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# The derivation of conservation laws for nonlinear Schrödinger equations with power type nonlinearities

By

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**Abstract**

For nonlinear Schrödinger equations with power type nonlinearities, a new approach to derive the conservation law of the momentum and the pseudo conformal conservation law is obtained. The approach involves properties of solutions provided by the Strichartz estimate.

## § 1. Introduction

In this paper, we consider the nonlinear Schrödinger equations with power type nonlinearities

$$(1.1) \qquad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = f(u), \quad (t,x) \in \mathbb{R}^{1+n}, \\ u(0,x) = \phi(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $u(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}$, the initial data $\phi$ is a complex valued function on $\mathbb{R}^n$, and $f$ is a nonlinear term of gauge invariance. The equation (1.1) has been extensively studied both in physical and mathematical literatures (see [1], [5]). The conservation law of the momentum and the pseudo conformal conservation law play a role to investigate asymptotic behavior of the solution to (1.1). For example, using the conservation law of the momentum, we study blow-up in finite time and dynamics of blow-up solutions (see

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[3]). Moreover, the pseudo conformal conservation law implies that if $f(u) = \lambda|u|^{p-1}u$ with $p \geq 1 + 4/n$ and $\lambda > 0$, then we have a time decay estimate for the solution $u \in C([0,T],\Sigma)$ of (1.1) such that $\|u(t)\|_{L^r} \leq C|t|^{-n(\frac{1}{2}-\frac{1}{r})}$ for any $r \in [2, \alpha(n)]$, where $\alpha(n) = 1+4/(n-2)$ if $n \geq 3$, $\alpha(n) = \infty$ if $n = 1, 2$ and $\Sigma = \{u \in H^1(\mathbb{R}^n) \mid xu \in L^2(\mathbb{R}^n)\}$ (see §7 of [1]). According to [1, 4], we set the assumptions of nonlinearity $f(u)$ as follows:

(A1) $f \in C^1(\mathbb{C},\mathbb{C})$, $f(0) = 0$ and for some $p > 1$, $f$ satisfies

$$|f'(z)| \leq C(1 + |z|^{p-1})$$

for any $z \in \mathbb{C}$, where $|f'(z)| = \max\left(\left|\frac{\partial f}{\partial z}\right|, \left|\frac{\partial f}{\partial \bar{z}}\right|\right)$,

(A2) $\mathrm{Im}(\bar{z}f(z)) = 0$ for all $z \in \mathbb{C}$,

(A3) There exists $V \in C^1(\mathbb{C},\mathbb{R})$ such that $V(0) = 0$ and $f(z) = \partial V/\partial \bar{z}$.

Note that if $f$ satisfies (A2) and (A3), then $V(z) = V(|z|)$. Hence, we simply write $V'(z) = \frac{dV}{dr}(r)\big|_{r=|z|}$.

We can obtain formally the conservation law of the momentum $P(u)$ by multiplying the equation (1.1) by $\nabla\bar{u}$, integrating over $\mathbb{R}^n$, and taking the real part as follows:

$$\begin{aligned} 0 &= 2\mathrm{Re}\left(i\partial_t u + \frac{1}{2}\Delta u - f(u), \nabla u\right)_{L^2} \\ &= 2\mathrm{Re}(i\partial_t u, \nabla u)_{L^2} - 2\mathrm{Re}(f(u), \nabla u)_{L^2} = \frac{d}{dt}P(u(t)), \end{aligned}$$

where

$$P(u) = \mathrm{Im}\int_{\mathbb{R}^n} u\nabla\overline{u}dx.$$

We present a formal argument to derive the pseudo conformal conservation law

$$\begin{aligned} &\frac{1}{2}\|J(t)u(t)\|_{L^2}^2 + t^2\int_{\mathbb{R}^n} V(u(t))dx \\ &= \frac{1}{2}\|x\phi\|_{L^2}^2 + \int_0^t s\left(\int_{\mathbb{R}^n} (n+2)V(u(s)) - \frac{n}{2}V'(u(s))|u(s)|dx\right)ds, \end{aligned}$$

where $J(t) = x + it\nabla$. Applying the operator $J$ to (1.1), we have a equation

$$\left(i\partial_t + \frac{1}{2}\Delta\right)J(t)u = J(t)f(u). \tag{1.2}$$

Furthermore we can obtain the pseudo conformal conservation law by multiplying (1.2) by $\overline{J(t)u(t)}$, integrating over $\mathbb{R}^n$, and taking the imaginary part as follows:

