

Classical and quantum vertex algebras

By

Shintarou YANAGIDA *

Abstract

We give a review on Borcherds' theory of (\mathcal{A}, H, S) -vertex algebras and its deformed version with several concrete examples. We explain that deformed chiral algebras in the sense of E. Frenkel and Reshetikhin are examples of quantum (\mathcal{A}, H, S) -vertex algebras. We also investigate the relationship between (\mathcal{A}, H, S) -vertex algebras and chiral algebras introduced by Beilinson and Drinfeld.

§ 0. Introduction

§ 0.1.

The notion of vertex algebra [B86] was introduced by Borcherds for a formulation of two-dimensional conformal field theory. Although this formulation is successful for encoding algebraic structure of conformal field theory and giving representation theoretic treatment, it involves somewhat complicated axioms and lacks geometric interpretation of the quantum field theory.

In [B98] and [B01], Borcherds himself reformulated the axiom of vertex algebras and built the theory of (\mathcal{A}, H, S) -vertex algebras. One of the motivation of his reformulation was the simplification of the axioms of vertex algebras, Another motivation seems to relate deformations of vertex algebras with the non-deformed vertex algebras in a simple way.

In this note we give a review on Borcherds' formulation of (\mathcal{A}, H, S) -vertex algebras and its deformed version called quantum (\mathcal{A}, H, S) -vertex algebras. We will give several concrete examples for both non-quantum and quantum algebras. In particular, we

Received November 30, 2013. Revised November 18, 2016.

2010 Mathematics Subject Classification(s): 16T10,17B37

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan, 606-8502

(current: Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Japan, 464-8602)

e-mail: yanagida@math.nagoya-u.ac.jp

© 2017 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

explain that deformed chiral algebras in the sense of Frenkel and Reshetikhin [FR96] can be considered as quantum (\mathcal{A}, H, S) -vertex algebras (Theorem 3.7).

We also investigate the relationship between (\mathcal{A}, H, S) -vertex algebras and chiral algebras due to Beilinson and Drinfeld [BD04]. We show that an (\mathcal{A}, H, S) -vertex algebra in a geometric setting gives the reformulation of the chiral algebra (Theorem 4.2). Since the notion of chiral algebra has geometric flavor, we may say that the (\mathcal{A}, H, S) -vertex algebra help us in geometric investigation of conformal field theory and its quantum deformations.

§ 0.2. Organization

Let us explain the organization of this note briefly. In §1, we review the theory of Borchers' (\mathcal{A}, H, S) -vertex algebras, so the readers who are familiar with the discussion in [B98] and [B01] may skip the details in this part. Let us mention that in §1.3 we included a slightly generalized treatment of twisting construction. In §2 we give a few examples of (\mathcal{A}, H, S) -vertex algebras. In §3 we recall the notion of quantum (\mathcal{A}, H, S) -vertex algebras. Its relationship with the deformed chiral algebra is stated in §3.3. In the final §4 we investigate the relationship between (\mathcal{A}, H, S) -vertex algebras and the Beilinson-Drinfeld chiral algebras.

§ 0.3. Notations

Let us explain some global notations in this note.

- For a category \mathcal{C} , the class of objects is denoted by $\text{Ob}(\mathcal{C})$ or $\text{Ob}\mathcal{C}$, and the class of morphisms between objects A, B is denoted by $\mathcal{C}(A, B)$ or $\text{Hom}_{\mathcal{C}}(A, B)$.
- The composition of morphisms $f : A \rightarrow B$ and $g : B \rightarrow C$ is denoted by $g \circ f$.
- Functors between categories means covariant functors.
- For a category \mathcal{C} , its opposite category is denoted by \mathcal{C}^{op} .
- For a bialgebra B over a commutative ring R let us denote by Δ_B and ε_B the comultiplication $B \otimes_R B \rightarrow B$ and the counit $B \rightarrow R$.
- For an element a of a bialgebra B we express the comultiplication of a by

$$\Delta_B(a) = \sum a' \otimes a'' = \sum_{(a)} a' \otimes a'' = \sum_{(a)} a^{(1)} \otimes a^{(2)}.$$

Vertex algebras in the sense of [B86] will be called *ordinary vertex algebras* (see Definition 1.12 for the precise definition).

Acknowledgement

This work is partially supported by the JSPS Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation “Deepening and Evolution of Mathematics and Physics, Building of International Network Hub based on OCAMI”.

§ 1. Borcherds’ formulation of (\mathcal{A}, H, S) -vertex algebras

In this section we review the formulation of (\mathcal{A}, H, S) -vertex algebra introduced by Borcherds [B01], which is a generalization (and simplification) of the classical axiom of vertex algebras [B86].

§ 1.1. Categorical setting

Definition.

1. Consider a category whose objects are finite sets and whose morphisms are arbitrary maps between them. Denote its skeleton by Fin .
2. Consider a category whose objects are finite sets and an equivalence relation \equiv , and whose morphisms are the maps f preserving inequivalence, i.e., we have $a \equiv b$ if $f(a) = f(b)$. Denote its skeleton by Fin^{\neq} .

Note that both Fin and Fin^{\neq} are small.

Although these categories are defined as skeletons of some other categories and objects should be called ‘isomorphic classes of sets’, we call them just by ‘sets’ for simplicity.

Objects of Fin will be expressed by $\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \dots$, in form of finite sets. We will also use the symbols $\{2\}, \{3\}$ for plain explanations in the discussion later, although these objects are the same as $\{1\}$.

When denoting an object of Fin^{\neq} , we will use colons to separate equivalence classes. For example, $\{1; 2\}$ means a set consisting of two objects with two equivalent classes, and $\{1, 2\}$ means a set consisting of two objects with one equivalent class.

The disjoint union is a coproduct on the category Fin , and it makes Fin into a symmetric monoidal category (in the sense of [M98]). We denote the disjoint union in Fin by the symbol \sqcup . We may define an analogue of the disjoint union for Fin^{\neq} as follows.

Definition. For objects I and J in Fin^{\neq} , we define $I \sqcup J$ to be the disjoint union of I and J as sets with the equivalence relation where an element of I and another of J are inequivalent and the other cases are determined by the equivalence relations in I and J . We call this \sqcup on Fin^{\neq} simply by disjoint union.

Then the disjoint union \sqcup on Fin^{\neq} gives a symmetric monoidal structure on Fin^{\neq} , although it is not a coproduct on Fin^{\neq} as mentioned in [B01, §3].f

Also note that Fin can be considered as a full subcategory of Fin^{\neq} by imposing the indiscrete equivalence on each set (all the elements in a set are defined to be equivalent). This embedding is denoted by

$$(1.1) \quad \iota : \text{Fin} \hookrightarrow \text{Fin}^{\neq}$$

In the following we fix a category \mathcal{A} which is additive, symmetric monoidal, co-complete and such that colimits commute with tensor products. We denote by \otimes the bifunctor $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ giving the monoidal structure of \mathcal{A} , and by $\mathbf{1}$ the unit object. When emphasizing that we are considering the monoidal category \mathcal{A} , we sometimes denote the tensor product by $\otimes_{\mathcal{A}}$. The isomorphism $M \otimes_{\mathcal{A}} N \rightarrow N \otimes_{\mathcal{A}} M$ giving the symmetric monoidal structure on \mathcal{A} will be denoted by $\sigma_{M,N}$ and called symmetry.

The main example of \mathcal{A} we consider is the category $R\text{-Mod}$ of modules over a commutative ring R . The tensor product is given by the tensor product \otimes_R of modules over R , and the symmetry is given by the transposition map $\sigma_{M,N} : M \otimes_R N \rightarrow N \otimes_R M$ of R -modules.

Definition. For a category \mathcal{C} let us denote by $\text{Fun}(\mathcal{C}, \mathcal{A})$ the category of functors from \mathcal{C} to \mathcal{A} . By the additive monoidal structure on \mathcal{A} , the category $\text{Fun}(\mathcal{C}, \mathcal{A})$ is an additive symmetric monoidal structure, where the tensor product is given by $(U \otimes V)(I) := U(I) \otimes_{\mathcal{A}} V(I)$ for $I \in \text{Ob}(\mathcal{A})$ and $U, V \in \text{Ob}(\text{Fun}(\mathcal{C}, \mathcal{A}))$.

Let us recall the notion of *rings (or algebras) in monoidal categories*. A ring object A of a monoidal category $(\mathcal{D}, \otimes, \mathbf{1})$ is an object of \mathcal{D} such that for any $X \in \text{Ob}(\mathcal{D})$ the set of morphisms $\text{Hom}_{\mathcal{D}}(X, A)$ is a ring, and the correspondence $Y \rightarrow \text{Hom}_{\mathcal{D}}(X, A)$ is a functor from \mathcal{D} to the category of rings. Here a ring means an associative unital ring.

If \mathcal{D} has finite products and a terminal object T , then a ring object can be defined similarly as the usual ring: there exist morphisms $a : A \otimes A \rightarrow A$ (addition), $r : A \rightarrow A$ (inversion), $z : T \rightarrow A$ (zero), $m : A \otimes A \rightarrow A$ (multiplication) and $u : \mathbf{1} \rightarrow A$ (unit), satisfying the sets of axioms.

One can define a commutative ring object as a ring object with the multiplication m satisfying the commutative axiom. We omit the detail.

A coalgebra object is defined in a similar way, as an object with morphisms $a, r, z, \Delta : A \rightarrow A \otimes A$ (comultiplication) and $\varepsilon : A \rightarrow \mathbf{1}$ (counit) satisfying several sets of axioms. A cocommutative coalgebra object is defined in a similar way.

Similarly we can define a bialgebra object, a module object over a ring object, a comodule object over a coalgebra object and so on in a given category.

Hereafter the symbol \mathcal{C} means the category Fin or Fin^{\neq} .

Definition 1.1. Let \mathcal{C} be the category Fin or Fin^{\neq} . Let A be a ring object in \mathcal{A} . Define an object $T_*(A)$ in $\text{Fun}(\mathcal{C}, \mathcal{A})$ by $T_*(A)(I) := \otimes_{i \in I} A$ for $I \in \text{Ob}(\mathcal{C})$, and for a morphism $f : I \rightarrow J$ in \mathcal{C} define $T_*(A)(f) : T_*(A)(I) \rightarrow T_*(A)(J)$ in a natural way by the multiplication and the unit of A . We sometimes write $f_* := T_*(A)(f)$ for simplicity.

Let us explain the “natural way” in the definition above by examples.

Example 1.2.

1. For the identity morphism $\text{id}_I : I = \{1, 2, \dots, n\} \rightarrow I$ in Fin , $\text{id}_{I,*} : A^{\otimes n} \rightarrow A^{\otimes n}$ is given by $\text{id}_{A^{\otimes n}}$.

2. Consider a surjective morphism $p : \{1, 2\} \rightarrow \{1\}$ in Fin . Then $p_* : A \otimes_{\mathcal{A}} A \rightarrow A$ is defined to be the multiplication morphism $m : A \otimes_{\mathcal{A}} A \rightarrow A$ of A .

For the morphism $p : \{1, 2, 3\} \rightarrow \{1\}$ in Fin , $p_* : A \otimes_{\mathcal{A}} A \otimes_{\mathcal{A}} A \rightarrow A$ is defined to be the composition of multiplication morphisms $m^2 := m \circ (m \otimes \text{id}_{\mathcal{A}}) = m \circ (\text{id}_{\mathcal{A}} \otimes m)$.

In general, for the surjective morphism $p_I : I = \{1, 2, \dots, n\} \rightarrow \{1\}$ in Fin , $p_{I,*}$ is defined by the n -times composition of multiplication morphisms

3. For the morphism $i : \emptyset \rightarrow \{1\}$, $i_* : T \rightarrow A$ is defined to be the unit morphism $u : T \rightarrow A$.

Similarly, for the morphism $i_I : \emptyset \rightarrow I = \{1, 2, \dots, n\}$, $i_{I,*} : T \rightarrow A^{\otimes n}$ is given by $u^{\otimes n}$.

4. For the morphism $s : \{1, 2\} \rightarrow \{1, 2\}$ in Fin with $s(1) = 2$ and $s(2) = 1$, $s_* : A \otimes_{\mathcal{A}} A \rightarrow A \otimes_{\mathcal{A}} A$ is defined to be the isomorphism $\sigma_{A,A} : A \otimes_{\mathcal{A}} A \rightarrow A \otimes_{\mathcal{A}} A$ given by the symmetric monoidal structure of \mathcal{A} .

Since any morphism f in Fin can be decomposed into id_I , p_J and s given in Example above, we can compute f_* by combining the rules given above. The ways of decomposition are not unique, but the resulting f_* is determined uniquely by the symmetric monoidal structure of \mathcal{A} .

Here we give a few more examples for f_* .

Example.

1. For the morphism $i : \{1\} \rightarrow \{1, 2\}$ with $i(1) = 1$, $i_* : A \rightarrow A \otimes A$ is defined to be $(\text{id}_A, u \circ t_A)$, where $t_A : A \rightarrow T$ is the canonical morphism from A to the terminal object T .

In general, for the injective morphism $i : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$ ($m \leq n$) with $i(j) = j$, $i_* : A^{\otimes m} \rightarrow A^{\otimes n}$ is defined to be $\text{id}_A^{\otimes m} \otimes u^{\otimes (n-m)}$.

2. For the morphism $f : \{1, 2\} \rightarrow \{1, 2\}$ in Fin with $f(1) = f(2) = 1$, $f_* : A \otimes_{\mathcal{A}} A \rightarrow A \otimes_{\mathcal{A}} A$ is defined to be $m \otimes u$.

