RIMS Kôkyûroku Bessatsu
B63 (2017), 1–11

Asymptotic analysis based on the inverse scattering method

By

Hideshi YAMANE*

Abstract

This is a survey article concerning the asymptotic study of integrable systems. Technical details are often omitted in favor of informal explanations of fundamental ideas. First we briefly recall the inverse scattering method. Our presentation is based on Riemann-Hilbert problems (a kind of boundary value problems on the complex plane) rather than the Gelfand-Levian-Marchenko integral equations. We explain the method of nonlinear steepest descent, which is a powerful tool for analyzing the long-time behavior of integrable systems. Two examples are given: the defocusing integrable nonlinear Schrödinger equation (NLS) and its discrete version (IDNLS) due to Ablowitz-Ladik.

§ 1. Inverse scattering

For the details about the facts stated in this section and §3, the reader is referred to [5]. It is an easy-to-read introduction to the inverse scattering transform for the integrable nonlinear Schrödinger equation and its variants.

§ 1.1. Defocusing NLS

The defocusing integrable nonlinear Schrödinger equation (NLS)

$$iy_t + y_{xx} - 2|y|^2 y = 0 \tag{1.1}$$

Received July 1, 2015. Revised March 24, 2016.
2010 Mathematics Subject Classification(s): Primary 35Q55; Secondary 35Q15
Key Words: *discrete nonlinear Schrödinger equation, soliton, nonlinear steepest descent.*
This work was partially supported by JSPS KAKENHI Grant Number 26400127.
*Department of Mathematical Sciences, Kwansei Gakuin University, Gakuen 2-1, Sanda, Hyogo 669-1337, Japan.
e-mail: yamane@kwansei.ac.jp

can be solved by the inverse scattering transform as was discovered by [16][1]. Let z be the spectral parameter and introduce two matrices P and Q by

$$P = -iz\sigma_3 + \begin{bmatrix} 0 & y \\ \bar{y} & 0 \end{bmatrix}, \quad \sigma_3 := \mathrm{diag}(1, -1),$$
$$Q = -2iz^2\sigma_3 + 2z \begin{bmatrix} 0 & y \\ \bar{y} & 0 \end{bmatrix} + \begin{bmatrix} -i|y|^2 & iy_x \\ -i\bar{y}_x & i|y|^2 \end{bmatrix}.$$

Then (1.1) is nothing but the compatibility condition $\Psi_{xt} = \Psi_{tx}$ of the Lax pair $\Psi_x = P\Psi, \Psi_t = Q\Psi$. One can construct solutions ψ and ψ^* of the x-part $\Psi_x = P\Psi$ such that

$$\psi \sim {}^t[0,1]e^{izx},\ \psi^* \sim {}^t[1,0]e^{-izx} \text{ as } x \to \infty.$$

The *reflection coefficient* r_1 is defined by

$$\underbrace{r_1\psi}_{\text{reflection}} + \underbrace{\psi^*}_{\text{incidence}} \sim \underbrace{\text{const.}{}^t[1,0]e^{-izx}}_{\text{transmission}} \text{ as } x \to -\infty.$$

It can be proved that $0 \le |r_1| < 1$. The t-part $\Psi = Q\Psi$ determines the time evolution of $r_1 = r_1(z,t), z \in \mathbb{R}$. We have

$$r_1(z,t) = e^{-4iz^2t}r_1(z,0).$$

The initial value problem for (1.1) can be solved by the following procedure. Let the initial value $y(x,0)$ be given (typically it is in the Schwartz class). Then $r_1(z,0)$ and $r_1(z,t)\,(t>0)$ are determined. There are two ways to *reconstruct* the potential $y(x,t)$ from $r_1(z,t)$: the use of a Gelfand-Levitan-Marchenko integral equation or of a Riemann-Hilbert problem. In the present article, the latter is employed. Its main advantage is that it admits contour deformation.