$$0 = 2\mathrm{Im}\left(i\partial_t J(t)u(t) + \frac{1}{2}\Delta J(t)u(t) - J(t)f(u(t)), J(t)u(t)\right)_{L^2}$$

$$
\begin{aligned}
&= 2\mathrm{Im}(i\partial_t J(t)u(t), J(t)u(t))_{L^2} - 2\mathrm{Im}(J(t)f(u(t)), J(t)u(t))_{L^2} \\
&= \frac{d}{dt}\|J(t)u(t)\|_{L^2}^2 - 2\mathrm{Im}(f(u), J(t)^2 u)_{L^2} \\
&= \frac{d}{dt}\|J(t)u(t)\|_{L^2}^2 + 4t\mathrm{Re}(xf(u), \nabla u)_{L^2} \\
&+ 2nt\mathrm{Re}(f(u(t)), u(t))_{L^2} + 4t^2\mathrm{Re}(f(u(t)), \partial_t u(t))_{L^2} \\
&= \frac{d}{dt}\|J(t)u(t)\|_{L^2}^2 - 2nt\int_{\mathbb{R}^n} V(u(t))dx \\
&+ nt\int_{\mathbb{R}^n} V'(u(t))|u(t)|dx + 2t^2\frac{d}{dt}\int_{\mathbb{R}^n} V(u(t))dx.
\end{aligned}
$$

To justify the procedures above, we require that at least $u$ is an $H^2$-solution and $H^{2,1}$-solution, respectively, where $H^{2,1} := \{u \in H^2(\mathbb{R}^n); (1+|x|^2)^{1/2}u \in L^2(\mathbb{R}^n)\}$. For an $H^s$-solution with $s < 2$, there are basically two methods to justify the procedure. One is that from the continuous dependence of solutions on the initial data, the solution is approximated by a sequence of $H^2$ solutions. Other is to use a sequence of regularized equations of (1.1) whose solutions have enough regularities to perform the procedure above (see [4]). However, these two methods involve a limiting procedure on approximate solutions.

On the other hand, Ozawa [4] gave a direct proof of conservation laws of the charge and the energy by using solutions of the associated integral equation. Especially, he justifies the computation only by the Strichartz estimates. In order to state the Strichartz estimates, we introduce the following notations:

**Notation.** For a Banach space $X$, $p \in [1,\infty]$ and an interval $I \subset \mathbb{R}$, $L_t^p X$ denotes the Banach space $L^p(I,X)$ equipped with its natural norm. Let $U(t)$ be the Schrödinger operator $e^{\frac{it}{2}\Delta}$. For $s \in \mathbb{R}$ and $p,q \in [1,\infty]$, let $B^s_{p,q}(\mathbb{R}^n) = B^s_{p,q}$ be the inhomogeneous Besov space in $\mathbb{R}^n$.

**Definition 1.1.**

1. A positive exponent $p'$ is called the dual exponent of $p$ if $p$ and $p'$ satisfy $1/p + 1/p' = 1$.

2. A pair of two exponents $(p,q)$ is called an admissible pair if $(p,q)$ satisfies $2/p + n/q = n/2$, $p \geq 2$ and $(p,q) \neq (2,\infty)$.

The Strichartz estimates are described as the following lemma:

**Lemma 1.2** (Strichartz estimates, see [1]). *Let $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ be an interval with $0 \in \bar{I}$. $(p_1,q_1)$ and $(p_2,q_2)$ denote admissible pairs. Let $t_0 \in \bar{I}$. Then*

*1. for all $f \in L^2(\mathbb{R}^n)$,*

$$
\|U(t)f\|_{L^{p_1}(\mathbb{R},L^{q_1}(\mathbb{R}^n))} \leq C\|f\|_{L^2(\mathbb{R}^n)},
$$

*2. for any* $\varphi \in H^s(\mathbb{R}^n)$,

$$\|U(t)\varphi\|_{L^\infty(\mathbb{R},H^s(\mathbb{R}^n))} \le C\|\varphi\|_{H^s(\mathbb{R}^n)},$$

$$\|U(t)\varphi\|_{L^{p_1}(\mathbb{R},B^s_{q_1,2}(\mathbb{R}^n))} \le C\|\varphi\|_{H^s(\mathbb{R}^n)},$$