The case Fin^{\neq} is quite similar, and we omit the detail.

Remark. The axiom of ring object implies that for commutative diagrams

$$\begin{array}{ccc} \{1, 2, 3\} & \xrightarrow{f} & \{1, 2\} \\ g \downarrow & & \downarrow h \\ \{1, 2\} & \xrightarrow{h} & \{1\} \end{array} \qquad \begin{array}{ccc} \{1\} & \xrightarrow{i} & \{1, 2\} \\ j \downarrow & & \downarrow h \\ \{1, 2\} & \xrightarrow{h} & \{1\} \end{array}$$

in Fin with

$$\begin{aligned} f(1) = f(2) = 1, \quad f(3) = 2, \quad g(1) = 1, \quad g(2) = g(3) = 1, \quad h(1) = h(2) = 1, \\ i(1) = 1, \quad j(1) = 2, \end{aligned}$$

the diagrams

$$\begin{array}{ccc} A^{\otimes 3} & \xrightarrow{f_* = m \otimes \text{id}_A} & A^{\otimes 2} \\ g_* = \text{id}_A \otimes m \downarrow & & \downarrow h_* = m \\ A^{\otimes 2} & \xrightarrow{h_* = m} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{i_* = (\text{id}_A, u \otimes \text{id}_A)} & A^{\otimes 2} \\ j_* = (u \otimes \text{id}_A, \text{id}_A) \downarrow & & \downarrow h_* = m \\ A^{\otimes 2} & \xrightarrow{h_* = m} & A \end{array}$$

in \mathcal{A} commute.

One can check that

Lemma 1.3. Let A be a commutative ring object in \mathcal{A} . Then the object $T_*(A)$ is a commutative ring object in $\text{Fun}(\text{Fin}, \mathcal{A})$.

Remark. If A is not commutative, then $T_*(A)$ is not a ring object in the category $\text{Fun}(\text{Fin}, \mathcal{A})$. Assume A is a ring object in \mathcal{A} . If $T_*(A)$ is a ring object in $\text{Fun}(\text{Fin}, \mathcal{A})$, then there is a morphism $m_* : T_*(A) \otimes T_*(A) \rightarrow T_*(A)$ giving a multiplicative structure on $T_*(A)$. It means that for any morphism $f : I \rightarrow J$ in Fin we have a commuting diagram

$$\begin{array}{ccc} T_*(A)(I) \otimes_{\mathcal{A}} T_*(A)(I) & \xrightarrow{m_*(I)} & T_*(A)(I) \\ T_*(A)(f) \otimes T_*(A)(f) \downarrow & & \downarrow T_*(A)(f) \\ T_*(A)(J) \otimes_{\mathcal{A}} T_*(A)(J) & \xrightarrow{m_*(J)} & T_*(A)(J) \end{array}$$

in \mathcal{A} . Consider, for example, the morphism $f : \{1, 2\} \rightarrow \{1\}$. Then the above diagram becomes

$$\begin{array}{ccc} A^{\otimes 2} \otimes A^{\otimes 2} & \xrightarrow{m_*({1,2})} & A^{\otimes 2} \\ \downarrow f_* \otimes f_* = m \otimes m & & \downarrow f_* = m \\ A \otimes A & \xrightarrow{m_*({1})} & A \end{array}$$

Unless A is commutative, there is no canonical way of defining m_* such that the above diagram commutes.

In a dual way, one can consider

Definition. Let \mathcal{C} be the category Fin or Fin^{\neq} . For a coalgebra object C in \mathcal{A} we define an object $T^*(C)$ in $\text{Fun}(\mathcal{C}^{op}, \mathcal{A})$ by $T^*(C)(I) := \otimes_{i \in I} C$ for $I \in \text{Ob}(\mathcal{C})$, and $T^*(C)(f) : T^*(C)(J) \rightarrow T^*(C)(I)$ for a morphism $f : I \rightarrow J$ in \mathcal{C} in a natural way by the comultiplication and the counit of H . We sometimes use the symbol $f^* := T^*(C)(f)$ for simplicity.

Lemma. For a cocommutative coalgebra object C in \mathcal{A} , $T^*(C)$ is a cocommutative coalgebra object in $\text{Fun}(\text{Fin}, \mathcal{A})$.

As for a bialgebra object, let us mention the following remark.

Remark. For a bialgebra object H in \mathcal{A} , we can consider $T_*(H)$ in $\text{Fun}(\mathcal{C}, \mathcal{A})$ and $T^*(H)$ in $\text{Fun}(\mathcal{C}^{op}, \mathcal{A})$ using the algebra and coalgebra structure on H . By the axiom of bialgebra object, for a commutative diagram

$$\begin{array}{ccccc} \{1, 2\} & \xrightarrow{f} & \{1\} & \xleftarrow{f} & \{1, 2\} \\ \uparrow g & & & & \parallel \\ \{1, 2, 3, 4\} & \xrightarrow{h} & \{1, 2, 3, 4\} & \xrightarrow{g} & \{1, 2\} \end{array}$$

in Fin with

$$\begin{aligned} f(1) = f(2) = 1, \quad g(1) = g(2) = 1, \quad g(3) = g(4) = 2, \\ h(1) = 1, \quad h(2) = 3, \quad h(3) = 2, \quad h(4) = 4, \end{aligned}$$

we have a commutative diagram

$$(1.2) \quad \begin{array}{ccccc} H^{\otimes 2} & \xrightarrow{f_* = m} & H & \xleftarrow{f^* = \Delta} & H^{\otimes 2} \\ \downarrow g^* = \Delta \otimes \Delta & & & & \parallel \\ H^{\otimes 4} & \xrightarrow{h_* = \text{id}_H \otimes \sigma_{H,H} \otimes \text{id}_H} & H^{\otimes 4} & \xrightarrow{g_* = m \otimes m} & H^{\otimes 2} \end{array}$$

One can introduce a module on a ring object in $\text{Fun}(\mathcal{C}, \mathcal{A})$, although we don't write it down. We will focus on modules of bialgebra objects in the following sense:

Definition 1.4. *Let \mathcal{C} be Fin or Fin^{\neq} , and H be a bialgebra object in \mathcal{A} . Define a $T^*(H)$ -module in $\text{Fun}(\mathcal{C}, \mathcal{A})$ to be an object M of $\text{Fun}(\mathcal{C}, \mathcal{A})$ such that $M(I)$ is a module of the ring object $T^*(H)(I) = \otimes_{i \in I} H$ (with component-wise multiplication) for any $I \in \text{Ob}(\mathcal{C})$ and such that the diagram*

$$(1.3) \quad \begin{array}{ccccc} T^*(H)(I) \otimes M(I) & \xleftarrow{\ell(f)} & T^*(H)(J) \otimes M(I) & \xrightarrow{r(f)} & T^*(H)(J) \otimes M(J) \\ a(I) \downarrow & & & & \downarrow a(J) \\ M(I) & \xrightarrow{M(f)} & & & M(J) \end{array}$$

in the category \mathcal{A} commutes for any morphism $f : I \rightarrow J$ in \mathcal{C} . Here the arrows $a(I)$ and $a(J)$ indicate the H -action on M , and the H -action on the tensor product of modules is given by the comultiplication Δ of H as usual. Moreover $\ell(f) := T^*(H)(f) \otimes \text{id}_{M(I)}$ and $r(f) := \text{id}_{M(J)} \otimes M(f)$.

If $\mathcal{A} = R\text{-Mod}$, then the commutativity of the diagram (1.3) can be written as

$$(1.4) \quad f_*(f^*(g) \cdot m) = g \cdot f_*(m)$$

for any $g \in T^*(H)(J)$ and $m \in M(I)$, where we denoted by \cdot the M -action and $f_* = M(f)$.

Example 1.5.

1. For a bialgebra object H in \mathcal{A} , $T_*(H) \in \text{Ob Fun}(\mathcal{C}, \mathcal{A})$ is an H -module, since $T_*(H)(I) = \otimes_{i \in I} H$ is a module of $T^*(H)(I)$ by the component-wise product, and since the commutativity of the diagram (1.3) can be checked by the bialgebra axiom. For example, the case $f : \{1, 2\} \rightarrow \{1\}$ follows from (1.2).
2. If M is a ring object in \mathcal{A} with an action of a bialgebra object H , then $T_*(M)$ is a $T^*(H)$ -module in $\text{Fun}(\mathcal{C}, \mathcal{A})$.

One can check that H -modules in $\text{Fun}(\mathcal{C}, \mathcal{A})$ form an additive monoidal category. Let us introduce

Definition. *Define $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H))$ to be the additive monoidal category of $T^*(H)$ -modules in $\text{Fun}(\mathcal{C}, \mathcal{A})$.*

If H is cocommutative, then $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H))$ becomes a symmetric monoidal category.

Example. If M is a commutative ring object in \mathcal{A} with action of a cocommutative bialgebra object H , then $T_*(M)$ is a commutative ring object in $\text{Fun}(\text{Fin}, \mathcal{A}, T^*(H))$.

Remarking that one can define the category of modules over a commutative ring object in an additive symmetric monoidal category, and that it is again an additive symmetric monoidal category, we introduce

Definition. Let H be a cocommutative bialgebra object in \mathcal{A} and let S be a commutative ring object in $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H))$. Define $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H), S)$ to be the additive symmetric monoidal category of S -modules.

The letter S means ‘singular’, and the object S encodes the singular parts of OPEs of the fields considered. The (\mathcal{A}, H, S) -vertex algebra is defined to be a *singular* commutative ring object in $\text{Fun}(\text{Fin}, T^*(H), S)$. The term *singular* is clarified by the following notion.

Definition. Let \mathcal{C} be Fin or Fin^{\neq} . Let H be a cocommutative bialgebra object in \mathcal{A} and let S be a commutative ring object in $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H))$. For objects U_1, U_2, \dots, U_n and V of $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H), S)$, define the singular multilinear map to be a family of maps

$$U_1(I_1) \otimes_{\mathcal{A}} U_2(I_2) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} U_n(I_n) \longrightarrow V(I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n)$$

for any $I_1, I_2, \dots, I_n \in \text{Ob}(\mathcal{C})$ satisfying the following conditions.

1. The maps commute with the action of $T^*(H)$.
2. The maps commute with the actions of $S(I_1), S(I_2), \dots, S(I_n)$.
3. For morphisms $I_1 \rightarrow I'_1, I_2 \rightarrow I'_2, \dots, I_n \rightarrow I'_n$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} U_1(I_1) \otimes U_2(I_2) \otimes \cdots \otimes U_n(I_n) & \longrightarrow & V(I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n) \\ \downarrow & & \downarrow \\ U_1(I'_1) \otimes U_2(I'_2) \otimes \cdots \otimes U_n(I'_n) & \longrightarrow & V(I'_1 \sqcup I'_2 \sqcup \cdots \sqcup I'_n) \end{array}$$

in \mathcal{A} commutes.

Since we assumed that \mathcal{A} is cocomplete and colimits commute with tensor products, the singular multilinear maps are representable. Thus the following definition makes sense.

Definition. For objects U_1, U_2, \dots, U_n of $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H), S)$, the singular tensor product $U_1 \odot U_2 \odot \cdots \odot U_n$ is the object in $\text{Fun}(\mathcal{C}, \mathcal{A}, T^*(H), S)$ representing the singular multilinear maps $U_1(I_1) \otimes \cdots \otimes U_n(I_n) \rightarrow V(I_1 \sqcup \cdots \sqcup I_n)$.

The singular tensor product can be expressed explicitly as

$$(U_1 \odot U_2 \odot \cdots \odot U_n)(I) := \varinjlim_{\sqcup_{i=1}^n I_i \rightarrow I} (U_1(I_1) \otimes U_2(I_2) \otimes \cdots \otimes U_n(I_n)) \bigotimes_{S(I_1) \otimes S(I_2) \otimes \cdots \otimes S(I_n)} S(I),$$

where the colimit is taken over the following category. An object $\sqcup_{i=1}^n I_i \rightarrow I$ consists of $I_1, I_2, \dots, I_n \in \text{Ob}(\mathcal{C})$ with a morphism from $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n$ to I in \mathcal{C} , and a morphism from $\sqcup_{i=1}^n I_i \rightarrow I$ to $\sqcup_{i=1}^n I'_i \rightarrow I$ consists of morphisms $I_i \rightarrow I'_i$ ($i = 1, 2, \dots, n$) making the diagram

$$\begin{array}{ccc} I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n & \longrightarrow & I \\ & \downarrow & \parallel \\ I'_1 \sqcup I'_2 \sqcup \cdots \sqcup I'_n & \longrightarrow & I \end{array}$$

in \mathcal{C} commutative.

One can check that the category appearing above is a filtered (in the sense of [M98, Chap. IX]) small category, so that the colimit is in fact the filtered inductive limit (or the direct limit).

For $\mathcal{C} = \text{Fin}$, the disjoint union \sqcup is a coproduct, which implies that the singular tensor product \odot is the same as the ordinary tensor product \otimes .

By the definition of \odot , there is a canonical morphism from $U_1 \odot U_2$ to $U_1 \otimes U_2$, so that any ring object automatically has another ring structure with multiplication given by singular tensor products. Thus the following definition makes sense.

Definition. A singular ring object in $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A}, T^*H, S)$ is a ring object whose multiplicative structure is given by the singular tensor product \odot .

