RIMS Kôkyûroku Bessatsu
B63 (2017), 31–50

Area-constrained Willmore surfaces of small area in Riemannian three-manifolds: an approach via Lyapunov-Schmidt reduction

By

Norihisa Ikoma,* Andrea Malchiodi**and Andrea Mondino***

Abstract

The goal of the present note is to survey and announce recent results by the authors about existence of embedded Willmore surfaces with small area constraint in Riemannian three-manifolds. The common feature about the results presented here is that the constructions rely on suitable Lyapunov-Schmidt reductions.

§ 1. Introduction

The goal of the present note is to survey and announce recent results by the authors about existence of embedded Willmore surfaces with small area constraint in Riemannian three-manifolds. First of all, we introduce the Willmore functional. Let Σ be a closed (compact, without boundary) two-dimensional surface and (M, g) a Riemannian 3-manifold. Let us consider an immersion $f : \Sigma \to M$. Then for f, we define the *Willmore functional* $W(f)$ by

$$W(f) := \int_\Sigma H^2 \, d\sigma. \tag{1.1}$$

Received December 17, 2015. Revised July 4, 2016.
2010 Mathematics Subject Classification(s): 49Q10, 53C21, 53C42, 35J60, 58E05
Key Words: *Willmore functional, Hawking mass, non linear PDE, fourth order PDE.*
The second author has been supported by the project PRIN 2015 2015KB29WPT_001 and by the grant "Geometric Variational Problems" from Scuola Normale Superiore. He is also a member of INdAM.
*Kanazawa University, Ishikawa, 9201192, Japan.
e-mail: ikoma@se.kanazawa-u.ac.jp
**Scuola Normale Superiore, 56126 Pisa, Italy.
e-mail: andrea.malchiodi@sns.it
***University of Warwick, CV4 7AL Coventry, UK
e-mail: A.Mondino@warwick.ac.uk

Here $d\sigma$ is the area form induced by f, $H := \bar{g}^{ij} A_{ij}$ the mean curvature, $\bar{g}_{ij}$ the induced metric and A_{ij} the second fundamental form.

When an immersion f is a critical point of W with respect to normal variations, f is said to be a *Willmore surface* (or *Willmore immersion*). It is known that f is a Willmore surface if and only if f satisfies the following Euler-Lagrange equation

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(n,n) = 0. \tag{1.2}$$

See, for instance, Lamm-Metzger-Schulze [23]. In the above equation, $\Delta_{\bar{g}}$ is the Laplace-Beltrami operator, $\mathring{A}_{ij} := A_{ij} - \frac{1}{2}H\bar{g}_{ij}$ the trace free second fundamental form, n a unit normal to f and Ric the Ricci tensor of (M,g). We remark that (1.2) is a fourth-order nonlinear elliptic PDE in the immersion map f.

We point out that the Willmore functional appears not only in mathematics but also in various fields. For example, in a biology, the Willmore functional appears as a special case of *Helfrich energy* ([16, 17, 45]). In general relativity, the *Hawking mass* contains the Willmore functional as the main term and see below for the definition of the Hawking mass. There are other examples, that is, Polyakov's extrinsic action, free energy of the nonlinear plate Birkhoff theory and so on.

In mathematics, the Willmore functional was studied by Blaschke and Thomsen in the 1920s and 1930s in the case where (M,g) is the Euclidean space. They tried to find a conformally invariant theory which contains minimal surfaces. Here we remark that minimal surfaces are solutions of (1.2) due to $H \equiv 0$, and the Willmore functional W in the Euclidean space is conformally invariant. For a proof, see Willmore [51]. Therefore, they detected the class of Willmore surfaces as a conformally invariant generalization of minimal surfaces, and Willmore surfaces were called conformal minimal surfaces.