§ 1.2. Defocusing integrable discrete NLS

The defocusing integrable *discrete* nonlinear Schrödinger equation (IDNLS)

$$(1.2) \qquad i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2(R_{n+1} + R_{n-1}) = 0 \ \ (t \ge 0, n \in \mathbb{Z}).$$

was introduced by Ablowitz-Ladik ([3, 4]). The nonlinear term is so chosen that the equation admits a Lax pair representation and can be solved by the inverse scattering

[1]This article studies the focusing case, but the essential ideas are the same. The defocusing case is easier because of the absence of (bright) solitons.

transform. The Lax pair is

$$X_{n+1} = \begin{bmatrix} z & \overline{R}_n \\ R_n & z^{-1} \end{bmatrix} X_n,$$

n-part (Ablowitz-Ladik scattering problem),

$$\frac{d}{dt}X_n = \begin{bmatrix} iR_{n-1}\overline{R}_n - \frac{i}{2}(z-z^{-1})^2 & -i(z\overline{R}_n - z^{-1}\overline{R}_{n-1}) \\ i(z^{-1}R_n - zR_{n-1}) & -iR_n\overline{R}_{n-1} + \frac{i}{2}(z-z^{-1})^2 \end{bmatrix} X_n,$$

t-part (time evolution).

Notice that (IDNLS) is the compatibility condition $\frac{d}{dt}X_{n+1} = \left(\frac{d}{dt}X_m\right)_{m=n+1}$. If the potential $R_n = R_n(t)$ is identically zero, then the n-part has two linearly independent solutions ${}^t[z^n, 0]$ and ${}^t[0, z^{-n}]$. For a nontrivial potential, one can construct eigenfunctions which behave like these functions as $n \to \infty$ or $n \to -\infty$. Let $\psi_n(z,t)$ and $\psi_n^*(z,t)$ be solutions with the following behavior:

$$\psi_n(z,t) \sim z^{-n} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \psi_n^*(z,t) \sim z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ as } n \to \infty.$$

The reflection coefficient $r_2 = r_2(z,t)$ is defined by

$$r_2\psi_n + \psi_n^* \sim \text{const.}z^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ as } n \to -\infty.$$

We can show that $0 \le |r_2| < 1$ and that r_2 is smooth if $\{R_n\}$ decreases rapidly as $|n| \to \infty$. The time evolution is

$$r_2(z,t) = r_2(z)\exp\left(it(z-z^{-1})^2\right), \text{where}\, r_2(z) = r_2(z,0)$$

Notice that the time evolution formula is much more complicated than that for the continuous NLS.

§ 2. Riemann-Hilbert problems

§ 2.1. Formulation and properties

Let Γ be an oriented (reasonably good) contour in the complex plane. Its left-hand side is called the + side. Let $v(z)$ be a given 2×2 matrix on Γ. We often consider the case where $v(z)$ admits analytic continuation. For an unknown 2×2 matrix $m(z)$ whose components are holomorphic in $\mathbb{C} \setminus \Gamma$, its boundary values on Γ from the $\pm$ sides are denoted by $m_\pm(z)$. We consider the boundary value problem

$$m_+(z) = m_-(z)v(z) \text{ on } \Gamma. \tag{2.1}$$

Its solution is not unique. Indeed, if $m(z)$ is a solution, then so is $a(z)m(z)$ for any $a(z)$. It is customary to impose the normalization condition

$$m(z) \to I \text{ (the identity matrix) as } z \to \infty \tag{2.2}$$

which is good enough to ensure uniqueness in many cases. This kind of boundary value problem is called a Riemann-Hilbert problem (RHP). RHPs can replace Gelfand-Levitan-Marchenko integral equations in the study of integrable systems. They are useful because they 'behave like integrals'. More precisely,

1. [Contour deformation] RHPs admit contour deformation.
2. [Deletion of an insignificant part of the contour] If $v = I$ on $\hat{\Gamma} \subset \Gamma$, then $\hat{\Gamma}$ can be deleted.
3. [Continuity] The mapping $v \mapsto m$ is continuous (under some reasonable assumptions). In other words, if $v \approx v'$ (i.e. v is close to v' in an unspecified topology), then the solution corresponding to v is approximated by that corresponding to v'.
4. [Consequence of the preceding two facts] If $v \approx I$ in a certain sense on $\hat{\Gamma} \subset \Gamma$, then the solution to the original problem is approximated by that to the revised problem with $\hat{\Gamma}$ deleted.

