta}_0,j}$. In terms of (5.8) we have

$$\Re\,\Phi_j(\zeta,\zeta_0) + \Re\left(\mu_j \log\zeta_0\right) = \Re\,\tilde{\zeta} + \Re\,\tilde{R}(\tilde{\zeta}) = \tilde{x} + \Re\,\tilde{R}(\tilde{\zeta}). \tag{5.15}$$

By the proof of Lemma 5.2 we see that $\tilde{R}(\tilde{\zeta}) = R(\zeta)$ and its derivative with respect to $\tilde{x}$ tends to zero. Thus, with $\tilde{\zeta} = \tilde{x} + i\tilde{y}$ and $\tilde{y} = \tilde{y}(\tilde{x})$, we see that $\left|(d/d\tilde{x})\Re\,\tilde{R}(\tilde{\zeta})\right|$ can be made small when $|\tilde{x}| \geq |\tilde{x}_0|$ for some $\tilde{x}_0$. Therefore we have $(d/d\tilde{x})\Re\,\Phi_j(\zeta,\zeta_0) > 0$, and the assertion follows. □

Lemma 5.4. *Assume (2.7) and (2.8). Let $g = g(x,y) = (g_1, \ldots, g_N)$, $g_j \in \mathcal{H}_c(D,\Omega)$. Then the solution of (5.4) is given by*

$$w = P_0 g := (P_{0,1}g_1, \ldots, P_{0,N}g_N). \tag{5.16}$$

Here, for every j, $1 \leq j \leq N$ and $\zeta \neq 0$ in a neigborhood of the origin we take ζ_0 such that $\zeta \in \gamma_{\zeta_0,j}$ and $P_{0,j}$ is given by

$$P_{0,j}g_j := \int_{\zeta_0}^{\zeta} g_j(x_1(s), \cdots, x_{n-1}(s), s; y_0 - \Phi_j(s,b))s^{-1}ds, \tag{5.17}$$

where the integral is taken along the curve $\gamma_{\zeta_0,j}$ from ζ_0 to $\zeta \in \gamma_{\zeta_0,j}$ and $x_j(s)$ is given by (5.6) with $\zeta = s$. The independent variables x_k and y in (5.17) are related to c_k and y_0 via (5.6).

Proof. We show that the integrand in (5.17) is well-defined. By (5.6) and (5.7) we have

$$y_0 - \Phi_j(s,b) = y - \Phi_j(s,b) + \Phi_j(\zeta,b) = y + \Phi_j(\zeta,s). \tag{5.18}$$

By Lemma 5.2 we see that $\Im\Phi_j(\zeta,s) = 0$ if $s \in \gamma_{\zeta_0,j}$ because $\zeta \in \gamma_{\zeta_0,j}$. In view of the assumption on Ω we have $y + \Re\,\Phi_j(\zeta,s) \in \Omega$ for every $y \in \Omega$.

Next we take the neighborhood U_0 of the origin such that the formal solution is holomorphic in U_0. Consider the substitution $x_k = x_k(s)$ into the integrand of (5.17) where $x_k(s)$ is given by (5.6) with $\zeta = s$. This is possible for s on the segment of $\gamma_{\zeta_0,j}$ between ζ_0 and ζ if c_k is sufficiently small. Indeed, since $\Re\,\lambda_k > 0$, the definition of $\gamma_{\zeta_0,j}$ implies that $s^{\lambda_k} c_k = \exp\left(\lambda_j(\log|s| + i\arg s)\right) c_k$ is small if c_k is sufficiently small.