A ring object S in $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A}, T^*(H))$ can be seen as a ring object in

$$\text{Fun}(\text{Fin}, \mathcal{A}, T^*(H))$$

by restriction under the embedding (1.1) of Fin into Fin^{\neq} . Then we can embed $\text{Fun}(\text{Fin}, \mathcal{A}, T^*(H), S)$ into $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A}, T^*(H), S)$ by defining

$$(1.5) \quad \begin{aligned} & V(I_1 : I_2 : \cdots : I_n) \\ & := V(I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n) \bigotimes_{S(I_1) \otimes S(I_2) \otimes \cdots \otimes S(I_n)} S(I_1 : I_2 : \cdots : I_n) \end{aligned}$$

for V in $\text{Fun}(\text{Fin}, \mathcal{A}, T^*(H), S)$. Here $I_1 : I_2 : \cdots : I_n$ is an object of Fin^{\neq} , which is the disjoint union of I_j 's as a set, and where the equivalence relation is defined so that each I_j is the equivalence class. For example, for $I_1 = \{1\}$ and $I_2 = \{1, 2\}$, we have $I_1 : I_2 = \{1 : 2, 3\}$.

Thus the following definition makes sense.

Definition 1.6. A singular commutative ring object in $\text{Fun}(\text{Fin}, \mathcal{A}, T^*H, S)$ is an object such that its extension (1.5) gives a singular commutative ring object in $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A}, T^*H, S)$.

Now we can introduce the main object.

Definition 1.7. Let \mathcal{A} be an additive symmetric monoidal category, H be a cocommutative bialgebra object in \mathcal{A} , and S be a commutative ring object in the additive symmetric monoidal category $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A}, T^*(H))$. Define an (\mathcal{A}, H, S) -vertex algebra to be a singular commutative ring in the category $\text{Fun}(\text{Fin}, \mathcal{A}, T^*(H), S)$.

An (\mathcal{A}, H, S) -vertex algebra V is thus an object in $\text{Fun}(\text{Fin}, \mathcal{A})$, although we often consider it as an object in $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A})$ by the extension (1.5).

§ 1.2. Relation to ordinary vertex algebras

Let R be a commutative ring. In the case $\mathcal{A} = R\text{-Mod}$, one can consider the following bialgebra.

Definition 1.8. Let H_a be the commutative cocommutative bialgebra over R with basis $\{D^{(i)} \mid i \in \mathbb{Z}_{\geq 0}\}$, multiplication $D^{(i)}D^{(j)} = \binom{i+j}{i}D^{(i+j)}$ and comultiplication $\Delta(D^{(i)}) = \sum_{j=0}^i D^{(i)} \otimes D^{(i-j)}$.

H_a is the formal group ring of the one-dimensional additive formal group corresponding to the formal group law $F(X, Y) = X + Y$. Symbolically one has $D^{(i)} = D^i/i!$.

An important example of a commutative ring object S in the category

$$\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod}, T^*(H_a))$$

is

Definition 1.9. Define an object S_0 in $\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod})$ by

$$(1.6) \quad S_0(I) := R[(x_i - x_j)^{\pm 1} \mid i \neq j \text{ in } I]$$

for $I \in \text{Ob}(\text{Fin}^{\neq})$, and

$$S_0(f) : S_0(I) \longrightarrow S_0(J), \quad (x_i - x_j) \longmapsto (x_{f(i)} - x_{f(j)})$$

for $f \in \text{Fin}^{\neq}(I, J)$.

One can easily check that S_0 is indeed an object of $\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod})$. One further has

Lemma 1.10. S_0 is a commutative ring object in $\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod}, T^*(H_a))$, where the action of H_a on S_0 is given by the derivation. More explicitly, one has $D^{(i)}(x^m) = \binom{m}{i} x^{m-i}$.

Let V be an $(R\text{-Mod}, H_a, S_0)$ -vertex algebra, or an object of the category

$$\text{Fun}(\text{Fin}, R\text{-Mod}, T^*(H_a), S_0),$$

so $V(I)$ is just an R -module for each $I \in \text{Ob}(\text{Fin})$. Let us look at the definition of singular tensor product for two V 's:

$$(V \odot V)(I) = \varinjlim_{\sqcup_{i=1}^2 I_i \rightarrow I} (V(I_1) \otimes V(I_2)) \bigotimes_{S_0(I_1) \otimes S_0(I_2)} S_0(I).$$

Fix objects $I_1, I_2 \in \text{Ob}(\text{Fin})$ and take arbitrary elements $v_1 \in V(I_1)$ and $v_2 \in V(I_2)$. The ordinary product $v_1 v_2$ is defined in $V(I_1 \sqcup I_2)$. By Definition 1.6 and the extension (1.5), the singular tensor product $v_1 \odot v_2$ is defined in $(V \odot V)(I_1 : I_2) \subset V(I_1 : I_2)$ with $V(I_1 : I_2) = V(I) \otimes_{S_0(I_1) \otimes S_0(I_2)} S_0(I_1 : I_2)$, and the singular commutativity of V means $v_1 \odot v_2 = v_2 \odot v_1$ in $V(I_1 : I_2)$.

In particular, setting $I_1 = \{1\}$ and $I_2 = \{2\}$, we have $I = I_1 \sqcup I_2 = \{1, 2\}$ and $I_1 : I_2 = \{1 : 2\}$, so that $S(I_1) = S(I_2) = R$ and $S(I_1 : I_2) = R[(x_1 - x_2)^{\pm 1}]$, hence we have $V(I_1 : I_2) = V(\{1, 2\})[(x_1 - x_2)^{\pm 1}]$ and in this module the equation $v_1 \odot v_2 = v_2 \odot v_1$ holds.

Now we can recall the following main theorem in [B01]:

Fact 1.11 ([B01, Theorem 4.3]). *Let V be an $(R\text{-Mod}, H_a, S_0)$ -vertex algebra. Then $V(\{1\})$ has a structure of ordinary vertex algebra over the ring R .*

For the sake of completeness, let us write down the axiom of ordinary vertex algebra here.

Definition 1.12. *An ordinary vertex algebra defined over a commutative ring R is a collection of data*

- (space of fields) an R -module V
- (vacuum) an element $|0\rangle \in V$
- (translation) an R -linear operator $T : V \rightarrow V$
- (vertex operators) an R -linear operation $Y(\cdot, z) : V \rightarrow \text{End}(V)[[z^{\pm 1}]]$

satisfying the following axioms.

- (vacuum axiom) $Y(|0\rangle, z) = \text{id}_V$ and $Y(A, z)|0\rangle \in A + zR[[z]]$ for any $A \in V$.

- (translation axiom) $[T, Y(A, z)] = \partial_z Y(A, z)$ for any $A \in V$
- (locality axiom) $\{Y(A, z) \mid A \in V\}$ are mutually local, that is, for any $A, B \in V$ there exists $N \in \mathbb{Z}_{\geq 0}$ such that $(z - w)^N [Y(a, z), Y(b, w)] = 0$ as a formal power series in $\text{End}(V)[[z^{\pm 1}, w^{\pm 1}]]$.

Let us sketch the proof of Fact 1.11 briefly. For details see [B01, Proof of Theorem 4.3]. [P09, (4.3) Proof of Theorem 1] also gives a nice demonstration.

Proof of Fact 1.11. The vacuum $|0\rangle$ is defined to be 1 in the R -algebra $V(\{1\})$.

The translation T is defined by the action of H_a on $V(\{1\})$. In other words, $T := D^{(1)}$.

We want to make an R -linear map

$$Y(\cdot, x_1) : V(\{1\}) \rightarrow \text{End}_R(V(\{1\}))[[x_1]][x_1^{-1}]$$

satisfying the axiom of ordinary vertex algebra. For $u_1, u_2 \in V(\{1\})$, we have $u_1 \odot u_2 = u_2 \odot u_1$ in $V(\{1 : 2\}) = V(\{1, 2\})[(x_1 - x_2)^{\pm 1}]$ as remarked in the paragraph before Fact 1.11. Recalling the action of H_a on S_0 , we may consider the ‘‘Taylor series expansion’’

$$(1.7) \quad V(\{1, 2\}) \longrightarrow V(\{1\})[[x_1, x_2]], \quad w \longmapsto \sum_{i, j \geq 0} f_*(D_1^{(i)} D_2^{(j)} w) x_1^i x_2^j,$$

where $f : \{1, 2\} \rightarrow \{1\}$ is a morphism in Fin and D_1, D_2 indicate the two different actions of H_a on $V(\{1, 2\})$. Combining this expansion with the extension (1.5), we have an R -linear map

$$V(\{1, 2\})[(x_1 - x_2)^{\pm 1}] \rightarrow V(\{1\})[[x_1, x_2]][(x_1 - x_2)^{-1}],$$

and we denote the image of $u_1 \odot u_2$ under this map by $u_1(x_1)u_2(x_2)$. Then define $Y(v_1, x_1)$ by

$$u_2 \longmapsto u_1(x_1)u_2(0) \in V(\{1\})[[x_1, x_2]][(x_1 - x_2)^{-1}]|_{x_2=0} = V(\{1\})[[x_1]][x_1^{-1}].$$

As for the check of vertex algebra axioms, the most non-trivial part is the locality axiom, which is a consequence of the singular commutativity $u_1 \odot u_2 = u_2 \odot u_1$. Indeed, the singular commutativity implies

$$(x_1 - x_2)^N (u_1(x)u_2(x) - u_2(x)u_1(x))u_3 = 0$$

with some N , which depends only on u_1 and u_2 . This is nothing but the locality.

The translation axiom comes from the action of H_a . The vacuum axiom is the consequence of the singular commutativity with respect to $A \in V(\{1\})$ and $1 \in V(\{1\})$. We omit the detailed discussion. \square

Definition. For an (\mathcal{A}, H, S) -vertex algebra V , we call $V(\{1\})$ the ordinary vertex algebra associated to V .

(\mathcal{A}, H, S) -vertex algebras form an abelian category. Moreover they form a symmetric monoidal category under the tensor product \otimes . These structures induces the same ones on the ordinary vertex algebras, which are described in [FB04, §1.3] for example.

Remark. As mentioned in [B01, Example 4.9], $\text{Fun}(\text{Fin}, \mathcal{A}, T^*(H), S)$ is not closed under the singular tensor product \odot , so that one should consider \otimes for the monoidal structure on (\mathcal{A}, H, S) -vertex algebras.

§ 1.3. Twisting construction

In the next section we will reconstruct several (\mathcal{A}, H, S) -vertex algebras, to which the associated ordinary vertex algebras are well-known ones: Heisenberg algebras, affine Kac-Moody Lie algebras, the lattice vertex algebras and so on. For this purpose, let us recall the twisted group construction of (\mathcal{A}, H, S) -vertex algebra, which was introduced in [B01] and explained in detail in [P09].

Definition. Let R be a commutative ring, M and N be bialgebras over R , and S be a commutative algebra over R .

1. A bimultiplicative map from $M \otimes_R N$ to S is an R -linear map $r : M \otimes_R N \rightarrow S$ such that

$$\begin{aligned} r(a \otimes 1) &= \varepsilon_M(a), & r(1 \otimes a) &= \varepsilon_N(a), \\ r(ab \otimes c) &= \sum r(a \otimes c')r(b \otimes c''), \\ r(a \otimes bc) &= \sum r(a' \otimes b)r(a'' \otimes c) \end{aligned}$$

hold for any $a, b, c \in M$.

2. A bimultiplicative map on $M \otimes_R M$ to S is called an S -valued bicharacter.
3. A bicharacter r is called symmetric if

$$r(a \otimes b) = r(b \otimes a)$$

holds for any $a, b \in M$.

The following lemma is due to [B01, Lemma/Definition 2.6], where M is assumed to be commutative.

Lemma 1.13. Suppose r is an S -valued bicharacter of a cocommutative bialgebra M over R .

1. The operation

$$(1.8) \quad a \circ_r b := \sum a'b'r(a'' \otimes b'')$$

defines a unital associative algebra $(M \otimes_R S, \circ_r, 1_M)$ over R , where 1_M is the unit of the original algebra structure on M .

 2. If r is symmetric and M is commutative, then the new algebra $(M \otimes_R S, \circ_r, 1_M)$ is commutative.

Proof. We only indicate the proof of the associativity. On one side we have

$$\begin{aligned} (a \circ_r b) \circ_r c &= \left(\sum a'b'r(a'' \otimes b'') \right) \circ_r c \\ &= \sum (a'b')'c'r((a'b'')'' \otimes c'')r(a'' \otimes b'') \\ &= \sum (a'b')c'r(a''b'' \otimes c'')r(a''' \otimes b''') \\ &= \sum (a'b')c'r(a'' \otimes c'')r(b'' \otimes c''')r(a''' \otimes b'''), \end{aligned}$$

where in the third line we used the notation $((\Delta \otimes 1) \circ \Delta)(a) = \sum a' \otimes a'' \otimes a'''$. On the other side we have

$$\begin{aligned} a \circ_r (b \circ_r c) &= a \circ_r \left(\sum b'c'r(b'' \otimes c'') \right) \\ &= \sum a'(b'c')'r(a'' \otimes (b'c'')''r(b'' \otimes c'') \\ &= \sum a'(b'c')r(a'' \otimes (b''c'')r(b''' \otimes c''')) \\ &= \sum a'(b'c')r(a'' \otimes b'')r(a''' \otimes c'')r(b''' \otimes c'''). \end{aligned}$$

Since M is an associative algebra, we have

$$(a'b')c' = a'(b'c').$$

Since M is a cocommutative coalgebra, we have

$$\begin{aligned} &\sum (a'b')c'r(a'' \otimes c'')r(b'' \otimes c''')r(a''' \otimes b''') \\ &= \sum (a'b')c'r(a'' \otimes b'')r(a''' \otimes c'')r(b''' \otimes c'''). \end{aligned}$$

Therefore we have the conclusion. □

Definition. The algebra $(M \otimes_R S, \circ_r, 1_M)$ constructed in Lemma 1.13 is called the twisting of M by r and denoted by \widetilde{M} or M^r .