*3. for all* $f \in L^{p_1'}(I, L^{q_1'}(\mathbb{R}^n))$,

$$\left\|\int_{t_0}^t U(t-\tau)f(\tau)d\tau\right\|_{L^{p_2}(I,L^{q_2}(\mathbb{R}^n))} \le C\|f\|_{L^{p_1'}(I,L^{q_1'}(\mathbb{R}^n))},$$

*4. for any* $f \in L^1(I, H^s(\mathbb{R}^n))$,

$$\left\|\int_{t_0}^t U(t-\tau)f(\tau)d\tau\right\|_{L^\infty(I,H^s(\mathbb{R}^n))} \le C\|f\|_{L^1(I,H^s(\mathbb{R}^n))},$$

*5. for all* $f \in L^{p_1'}(I, B^s_{q_1',2}(\mathbb{R}^n))$,

$$\left\|\int_{t_0}^t U(t-\tau)f(\tau)d\tau\right\|_{L^{p_2}(I,B^s_{q_2,2}(\mathbb{R}^n))} \le C\|f\|_{L^{p_1'}(I,B^s_{q_1',2}(\mathbb{R}^n))},$$

*where* $p_1'$ *and* $p_2'$ *are the dual exponents of* $p_1$ *and* $p_2$, *respectively.*

## § 2. Main results

The aim of this paper is to revisit the conservation law of the momentum and the pseudo conformal conservation law with solutions of the following integral equation:

$$u(t) = U(t)\phi - i\int_0^t U(t-\tau)f(u(\tau))d\tau. \tag{2.1}$$

Firstly, the momentum is computed as follows:

**Proposition 2.1.** *Assume that* $f$ *satisfies (A1) - (A3). Let* $1/2 \le s < \min\{1, n/2\}$. *Let an admissible pair* $(\gamma, \rho)$ *be as follows:*

$$\rho = \frac{n(p+1)}{n+s(p-1)}, \quad \gamma = \frac{4(p+1)}{(p-1)(n-2s)}. \tag{2.2}$$

*Let* $u \in C([0,T], H^s(\mathbb{R}^n)) \cap L^\gamma(0,T; B^s_{\rho,2}(\mathbb{R}^n))$ *be a solution of the integral equation* (2.1) *for some* $\phi \in H^s(\mathbb{R}^n)$ *and* $T > 0$. *Then*

$$P(u(t)) = P(\phi) - 2\int_0^t \mathrm{Re}\left\langle f(u(\tau)), \overline{\nabla u(\tau)}\right\rangle d\tau$$

*for all* $t \in [0,T]$, *where the time integral of the scalar product to the RHS in the above is understood as the duality coupling on* $(L^\infty_t H^{1/2} + L^{\gamma'}_t B^{1/2}_{\rho',2}) \times (L^\infty_t H^{-1/2} \cap L^\gamma_t B^{-1/2}_{\rho,2})$.

*Remark* 1. Cazenave-Weissler [2] proved that if $0 < s < \min\{1, n/2\}$, $\phi \in H^s$ and $1 \le p < 1 + 4/(n-2s)$, then the Cauchy problem (1.1) have an unique solution $u \in C([0,T], H^s) \cap L^\gamma(0,T; B^s_{\rho,2})$ with some admissible pair $(\gamma, \rho)$ as in (2.2). We remark that $u \in L^\infty_t H^s \cap L^\gamma_t B^s_{\rho,2}$ implies $f(u) \in L^\infty_t H^s + L^{\gamma'}_t B^s_{\rho',2}$.

It seems difficult to cancel out the error term

$$\int_0^t \mathrm{Re} \left\langle f(u(\tau)), \overline{\nabla u(\tau)} \right\rangle d\tau$$

without any approximation. However, since we have

$$\|f(u) - f(\widetilde{u})\|_{L^2 + L^{\rho'}} \le \left(1 + \|u\|^{p-1}_{B^s_{\rho,2}} + \|\widetilde{u}\|^{p-1}_{B^s_{\rho,2}}\right) \|u - \widetilde{u}\|_{L^2 \cap L^\rho},$$

if $\mathscr{S} \ni u_j \to u$ in $H^s \cap B^s_{\rho,2}$ as $j \to \infty$, then $f(u_j) \to f(u)$ in $L^2 + L^{\rho'}$. Hence, noting that $\mathrm{Re}\langle f(g), \nabla g \rangle = 0$ for any $g \in \mathscr{S}$, the error term is computed to be 0, where $\mathscr{S}$ is the Schwartz space in $\mathbb{R}^n$. Therefore, we obtain the conservation law of the momentum as a corollary of Proposition 2.1.