After that, Willmore rediscovered this topic in 1960s. He proved that round spheres are only the global minimizers of W among all closed immersed surfaces into the Euclidean space. See Willmore [51]. Furthermore, he conjectured that the Clifford torus and its images under Möbius transformations are the global minimizers among surfaces with higher genus. This Willmore conjecture was recently solved by Marques-Neves [28] through minimax techniques. We refer to previous results toward the Willmore conjecture obtained by Li-Yau [26], Montiel-Ros [38], Ros [44], Topping [47] and others. We also refer to a result by Urbano [48] which plays a crucial role in the proof of the Willmore conjecture. Let us also mention other fundamental works on the Willmore functional. Simon [46] proved the existence of a smooth genus one minimizer of W in $\mathbb{R}^m$. Later the result was generalized to the higher genus case by Bauer-Kuwert [5], Kusner [18] and Rivière [42, 43]. We also wish to mention the work by Bernard-Rivière [6] on bubbling and energy-identities phenomena and by Kuwert-Schätzle [20] on the

Willmore flow.

Here it is worth to emphasizing that all the aforementioned results about Willmore surfaces treat immersions into the Euclidean space or equivalently into a round sphere due to the conformal invariance. On the other hand, Willmore immersions into curved Riemannian manifolds are paid much attentions recently. The first existence result was [32] in which the third author showed the existence of embedded Willmore spheres (Willmore surface with genus equal to 0) in a perturbative setting. We also refer to [33] and [8] in collaboration with Carlotto for related topics. Under the area constraint condition, the existence of Willmore type spheres and their properties have been investigated by Lamm-Metzger-Schulze [23], Lamm-Metzger [21] and the third author in collaboration with Laurain [24].

In addition, the global problem, i.e. the existence of smooth immersed spheres minimizing quadratic curvature functionals in compact Riemannian three-manifolds, was also studied by Lamm-Metzger [22] and the third author in collaboration with Kuwert and Schygulla in [19]. We also mention the work [37] for the non-compact case. Moreover, in collaboration with Rivière [35, 36], the third author developed the necessary tools for the calculus of variations of the Willmore functional in Riemannian manifolds and proved the existence of area-constrained Willmore spheres in homotopy classes as well as the existence of Willmore spheres under various assumptions and constraints.

As we already mentioned, some of the above results [21, 22, 23, 24, 35, 36] regard the existence of Willmore spheres under area constraint. Such immersions satisfy the equation

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(n,n) = \lambda H, \tag{1.3}$$

for some $\lambda \in \mathbb{R}$ playing the role of Lagrange multiplier. Seeking critical points of W under the area constraint condition is linked to the Hawking mass

$$m_H(f) := \frac{\sqrt{Area(f)}}{64\pi^{3/2}} \left(16\pi - W(f)\right)$$

in the sense that critical points of the Hawking mass under the area constraint condition are equivalent to the area-constrained Willmore immersions. Here we refer to [10, 23] and the references therein for more material about the Hawking mass.

The aim of this note is to survey and announce three papers of the authors [12, 13, 14]. In those papers, we aimed to understand the following questions:

- Genus 0: In the aforementioned results it was established the existence of Willmore spheres (possibly under area constraint). What about their multiplicity? Can we show there is a foliation by Willmore spheres under some geometric conditions on the ambient manifold?

- Genus 1: since all the above results are about spherical Willmore surfaces in manifolds, what about the existence (and multiplicity) of Willmore tori (possibly under area constraint)?

§ 2. Foliation of area-constrained Willmore spheres and multiplicity

As mentioned above, the literature about (both area-constrained and free) Willmore spheres in Riemannian 3-manifolds has seen a fast development in the last years [8, 19, 21, 22, 23, 24, 32, 33, 35, 36, 37]. In particular let us mention those works which are particularly related to our new results:

- Lamm-Metzger [22] showed that, given a closed 3-dimensional Riemannian manofold (M, g), there exists $\varepsilon_0 > 0$ with the following property: for every $\varepsilon \in (0, \varepsilon_0]$ there exists an area-constrained Willmore sphere minimizing the Willmore energy among immersed spheres of area equal to $4\pi\varepsilon^2$. Moreover, as $\varepsilon \to 0$, such area-constrained Willmore spheres concentrate to a critical point of the scalar curvature and, after suitable rescaling, they converge in $W^{2,2}$-sense to a round sphere.