As in §1.2, we will consider the case where M has an action of a cocommutative coalgebra (or bialgebra) H . There is a universal ring with H -action in the following sense.

Fact 1.14 ([B01, Lemma/Definition 2.10]). *Suppose M is an R -algebra and H is an R -coalgebra. Then there is a universal R -algebra $H(M)$ such that there is a map*

$$H \otimes M \longrightarrow H(M), \quad h \otimes m \mapsto h(m)$$

with

$$h(mn) = \sum h'(m)h''(n), \quad h(1) = \varepsilon_H(h).$$

If M is commutative and H is cocommutative, then $H(M)$ is commutative. If H is a bialgebra, then H acts on the algebra $H(M)$. If M is a bialgebra, then $H(M)$ is also a bialgebra.

Proof. $H(M)$ is defined to be the quotient of the tensor algebra of $H \otimes M$ by the ideal generated by the desired relations. The rest statements are easy to check. \square

Then it is natural to introduce

Definition 1.15. *Let M be an R -bialgebra and S be a commutative R -algebra. Suppose that an R -coalgebra H acts on M and $H \otimes H$ acts on S . An S -valued bicharacter r on M is called H -invariant if*

$$r((ga) \otimes (hb)) = (g \otimes h)(r(a \otimes b))$$

holds for any $g, h \in H$ and $a, b \in M$.

We also have

Fact 1.16 ([B01, Lemma 2.15]). *Let H be a cocommutative bialgebra, S be a commutative algebra acted on by $H \otimes H$, and M be a commutative cocommutative bialgebra with an S -valued bicharacter r . Then r extends uniquely to an H -invariant S -valued bicharacter on $H(M)$.*

The discussion above can be generalized to the categorical setting given in §1.1. For example, one can define an S -valued bicharacter on $M \otimes_{\mathcal{A}} M$, where M is a bialgebra object in the additive monoidal category \mathcal{A} and S is a commutative ring object in \mathcal{A} .

Now we recall the construction of (\mathcal{A}, H, S) -vertex algebra using bicharacter, which was introduced in [B01, Lemma 4.1, Theorem 4.2].

Lemma 1.17 ([B01, Lemma 4.1]). *Let M be a commutative and cocommutative bialgebra object in \mathcal{A} , H be a cocommutative bialgebra object \mathcal{A} , and S be a commutative ring object in $\text{Fun}(\text{Fin}^{\neq}, T^*(H))$. If r is an $S(\{1 : 2\})$ -valued H -invariant bicharacter on a commutative cocommutative bialgebra $H(M)$ in \mathcal{A} , then one can extend r to a singular bicharacter of $T_*(H(M))$.*

Note that $S(\{1 : 2\})$ has an $H \otimes H$ -action since S is a $T^*(H)$ -module so that $S(\{1 : 2\})$ is a module over $T^*(H)(\{1 : 2\}) = H \otimes H$ by Definition 1.4. So the term ‘ H -invariant’ makes sense by Definition 1.15.

Let us briefly sketch the proof of Lemma 1.17. We define the extended r on $T_*(H(M))(I \sqcup J)$ by

$$r\left(\bigotimes_{i \in I} a_i \otimes \bigotimes_{j \in J} b_j\right) := \sum \prod_{i \in I} \prod_{j \in J} r(a_i^{(j)} \otimes b_j^{(i)})$$

with $\Delta_M^{|J|-1}(a_i) = \sum \bigotimes_{j \in J} a_i^{(j)}$ and $\Delta_M^{|I|-1}(b_j) = \sum \bigotimes_{i \in I} b_j^{(i)}$. Each $r(a_i^{(j)} \otimes b_j^{(i)})$ is considered as an element of $S(I \sqcup J)$ using the natural map from $S(\{i : j\})$ to $S(I \sqcup J)$.

Fact 1.18 ([B01, Theorem 4.2]). *Suppose that H is a cocommutative bialgebra in \mathcal{A} and that S is a commutative ring in $\text{Fun}(\text{Fin}^\neq, \mathcal{A}, T^*(H))$. Assume that r is a symmetric $S(\{1 : 2\})$ -valued bicharacter of a commutative and cocommutative bialgebra M in \mathcal{A} . Then the twisting $T_*(H(M))^r$ of $T_*(H(M))$ by the singular bicharacter constructed by the extension of r in Lemma 1.17 is (\mathcal{A}, H, S) -vertex algebra.*

The following remark by Patnaik [P09, (4.2)] is useful when one checks the axiom in particular examples.

Lemma. *Let $T_*(H_a(M))$ be an $(R\text{-Mod}, H_a, S_0)$ -vertex algebra arising from a universal commutative cocommutative bialgebra $H_a(M)$. Let us consider the twisting $T_*(H_a(M))^r$ of $T_*(H_a(M))$ by the (singular) bicharacter r . Then in the ordinary vertex algebra associated to $T_*(H_a(M))^r$ we have*

$$Y(a, x_1)Y(b, x_2) | 0 \rangle = \Phi_r(a, b) \in R[(x_1 - x_2)^{\pm 1}]$$

for $a, b \in T_*(H_a(M))^r(\{1\}) = H_a(M)$ with

$$\Phi_r(a, b) := \sum_{i, j \geq 0, (a), (b)} x_1^i x_2^j D^{(i)}(a') D^{(j)}(b'') r(a'', b'')$$

Proof. This is the direct consequence of the formula (1.7) and the definition of the twisted product (1.8). □

§ 2. Examples of (\mathcal{A}, H, S) -vertex algebras

After the review of (\mathcal{A}, H, S) -vertex algebras in the previous section, we now answer a problem stated in [B01, §5, Problem 5.5]: Construct $(R\text{-mod}, H, S)$ vertex algebras corresponding to the other standard examples of vertex algebras, such as the vertex algebras of affine and Virasoro algebras.

§ 2.1. Heisenberg algebra

We introduce a typical example of (\mathcal{A}, H, S) -vertex algebra, whose ordinary vertex algebra will be the Heisenberg vertex algebra. Let R be a fixed commutative ring, and let us set $\mathcal{A} = R\text{-Mod}$, $H = H_a$ and $S = S_0$ as in §1.2.

Consider the Laurent polynomial ring $R[t^{\pm 1}]$ of one variable. It is (trivially) a commutative ring object in $R\text{-Mod}$, and has an action of $H_a = R[D^{(i)} \mid i \in \mathbb{Z}_{\geq 0}]$ defined as

$$D^{(i)}t^n = \binom{n}{i}t^{n-i}.$$

By Lemma 1.3, the object $T_*(R[t^{\pm 1}])$ in $\text{Fun}(\mathcal{C}, R\text{-Mod})$ is a commutative ring object. Hereafter let us denote $T_*(R[t^{\pm 1}])(\{1, \dots, n\}) = R[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

We also have

Lemma 2.1. $T_*(R[t^{\pm 1}])$ is a $T^*(H_a)$ -module.

Proof. We only need to check the formula (1.4) with f given by each case in Example 1.2. The case $f = \text{id}$ is trivial. In the case $f : \{1, 2\} \rightarrow \{1\}$, we may set $g = D^{(k)}$ and $m = t_1^m \otimes t_2^n$. Then

$$\begin{aligned} f_*(f^*(g).m) &= f_*\left(\sum_{i+j=k} (D^{(i)} \otimes D^{(j)})(t_1^m \otimes t_2^n)\right) \\ &= f_*\left(\sum_{i+j=k} \binom{m}{i} \binom{n}{j} t_1^{m-i} \otimes t_2^{n-j}\right) \\ &= \sum_{i+j=k} \binom{m}{i} \binom{n}{j} t_1^{m+n-k} = \binom{m+n}{k} t_1^{m+n-k} = g.f_*(m). \end{aligned}$$

In the second from last equality is the result of the binomial formula. The case $f : \{1, \dots, n\} \rightarrow \{1\}$ can be treated similarly. The case $f : \emptyset \rightarrow \{1\}$ is the result of $D^{(i)}1 = 0$ for $i > 0$. The last case $f = s : \{1, 2\} \rightarrow \{1, 2\}$ is the consequence of the commutativity of the polynomial ring. \square

$T_*(R[t^{\pm 1}])$ is obviously an S_0 -module (with the action given by the multiplications of rational functions), and $T_*(R[t^{\pm 1}])$ is a commutative ring object in

$$\text{Fun}(\mathcal{C}, R\text{-Mod}, T^*(H_a), S_0).$$

We also have the commutative subalgebra $T_*(R[t^{-1}])$ in $\text{Fun}(\mathcal{C}, R\text{-Mod}, T^*(H_a), S_0)$.

Next we consider the one-dimensional commutative Lie algebra $\mathfrak{c} = Rb$. Its universal enveloping algebra $U(\mathfrak{c})$ is isomorphic to the polynomial ring $R[b]$, or the symmetric algebra $S^\bullet b$. $U(\mathfrak{c})$ has a cocommutative bialgebra structure with $\Delta(b) = b \otimes 1 + 1 \otimes b$.

The tensor product $L\mathfrak{c} \equiv \mathfrak{c}[t^{\pm 1}] := \mathfrak{c} \otimes_R R[t^{\pm 1}]$ also has the structure of commutative Lie algebra, which may be called the loop Lie algebra (attached to \mathfrak{c}). It has a Lie subalgebra $\mathfrak{c}[t] := \mathfrak{c} \otimes_R R[t]$. One may consider the universal enveloping algebras $U(L\mathfrak{c}) \supset U(\mathfrak{c}[t])$. The trivial representation $R_0 := Rv_0$ of $\mathfrak{c}[t]$, where v_0 is a basis of the representation space, induces the Verma module $\pi_0 := \text{Ind}_{\mathfrak{c}[t]}^{L\mathfrak{c}}$ of $U(L\mathfrak{c})$.

Hereafter we use the notation

$$b_i := b \otimes t^i$$

for $i \in \mathbb{Z}$. π_0 has a basis consisting of the monomials

$$b_{-i_1} b_{-i_2} \cdots b_{-i_n} v_0, \quad n \geq 0, \quad i_1 \geq i_2 \geq \cdots \geq i_n > 0.$$

Since \mathfrak{c} is commutative, one has a unital associative commutative ring structure on π_0 defined by

$$b_{-i} v_0 \otimes b_{-j} v_0 \mapsto b_{-i} b_{-j} v_0.$$

Then, as in the case of $R[t^{-1}]$, Lemma 1.3 says that $T_*(\pi_0)$ is a commutative ring object in $\text{Fun}(\mathcal{C}, R\text{-Mod})$. The $T^*(H_a)$ -module structure on $T_*(R[t^{-1}])$ induces one on $T_*(\pi_0)$. Thus $T_*(\pi_0)$ is an object of $\text{Fun}(\mathcal{C}, R\text{-Mod}, T^*(H_a))$. Similarly one can see that $T_*(\pi_0)$ is an S_0 -module, and it is also a (singular) commutative ring object in $\text{Fun}(\mathcal{C}, R\text{-Mod}, T^*(H_a), S_0)$. Therefore we have an (\mathcal{A}, H, S) -vertex algebra $T_*(\pi_0)$.

Let us describe the ordinary vertex algebra associated to $T_*(\pi_0)$. We use the notations

$$T_*(\pi_0)(\{1\}) = R[b_{-i} \mid i \geq 0]v_0 = R[b_{-i}^{(1)} \mid i \geq 0]v_0$$

and

$$T_*(\pi_0)(\{1, \dots, n\}) = R[b_{-i}^{(j)} \mid i \geq 0, n \geq j \geq 1]v_0.$$

Recalling the proof of Fact 1.11, we compute several vertex operators $Y(\ , z)$.

$$\begin{aligned} & Y(b_{-n}v_0, x_1)b_{-k}v_0 \\ &= \sum_{i,j \geq 0} x_1^i x_2^j f_* \left((D^{(i)} \otimes D^{(j)})(b_{-n}^{(1)}v_0 \otimes b_{-k}^{(2)}v_0) \right) \Big|_{x_2=0} \\ &= \sum_{i,j \geq 0} x_1^i x_2^j \binom{n+i-1}{i} \binom{k+j-1}{j} f_* (b_{-n-i}^{(1)}v_0 \otimes b_{-k-j}^{(2)}v_0) \Big|_{x_2=0} \\ &= \sum_{i,j \geq 0} x_1^i x_2^j \binom{n+i-1}{i} \binom{k+j-1}{j} b_{-n-i}^{(1)} b_{-k-j}^{(1)} v_0 \Big|_{x_2=0} \\ &= \sum_{i \geq 0} x_1^i \binom{n+i-1}{i} b_{-n-i} b_{-k} v_0 \end{aligned}$$

Here we used the notation $f : \{1, 2\} \rightarrow \{1\}$, a morphism in Fin .