The second assertion is trivial. We explain the first and third assertions.

Contour deformation

One can deform the contour in a Riemann-Hilbert problem. Usually contour deformation is coupled with a factorization of a jump matrix.

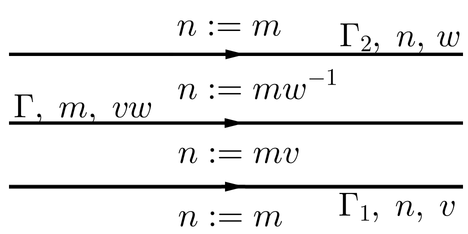

Figure 1. deformation

We consider $m_+ = m_- vw$ on Γ in Figure 1. We assume that $v = v(z)$ and $w = w(z)$ can be analytically continued up to sufficiently large open sets. The the original RHP is equivalent to $n_+ = n_- v$ on Γ_1 and $n_+ = n_- w$ on Γ_2, where the new unknown n is defined as in the figure.

Continuity

If the jump matrix admits a certain kind of factorization, there is an integral representation (due to [6]) for the solution of (2.1) and (2.2). It is continuous with respect to small perturbations of the coefficients.

Let C be the Cauchy integral along Γ: for an arbitrary function f on Γ, set

$$Cf(z) = \int_\Gamma \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}.$$

We denote its boundary values on Γ by $C_\pm f$. Notice that $C_+ - C_-$ is the identity. Assume that v in (2.1) admits a factorization of the form $v = (I - w_-)^{-1}(I + w_+)$. (This is usually the case with RHPs associated with integrable systems.) Set $w = w_+ + w_-$ and define the operator C_w by $C_w f = C_+(f w_-) + C_-(f w_+)$. If $\mu = (1 - C_w)^{-1} I$, we set

$$m(z) = I + (C(\mu w))(z) = I + \int_\Gamma \frac{\mu(\zeta) w(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}. \tag{2.3}$$

Then $m(z)$ satisfies (2.2). Moreover, (2.1) also holds because

$$\begin{aligned} m_+ &= I + C_+(\mu w) = I + C_+(\mu w_+) + C_+(\mu w_-) \\ &= I + \mu w_+ + C_-(\mu w_+) + C_+(\mu w_-) = I + C_w \mu + \mu w_+ = \mu(I + w_+), \end{aligned}$$

and similarly $m_- = \mu(I - w_-)$. Since the solution m is written in terms of integrals, the mapping $(w_+, w_-) \mapsto m$ is continuous. We can justifiably say, at least if we fix a certain canonical way of factorization, that $v \mapsto m$ is continuous.

It is possible to calculate the integral in (2.3) in closed form if v has a certain property. When v is perturbed, the closed form expression gives an asymptotic expansion of the perturbed m because of the continuity of $v \mapsto m$. This observation justifies the approximation in the next subsection.

§ 2.2. Steepest descent arguments

Classical method of steepest descent

Let us consider the asymptotic behavior of

$$I(t) = \int_\Gamma e^{it\psi(z)} f(z)\, dz$$

as $t \to \infty$. Here Γ is a contour and $\psi(z)$ and $f(z)$ are holomorphic functions. We assume that ψ is real at a saddle point $S \in \Gamma$. We deform the contour into $\operatorname{Im} \psi > 0$. Then $e^{it\psi} \to 0$ as $t \to \infty$ on $\Gamma \setminus S$ and we have for large t