Next, we compute the pseudo conformal conservation law as follows:

**Proposition 2.2.** *Assume that $f$ satisfies (A1) - (A3). Denote an admissible pair $(q,r)$ by $(q,r) = (4(p+1)/n(p-1), p+1)$. Let $u \in C([0,T], H^1(\mathbb{R}^n)) \cap L^q(0,T; W^{1,r}(\mathbb{R}^n))$ be a solution of the integral equation* (2.1) *for some $\phi \in \Sigma$ and $T > 0$. Then*

$$\|J(t)u(t)\|^2_{L^2} = \|x\phi\|^2_{L^2} + 2\mathrm{Im} \int_0^t \left\langle J(s)f(u(s)), \overline{J(s)u(s)} \right\rangle ds$$

*for all $t \in [0,T]$, where the time integral of the scalar product to the RHS in the above is understood as the duality coupling on $(L^1_t L^2 + L^{q'}_t L^{r'}) \times (L^\infty_t L^2 \cap L^q_t L^r)$.*

*Remark* 2. It is also known that if $f$ satisfies (A1) - (A3) and $1 < p < \alpha(n)$, where $\alpha(n) = 1 + 4/(n-2)$ if $n \ge 3$, $\alpha(n) = \infty$ if $n = 1$, 2, then for any $\phi \in H^1$, there exists $T > 0$ such that (1.1) has a unique solution $u \in C([0,T], H^1) \cap L^q(0,T; W^{1,r})$, where a pair $(q,r)$ is the same as in Proposition 2.2. Furthermore, if $\phi \in \Sigma$, then $u \in C([0,T], \Sigma)$. For detail, we refer the reader to [1].

*Remark* 3. Immediately, under the assumption of Proposition 2.2, we see that $Ju \in C([0,T], L^2) \cap L^q(0,T; L^r)$ and $Ju$ satisfies a integral equation

$$J(t)u(t) = U(t)x\phi - i \int_0^t U(t-s)J(s)f(u(s))ds. \tag{2.3}$$

Note that $Jf(u) \in L^\infty_t L^2 + L^{q'}_t L^{r'}$.

It also seems difficult to calculate the error term

$$2\mathrm{Im}\int_0^t \left\langle J(s)f(u(s)), \overline{J(s)u(s)}\right\rangle ds,$$

further without any approximation. However, for any $s \in \mathbb{R}$ and $g \in \mathscr{S}$,

$$\mathrm{Im}\langle J(s)f(g), J(s)g\rangle = -2t\mathrm{Re}\langle xf(g), \nabla g\rangle - nt\mathrm{Re}\langle f(g), g\rangle - t^2\mathrm{Im}\langle f(g), \Delta g\rangle. \tag{2.4}$$

Hence, noting that there exists $\{u_j\}_{j\in\mathbb{N}}^{\infty} \subset \mathscr{S}$ such that $u_j \to u$ in $H^1 \cap W^{1,r}$, from (2.4), the error term is computed as

$$\begin{aligned}
&2\mathrm{Im}\int_0^t \left\langle J(s)f(u(s)), \overline{J(s)u(s)}\right\rangle ds \\
&= -4\mathrm{Re}\int_0^t s\left\langle xf(u(s)), \overline{\nabla u(s)}\right\rangle ds \\
&- 2n\mathrm{Re}\int_0^t s\left\langle f(u(s)), \overline{u(s)}\right\rangle ds - 4\mathrm{Re}\int_0^t s^2\left\langle f(u(s)), \overline{\partial_t u(s)}\right\rangle ds,
\end{aligned} \tag{2.5}$$

where the time integral of the scalar product to each term of the RHS to (2.5) is understood as the duality coupling on $(L_t^1L^2 + L_t^{q'}L^{r'}) \times (L_t^\infty L^2 \cap L_t^q L^r)$, $(L_t^1L^2 + L_t^{q'}L^{r'}) \times (L_t^\infty L^2 \cap L_t^q L^r)$ and $(L_t^1 H^1 + L_t^{q'}W^{1,r'}) \times (L_t^\infty H^{-1} \cap L_t^q W^{-1,r})$, respectively, and from the equation (1.1), it holds that