A similar calculation gives

$$\begin{aligned} & Y(b_{-m_1} \cdots b_{-m_k} v_0, z) b_{-n_1} \cdots b_{-n_l} v_0 \\ &= \sum_{i, j_1, \dots, j_k \geq 0} x_1^i \binom{i}{j_1, \dots, j_k} \binom{m_1 + j_1 - 1}{j_1} \cdots \binom{m_k + j_k - 1}{j_k} \\ & \quad \cdot b_{-m_1 - j_1} \cdots b_{-m_k - j_k} b_{-n_1} \cdots b_{-n_l} v_0 \end{aligned}$$

These formulas correspond to the OPE

$$\partial_z^i b(z) \partial_w^j b(w) = \underset{\circ}{\circ} \partial_z^i b(z) \partial_w^j b(w) \underset{\circ}{\circ}$$

with $b(z) := \sum_{n \in \mathbb{Z}} b_n z^{-n-1}$, where $\underset{\circ}{\circ}$ is the usual normal ordering. Goddard's uniqueness theorem [FB04, §3.1.1] and the reconstruction theorem [FB04, §2.3.11] imply that the ordinary vertex algebra attached to π_0 coincides with the Heisenberg vertex algebra without central extension. In order to construct the usual Heisenberg algebra with central extension, where the OPE reads

$$b(z)b(w) = \frac{k}{(z-w)^2} + \underset{\circ}{\circ} b(z)b(w) \underset{\circ}{\circ}$$

we need to recall the twisting construction reviewed in §1.3.

Lemma 2.2. π_0 coincides with the universal algebra $H_a(U(\mathfrak{c}))$.

Proof. $H_a(U(\mathfrak{c}))$ is isomorphic to $R[D^{(i)}] \otimes_R R[b]$, so it is isomorphic to π_0 under the map

$$D^{(i)} b^n \mapsto \sum b_{-i_1-1} \cdots b_{-i_n-1} v_0.$$

Here the summation is over $i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}$ with $i_1 + \cdots + i_n = i$. \square

Since H_a is cocommutative and $U(\mathfrak{c})$ is a cocommutative commutative bialgebra, we may apply the twisting construction. Let us consider the following bicharacter. Fix an element c in R . The R -bilinear map $b \otimes b \mapsto c$ on $R[b]$ induces an R -valued bicharacter on $U(\mathfrak{c})$ given by $b^m \otimes b^n \mapsto m! c^m \delta_{m,n}$. Now consider the $S_0(\{1:2\})$ -valued bicharacter

$$r(b^m \otimes b^n) = \frac{m! c^m \delta_{m,n}}{(x_1 - x_2)^2} \in S_0(\{1:2\} = R[(x_1 - x_2)^{\pm 1}])$$

Then Fact 1.16 claims that r lifts to an H_a -invariant $S_0(\{1:2\})$ -valued bicharacter on $H_a(U(\mathfrak{c}))$. It can be written down as

$$(2.1) \quad r(D^{(i)} b^m \otimes D^{(j)} b^n) = \frac{\partial_{x_1}^i \partial_{x_2}^j}{i! j!} \frac{m! c^m \delta_{m,n}}{(x_1 - x_2)^2}.$$

Lemma. *Consider the twisting $V := T_*(\pi_0)^r = T_*(H_a(U(\mathfrak{c})))^r$ of $T_*(\pi_0) = T_*(H_a(U(\mathfrak{c})))$ by the bicharacter (2.1), which is an $(R\text{-Mod}, H_a, S_0)$ -vertex algebra by Fact 1.11. Then in the ordinary vertex algebra associated to V we have*

$$Y(b_{-1}v_0, x_1)Y(b_{-1}v_0, x_2) = \frac{1}{(x - y)^2} + \circ b(x_1)b(x_2)\circ.$$

Thus we have

Proposition. *The ordinary vertex algebra associated to V coincides with the Heisenberg vertex algebra.*

§ 2.2. Formal delta functions

For later use, we now recall some facts on the formal delta function following [K98, Chap. 2]. As before let us fix a commutative ring R .

Let us call elements of $R[[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}]]$, that is, formal Laurent series

$$\sum_{m_1, m_2, \dots, m_n \in \mathbb{Z}} a_{m_1, m_2, \dots, m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$$

by (R -valued) formal distributions. For a formal distribution $f(z) = \sum_{n \in \mathbb{Z}} f_n z^n$, the residue is given by

$$\text{Res}_z f(z) := f_{-1}.$$

It induces a non-degenerate pairing

$$(2.2) \quad \langle \cdot, \cdot \rangle : R[[z^{\pm 1}]] \times R[[z^{\pm 1}]] \longrightarrow R, \quad \langle f, g \rangle := \text{Res}_z (f(z)g(z)).$$

Definition. *The formal delta function $\delta(z, w)$ is the formal distribution*

$$\delta(z, w) := \sum_{n \in \mathbb{Z}} z^{-n-1} w^n \in R[[z^{\pm 1}, w^{\pm 1}]].$$

We have the following well-known facts.

Fact. *The formal delta function enjoys the following properties.*

1. *For any formal distribution $f(z) \in R[[z^{\pm 1}]]$, the product $f(z)\delta(z, w)$ is well-defined in $R[[z^{\pm 1}, w^{\pm 1}]]$, and one has*

$$(2.3) \quad \text{Res}_z f(z)\delta(z, w) = f(w).$$

2. *One has*

$$(2.4) \quad \delta(z, w) = \delta(w, z).$$

3. For $j \in \mathbb{Z}_{\geq 0}$,

$$(2.5) \quad (z-w)^{j+1} \partial_w^j \delta(z, w) = 0$$

holds in $R[[z^{\pm 1}, w^{\pm 1}]]$.

4. $\delta(z, t)\delta(w, t)$ and $\delta(w, t)\delta(z, t)$ are well defined in $R[[z^{\pm 1}, w^{\pm 1}, t^{\pm 1}]]$, and one has

$$(2.6) \quad \delta(z, t)\delta(w, t) = \delta(w, t)\delta(z, t).$$

As a preliminary of the next subsection, we introduce several well-known notions.

Definition.

1. For an R -module M , an $\text{End}(M)$ -valued formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in \text{End}(M)[[z^{\pm 1}]]$ is called a field on M if for any $v \in V$ we have $a_j.v = 0$ for large enough j .
2. For two distributions $f(z)$ and $g(z)$ in $R[[z^{\pm 1}]]$, we define the normal ordering by

$$\circ f(z)g(w) \circ := f(z)_+ g(w) + g(w) f(z)_-,$$

where for $f(z) = \sum_{n \in \mathbb{Z}} f_n z^n$ we used the symbols

$$f(z)_+ := \sum_{n \geq 0} f_n z^n, \quad f(z)_- := \sum_{n < 0} f_n z^n.$$

3. For distributions $f_i(z)$ ($i = 1, 2, \dots, m$), we define the normal ordering by

$$\begin{aligned} \circ f_1(z_1) f_2(z_2) \cdots f_m(z_m) \circ &:= \\ \circ f_1(z_1) \circ f_2(z_2) \cdots \circ f_{m-1}(z_{m-1}) f_m(z_m) \circ &\cdots \circ \circ. \end{aligned}$$

As is well-known, we have

Fact. For two fields $a(z), b(z)$ on an R -module M , the specialization $z = w$ of the normal ordering $\circ a(z)b(w) \circ$, that is, $\circ a(z)b(z) \circ$, is a well-defined field on M .

Thus the following definition makes sense.

Definition. For fields $f_i(z)$ ($i = 1, 2, \dots, m$) on an R -module M , define the (specialized) normal ordering by

$$\circ f_1(z) f_2(z) \cdots f_m(z) \circ := \circ f_1(z) \circ f_2(z) \cdots \circ f_{m-1}(z) f_m(z) \circ \cdots \circ \circ.$$

Now we have

Lemma. Consider an object V_δ of $\text{Fun}(\text{Fin}^\neq, R\text{-Mod})$ defined by

$$V_\delta(I) := S_0(I)[\partial_{x_i}^n \delta(x_i, x_j) \mid i \neq j \text{ in } I, n \in \mathbb{Z}_{\geq 0}]$$

for $I \in \text{Ob}(\text{Fin}^\neq)$ and

$$V_\delta(f) : V_\delta(I) \longrightarrow V_\delta(J), \quad x_i \longmapsto x_{f(i)}$$

for $f \in \text{Fin}^\neq(I, J)$. Here the multiplication of $\partial_z^j \delta(z, w)$'s are given in terms of normal orderings, and we assume that those normal orderings make sense. Then V_δ is a $T^*(H_a)$ -module, and also an S_0 -module in $\text{Fun}(\text{Fin}^\neq, R\text{-Mod}, T^*(H_a))$. Finally, V_δ is a singular commutative in $\text{Fun}(\text{Fin}^\neq, R\text{-Mod}, T^*(H_a), S_0)$, that is, an (\mathcal{A}, H, S) -vertex algebra.

Proof. Well-definedness as an object of $\text{Fun}(\text{Fin}^\neq, R\text{-Mod})$ is easily checked. The $T^*(H_a)$ -module structure is given by derivation, that is, $D_i^{(n)} \delta(x_i, x_j) = \partial_i^n \delta(x_i, x_j)/n!$. (Although we used the fractional symbol $1/n!$, the coefficients are always in the commutative ring R .) The S_0 -module structure is obviously given. The singular commutativity follows from (2.6). □

Since $V_\delta(\{1\}) = R$, the associated ordinary vertex algebra is the trivial one.

§ 2.3. Vertex algebras of loop Lie algebras

In §2.4 we construct the (ordinary) vertex algebras of affine Kac-Moody Lie algebras in the formulation of Borchers reviewed in the previous subsections. Before doing so, we first construct the ordinary vertex algebras of loop Lie algebras, that is, affine Lie algebra without the central extension.

Let R be a fixed commutative ring containing \mathbb{Q} .

For a Lie algebra \mathfrak{g} defined on R , its universal enveloping algebra is denoted by $U(\mathfrak{g})$ as usual. It is a cocommutative bialgebra with the comultiplication given by $\Delta(A) = A \otimes 1 + 1 \otimes A$ for $A \in \mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g})$ has a basis arising from a fixed totally ordered basis of \mathfrak{g} . Hereafter we fix a total order \leq on a basis of \mathfrak{g} .

The loop algebra of \mathfrak{g} is an R -vector space

$$L\mathfrak{g} := \mathfrak{g} \otimes_R R[t^{\pm 1}]$$

with Lie algebra structure given by

$$[A \otimes t^m, B \otimes t^n] := [A, B] \otimes t^{m+n}$$

for $A, B \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

Consider the one-dimensional trivial representation $R_0 = Rv_0$ of $\mathfrak{g} \otimes R[t]$. Here v_0 is the basis vector of R_0 . The induced representation

$$V_0(\mathfrak{g}) := \text{Ind}_{\mathfrak{g} \otimes R[t]}^{L\mathfrak{g}} R_0 = U(L\mathfrak{g}) \otimes_{U(\mathfrak{g} \otimes R[t])} R_0$$

is called the Verma module of $L\mathfrak{g}$. $V_0(\mathfrak{g})$ has a cocommutative bialgebra structure induced from that on $U(L\mathfrak{g})$.

The Poincaré-Birkhoff-Witt theorem gives the isomorphism

$$V_0(\mathfrak{g}) \simeq U(\mathfrak{g} \otimes t^{-1}R[t^{-1}])$$

of R -vector spaces. In particular, using a basis $\{a_i \mid i = 1, 2, \dots, \dim \mathfrak{g}\}$ of \mathfrak{g} and denoting

$$(2.7) \quad J_n^a := a \otimes t^n,$$

for $a \in \mathfrak{g}$, we have a basis

$$(2.8) \quad \left\{ J_{n_1}^{a_1} J_{n_2}^{a_2} \cdots J_{n_j}^{a_j} v_0 \mid j \in \mathbb{Z}_{\geq 0}, n_1 \leq n_2 \leq \cdots \leq n_j < 0, \right. \\ \left. \text{if } n_i = n_{i+1} \text{ then } a_i \leq a_{i+1} \right\}$$

for $V_0(\mathfrak{g})$.

Let us recall the commutative cocommutative bialgebra $H_a = R[D^{(i)}]$ given in Definition 1.8. The action of H_a on the polynomial ring as derivation induces another action on $L\mathfrak{g}$. Written explicitly, H_a acts on $L\mathfrak{g}$ via

$$D^{(i)}(A \otimes t^{-m}) = \binom{m+i-1}{i} A \otimes t^{-m-i}.$$

This action extends to $U(L\mathfrak{g})$ and then restricts to $V_0(\mathfrak{g})$.

Remark 2.3. *Under this H_a -action, $V_0(\mathfrak{g})$ is generated over R by the subset $\{J_{-1}^a \mid a = 1, 2, \dots, \dim \mathfrak{g}\}$.*

Now we construct an (\mathcal{A}, H_a, S_0) -vertex algebra $V_{L\mathfrak{g}}$ from $V_0(\mathfrak{g})$. The resulting ordinary vertex algebra (see Fact 1.11) turns out to be the vertex algebra of affine Lie algebra with level $k = 0$.

Recalling the functor T_* in Definition 1.1, let us set

$$V_{L\mathfrak{g}} := T_*(V_0(\mathfrak{g})).$$

In particular, $V_{L\mathfrak{g}}(\{1\})$ is the R -vector space $V_0(\mathfrak{g})$, and $V_{L\mathfrak{g}}(\{1, 2, \dots, n\})$ is the n -th tensor product of $V_{L\mathfrak{g}}(\{1\})$. By Example 1.5 (2), $V_{L\mathfrak{g}}$ is an object of

$$\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod}, T^*(H_a)),$$

although it is not a ring object since the multiplicative structure on $V_0(\mathfrak{g})$ is not commutative.