Next we shall show that $w_j := P_{0,j}g_j$ $(j = 1,2,\ldots,N)$ satisfies the equation (5.4), namely

$$\mathcal{L}w_j - (\nabla_u f_0)_j(x,0)\frac{\partial w_j}{\partial y} = g_j(x,y). \tag{5.19}$$

Indeed, by (5.5) and (5.6) we have

$$\begin{aligned} g_j(x,y)x_n^{-1} = \frac{dw_j}{d\zeta} &= \sum_{k=1}^{n} \frac{\partial x_k}{\partial \zeta}\frac{\partial w_j}{\partial x_k} + \frac{\partial y}{\partial \zeta}\frac{\partial w_j}{\partial y} \\ &= \sum_{k=1}^{n} \frac{\lambda_k x_k}{\zeta}\frac{\partial w_j}{\partial x_k} - \frac{(\nabla_u f_0)_j(x,0)}{\zeta}\frac{\partial w_j}{\partial y}. \end{aligned} \tag{5.20}$$

Multiplying both sides with ζ and setting $\zeta = x_n$ we have (5.19). This completes the proof. $\square$

Let ζ_0 satisfy $|\zeta_0| = r_0 > 0$. In the following we assume that there exists an $\varepsilon_0 > 0$ such that $|\zeta|/|\zeta_0| \geq \varepsilon_0$ for ζ and ζ_0 in D, where ζ and ζ_0 are related by $\zeta \in \gamma_{\zeta_0,j}$. Then we have

Lemma 5.5. *Assume (2.7) and (2.8). Then, there exists a constant c_1 such that, for every $1 \leq j \leq N$, $g_j \in \mathcal{H}(D,\Omega)$, we have*

$$\|P_{0,j}g_j\|_c \leq c_1\|g_j\|_c, \quad \left\|\frac{\partial}{\partial y}(P_{0,j}g_j)\right\|_c \leq c_1\|g_j\|_c. \tag{5.21}$$

The constant c_1 is independent of ζ_0, $|\zeta_0| = r_0 > 0$.

Proof. Let $\zeta \in \gamma_{\zeta_0,j}$ and consider the integral (5.17). Noting that $y_0 - \Phi_j(s,b) = y + \Phi_j(\zeta,s)$ we make the change of variable $\sigma = y + \Phi_j(\zeta,s)$ in (5.17) from s to σ. We have $d\sigma = -\frac{(\nabla_u f)_j}{s}ds$. We have $\sigma = y$ for $s = \zeta$ and $\sigma = y + \tilde{\zeta_0}$ for $s = \zeta_0$, where

$\tilde{\zeta_0} = \Phi_j(\zeta, \zeta_0)$. Clearly, if s moves on $\gamma_{\zeta_0,j}$, then σ moves on $y + \gamma_{\tilde{\zeta_0},j}$, where $\gamma_{\tilde{\zeta_0},j}$ is the straight line connecting 0 and $\tilde{\zeta_0}$. Then (5.17) is written as

$$w = -\int_{y+\gamma_{\tilde{\zeta_0},j}} g(x_1(s), \cdots, x_{n-1}(s), s; \sigma)\frac{d\sigma}{(\nabla_u f_0)_j}. \tag{5.22}$$

Note that $(\nabla_u f_0)_j$ is bounded from the below by (2.3).

We next estimate the growth of $y_0 - \Phi_j(s, b)$. In terms of (5.18) we have

$$\begin{aligned}\exp\left(c|\Re\left(y_0 - \Phi_j(s, b)\right)|\right) &= \exp\left(c|\Re\left(y + \Phi_j(\zeta, s)\right)|\right) \\ &\le \exp\left(c|\Re\, y| + c|\Re\,\Phi_j(\zeta, s)|\right).\end{aligned} \tag{5.23}$$

Hence we need to estimate $e^{c|\Re\,\Phi_j(\zeta,s)|}$, namely we shall estimate $|\Re\,\Phi_j(\zeta, s)|$ from the above.