$$I(t) = \int_\Gamma \approx \int_{\text{near } S}.$$

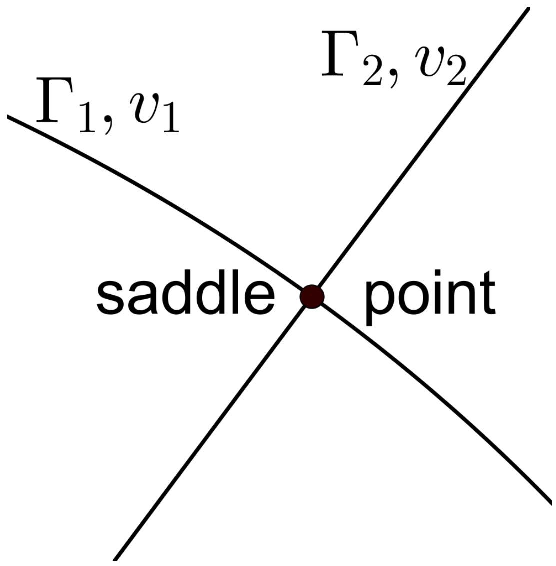

Figure 2. saddle point and a new contour

We can approximate $\psi(z)$ and $f(z)$ by simpler functions near S and obtain

$$I(t) \approx \int_{\text{near } S} (\text{simpler integrand})\, dz.$$

Nonlinear steepest descent

The method of steepest descent is based on the three properties of complex integrals: contour deformation, continuous dependence on integrands and the trivial fact that a zero integrand implies the vanishing an integral. As we have seen above, RHPs have analogous properties and Deift-Zhou ([8]) managed to formulate an RHP version of the method of steepest descent. Since it is usually applied to the study of integral systems, it is called *nonlinear steepest descent.*

We consider a Riemann-Hilbert problem $m_+ = m_- v$ on a contour Γ. Assume that v involves $\exp(\pm it\psi)$ and that the function ψ is real at its saddle point $S \in \Gamma$. Let $\Gamma_1 \cup \Gamma_2$ be a new contour as in Figure 2. The contours Γ_1 and Γ_2 are so drawn that $\Gamma_1 \setminus \{S\} \subset \{\operatorname{Im} \psi > 0\}$ and $\Gamma_2 \setminus \{S\} \subset \{\operatorname{Im} \psi < 0\}$. It follows that $\exp(it\psi(z)) \to 0$ as $t \to \infty$ on $\Gamma_1 \setminus \{S\}$ and that $\exp(-it\psi(z)) \to 0$ on Γ_2. Assume that the original RHP is equivalent to $n_+ = n_- v_j$ on Γ_j $(j = 1, 2)$ by the deformation argument in §2. Additionally, suppose that $v_1 - I$ and $v_2 - I$ are of orders $O(\exp(it\psi))$ and $O(\exp(-it\psi))$ respectively. Then we have $v_1, v_2 \approx I$ except in a small neighborhood of S for t sufficiently large. It implies that $n(z)$ and $m(z)$ are determined by $v(z)$, where z is near S, up to small errors. We can approximate $v(z)$ by a constant matrix $v(S)$. If a solution of an integrable equation can be reconstructed from $m(z)$, the steepest descent argument as above enables us to derive the asymptotic expansion of the solution. This method works even if there are multiple saddle points, although constructing suitable contours can be a delicate task.

Remark. Some authors study oscillatory RHPs with non-analytic phases (independently of integrable systems). Saddle point arguments are complex-analytic in nature and are not valid in this context. However, it is still possible to obtain asymptotic results by using stationary phase techniques. See [10, 12].

§ 3. RHPs and inverse scattering

§ 3.1. Defocusing NLS

The time evolution of the reflection coefficient for the defocusing NLS is $r_1(z,t) = e^{-4iz^2t} r_1(z,0)$. It leads to the following RHP:

$$\begin{aligned} m^1_+(z) &= m^1_-(z) v_1(z),\ z \in \mathbb{R}, \\ v_1(z) &= \begin{bmatrix} 1-|r_1(z)|^2 & -\overline{r_1(z)} e^{-2it\psi_1(z)} \\ r_1(z) e^{2it\psi_1(z)} & 1 \end{bmatrix},\ \psi_1(z) = 2z^2 + (x/t)z, \\ m^1(z) &\to I \quad \text{as } z \to \infty. \end{aligned}$$

The solution $y(x,t)$ can be reconstructed from the solution $m^1(z) = m^1(z;x,t)$. Indeed, we have

$$y = 2i \lim_{z\to\infty} z \left(m^1(z;x,t) \right)_{12},$$

where the subscript 12 means the $(1,2)$-component of a 2×2 matrix.