- The above result has been generalized in two ways. On the one hand Rivière and the third author [35, 36] proved that it is possible to minimize the Willmore energy among (bubble trees of possibly branched weak) immersed spheres of fixed area, for every positive value of the area. On the other hand Laurain and the third author [24] showed that any sequence of area-constrained Willmore spheres with areas converging to zero and Willmore energy strictly below 32π (no matter if they minimize the Willmore energy) have to concentrate to a critical point of the scalar curvature and, after suitable rescaling, they converge *smoothly* to a round sphere.

A natural question then arises: Is it true that around any critical point P of the scalar curvature we can find a sequence of area-constrained Willmore spheres having area equal to $4\pi\varepsilon_n^2 \to 0$ and concentrating at P?

§ 2.1. Main results

The goal of our paper [14] is exactly to investigate this kind of question above. More precisely, on the one hand we reinforce the assumption by asking that P is a *non-degenerate* critical point on the scalar curvature (in the sense that the Hessian expressed in local coordinates is an invertible matrix); on the other hand we do not just prove existence of area-constrained Willmore spheres concentrating at P but we show that there exists a *regular foliation* of a neighborhood of P made by area-constrained Willmore spheres. The precise statement is the following.

Theorem 2.1. *Let (M, g) be a 3-dimensional Riemannian manifold and let $P \in M$ be a non-degenerate critical point of the scalar curvature* Sc*. Then there exist $\varepsilon_0 > 0$ and a neighborhood U of P such that $U \setminus \{P\}$ is foliated by area-constrained Willmore spheres Σ_ε having area $4\pi\varepsilon^2$, $\varepsilon \in (0, \varepsilon_0)$. More precisely, there is a diffeomorphism $F : S^2 \times (0, \varepsilon_0) \to U \setminus \{P\}$ such that $\Sigma_\varepsilon := F(S^2, \varepsilon)$ is an area-constrained Willmore sphere having area equal to $4\pi\varepsilon^2$. Moreover*

- *If the index of P as critical point of* Sc *is equal to $3 - k$* [1]*, then each surface Σ_ε is an area-constrained critical point of W of index k.*
- *If $\mathrm{Sc}_P > 0$ then the surfaces Σ_ε have strictly positive Hawking mass.*
- *The foliation is* regular *at $\varepsilon = 0$ in the following sense. Fix a system of normal coordinates of U centred at P and indentify U with an open subset of $\mathbb{R}^3$; then, called $F_\varepsilon := \frac{1}{\varepsilon} F(\cdot, \varepsilon) : S^2 \to \mathbb{R}^3$, as $\varepsilon \to 0$ the immersions F_ε converge smoothly to the round unit sphere of $\mathbb{R}^3$.*
- *The foliation is* unique *in the following sense. Let $V \subset U$ be another neighborhood of $P \in M$ such that $V \setminus \{P\}$ is foliated by area-constrained Willmore spheres Σ'_ε having area $4\pi\varepsilon^2$, $\varepsilon \in (0, \varepsilon_1)$, and satisfying $\sup_{\varepsilon \in (0,\varepsilon_1)} W(\Sigma'_\varepsilon) < 32\pi$. Then there exists $\varepsilon_2 \in (0, \min(\varepsilon_0, \varepsilon_1))$ such that $\Sigma_\varepsilon = \Sigma'_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_2)$.*
- *The foliation F can be obtained by a smooth deformation of the foliation of $\mathbb{R}^3$ by round spheres: there exists a differentiable map $G : S^2 \times (0, \varepsilon_0) \times [0, 1] \to M$ such that the surfaces $G(S^2, \varepsilon, \tau)$ are area-constrained Willmore spheres in metric $(1 - \tau) g_\varepsilon + \tau\delta$, and $G(S^2, \varepsilon, 0)$ is a round sphere in $\mathbb{R}^3$ of area $4\pi\varepsilon^2$. We used the notation that δ_{ij} is the euclidean metric in $\mathbb{R}^3$ and $(g_\varepsilon)_{ij} = \varepsilon^{-2} g_{ij}$ is the natural rescaling of the metric g expressed in normal coordinates on U centered at P.*