$$\begin{aligned}
&\mathrm{Im}\left\langle f(u), \overline{\Delta u}\right\rangle_{(H^1+W^{1,r'})\times(H^{-1}\cap W^{-1,r})} \\
&= \lim_{\varepsilon\to 0}\mathrm{Im}\left\langle (1-\varepsilon\Delta)^{-1}f(u), \overline{\Delta u}\right\rangle_{H^1\times H^{-1}} \\
&= \lim_{\varepsilon\to 0} 2\mathrm{Im}\langle (1-\varepsilon\Delta)^{-1}f(u), \overline{f(u) - i\partial_t u}\rangle_{H^1\times H^{-1}} \\
&= 2\mathrm{Re}\langle f(u), \overline{\partial_t u}\rangle_{(H^1+W^{1,r'})\times(H^{-1}\cap W^{-1,r})}.
\end{aligned}$$

The identity (2.5) implies

$$\begin{aligned}
&2\mathrm{Im}\int_0^t \left\langle J(s)f(u(s)), \overline{J(s)u(s)}\right\rangle ds \\
&= 2\int_0^t s\left(\int_{\mathbb{R}^n}(n+2)V(u(s)) - \frac{n}{2}V'(u(s))|u(s)|dx\right) ds - 2t^2\int_{\mathbb{R}^n} V(u(t))dx.
\end{aligned}$$

Therefore, we obtain the pseudo conformal conservation law as a corollary of Proposition 2.2.

## § 3. The proof of main results

*Proof of Proposition 2.1.* Note that $\int_{\mathbb{R}^n} u\nabla\bar{u}dx$ is a pure imaginary number. For all $t \in [0,T]$, we obtain

$$iP(u(t)) = \left\langle u, \overline{\nabla u}\right\rangle_{H^{1/2}\times H^{-1/2}}$$

$$
\begin{aligned}
&= \left\langle U(-t)u, \overline{U(-t)\nabla u} \right\rangle_{H^{1/2}\times H^{-1/2}} \\
&= \langle \phi, \overline{\nabla\phi} \rangle_{H^{1/2}\times H^{-1/2}} \\
&+ \left\langle \phi, \overline{-i\int_0^t U(-\tau)\nabla f(u(\tau))d\tau} \right\rangle_{H^{1/2}\times H^{-1/2}} \\
&+ \left\langle -i\int_0^t U(-\widetilde{\tau})f(u(\widetilde{\tau}))d\widetilde{\tau}, \overline{\nabla\phi} \right\rangle_{H^{1/2}\times H^{-1/2}} \\
&+ \left\langle -i\int_0^t U(-\widetilde{\tau})f(u(\widetilde{\tau}))d\widetilde{\tau}, \overline{-i\int_0^t U(-\tau)\nabla f(u(\tau))d\tau} \right\rangle_{H^{1/2}\times H^{-1/2}} \\
&= iP(\phi) \\
&- \int_0^t \left\langle U(\tau)\phi, \overline{i\nabla f(u(\tau))} \right\rangle d\tau - \int_0^t \left\langle if(u(\widetilde{\tau})), \overline{U(\widetilde{\tau})\nabla\phi} \right\rangle d\widetilde{\tau} \\
&- \int_0^t \left\langle if(u(\widetilde{\tau})), \overline{-i\int_0^{\widetilde{\tau}} U(\widetilde{\tau}-\tau)\nabla f(u(\tau))d\tau} \right\rangle d\widetilde{\tau} \\
&- \int_0^t \left\langle -i\int_0^{\tau} U(\tau-\widetilde{\tau})f(u(\widetilde{\tau}))d\widetilde{\tau}, \overline{i\nabla f(u(\tau))} \right\rangle d\tau,
\end{aligned}
$$

where concatenating the Strichartz estimates and Remark 1, the time integral of the scalar product to each term of the last line in the above is understood as the duality coupling on $(L_t^\infty H^{1/2} \cap L_t^\gamma B_{\rho,2}^{1/2}) \times (L_t^\infty H^{-1/2} + L_t^{\gamma'} B_{\rho',2}^{-1/2})$, $(L_t^\infty H^{1/2} + L_t^{\gamma'} B_{\rho',2}^{1/2}) \times (L_t^\infty H^{-1/2} \cap L_t^\gamma B_{\rho,2}^{-1/2})$, $(L_t^\infty H^{1/2} + L_t^{\gamma'} B_{\rho',2}^{1/2}) \times (L_t^\infty H^{-1/2} \cap L_t^\gamma B_{\rho,2}^{-1/2})$, and $(L_t^\infty H^{1/2} \cap L_t^\gamma B_{\rho,2}^{1/2}) \times (L_t^\infty H^{-1/2} + L_t^{\gamma'} B_{\rho',2}^{-1/2})$, respectively. Using the integral equation (2.1), we compute