We define the singular tensor product on $V_{L\mathfrak{g}}$ with the help of the trivial (\mathcal{A}, H, S) -vertex algebra V_δ constructed in the previous subsection. We will use the notation (2.7) and (2.8) for elements of $V_{L\mathfrak{g}}(\{1\}) = V_0(\mathfrak{g})$.

Lemma.

1. *The \mathfrak{g} -valued distribution*

$$\partial_x^i J^a(x) := \partial_x^i \sum_{n \in \mathbb{Z}} J_{-n-1}^a x^n = a \otimes \partial_x^i \delta(t, x) \in U(\mathfrak{g}) \otimes V_\delta(\{0 : 1\})$$

with $a \in \mathfrak{g}$ and $i \in \mathbb{Z}_{\geq 0}$ is a field on $V_0(\mathfrak{g})$. (Here we used $\{0 : 1\}$ to indicate the set of two elements with two equivalent classes, and the associated indeterminates are t and x .)

2. *The correspondence*

$$J_{-n_1}^{a_1} J_{-n_2}^{a_2} \cdots J_{-n_j}^{a_j} v_0 \longmapsto \frac{1}{(n_1 - 1)! \cdots (n_j - 1)!} \circ \partial_x^{n_1-1} J^{a_1}(x) \partial_x^{n_2-1} J^{a_2}(x) \cdots \partial_x^{n_j-1} J^{a_n}(x) \circ$$

gives an isomorphism of R -modules

$$\theta : V_0(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes V_\delta(\{0 : 1\}).$$

3. *θ extends to an isomorphism*

$$\theta : V_{L\mathfrak{g}} \longrightarrow T_*U(\mathfrak{g}) \otimes V_\delta$$

of objects in $\text{Fun}(\text{Fin}^\neq, R\text{-Mod}, T^*(H_a))$. In the right hand side $T_*U(\mathfrak{g})$ is regarded as a trivial $T^*(H_a)$ -module in $\text{Fun}(\text{Fin}^\neq, R\text{-Mod})$.

Proof. The first part is well-known, and the second part is obvious from the description of the basis on $V_0(\mathfrak{g})$. For the third part, it is enough to notice that the isomorphism θ is equivalent with respect to the H_a -actions. □

A typical element of $V_{L\mathfrak{g}}(\{1, 2\})$ is $J_{-1}^a v_0 \otimes J_{-1}^b v_0 = J^a(x_1) \otimes J^b(x_1)$, and one of $V_{L\mathfrak{g}}(\{1 : 2\})$ is $J^a(x_1) J^b(x_2)$. Here the expression $J^a(x_1) J^b(x_2)$ means the product (or composition) of fields on $V_0(\mathfrak{g})$. The strict definition is given by the following lemma.

Lemma 2.4. *Define a bioperator*

$$\bullet : V_{L\mathfrak{g}}(\{1\}) \underset{R\text{-Mod}}{\otimes} V_{L\mathfrak{g}}(\{2\}) \longrightarrow V_{L\mathfrak{g}}(\{1, 2\})$$

by

$$J_{-1}^a v_0 \bullet J_{-1}^b v_0 := \theta^{-1} \left(\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} J^a(x_1) J^b(x_2) \begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right)$$

with $a, b \in \mathfrak{g}$. Then it extends to a bioperator

$$\bullet : V_{L\mathfrak{g}} \otimes_{\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod})} V_{L\mathfrak{g}} \longrightarrow V_{L\mathfrak{g}}$$

Lemma. Define the singular tensor product

$$\odot : V_{L\mathfrak{g}}(\{1\}) \otimes_{R\text{-Mod}} V_{L\mathfrak{g}}(\{2\}) \longrightarrow V_{L\mathfrak{g}}(\{1:2\}) = V_{L\mathfrak{g}}(\{1,2\})[(x_1 - x_2)^{\pm 1}]$$

by

$$(2.9) \quad J_{-1}^a v_0 \odot J_{-1}^b v_0 := \frac{1 \otimes J_{-1}^{[a,b]} v_0}{x_1 - x_2} + J_{-1}^a v_0 \bullet J_{-1}^b v_0$$

for $a, b \in \mathfrak{g}$. Then it extends to the singular tensor product on $V_{L\mathfrak{g}} = T_*(V_0(\mathfrak{g}))$.

Proof. Recall that in the definition of the singular tensor product we have the compatibility of $T^*(H_a)$ -action and S_0 -action. Since $V_0(\mathfrak{g})$ is generated by J_{-1}^a under the H_a -action, we immediately have the conclusion. \square

Remark. The singular tensor product \odot gives a composition of fields on $V_0(L\mathfrak{g})$. It looks as

$$J^a(z) \odot J^b(w) = \frac{J^{[a,b]}(w)}{z - w} + \begin{smallmatrix} \circ \\ \circ \end{smallmatrix} J^a(z) J^b(w) \begin{smallmatrix} \circ \\ \circ \end{smallmatrix},$$

which is the OPE usually used in calculations by physicists.

Lemma. The singular tensor product on $V_{L\mathfrak{g}}$ is commutative, so that $V_{L\mathfrak{g}}$ is an $(R\text{-Mod}, H_a, S_0)$ -vertex algebra.

Proof. By the definition of S_0 , it is sufficient to show $(x_1 - x_2)^N v_1 \odot v_2 = (x_1 - x_2)^N v_2 \odot v_1$ with some N for any $v_1, v_2 \in V_{L\mathfrak{g}}(\{1\})$. We demonstrate only for the case $v_1 = J_{-1}^a v_0$ and $v_2 = J_{-1}^b v_0$, since the other cases follows by the H_a action (as derivation) and the normal ordering (that is, by usual field calculus). By the formula (2.9), we have

$$\begin{aligned} & \theta(J_{-1}^a v_0 \odot J_{-1}^b v_0 - J_{-1}^b v_0 \odot J_{-1}^a v_0) \\ &= \left(\frac{J^{[a,b]}(w)}{x_1 - x_2} + \begin{smallmatrix} \circ \\ \circ \end{smallmatrix} J^a(x_1) J^b(x_2) \begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right) - \left(\frac{J^{[b,a]}(z)}{x_2 - x_1} + \begin{smallmatrix} \circ \\ \circ \end{smallmatrix} J^a(x_2) J^b(x_1) \begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right) \\ &= \frac{J^{[a,b]}(x_2) - J^{[b,a]}(x_1)}{x_1 - x_2} + [J^a(x_1)_+, J^b(x_2)_+] + [J^b(x_2)_-, J^a(x_1)_-] \end{aligned}$$

$$\begin{aligned}
 &= \frac{J^{[a,b]}(x_2) - J^{[b,a]}(x_1)}{x_1 - x_2} - \frac{J^{[a,b]}(x_1)_+ - J^{[a,b]}(x_2)_+}{x_1 - x_2} - \frac{J^{[b,a]}(x_2)_- - J^{[b,a]}(x_1)_-}{x_2 - x_1} \\
 &= 0.
 \end{aligned}$$

Thus we have the conclusion. □

Next we study the ordinary vertex algebra associated to $V_{L\mathfrak{g}}$. Recall the proof of Fact 1.11, in particular the construction of vertex operator $Y(A, x)$ using the Taylor expansion formula (1.7). For $A = J_{-1}^a v_0$, we can compute

$$\begin{aligned}
 Y(J_{-1}^a v_0, x_1) J_{-n}^b v_0 &= \sum_{i,j \geq 0} x_1^i x_2^j f_*(D_1^{(i)} D_2^{(j)} (J_{-1}^a v_0 \odot J_{-n}^b v_0)) \Big|_{x_2=0} \\
 &= \sum_{i,j \geq 0} \binom{n+j-1}{j} x_1^i x_2^j f_*(J_{-1-i}^a v_0 \odot J_{-n-j}^b v_0) \Big|_{x_2=0} \\
 &= \sum_{i,j \geq 0} \frac{x_1^i x_2^j}{i! j! (n-1)!} \theta^{-1} f_* \left(\partial_{x_1}^i \partial_{x_2}^{j+n-1} \frac{J^{[a,b]}(x_2)}{x_1 - x_2} + \circ \partial_{x_1}^i J^a(x_1) \partial_{x_2}^{j+n} J^b(x_2) \circ \right) \Big|_{x_2=0} \\
 &= \sum_{i,j \geq 0} \frac{x_1^i x_2^j}{j! (n-1)!} \theta^{-1} \left(\partial_{x_1}^{j+n-1} \frac{J^{[a,b]}(x_1)}{(x_1 - x_2)^{i+1}} + \frac{1}{i!} \circ \partial_{x_1}^i J^a(x_1) \partial_{x_1}^{j+n} J^b(x_1) \circ \right) \Big|_{x_2=0} \\
 &= \sum_{0 \leq k \leq n-1} \frac{1}{(n-k-1)!} x_1^{-k-1} \theta^{-1} \left(\partial_{x_1}^{n-k-1} J^{[a,b]}(x_1) \right) \\
 &\quad + \sum_{i \geq 0} \frac{1}{i! (n-1)!} x_1^{-j} \theta^{-1} \left(\circ \partial_{x_1}^i J^a(x_1) \partial_{x_1}^n J^b(x_1) \circ \right) \\
 &= \sum_{i < 0} x_1^i J_{-i-n-1}^{[a,b]} v_0 + \sum_{i \geq 0} x_1^i J_{-i-1}^a J_{-n}^b v_0.
 \end{aligned}$$

On the other hand, in the ordinary vertex algebra of loop Lie algebra, one associates to $J_{-1}^a v_0$ the field $J^a(z)$, which acts on $J_{-n}^b v_0$ ($n > 0$) as

$$J^a(z) J_{-n}^b v_0 = \sum_{i \geq -n} z^i J_{-i-1}^a J_{-n}^b v_0 = \sum_{i < 0} z^i J_{-i-n-1}^{[a,b]} v_0 + \sum_{i \geq 0} z^i J_{-i-1}^a J_{-n}^b v_0.$$

Thus we have

$$Y(J_{-1}^a v_0, z) J_n^b v_0 = J^a(z) J_n^b v_0.$$

Similarly we have

$$Y(J_{-1}^a v_0, z) A = J^a(z) A$$

for any $A \in V_0(\mathfrak{g})$. Then by Goddard's uniqueness Theorem we have

$$Y(J_{-1}^a v_0, z) = J^a(z)$$

as fields. Finally by the reconstruction theorem [FB04, §2.3.11], we conclude

Proposition. *The ordinary vertex algebra structure on $V_{L\mathfrak{g}}(\{1\})$ coincides with the ordinary vertex algebra $V_0(\mathfrak{g})$.*

§ 2.4. Vertex algebras of affine Kac-Moody Lie algebras

Let us return to the $(R\text{-Mod}, H_a, S_0)$ -vertex algebra $V_{L\mathfrak{g}}$ constructed from the Verma module $V_0(\mathfrak{g})$ of the loop algebra $L\mathfrak{g}$. We will use the twisting operation reviewed in §1.3 to construct another $(R\text{-Mod}, H_a, S_0)$ -vertex algebra such that it corresponds to the ordinary vertex algebra $V_k(\mathfrak{g})$ of affine Kac-Moody Lie algebra with arbitrary level k .

Let R be a commutative field again, and k be an arbitrary element of R , which will be the level of affine Lie algebra $\widehat{\mathfrak{g}}$. Let us fix an invariant symmetric bilinear form on \mathfrak{g} and denote it by $(\ , \)$.

Definition. *Let r be an H_a -invariant $S(\{1 : 2\})$ -valued R -bicharacter of $V_0(\mathfrak{g})$ such that*

$$(2.10) \quad r(J_{-1}^a v_0 \otimes J_{-1}^b v_0) = \frac{k(J^a, J^b)}{(x_1 - x_2)^2}.$$

The H_a -invariant bicharacter r is uniquely determined from this formula since $V_0(\mathfrak{g})$ is H_a -generated by $\{J_{-1}^a \mid a = 1, 2, \dots, \dim \mathfrak{g}\}$, as we noted in Remark 2.3.

Then by Fact 1.18, the twisting of the $(R\text{-Mod}, H_a, S_0)$ -vertex algebra $V_{L\mathfrak{g}}$ by the singular bicharacter defined by (2.10) is another $(R\text{-Mod}, H_a, S_0)$ -vertex algebra. Let us denote this new one by $V_{\widehat{\mathfrak{g}}, k}$. $V_{\widehat{\mathfrak{g}}, k}(\{1\})$ is an R -vector space with a basis (2.8). To distinguish it from the old $V_{L\mathfrak{g}}(\{1\})$, let us denote the vacuum vector in the new one by v_k , and denote the basis as

$$(2.11) \quad \left\{ J_{n_1}^{a_1} J_{n_2}^{a_2} \cdots J_{n_l}^{a_l} v_k \mid l \in \mathbb{Z}_{\geq 0}, n_1 \leq n_2 \leq \cdots \leq n_l < 0, \right. \\ \left. \text{if } n_i = n_{i+1} \text{ then } a_i \leq a_{i+1} \right\}.$$

Recalling the formula (2.9) for the singular tensor product in $V_{\widehat{\mathfrak{g}}, k}$ and the twisted product (1.8) yields

$$[Y(J_{-1}^a v_k, z), Y(J_{-1}^b v_k, w)] = \frac{k(J^a, J^b)}{(z - w)^2} + \frac{[J^a, J^b](w)}{z - w},$$

which coincides with the formula in the vertex algebra of affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$ with level k . Therefore we get

Proposition. *For the twisting $V_{\widehat{\mathfrak{g}}, k}$ of the $(R\text{-Mod}, H_a, S_0)$ -vertex algebra V by the singular bicharacter defined by (2.10), the associated ordinary vertex algebra $V_{\widehat{\mathfrak{g}}, k}(\{i\})$ coincides with the vertex algebra $V_k(\mathfrak{g})$ of Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$ with level k .*

Applying our construction to the case where \mathfrak{g} is the one-dimensional commutative Lie algebra, one gets as $V_{\widehat{\mathfrak{g}},k}(\{1\})$ the Heisenberg vertex algebra (denoted as π_0 in [FB04, §§2.1 – 2.4]). Similarly from the positive-definite even lattice one gets the lattice vertex algebra. These two cases were investigated in [P09].