Foliations by area-constrained Willmore spheres have been recently investigated by Lamm-Metzger-Schulze [23] who proved that a non-compact 3-manifold which is asymptotically Schwartzschild with positive mass is foliated at infinity by area-constrained Willmore spheres of large area. Even if both ours and theirs construction rely on a suitable application of the Implicit Function Theorem, the two results and proofs are actually quite different: the former is a local foliation in a small neighborood of a point and the driving geometric quantity is the scalar curvature. On the other hand, the latter

[1] The index of a non-degenerate critical point P of a function $h : M \to \mathbb{R}$ is the number of negative eigenvalues of the Hessian of h at P

is a foliation at infinity and the driving geometric quantity is the ADM mass of the manifold. Let us also mention that local foliations by spherical surfaces in manifolds have already been investigated in literature, but mostly by constant mean curvature spheres. In particular we have been inspired by the seminal paper of Ye [52] where the author constructed a local foliation of constant mean curvature spheres near a non-degenerate critical point of the scalar curvature. On the other hand let us stress the difference between the two problems: finding a foliation by constant mean curvature spheres is a *second* order problem since the mean curvature is a second order elliptic operator, while finding a foliation by area-constrained Willmore spheres is a *forth* order problem since the area-constrained Willmore equation (1.3) is of order four.

Let us also discuss the relevance of Theorem 2.1 in connection with the Hawking mass. Recall that, from the note of Christodoulou and Yau [10], if (M, g) has non negative scalar curvature then isoperimetric spheres (and more generally stable CMC spheres) have positive Hawking mass; on the other hand it is known (see for instance [11] or [40]) that, if M is compact, then small isoperimetric regions converge to geodesic spheres centered at a maximum point of the scalar curvature as the enclosed volume converges to 0. Moreover, from the aforementioned paper of Ye [52] it follows that near a non-degenerate maximum point of the scalar curvature we can find a a foliation by stable CMC spheres, which in particular by [10] will have positive Hawking mass. Therefore a link between Hawking mass and critical points of the scalar curvature was already present in literature, but Theorem 2.1 expresses this relation precisely.

In the paper [14] we also investigate multiplicity of area-constrained Willmore spheres and generic multiplicity of foliations. Let us mention that, despite the rich literature about existence of area-constrained Willmore spheres, this is the first multiplicity result in general Riemannian manifolds.

Theorem 2.2. *Let (M, g) be a compact 3-dimensional Riemannian manifold. Let*

- *k=2, if M is simply connected (i.e. if and only if M is diffeomorphic to S^3 by the recent proof of the Poincaré conjecture);*
- *k=3, if $\pi_1(M)$ is free and not trivial;*
- *k=4, otherwise.*

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exist at least k distinct area-constrained Willmore spheres of area $4\pi\varepsilon^2$.

Examples of manifolds having non-trivial free fundamental group are for instance $M = S^1 \times S^1 \times S^1$ or $M = S^1 \times S^2$; the 3-dimensional real projective space $\mathbb{RP}^3$ is instead an example of manifold where $k = 4$. An expert reader will notice that $k = Cat(M) + 1$, where $Cat(M)$ is the Lusternik-Schnirelmann category of M. This is not a case, indeed Theorem 2.2 is proved by combining a Lyapunov-Schmidt reduction with the celebrated Lusternik-Schnirelmann theory.

We conclude by stating the generic multiplicity of foliations. First notice that, fixed a compact manifold M, for generic metrics the scalar curvature is a Morse function.

Remark 2.3. Let (M, g) be a compact 3-dimensional manifold such that the scalar curvature $\mathrm{Sc} : M \to \mathbb{R}$ is a Morse function and denote with $b_k(M)$ the k^{th} Betti number of M, $k = 0, \dots, 3$. Then, by the Morse inequalities, Sc has at least $b_k(M)$ non-degenerate critical points of index k and, by Theorem 2.1, each one of these points has an associated foliation by area-constrained Willmore spheres of index $3-k$. In particular there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, there exist $b_k(M)$ distinct area-constrained Willmore spheres of area $4\pi\varepsilon^2$ and index $3 - k$, for $k = 0, \dots, 3$; therefore there exist at least $\sum_{k=0}^{3} b_k(M)$ distinct area-constrained Willmore spheres of area $4\pi\varepsilon^2$.