Because $R(\zeta)$ in (5.8) is bounded, $\Phi_j(\zeta, s)$ has asymptotic behavior

$$\Phi_j(\zeta, s) = \tilde{\mu}_j \log(\zeta/s)\,(1 + O(1)), \tag{5.24}$$

for $s \in \gamma_{\zeta_0,j}$ if $|\log \zeta_0|$ is sufficiently large. Set $\log(\zeta/s) = x + iy$ and $\tilde{\mu}_j = \alpha + i\beta$ with $\alpha \ne 0$. Then we have $\Re\left(\tilde{\mu}_j \log(\zeta/s)\right) = \alpha x - \beta y$ and $\Im\left(\tilde{\mu}_j \log(\zeta/s)\right) = \alpha y + \beta x$. It follows that $|\Re\,\Phi_j(\zeta, s)| \le \gamma_0|\alpha x - \beta y|$ for some $\gamma_0 > 0$ if $|\log \zeta_0|$ is sufficiently large. The condition holds if ζ and ζ_0 are in a sector of a neighborhood of the origin.

By (5.24) there exist $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\gamma_1|\alpha y + \beta x| \le |\Im\,\Phi_j(\zeta, s)| \le \gamma_2|\alpha y + \beta x|, \tag{5.25}$$

if $|\log \zeta_0|$ is sufficiently large. Because $\Im\,\Phi_j(\zeta, s)$ is constant when $\zeta \in \gamma_{\zeta_0,j}$, there exist $\tilde{\gamma}_1 > 0$ and $\tilde{\gamma}_2 > 0$ such that $\tilde{\gamma}_2 \le |\alpha y + \beta x| \le \tilde{\gamma}_1$, if $\zeta \in \gamma_{\zeta_0,j}$. It follows that $|y + \beta\alpha^{-1}x| \le |\alpha^{-1}|\tilde{\gamma}_1$. Therefore we have $|\beta y| \le |\beta^2\alpha^{-1}x| + |\beta\alpha^{-1}|\tilde{\gamma}_1$. Hence we have

$$|\alpha x - \beta y| \le |\alpha x| + |\beta y| \le (|\alpha| + |\beta^2\alpha^{-1}|)|x| + \tilde{\gamma}_1|\beta\alpha^{-1}|.$$

Noting that $x = \log(|\zeta|/|s|) > \log(|\zeta|/|\zeta_0|) > \log\varepsilon_0$, we have

$$\begin{aligned}e^{c|\Re\,\Phi_j(\zeta,s)|} &\le e^{c|\alpha x - \beta y|} \le \exp(c(|\alpha| + |\beta^2\alpha^{-1}|)|x| + c\tilde{\gamma}_1|\beta\alpha^{-1}|) \\ &\le \exp\left(c(|\alpha| + |\beta^2\alpha^{-1}|)|\log\varepsilon_0^{-1}| + c\tilde{\gamma}_1|\beta\alpha^{-1}|\right) =: K_0.\end{aligned} \tag{5.26}$$

This proves

$$\exp\left(c|\Re\left(y_0 - \Phi_j(s, b)\right)|\right) \le K_0 \exp\left(c|\Re\, y|\right). \tag{5.27}$$

We shall estimate $|y_0 - \Phi_j(s, b)| = |y + \Phi_j(\zeta, s)|$ from the below. Because $\Im\,\Phi_j(\zeta, s) = 0$ and $|\Re\,\Phi_j(\zeta, s)|$ is bounded on $\gamma_{\zeta_0,j}$ by the argument in proving (5.26), there exists $C_1 > 0$ independent of ζ and s such that

$$(1 + |y_0 - \Phi_j(s, b)|)^{-2} \le C_1(1 + |y|)^{-2} \quad \text{for all } y \in \Omega. \tag{5.28}$$

Therefore we get, from (5.27) and (5.28) that

$$\begin{aligned}(5.29)\quad &\|w_j\|_c \le \\ &\le \sup\left((1+|y|)^2 \exp(-c|\Re y|)\int \|g_j\|_c \frac{\exp(-c|\Re(y_0-\Phi_j(s,b))|)}{(1+|y_0-\Phi_j(s,b)|)^2}|d\sigma|\right) \\ &\le C_2\|g_j\|_c \int |d\sigma| \le C_3\|g_j\|_c,\end{aligned}$$

for some $C_2 > 0$ and $C_3 > 0$.