§ 2.5. The Virasoro vertex algebra

The ordinary vertex algebra attached to Virasoro algebra can also be treated in the present formulation. Let us denote by Vir the Virasoro Lie algebra with generators $\{L_n \mid n \in \mathbb{Z}\}$ and the central element C defined over the complex number field \mathbb{C} . The commutation relation is given by

$$(2.12) \quad [L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C$$

as usual. We construct an $(R\text{-Mod}, H_a, S_0)$ -vertex algebra $V_{Vir,c}$ (with $R = \mathbb{C}$) as follows.

Fix a complex number $c \in \mathbb{C}$. Consider a Lie subalgebra

$$Vir_+ := \bigoplus_{n \in \mathbb{Z}_{\geq -1}} \mathbb{C}L_n \oplus \mathbb{C}C$$

of Vir and its one-dimensional representation $\mathbb{C}_c = \mathbb{C}v_c$ where L_n 's act trivially and C acts by c . Denote the induced representation of Vir by

$$Vir_c := \text{Ind}_{Vir_+}^{Vir} \mathbb{C}_c = U(Vir) \otimes_{U(Vir_+)} \mathbb{C}_c.$$

$U(Vir)$ has a cocommutative bialgebra structure, and it induces another structure on Vir_c . In particular, the comultiplication on $L_n v_c$ ($n < -1$) is given by $\Delta(L_n v_0) = (L_n \otimes 1 + 1 \otimes L_n)v_c \otimes v_c$.

We apply the construction of $V_{L\mathfrak{g}}$ to the derived algebra $[Vir, Vir]$, i.e, the Virasoro Lie algebra without central extension. The space of fields, that is the \mathbb{C} -vector space $V_{Vir,c}(\{1\})$, is given by Vir_0 . It has a basis

$$\{L_{n_1}L_{n_2} \cdots L_{n_l}v_0 \mid l \in \mathbb{Z}_{\geq 0}, n_1 \leq n_2 \leq \cdots \leq n_l < -2\}.$$

The action of the cocommutative bialgebra $H_a = R[D^{(i)}]$ on Vir_0 is given by

$$D^{(i)}A := \frac{1}{i!}L_{-1}^i A$$

for $A \in Vir_0$. By the commutation relation (2.12) one can check the formula

$$(2.13) \quad D^{(i)}L_{-n}v_0 = \binom{n+i-1}{i}L_{-n-i}v_0$$

for $n \in \mathbb{Z}_{>1}$ and $i \in \mathbb{Z}_{\geq 0}$.

As in the case of $V_{L\mathfrak{g}}$, we get an $(R\text{-Mod}, H_a, S_0)$ -vertex algebra V . Then we want to take a twist of V by some singular bicharacter. Consider the H_a -invariant $S_0(\{1 : 2\})$ -bicharacter r of Vir_0 such that

$$r(L_{-2} \otimes L_{-2}) = \frac{c/2}{(x_1 - x_2)^4}.$$

This formula determines r uniquely, since V is H_a -generated by L_{-1} by the action (2.13). Then Lemma 1.17 says that there is a singular bicharacter on V . By Fact 1.18 we have a twisted $(R\text{-Mod}, H_a, S_0)$ -vertex algebra, which is denoted by $V_{Vir,c}$.

As in $V_{\mathfrak{g},k}$, we rename the vacuum vector of $V_{Vir,c}(\{1\})$ as v_c . Then one can check that $T(z) := Y(L_{-2}v_c, z)$ satisfies

$$[T(z), T(w)] = \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \frac{c/2}{(z-w)^4},$$

which coincides with the OPE of conformal field of the Virasoro vertex algebra. Finally we have

Proposition. *The ordinary vertex algebra $V_{Vir,c}(\{1\})$ associated to $V_{Vir,c}$ coincides with the Virasoro vertex algebra with central charge c .*

§ 3. Quantum vertex algebra

In this section we recall quantum (\mathcal{A}, H, S) -vertex algebras given in [B01]. It can be considered as a deformation of (\mathcal{A}, H, S) -vertex algebras discussed in the previous section. We also give some examples of quantum (\mathcal{A}, H, S) -vertex algebras. In particular, deformed chiral algebras in the sense of Frenkel and Reshetikhin are such examples.

We remark that there are several formulations on deformation of ordinary vertex algebras, for example [FR96], [EK00], [L05], [AB09], [L10] and [L11].

§ 3.1. Borcherds' formulation

We begin with the introduction of *braided rings*. Let C be a fixed commutative ring.

Definition 3.1. *Let A be a unital associative C -algebra. A C -homomorphism $R : A \otimes A \rightarrow A \otimes A$ is called an R -matrix if it satisfies the following conditions:*

1. *Yang-Baxter equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.*
2. *$R_{12}m_{12} = m_{12}R_{23}R_{13}$ and $R_{12}m_{23} = m_{23}R_{12}R_{13}$ as C -homomorphisms $A \otimes A \otimes A \rightarrow A \otimes A$.*

3. $R(1 \otimes a) = 1 \otimes a$ and $R(a \otimes 1) = a \otimes 1$ for any $a \in A$.

The following lemma is a slight generalization of [B98, Lemma 10.1] where the ring A is assumed to be commutative.

Lemma. *Suppose A is a unital associative ring and R is an R -matrix for A . Then we have another unital associative ring $(A, m_{12}R_{12}, 1_A)$, where 1_A is the unit of the original ring structure on A .*

Proof. We only write down the proof of the associativity.

$$\begin{aligned} m_{12}R_{12}m_{23}R_{23} &= m_{12}m_{23}R_{12}R_{13}R_{23} = m_{12}m_{23}R_{23}R_{13}R_{12} \\ &= m_{12}m_{12}R_{23}R_{13}R_{12} = m_{12}R_{12}m_{12}R_{12}. \end{aligned}$$

□

Definition. *A braided ring A is a ring with an R -matrix R such that*

$$(3.1) \quad m_A R = m_A \tau : A \otimes A \longrightarrow A.$$

Here $\tau : a \otimes b \mapsto b \otimes a$ is the twist map and m_A is the multiplication of A .

The twisting construction gives us a family of braided rings. Let us give a remark before stating the construction of braided rings.

Remark.

1. *For a cocommutative bialgebra B , C -valued bicharacters of B form a commutative monoid under the multiplication*

$$(r * s)(a \otimes b) := \sum_{(a'), (b')} r(a' \otimes b') s(a'' \otimes b'')$$

and the unit

$$(3.2) \quad \varepsilon(a \otimes b) := \varepsilon(a)\varepsilon(b)$$

given by the counit ε of B . A bicharacter r is called invertible if it has its inverse r^{-1} in this monoid.

2. *If B is a Hopf algebra, then the inverse r^{-1} is given by*

$$r^{-1}(a \otimes b) = r(S_B(a) \otimes b)$$

with S_B the antipode of B .

The following lemma is a slight generalization of [B01, Lemma/Definition 2.6] where M is assumed to be commutative.

Lemma. *Consider the twisting \widetilde{M} of the commutative cocommutative bialgebra M by a C -valued bicharacter r . If r is invertible, then \widetilde{M} is a braided ring.*

Proof. We write down a proof for the sake of completeness. The R -matrix for the braided ring \widetilde{M} is given by

$$R(a \otimes b) := \sum_{(a),(b)} a' \otimes b' r'(b'' \otimes a''),$$

with r' a bicharacter defined to be

$$r'(a \otimes b) := \sum_{(a),(b)} r(a' \otimes b') r^{-1}(b'' \otimes a'').$$

We show that the above formula does define an R -matrix. For the Yang-Baxter equation, we have

$$\begin{aligned} & R_{12}R_{13}R_{23}(a \otimes b \otimes c) \\ &= R_{12}R_{13}\left(\sum a \otimes b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes b^{(2)})\right) \\ &= R_{12}\left(\sum a^{(1)} \otimes b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes a^{(2)}) r'(c^{(3)} \otimes b^{(2)})\right) \\ &= \sum a^{(1)} \otimes b^{(1)} \otimes c^{(1)} r'(b^{(2)} \otimes a^{(2)}) r'(c^{(2)} \otimes a^{(3)}) r'(c^{(3)} \otimes b^{(3)}) \end{aligned}$$

and

$$\begin{aligned} & R_{23}R_{13}R_{12}(a \otimes b \otimes c) \\ &= R_{23}R_{13}\left(\sum a \otimes b^{(1)} \otimes c^{(1)} r'(b^{(2)} \otimes a^{(2)})\right) \\ &= R_{23}\left(\sum a^{(1)} \otimes b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes a^{(2)}) r'(b^{(2)} \otimes a^{(3)})\right) \\ &= \sum a^{(1)} \otimes b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes b^{(2)}) r'(c^{(3)} \otimes a^{(2)}) r'(b^{(3)} \otimes a^{(3)}). \end{aligned}$$

These two equations are equal by the cocommutativity of M .

The first half of the second condition in Definition 3.1 can be checked by

$$\begin{aligned} R_{12}m_{12}(a \otimes b \otimes c) &= R_{12}(ab \otimes c) = \sum a^{(1)}b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes a^{(2)}b^{(2)}) \\ &= \sum a^{(1)}b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes a^{(2)}) r'(c^{(3)} \otimes b^{(2)}) \end{aligned}$$

and

$$m_{12}R_{23}R_{13}(a \otimes b \otimes c) = m_{12}R_{23}\left(\sum a^{(1)} \otimes b \otimes c^{(1)} r'(c^{(2)} \otimes a^{(2)})\right)$$

$$\begin{aligned}
 &= m_{12} \left(\sum a^{(1)} \otimes b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes b^{(2)}) r'(c^{(3)} \otimes a^{(2)}) \right) \\
 &= \sum a^{(1)} b^{(1)} \otimes c^{(1)} r'(c^{(2)} \otimes b^{(2)}) r'(c^{(3)} \otimes a^{(2)}).
 \end{aligned}$$

We used the cocommutativity and the bialgebra property in this demonstration. The last half is shown similarly.

The third condition in Definition 3.1 is easily checked. Note that we have not used the commutativity of M so far.

The braided commutativity condition (3.1) can be checked by

$$\begin{aligned}
 \tilde{m}R(a \otimes b) &= \tilde{m} \left(\sum a^{(1)} \otimes b^{(1)} r'(b^{(2)} \otimes a^{(2)}) \right) \\
 &= \sum a^{(1)} b^{(1)} r(a^{(2)} \otimes b^{(2)}) r'(b^{(3)} \otimes a^{(3)}) \\
 &= \sum a^{(1)} b^{(1)} r(b^{(2)} \otimes a^{(2)}) \\
 &= \tilde{m}\tau(a \otimes b),
 \end{aligned}$$

where \tilde{m} is the twisted multiplication on \tilde{M} , and at the last line we used the commutativity of M . □

Remark. *The twisting \tilde{M} by the unit bicharacter (3.2) is the original algebra M . In this case, $r' = \varepsilon$ and R is the identity operator.*

The notion of R -matrix can also be introduced in an additive symmetric monoidal category \mathcal{A} . Hereafter we switch to this categorical setting. Using the singular tensor product, Borchers introduced

Definition 3.2 ([B01]).

Let H be a cocommutative bialgebra object in \mathcal{A} , and S be a commutative ring object in the additive symmetric monoidal category $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A}, T^(H))$. Define a quantum (\mathcal{A}, H, S) -vertex algebra to be a singular braided ring in $\text{Fun}(\text{Fin}, \mathcal{A}, T^*(H), S)$.*

The twisting construction gives some examples of quantum (\mathcal{A}, H, S) -vertex algebra. The main theorem in [B01] was

Fact 3.3 ([B01, Theorem 4.2]). *Suppose that H is a cocommutative bialgebra in \mathcal{A} and that S is a commutative ring in $\text{Fun}(\text{Fin}^{\neq}, \mathcal{A}, T^*(H))$. Assume that r is an invertible $S(\{1 : 2\})$ -valued bicharacter of a commutative and cocommutative bialgebra M in \mathcal{A} . Then the twisting of $T_*(M)$ by r is a quantum (\mathcal{A}, H, S) -vertex algebra.*

§ 3.2. Yangian

To construct ordinary vertex algebras in the framework of Borchers, the commutative ring object S_0 (Definition 1.9) in $\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod}, T^*(H_a))$ was a key ingredient. It encodes the singular behavior of vertex operators $Y(\cdot, z)$ in vertex algebras.

In this subsection we consider another singular data. Fix an element $t \in R$.

Definition 3.4. Define an object S_t in $\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod})$ by

$$(3.3) \quad S_t(I) := R[(x_i - x_j - nt)^{\pm 1} \mid i \neq j \text{ in } I, n \in \mathbb{Z}]$$

for $I \in \text{Ob}(\text{Fin}^{\neq})$, and

$$S_t(f) : S_t(I) \longrightarrow S_t(J), \quad (x_i - x_j - nt) \longmapsto (x_{f(i)} - x_{f(j)} - nt)$$

for $f \in \text{Fin}^{\neq}(I, J)$.