Example 2.4. Since the Morse inequalities hold by taking the Betti numbers with coefficients in any field, we are free to choose $\mathbb{R}$ or $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ depending on convenience. Let us discuss some basic example to illustrate the last multiplicity statement.

- $M = S^3$. Then $b_0(M, \mathbb{R}) = b_3(M, \mathbb{R}) = 1$, $b_1(M) = b_2(M) = 0$ so generically there exits 2 distinct foliations of area-constrained Willmore spheres.
- $M = S^2 \times S^1$. Then $b_k(M, \mathbb{R}) = 1$ for $k = 0, \dots, 3$, so generically there exist 4 distinct foliations of area-constrained Willmore spheres.
- $M = \mathbb{RP}^3$. Then $b_k(M, \mathbb{Z}_2) = 1$ for $k = 0, \dots, 3$, so generically there exist 4 distinct foliations of area-constrained Willmore spheres.
- $M = S^1 \times S^1 \times S^1$. Then $b_k(M, \mathbb{R}) = 1$ for $k = 0, 3$ and $b_k(M, \mathbb{R}) = 3$ for $k = 1, 2$, so generically there exist 8 distinct foliations of area-constrained Willmore spheres.

§ 2.2. Outline of the strategy

The construction relies on a classical method in nonlinear analysis called the Lyapunov-Schmidt reduction and in what follows we summarize such a technique. As done by Ambrosetti-Badiale [1, 2], we incorporate here the variational structue of the problem. For details, we refer to the monograph of Ambrosetti and the second author [3].

Let us consider a family of functionals (I_ε) on an infinite dimensional Banach space or manifold X where $\varepsilon > 0$ is a small parameter and we would like to find critical points of (I_ε). We first suppose that for every small $\varepsilon > 0$, there exists a finite dimensional manifold $Z_\varepsilon \subset X$ such that each element of Z_ε is *almost critical points of* I_ε. This means that as $\varepsilon \to 0$, I'_ε converges to 0 on Z_ε in a suitable sense. Furthermore, we assume that the second differential I''_ε is non-degenerate on a topological complement of the tangent space of Z_ε in X (or the tangent bundle TX).

Under these conditions, one can solve the equation $I'_\varepsilon = 0$ up to a component in the tangent space of Z_ε by modifying elements of Z_ε slightly via the implicit function theorem. Using this modification, we can define a functional $\Phi_\varepsilon : Z_\varepsilon \to \mathbb{R}$ with the property that critical points of Φ_ε are critical points of the original functional I_ε. Therefore, we can reduce the existence of critical points of I_ε into finding critical points of Φ_ε and the advantage of this reduction is that Φ_ε is a function defined on the *finite dimensional manifold* Z_ε.

In our case, the functional I_ε is of course the Willmore energy W defined in (1.1) and X the space of smooth immersions (with area constraint $4\pi\varepsilon^2$) from the round 2-sphere S^2 into the Riemannian manifold (M, g). For a choice of Z_ε, we first remark that we deal with small scale objects due to the area constraint, that a Riemannian metric approaches the Euclidean one in such a scale and that the round spheres are critical points of the Euclidean Willmore functional $W_{\mathbb{R}^3}$. Hence, one naturally expects that the images of small round spheres via exponential map, the so called geodesic spheres, are *almost critical points* of W. This is exactly what we first prove, with quantitative estimates. Furthermore, it is known that the second derivative of $W_{\mathbb{R}^3}$ at the spheres is given by $\Delta(\Delta + 2)$ (for instance, see [23, 32]) and its kernel consists of the Jacobi fields of translations and dilations. Therefore, under the area constraint, one can check that W is non-degenerate in the above sense provided $\varepsilon > 0$ is small.