§ 3.2. Defocusing IDNLS

The time evolution of the reflection coefficient for the defocusing IDNLS is

$$r_2(z,t) = r_2(z) \exp\left(it(z-z^{-1})^2 \right), \text{where}\, r_2(z) = r_2(z,0).$$

It leads to the RHP:

$$\begin{aligned} m^2_+(z) &= m^2_-(z) v_2(z),\ |z| = 1, \\ v_2(z) &= \begin{bmatrix} 1-|r(z)|^2 & -e^{-2it\psi_2(z)}\overline{r(z)} \\ e^{2it\psi_2(z)} r(z) & 1 \end{bmatrix}, \\ \psi_2(z) &= \psi_2(z,n,t) = \frac{1}{2}(z-z^{-1})^2 + \frac{in}{t}\log z. \\ m^2(z) &\to I \quad \text{as } z \to \infty. \end{aligned}$$

The solution $R_n(t)$ can be reconstructed from the solution $m^2(z) = m^2(z; n, t)$ by

$$R_n(t) = -\lim_{z\to 0} \frac{1}{z} \left(m^2(z)\right)_{21} = -\left.\frac{d}{dz}\left(m^2(z)\right)_{21}\right|_{z=0}.$$

§ 4. Nonlinear steepest descent

§ 4.1. Defocusing NLS

The phase $\psi_1 = 2z^2 + (x/t)z$ has a unique saddle point $z_0 = -x/(4t)$ on the real line. Deift-Its-Zhou ([7], [9]) replaced the original contour by a cross like the one in Figure 2. They obtained

$$y(x,t) \sim t^{-1/2}\alpha(z_0)\exp\left[4itz_0^2 - i\nu(z_0)\log 8t\right] + \mathcal{O}(t^{-1}\log t),\ t\to\infty,$$

where

$$\begin{aligned}
\nu = \nu(z_0) &= -\frac{1}{2\pi}\log(1-|r(z_0)|^2),\\
|\alpha(z_0)| &= \nu(z_0)/2,\\
\arg\alpha(z_0) &= \frac{1}{\pi}\int_{-\infty}^{z_0}\log(z_0-\zeta)d\log(1-|r(\zeta)|^2) + \arg\frac{\Gamma(i\nu)}{r(z_0)} + \frac{\pi}{4}.
\end{aligned}$$

Let us compare this result with the well-known result about the linear case. (See, for example, [2].) Let $u(x,t)$ be the solution of the Cauchy problem

$$\begin{gathered}
iu_t + u_{xx} = 0, -\infty < x < \infty, t > 0,\\
u(x,0) = u_0(x) \in \mathcal{S}.
\end{gathered}$$

Then we have

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\exp(ikx - ik^2t)\,dk,\ b(k) = \int_{-\infty}^{\infty} u(x,0)e^{-ikx}\,dx.$$

By using the method of stationary phase, one can show that the asymptotic behavior as $t\to\infty$ is

$$u(x,t) \sim \sqrt{\frac{\pi}{t}}\, b\Big(\frac{x}{2t}\Big)\exp\left(it\left(\frac{x}{2t}\right)^2 - \frac{\pi i}{4}\right).$$