S_t with $t = 0$ is nothing but S_0 . Similarly as Lemma 1.10, one can check

Lemma 3.5. S_t is a commutative ring object in $\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod}, T^*(H_a))$, where the action of $T^*(H_a)$ on S_t is given by the derivation.

Thus we can consider a quantum $(R\text{-Mod}, H_a, S_0)$ -vertex algebra. Yangians (precisely speaking, the algebras of Drinfeld currents of Yangian) is an example of this setting.

§ 3.3. Deformed chiral algebras

In [FR96] Frenkel and Reshetikhin introduced the notion of deformed chiral algebras, in order to formulate the deformation of ordinary vertex algebras and treat the deformed W -algebras which emerged in the mid 1990s.

Definition 3.6. A deformed chiral algebra is a collection of the following data:

- A \mathbb{C} -vector space V called the space of fields.
- A \mathbb{C} -vector space $W = \cup_{n \geq 0} W_n$ called the space of states, which is union of finite dimensional subspaces W_n . We consider a topology on W in which $\{W_n \mid n \geq 0\}$ is the base of open neighborhoods of 0.
- A linear map $Y : V \rightarrow \text{End}(W) \widehat{\otimes} [[z, z^{-1}]]$ such that for each $A \in V$ each linear operator $A_n \in \text{End}(W)$ in the expansion $Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n}$ satisfies $A_n W_m \subset W_{m+N(n)}$ for any $m \in \mathbb{Z}_{\geq 0}$ with some $N(n) \in \mathbb{Z}$ depending only on A .
- A meromorphic function $S(x) : \mathbb{C}^\times \rightarrow \text{Aut}(V \otimes V)$, satisfying the Yang-Baxter equation

$$(3.4) \quad S_{12}(z)S_{13}(zw)S_{23}(w) = S_{23}(w)S_{13}(zw)S_{12}(z)$$

for any $z, w \in \mathbb{C}^\times$.

- A lattice $L \subset \mathbb{C}^\times$, which contains the poles of $S(x)$.

- An element $\Omega \in V$ such that $Y(\Omega, z) = \text{id}$.

These data should satisfy the following axioms:

1. For any $A_i \in V$ ($i = 1, \dots, n$), the composition $Y(A_1, z_1) \cdots Y(A_n, z_n)$ converges in the domain $|z_1| \gg \cdots \gg |z_n|$ and can be continued to a meromorphic operator valued function

$$R(Y(A_1, z_1) \cdots Y(A_n, z_n)) : (\mathbb{C}^\times)^n \rightarrow \text{Hom}(W, \overline{W}),$$

where \overline{W} is the completion of W with respect to its topology.

2. Denote $R(Y(A, z)Y(B, w))$ by $Y(A \otimes B; z, w)$. Then

$$Y(A \otimes B; z, w) = Y(S(w/z)(B \otimes A); w, z).$$

3. The poles of the meromorphic function $R(Y(A, z)Y(B, w))$ lie on the lines $z = w\gamma$ with $\gamma \in L$. For each such line and $n \in \mathbb{Z}_{\geq 0}$, there exists $C_n \in V$ such that

$$\text{Res}_{z=w\gamma} R(Y(A, z)Y(B, w))(z - w\gamma)^n \frac{dz}{z} = Y(C_n, w).$$

Now we explain that a deformed chiral algebra $(V, W, Y, S(x), L, \Omega)$ is actually an example of quantum (\mathcal{A}, H, S) -vertex algebra with appropriate choice of data. We begin with the singular data S for deformed chiral algebras.

Definition. For a lattice $L \subset \mathbb{C}^\times$. Define an object S_L in $\text{Fun}(\text{Fin}^\neq, R\text{-Mod})$ by

$$S_L(I) := R[(x_i/x_j - \gamma)^{\pm 1} \mid i \neq j \text{ in } I, \gamma \in L]$$

for $I \in \text{Ob}(\text{Fin}^\neq)$, and

$$S_L(f) : S_t(I) \longrightarrow S_t(J), \quad (x_i/x_j - \gamma) \longmapsto (x_{f(i)}/x_{f(j)} - \gamma)$$

for $f \in \text{Fin}^\neq(I, J)$.

Next we need a formal group ring.

Definition. Let H_m be the formal group ring of the one-dimensional multiplicative formal group corresponding to the formal group law $F(X, Y) = XY$.

As in the case of H_a , one can consider the action of $T^*(H_m)$ on S_L (by difference operators preserving L). Then one can show

Lemma. S_L is a commutative ring object in $\text{Fun}(\text{Fin}^{\neq}, R\text{-Mod}, T^*(H_m))$.

Thus we can consider a quantum $(R\text{-Mod}, H_m, S_L)$ -vertex algebra.

Theorem 3.7. Let V_1 be a \mathbb{C} -vector space and $S(x) : \mathbb{C}^\times \rightarrow \text{Aut}(V_1 \otimes V_1)$ be a meromorphic function satisfying the Yang-Baxter equation (3.4). Let V be a quantum $(\mathbb{C}\text{-Mod}, H_M, S_L)$ -vertex algebra given by the twisting using $S(x)$ (so that the underlying vector space of $V(\{1\})$ is V_1). Then $V(\{1\})$ has a structure of deformed chiral algebra.

The proof is similar as in Proof of Fact 1.11 for the non-quantum (or ordinary vertex algebra) case so we omit it.

§ 4. Chiral algebras

Let us briefly recall the formulation of *chiral algebras* due to Beilinson and Drinfeld [BD04]. We will use the notion of factorization algebra, which is equivalent to the chiral algebra when considered over smooth algebraic curves.

For an algebraic curve X defined over some field k and an object I in Fin , X^I denotes the symmetric product over k . $\text{QCoh}(X^I)$ denotes the category of quasi-coherent sheaves on X^I .

Definition. Let X be a smooth algebraic curve defined over \mathbb{C} . A factorization algebra over X consists of data $\{F_I \in \text{Ob } \text{QCoh}(X^I) \mid I \in \text{Ob } \text{Fin}\}$ such that

1. $F_I(\Delta) = 0$, where Δ is the (big) diagonal of X^I .
2. $\Delta_{J/I}^* F_J \xrightarrow{\sim} F_I$ for $p : J \twoheadrightarrow I$, where $\Delta_{J/I} : X^I \hookrightarrow X^J$ is the natural inclusion morphism induced by p .
3. $j_{J/I}^* F_J \simeq j_{J/I}^* (\boxtimes_{i \in I} F_{p^{-1}(i)})$ for $p : J \twoheadrightarrow I$, where $j_{J/I} : U^{J/I} \hookrightarrow X^J$ is the inclusion morphism from $U^{J/I} := \{(x_j) \in X^J \mid x_j \neq x_{j'} \text{ if } p(j) \neq p(j')\}$ to X^J .
4. There exists $1 \in F_1(X)$ such that for any $f \in F_1(U)$ (where $U \subset X$ is an arbitrary open subscheme) the element $1 \boxtimes f \in F_2(U^2 \setminus \Delta)$ extends across Δ and restricts to $f \in F_1(U) \simeq F_2(\Delta|_U)$.

A morphism between factorization algebras can be defined naturally. One of the fundamental results in the Beilinson-Drinfeld theory is

Fact 4.1. There exists an equivalence of categories between the category of quasi-conformal ordinary vertex algebras V and the category of factorization algebras $\{F_I\}$ such that $F_1 = \text{Aut}_X \times_{\text{Aut}_{\mathcal{O}_X}} V$.

Here we used the term *quasi-conformal* in the meaning of [FB04, §6.2]. Let us recall its definition briefly. The space $\mathcal{O} := \mathbb{C}[[z]]$ of formal series of one variable with complex coefficients may be considered as a complete topological \mathbb{C} -algebra (with the topology given by the unique maximal ideal). Let us also consider the Lie algebras

$$\text{Der } \mathcal{O} := \mathbb{C}[[z]]\partial_z \supset \text{Der}_0 \mathcal{O} := z\mathbb{C}[[z]]\partial_z \supset \text{Der}_+ \mathcal{O} := z^2\mathbb{C}[[z]]\partial_z.$$

Let us denote by $L_n := -z^{n+1}\partial_z \in \text{Der } \mathcal{O}$ for $n \in \mathbb{Z}_{\geq -1}$.

Definition. *An ordinary vertex algebra is called quasi-conformal if it has an action of $\text{Der } \mathcal{O}$ such that*

- *the formula*

$$\left[\sum_{n \geq -1} v_n L_n, Y(A, z) \right] = \sum_{m \geq -1} \frac{1}{(m+1)!} (\partial_w^{m+1} v(z)) Y(L_m A, z)$$

holds for any $A \in V$ and any $v(z)\partial_z = \sum_{n \geq -1} v_n z^{n+1}\partial_z \in \text{Der } \mathcal{O}$,

- *the element $L_{-1} = -\partial_z$ acts as the translation operator T ,*
- *$L_0 = -z\partial_z$ acts semisimply with integral eigenvalues,*
- *the Lie subalgebra $\text{Der}_+ \mathcal{O}$ acts locally nilpotently.*

It is known that a conformal ordinary vertex algebra (ordinary vertex algebra with a Virasoro element) is the canonical example of quasi-conformal ordinary vertex algebra.

Let us also mention that $\text{Lie}(\text{Aut } \mathcal{O}) = \text{Der}_0 \mathcal{O}$, where $\text{Aut } \mathcal{O}$ is the group of continuous automorphisms of \mathcal{O} . The axiom of quasi-conformal ordinary vertex algebra says that $\text{Aut } \mathcal{O}$ acts on the vertex algebra. Since $\text{Aut } \mathcal{O}$ describes the infinitesimal symmetry of an algebraic curve, the appearance of quasi-conformal ordinary vertex algebra in Fact 4.1 is natural.

Now we address the relationship between chiral algebras and (\mathcal{A}, H, S) -vertex algebras. Recall Fact 1.11 which says that (\mathcal{A}, H, S) -vertex algebras with $(\mathcal{A}, H, S) = (R\text{-Mod}, H_a, S_0)$ gives ordinary vertex algebras. As mentioned above, ordinary vertex algebras with quasi-conformal property are essentially the same as factorization algebras. Thus one can expect that under certain choice of the data (\mathcal{A}, H, S) -vertex algebras give factorization algebras directly. It is not difficult to find such data. The result is as follows.

Theorem 4.2. *Let us consider the (\mathcal{A}, H, S) -vertex algebra with the setting*

$$\mathcal{A} = \text{QCoh}(X), \quad H = \text{Der } \mathcal{O}_X, \quad S(J) = \mathcal{O}_{U^J/I}.$$

In the definition of $S(J)$ for $J \in \text{Ob Fin}^{\neq}$, I is uniquely determined by the surjection $J \twoheadrightarrow I$ corresponding to J .

Then the (\mathcal{A}, H, S) -vertex algebra has a structure of factorization algebra, and the associated vertex algebra is quasi-conformal.

References

- [AB09] Anguelova, I. I., Bergvelt, M. J., *H_D -quantum vertex algebras and bicharacters*, Commun. Contemp. Math., **11** (2009), no. 6, 937–991.
- [BD04] Beilinson, A., Drinfeld, V., *Chiral Algebras*, Colloquium Publications, vol. **51**, Amer. Math. Soc. (2004).
- [B86] Borcherds, R. E., *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Nat. Acad. Sci. U.S.A., **83** (1986), no. 10, 3068–3071.
- [B98] Borcherds, R. E., *Vertex algebras*, in *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, 35–77, Progr. Math., **160**, Birkhäuser (1998).
- [B01] Borcherds, R. E., *Quantum vertex algebras*, Taniguchi Conference on Mathematics Nara '98, 51–74, Adv. Stud. Pure Math., **31**, Math. Soc. Japan (2001).
- [EK00] Etingof, P., Kazhdan, D., *Quantization of Lie bialgebras. V. Quantum vertex operator algebras*, Selecta Math., (N.S.) **6**(1) (2000) 105–130.
- [FB04] Frenkel, E., Ben-Zvi, D., *Vertex algebras and algebraic curves*, 2nd ed., Mathematical Surveys and Monographs **88**, Amer. Math. Soc. (2004).
- [FR96] Frenkel, F., Reshetikhin, R., *Towards deformed chiral algebras*, in *Proceedings of Quantum Group Symposium at the XXith International Colloquium on Group Theoretical Methods in Physics*, Goslar, Germany (1996).
- [K98] Kac, V. *Vertex algebras for beginners*, 2nd ed., University Lecture Series **10**, Amer. Math. Soc. (1998).
- [L05] Li, H., *Nonlocal vertex algebras generated by formal vertex operators*, Selecta Math., (N.S.) **11** (2005), no. 3–4, 349–397.
- [L10] Li, H., *\hbar -adic quantum vertex algebras and their modules*, Comm. Math. Phys., **296** (2010), no. 2, 475–523.
- [L11] Li, H., *φ -coordinated quasi-modules for quantum vertex algebras*, Comm. Math. Phys., **308** (2011), no. 3, 703–741.
- [M98] Mac Lane, S., *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics **5**, Springer-Verlag (1998).
- [P09] Patnaik, M. M., *Vertex algebras as twisted bialgebras: on a theorem of Borcherds*, Contemp. Math. **479**, 223–238, Amer. Math. Soc. (2009).