$$
\begin{aligned}
iP(u(t)) &= iP(\phi) - \int_0^t \left\langle u(\tau), \overline{i\nabla f(u(\tau))} \right\rangle d\tau - \int_0^t \left\langle if(u(\widetilde{\tau})), \overline{\nabla u(\widetilde{\tau})} \right\rangle d\widetilde{\tau} \\
&= iP(\phi) - 2i\int_0^t \mathrm{Re} \left\langle f(u(\tau))\overline{\nabla u(\tau)} \right\rangle d\tau,
\end{aligned}
$$

which completes the proof. □

*Proof of Proposition 2.2.* We can give the proof in a way similar to Ozawa [4]. For all $t \in [0, T]$, from (2.3), we obtain

$$
\begin{aligned}
&\|J(t)u(t)\|_{L^2}^2 \\
&= \|U(-t)J(t)u(t)\|_{L^2}^2 \\
&= \|x\phi\|_{L^2}^2 - 2\mathrm{Im}\left( x\phi, \int_0^t U(-s)J(s)f(u(s))ds \right)_{L^2} \qquad (3.1) \\
&+ \left\| \int_0^t U(-s)J(s)f(u(s))ds \right\|_{L^2}^2 .
\end{aligned}
$$

The second term on the RHS of (3.1) satisfies the following equality:

$$
\begin{aligned}
&- 2\mathrm{Im}\left(x\phi, \int_0^t U(-s)J(s)f(u(s))ds\right)_{L^2} \\
&= -2\mathrm{Im}\int_0^t \left\langle U(s)x\phi, \overline{J(s)f(u(s))}\right\rangle ds,
\end{aligned}
\tag{3.2}
$$

where combining the Strichartz estimates with $Jf(u) \in L_t^1L^2 + L_t^{q'}L^{r'}$, the time integral of the scalar product is understood as the duality coupling on $(L_t^\infty L^2 \cap L_t^q L^r) \times (L_t^1 L^2 + L_t^{q'}L^{r'})$. For the last term on the RHS of (3.1), Fubini's theorem implies

$$
\begin{aligned}
&\left\|\int_0^t U(-s)J(s)f(u(s))ds\right\|_{L^2}^2 \\
&= 2\mathrm{Re}\int_0^t \left\langle J(s)f(u(s)), \overline{\int_0^s U(s-s')J(s')f(u(s'))ds'}\right\rangle ds,
\end{aligned}
\tag{3.3}
$$

where the time integral of the scalar product is understood as the duality coupling on $(L_t^1L^2 + L_t^{q'}L^{r'}) \times (L_t^\infty L^2 \cap L_t^q L^r)$. Concatenating (3.1) - (3.3), we compute

$$
\begin{aligned}
&\|J(t)u(t)\|_{L^2}^2 \\
&= \|x\phi\|_{L^2}^2 - 2\mathrm{Im}\int_0^t \left\langle U(s)x\phi, \overline{J(s)f(u(s))}\right\rangle ds \\
&+ 2\mathrm{Re}\int_0^t \left\langle J(s)f(u(s)), \overline{\int_0^s U(s-s')Jf(u(s'))ds'}\right\rangle ds \\
&= \|x\phi\|_{L^2}^2 + 2\mathrm{Im}\int_0^t \left\langle J(s)f(u(s)), \overline{U(s)x\phi}\right\rangle ds \\
&+ 2\mathrm{Im}\int_0^t \left\langle J(s)f(u(s)), \overline{-i\int_0^s U(s-s')J(s')f(u(s'))ds'}\right\rangle ds \\
&= \|x\phi\|_{L^2}^2 + 2\mathrm{Im}\int_0^t \left\langle J(s)f(u(s)), \overline{J(s)u(s)}\right\rangle ds,
\end{aligned}
$$

where the last equality in the above holds by (2.3). □

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