§ 4.2. Defocusing integrable discrete NLS

We assume $n \geq 0$ for the time being: the equation (1.2) is invariant under the reflection $n \mapsto -n$. The stationary points of $\psi_2 = \psi_2(z,n,t) = \frac{1}{2}(z-z^{-1})^2 + \frac{in}{t}\log z$ are $z = S_j$ $(j = 1,2,3,4)$, where

$$\begin{aligned}
&S_1 = e^{-\pi i/4}A,\ S_2 = e^{-\pi i/4}\bar{A},\ S_3 = -S_1,\ S_4 = -S_2,\\
&A = 2^{-1}\left(\sqrt{2+n/t} - i\sqrt{2-n/t}\,\right).
\end{aligned}$$

Their configuration is as follows:

- $n/t < 2$: four saddle points on $|z|=1$(simple zeros of ψ_2').
- $n/t = 2$: two stationary points (double zeros of ψ_2').
- $n/t > 2$: four saddle points *off* $|z| = 1$.

We calculate the asymptotic behavior of $R_n(t)$ by using the new contours shown in Figures 3-5. By the steepest descent argument, most parts of the contours can be neglected. For example, in the first case what remains is four small crosses near the saddle points. See Figure 6.

The contributions of antipodal stationary points coincide. In the first and second cases, the leading part consists of two terms or a single term respectively.

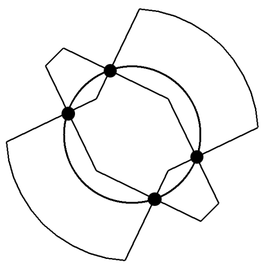

Figure 3. $n/t < 2$

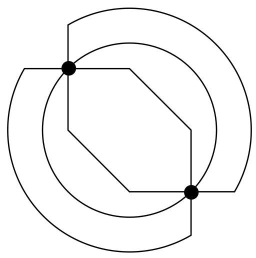

Figure 4. $n/t = 2$

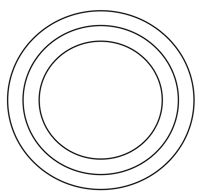

Figure 5. $n/t > 2$

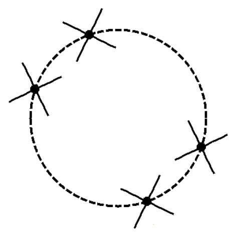

Figure 6. small crosses

Our result, including the case $n < 0$, is as follows (see [13, 14] for details).

Assume that $\sum_{n\in\mathbb{Z}} |n|^k |R_n(0)|$ is finite for any k and that $\sup_{n\in\mathbb{Z}} |R_n(0)| < 1$ holds. Then we have:

- In the region $|n|/t < 2$, there exist $C_j = C_j(n/t) \in \mathbb{C}$, $p_j = p_j(n/t) \in \mathbb{R}$ and $q_j = q_j(n/t) \in \mathbb{R}$ such that

$$R_n(t) = \sum_{j=1}^{2} C_j t^{-1/2} \exp\Big(-i(p_j t + q_j \log t)\Big) + O(t^{-1}\log t) \quad \text{as } t \to \infty.$$

- In the region $n/t \approx 2$, we consider a curve $2 - n/t = \text{const.}t^{-2/3}(6 - n/t)^{1/3}$. Then, up to a time shift $t \mapsto t - t_0$, we have $n/t \to 2$ and

$$R_n(t) = \text{const.}t^{-1/3}e^{i(-4t+\pi n)/2} + O(t^{-2/3}) \tag{4.1}$$

as $t \to \infty$ on this curve.

- In the region $n/t \approx -2$, the behavior is almost the same as (4.1): we have only to replace n by $-n$.

- In the region $|n|/t > 2$, we have $|R_n(t)| = O(n^{-j})$ for any j as $n \to \infty$.

Notice that a similar result was obtained for the *focusing* integrable discrete NLS by using an ansatz in ([11]).