After that, we shall move to the finite-dimensional reduction of the problem. In this case, for every (exponentiated) sphere, we will construct a graphical perturbation which will solve (1.3) up to some Lagrange multipliers given by the Jacobi fields of translations and introduce the reduced functional $\Phi_\varepsilon : Z_\varepsilon \to \mathbb{R}$. In order to take care of these Lagrange multipliers, we compute the expansion of Φ_ε for small ε and get (compare also with [21] and [32])

$$\Phi_\varepsilon(P) = 16\pi - \frac{8\pi}{3}\mathrm{Sc}_P\, \varepsilon^2 + o(\varepsilon^2). \tag{2.1}$$

The abstract reduction procedure explained above implies that P is a critical point of Φ_ε if and only if we can find a small perturbation of a geodesic sphere centered at P and with area $4\pi\varepsilon^2$ which is an area-constrained Willmore sphere. After careful estimates of the remainder $o(\varepsilon^2)$ in C^2-norm, the expansion (2.1) shows indeed that if P is a non-degenerate critical point of the scalar curvature then we can find critical points of Φ_ε near P and therefore we get the existence of small area-constrained Willmore spheres centered near P. Moreover, one can show indeed that they form a foliation of a neighborhood of P satisfying the claims of Theorem 2.1. The multiplicity results follows respectively by applying Lusternik-Schnirelmann theory to the reduced functional Φ_ε, and Morse theory to the function $\mathrm{Sc} : M \to \mathbb{R}$.

§ 3. Construction of small Willmore tori

In Section 2, we deal with the existence of Willmore surfaces with area constraint and genus equal to 0 in general curved spaces. Meanwhile, in this section, we concentrate on the second question in Introduction, that is, the existence (and multiplicity) of Willmore surfaces with genus equal to 1 (Willmore tori) in 3-Riemannian manifolds.

Before proceeding to the results of our papers [12, 13], we remark that when the ambient space (M, g) admits some symmetry property, the equation (1.2) is simplified and this enables us to obtain Willmore tori. Here we mention the works by Wang [49] and Barros-Ferrández-Lucas-Merono [4] who consider the case where (M, g) is a product and the metric is given by warped product, respectively. We also refer to Chen-Li [9] in which they study the existence of stratified weak branched immersions of arbitrary genus minimizing quadratic curvature functionals under various constraints.

The aim of our work [12, 13] is to construct smooth embedded Willmore tori with small area constraint in Riemannian three-manifolds, under curvature conditions. In contrast to the aforementioned papers, we do not assume any symmetry assumption on (M, g) here. We shall show the existence of such a surface by a Lyapunov-Schmidt reduction as in Section 2, but with some extra parameters. To find critical points of the reduced functional Φ_ε, we employ a minimization (or maximization) procedure in [12]. On the other hand, in [13], we shall utilize the Morse theory. We divide the discussion in two subsections corresponding to the two papers.

§ 3.1. Existence via a minimization procedure

The main result of the first paper [12] is the following:

Theorem 3.1. *Let (M, g) be a compact 3-dimensional Riemannian manifold. Denote by* Ric *and* Sc *the Ricci and the scalar curvature of (M, g) respectively, and suppose either*

$$3 \sup_{P \in M} \left(\mathrm{Sc}_P - \inf_{|\nu|_g = 1} \mathrm{Ric}_P(\nu, \nu) \right) > 2 \sup_{P \in M} \mathrm{Sc}_P, \tag{3.1}$$

or else

$$3 \inf_{P\in M}\left(\mathrm{Sc}_P - \sup_{|\nu|_g=1} \mathrm{Ric}_P(\nu,\nu)\right) < 2 \inf_{P\in M} \mathrm{Sc}_P. \tag{3.2}$$

Then there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$ there exists a smooth embedded Willmore torus in (M,g) with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$ ($4\sqrt{2}\pi^2$ is the area of the Clifford torus $\mathbb{T}$ in $\mathbb{R}^3$).

More precisely, these surfaces are obtained as normal graphs over exponentiated (Möbius transformations of) Clifford tori and the corresponding graph functions, once dilated by a factor $1/\varepsilon$, converge to 0 in $C^{4,\alpha}$-norm as $\varepsilon\to 0$ with decay rate $O(\varepsilon^2)$.

Now we give a remark on the conditions (3.1) and (3.2).