We shall show the latter inequality of (5.21). By (5.22) we have

$$\begin{aligned} w_y &= -g(x_1(\zeta_0),\cdots,x_{n-1}(\zeta_0),\zeta_0;y+\tilde{\zeta_0})\frac{1}{(\nabla_u f_0)_j} \\ &\quad + g(x_1(\zeta),\cdots,x_{n-1}(\zeta),\zeta;y)\frac{1}{(\nabla_u f_0)_j}. \end{aligned} \tag{5.30}$$

Using (5.30) we have the latter inequality of (5.21) by the same argument as $\|w\|_c$ since $(\nabla_u f_0)_j$ is bounded. □

We shall solve (5.3) in $\mathcal{H}_c(D,\Omega)$. Write

$$\nabla_u f_0(x,v_0)\frac{\partial \hat{u}}{\partial y} = \nabla_u f_0(x,0)\frac{\partial \hat{u}}{\partial y} + (\nabla_u f_0(x,v_0) - \nabla_u f_0(x,0))\frac{\partial \hat{u}}{\partial y}. \tag{5.31}$$

Since $\|\nabla_u f_0(x,v_0) - \nabla_u f_0(x,0)\| = O(\|v_0\|)$ when $\|v_0\| \to 0$, the second term is estimated by $K_4\varepsilon\|(\hat{u})_y\|_c$ for arbitrarily small ε and some constant K_4.

We define the approximate sequence $\hat{u}_k$ $(k = 0,1,2,\ldots)$ by $\hat{u}_0 = 0$ and

$$\hat{u}_1 = -P_0\mathcal{L}v_0 \tag{5.32}$$

$$\begin{aligned}\hat{u}_2 &= P_0\sum_{|\beta|\ge 2} r_\beta(x,v_0)\frac{\partial}{\partial y}(\hat{u}_1)^\beta_* - P_0\mathcal{L}v_0 + P_0R(x)\frac{\partial}{\partial y}\hat{u}_1 \\ &\quad + P_0(\nabla_u f_0(x,v_0) - \nabla_u f_0(x,0))\frac{\partial \hat{u}_1}{\partial y},\end{aligned} \tag{5.33}$$

$$\vdots$$

$$\begin{aligned}\hat{u}_{k+1} &= P_0\sum_{|\beta|\ge 2} r_\beta(x,v_0)\frac{\partial}{\partial y}(\hat{u}_k)^\beta_* - P_0\mathcal{L}v_0 + P_0R(x)\frac{\partial}{\partial y}\hat{u}_k \\ &\quad + P_0(\nabla_u f_0(x,v_0) - \nabla_u f_0(x,0))\frac{\partial \hat{u}_k}{\partial y},\end{aligned} \tag{5.34}$$

where $k = 1,2,\ldots$ Then we have

Lemma 5.6. *Assume (2.7) and (2.8). Let D be as in Lemma 5.5. Then there exists a constant $K_3 > 0$ independent of k such that*

$$\|\hat{u}_k\|_c \le C\varepsilon K_3, \quad \|(\hat{u}_k)_y\|_c \le C\varepsilon K_3,\ k = 0,1,2,\ldots \tag{5.35}$$

The proof of Lemma 5.6 is done by the same arugument in proving Lemma 5.6 of [3] once we have proved Lemmas 5.2, 5.3, 5.4 and 5.5. The following lemma implies the solvability of (5.3) in $\mathcal{H}_c(D, \Omega)$.

Lemma 5.7. *Under the same assumptions as in Lemma 5.6 we have that* $\hat{u}_k$ $(k = 1, 2, \ldots)$ *converges in* $\mathcal{H}_c(D, \Omega)$.

The proof is done by the same argument of that of Lemmas 5.7 of [3] by using Lemma 5.6.