Let us compare this result with the linear case (See [1]). Let $u_n = u_n(t)$ be the solution of the Cauchy problem

$$i\frac{d}{dt}u_n + u_{n+1} - 2u_n + u_{n-1} = 0, n \in \mathbb{Z}, t > 0,$$
$$u_n(t=0) = w_n$$

with $w_n \to 0$ sufficiently rapidly as $|n| \to \infty$. Set

$$\tilde{u}_0(k) = \sum_{n=-\infty}^{\infty} w_n e^{-ink}$$

Then we have

$$u_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{u}_0(e^{ik})e^{it\phi(k)}\,dk,\ \phi(k) = \frac{nk}{t} - 2(1 - \cos k).$$

Stationary points $k_0 \in [-\pi, \pi]$ of $\phi(k)$ are the solutions of $\sin k_0 = n/(2t)$. In the region $|n/t| < 2$, there are two values of k_0 and the asymptotic behavior as $t \to \infty$ is

$$u_n(t) \sim \sum_{k_0} \frac{\tilde{u}_0(k_0)}{2\sqrt{\pi t\,|\cos k_0|}} \exp\Big(i\phi(k_0) - i\nu\pi/4\Big),\ \nu = \operatorname{sgn}\cos k_0.$$

In $|n/t| > 2$, there are no (real) stationary points and $u_n(t)$ decays rapidly. Near $|n/t| = 2$, $u_n(t)$ decays in the rate $\mathcal{O}(t)^{-1/3}$.

References

[1] Ablowitz, M., *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons*, Cambridge University Press, 2011.

[2] Ablowitz, M. and Fokas, A., *Complex Variables: Introduction and Applications*, Cambridge University Press, 2003.

[3] Ablowitz, M. J. and Ladik, J. F., Nonlinear differential-difference equations, *J. Math. Phys.*, **16** (1975), 598-603.

[4] Ablowitz, M. J. and Ladik, J. F., Nonlinear differential-difference equations and Fourier analysis, *J. Math. Phys.*, **17** (1976), 1011-1018.

[5] Ablowitz, M. J., Prinari, B. and Trubatch, A. D., *Discrete and continuous nonlinear Schrödinger systems*, Cambridge University Press, 2004.

[6] Beals, R. and Coifman, R., Scattering and inverse scattering for first order systems, *Comm. Pure Appl. Math.*, **37** (1984), 39-90.

[7] Deift, P. A., Its, A. R. and Zhou, X., Long-time asymptotics for integrable nonlinear wave equations, *Important developments in soliton theory, 1980-1990* edited by A. S. Fokas and V. E. Zakharov, Springer-Verlag (1993), 181-204.

[8] Deift, P. A. and Zhou, X., A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. of Math.*(2), **137**(2) (1993), 295-368.

[9] Deift, P. A. and Zhou, X., *Long-time behavior of the non-focusing nonlinear* Schrödinger *equation – a case study*, Lectures in Mathematical Sciences No. 5, The University of Tokyo, 1994.

[10] Do, Y., A nonlinear stationary phase method for oscillatory Riemann-Hilbert Problems, *Int. Math. Res. Not. IMRN*, **12**, (2011), 2650-2765.

[11] Novokshönov, V. Yu., Asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem for a nonlinear differential-difference Schrödinger equation, *Differentsialnye Uravneniya*, **21** (1985), no. 11, 1915-1926. (in Russian); *Differential Equations*, **21** (1985), no. 11, 1288-1298.

[12] Varzugin, G. G., Asymptotics of oscillatory Riemann-Hilbert problems, *J. Math. Phys*, **37** (1996), 5869-5892.

[13] Yamane, H., Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation, *J. Math. Soc. Japan*, **66** (2014), 765 - 803.

[14] Yamane, H., Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation II, *SIGMA (Symmetry, Integrability and Geometry: Methods and Applications)* **11** (2015), 020, 17 pages.

[15] Yamane, H., Long-time asymptotics for the integrable discrete nonlinear Schrödinger equation: the focusing case, arXiv:1512.01760v4.

[16] Zakharov, V. E.; Shabat, A. B. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Soviet Physics JETP,* **34** (1972), no. 1, 62–69.; translated from *Z. Èksper. Teoret. Fiz.*, 61 (1971), no. 1, 118–134 (Russian).