Proof of Theorem 5.1. First we shall solve (5.3) in $\mathcal{H}_c(D, \Omega)$. We remark that the solvability of (5.3) yields the summability on D. Set $T_0 = \{\zeta \mid \varepsilon_0 r_0 < |\zeta| < r_0\}$, $r_0 = |\zeta_0|$ and take an open connected and simply connected set $A_0 \subset T_0$. Let D be such that $(x_1(\zeta), \ldots, x_{n-1}(\zeta), \zeta) \in D$ for every $\zeta \in A_0$, where $x_k(\zeta)$ is given by (5.6). By Lemma 5.7 we have the solvability of (5.3) on D. Next, take a set $A_1 \subset T_0$ with the same property as A_0 such that $A_0 \cap A_1 \neq \emptyset$. Then we have the summability on some D_1 corresponding to A_1. By virtue of the uniqueness of the Borel sum two sums corresponding to A_0 and A_1 coincide on the set $A_0 \cap A_1$. Hence we have an analytic continuation of the solution of (5.3) to the domain corresponding to $A_0 \cup A_1$. By repeating the arguement we have the solvability of (5.3) for D corresponding to T_0.

Next we take annulus T_1 with r_0 replaced by r_1 such that $T_0 \cap T_1 \neq \emptyset$. Then we have the summability on the domain corresponding to T_1. Moreover, in the proof of Lemma 5.5 the constant c_1 in the estimate in (5.21) depends on ϵ_0 and is independent of ζ_0. Hence we have the solvability of (5.3) in the same domain because we have the solvability in $\mathcal{H}_c(D_1, \Omega)$ for the same c and Ω. By the uniqueness of the Borel sum we can make analytic continuation with respect ζ. In order to show that the analytic continuation of the soution $\hat{u}$ is possible to a small neighborhood of the origin except for the origin $\zeta \neq 0$ we note that the length of the analytic continuation is given by $1 - \epsilon_0 + (1 - \epsilon_0)\epsilon_0 + (1 - \epsilon_0)\epsilon_0^2 + \cdots = 1$ because $0 < \epsilon_0 < 1$. We denote the solution by $\hat{u}_D$.

Let D' be any domain such that $D \cap D' \neq \emptyset$ and let $\hat{u}_D$ and $\hat{u}_{D'}$ be the corresponding solution in D and D', respectively. By the uniqueness for every x, we have that $\hat{u}_D = \hat{u}_{D'}$ on $D \cap D'$, from which we have an analytic continuation of $\hat{u}_D$ to $D \cup D'$. By choosing the sequence of open sets D we make an analytic continuation of $\hat{u}_D$ to the set $(\mathbb{C} \setminus 0)^n \cap B_0$, where B_0 is a small open ball centered at the origin. By the uniqueness the analytic continuation of $\hat{u}_D(x, y)$ with respect to x to the set $(\mathbb{C} \setminus 0)^n \cap B_0$, $y \in \Omega$ is single-valued. We also note that in view of the construction of $\hat{u}_D$ the growth estimate with respect to y of $\hat{u}_D(x, y)$ is uniform for $x \in (\mathbb{C} \setminus 0)^n \cap B_0$. Therefore we can define $\hat{u}(x, y) := \hat{u}_D(x, y)$ on $x \in (\mathbb{C} \setminus 0)^n \cap B_0$ and $y \in \Omega$ by taking $x \in D$.

The function $\hat{u}(x, y)$ may have singularity on $x \in (\mathbb{C}^n \setminus (\mathbb{C} \setminus 0)^n) \cap B_0$, $y \in \Omega$. We shall prove that the singularity is removable. First, consider the singularity with codimension 1. For simplicity, let us take $y_0 \in \Omega$, $x_0' = (x_2^0, \ldots, x_n^0)$ with $x_j^0 \neq 0$ and consider the expansion

$$\hat{u}(x,y) = \sum_{\nu \geq 0, j \geq 0} \hat{u}_{\nu,j}(x_1)(x' - x_0')^\nu (y - y_0)^j. \tag{5.36}$$

By what we have proved in the above, the right-hand side is convergent if $x' - x_0'$ and $y - y_0$ are sufficiently small and $x_1 \neq 0$. Moreover, by the boundedness of $\hat{u}(x,y)$ when $x_1 \to 0$ and the Cauchy's integral formula we have that $\hat{u}_{\nu,j}(x_1)$ is holomorphic and single-valued and bounded in some neighborhood of the origin except for $x_1 = 0$. Hence, its singularity is removable. In the same way, one can show that the singularity of codimension 1 is removable.

Next we consider the singularity of codimension 2. For the sake of simplicity, consider the one $x_1 = x_2 = 0$, $x_0'' = (x_3^0, \ldots, x_n^0)$ with $x_j^0 \neq 0$. By arguing in the same way as in the codimension-one case we have the expansion similar to (5.36) where $x' - x_0'$ and $\hat{u}_{\nu,j}(x_1)$ are replaced by $x'' - x_0''$ and $\hat{u}_{\nu,j}(x_1, x_2)$, respectively. Because $\hat{u}_{\nu,j}(x_1, x_2)$ is holomorphic and single-valued except for $x_1 = x_2 = 0$, we see that the singularity is removable by Hartogs' theorem. As for the singularity of higher codimension ≥ 3 we can argue in the same way by using Hartogs' theorem. We see that $\hat{u}(x,y)$ is holomorphic and single-valued on $x \in \mathbb{C}^n \cap B_0$, $y \in \Omega$.

The exponential growth of $\hat{u}(x,y)$ when $y \to \infty$ in $y \in \Omega$ for $x \in \mathbb{C}^n \cap B_0$ can be proved by putting some c_k to be zero when constructing $\hat{u}_D(x,y)$. Indeed, we have already proved the fact in the above argument. Hence we have proved the solvability of (5.3). This completes the proof of Theorem 5.1.

End of the proof of Theorem 2.1. We shall prove the summability in the direction $\arg \eta$ with $\pi/2 + \theta_1 < \arg \eta < 3\pi/2 - \theta_2$. By multiplying (2.2) with $e^{-i\theta}$ we see that η, λ_k, μ_j are replaced by $\eta e^{-i\theta}$, λ_k and $\mu_j e^{-i\theta}$, respectively. We choose $\theta \geq 0$ such that $\Re\,(\mu_j e^{-i\theta})$ and $\Re\,\mu_j$ have the same sign for every j. In view of the definition of $C_0^\pm$ we see that the requirement holds for θ such that $0 \leq \theta < \pi/2 - \theta_2^\pm$. Recalling that $\theta_2 = \max\{\theta_2^+, \theta_2^-\}$, these inequalities are equivalent to $0 \leq \theta < \pi/2 - \theta_2$. Therefore the summability follows for $\eta = e^{i(\pi+\theta)}$ with $0 \leq \theta < \pi/2 - \theta_2$, namely, for $\pi \leq \arg \eta < 3\pi/2 - \theta_2$.

Next we set $u = ve^{i\theta}$, and consider the equation of v. Clearly, η and μ_k are relaced by $\eta e^{i\theta}$ and $\mu_k e^{i\theta}$, respectively, while λ_k does not change. On the other hand, the reduced equation satisfies that $\Re\,(\mu_j e^{-i\theta})$ and $\Re\,\mu_j$ have the same sign for every j when $0 \leq \theta < \pi/2 - \theta_1$. It follows that the summability holds for $\pi/2 + \theta_1 < \arg \eta \leq \pi$. Therefore, the summability holds for $\pi/2 + \theta_1 < \arg \eta < 3\pi/2 - \theta_2$. In view of the definition of Borel sum we have the latter half of the assertion. This ends the proof of Theorem 2.1.

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