Remark 3.2. (i) When both conditions (3.1) and (3.2) hold in Theorem 3.1, then we can find at least two Willmore tori in (M,g) with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$.

(ii) It might be convenient to express the quantity $\mathrm{Sc}_P-\mathrm{Ric}_P(\nu,\nu)$ in (3.1) and (3.2) by the sectional curvatures. Let $\{e_1,e_2,e_3\}$ be an orthogonal basis of T_PM. Write K_{ij} for the sectional curvature at $P\in M$ spanned by $\{e_i,e_j\}$. We first recall the following relations between K_{ij}, R_{ij} and Sc_P:

$$R_{11}=K_{12}+K_{13},\quad R_{22}=K_{12}+K_{23},\quad R_{33}=K_{13}+K_{23},\quad \mathrm{Sc}_P=R_{11}+R_{22}+R_{33}.$$

From these relations, it is easily seen that

$$\mathrm{Sc}_P-\mathrm{Ric}_P(e_3,e_3)=\frac{1}{2}\mathrm{Sc}_P+K_{12}=2K_{12}+K_{13}+K_{23}.$$

In the rest of this subsection, we shall comment that we may apply Theorem 3.1 (and its arguments) for a large class of manifolds to find a smooth embedded Willmore torus with small constrained area. In this sense, our assumptions (3.1) and (3.2) are mild.

First, we consider compact 3-manifolds having constant scalar curvature.

Corollary 3.3. *Let (M,g) be a compact 3-dimensional manifold with constant scalar curvature. Then there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$ there exist at least two smooth embedded Willmore tori in (M,g) with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$.*

Proof. Suppose that $\mathrm{Sc}\equiv S\in\mathbb{R}$ and the sectional curvature of (M,g) is not constant. Then by Schur's lemma, we may find a $P\in M$ at which (M,g) is not isotropic. Noting the expressions in Remark 3.2 (ii) and $\mathrm{Sc}\equiv S$, one can observe that both of (3.1) and (3.2) hold. Hence, there are at least two Willmore tori with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$ according to Remark 3.2 (i).

On the other hand, let us assume that (M, g) has constant sectional curvature $\bar{K} \in \mathbb{R}$. Then it is conformally equivalent to the Euclidean space $\mathbb{R}^3$. Indeed either it is a quotient of the three-sphere $\mathbb{S}^3$ or of the Hyperbolic three-space $\mathbb{H}^3$, and both of them are conformally equivalent to the Euclidean three-space $\mathbb{R}^3$. Now it is well known that the functional

$$W_{cnf}(i) := \int_\Sigma [H^2 + 4\bar{K}] d\sigma = W(i) + 4\bar{K}\, Area(i)$$

is conformally invariant (see for instance [50]). Notice also that the area-constrained critical points of W are exactly the area-constrained critical points of W_{cnf}. By conformal invariance of W_{cnf} and the fact that the Clifford torus $\mathbb{T}$ and its images via the Möbius transformations are critical points of W_{cnf} in $\mathbb{R}^3$, rescaled Clifford tori are critical points of W_{cnf}. Thus these are also critical points of W under the area constraint and we complete the proof. □

Remark 3.4. The class of compact 3-manifolds with constant scalar curvature include many remarkable examples of ambient spaces which play an important role in contemporary surface theory. Trivial cases are manifolds with constant sectional curvature (notice that the same existence result applies to the standard non-compact space forms as explained above), but more generally any homogeneous three manifold has constant scalar curvature. Examples of compact homogeneous spaces are $\mathbb{S}^2 \times \mathbb{S}^1$, Berger spheres and any compact quotient of a three-dimensional Lie Group. The study of special surfaces (minimal, constant mean curvature, totally umbilic) in homogeneous spaces is a very active area of research, see for instance [27, 29, 30] and references therein. Let us mention that most of the results in this setting are for genus 0 surfaces and for second order problems, so the originality of our result lies in both exploring higher genus surfaces and higher order problems (recall that the Willmore equation is of fourth order, while minimal, CMC, and totally umbilical surface equations are of second order).

Second, we turn to the existence of Willmore tori with area constraint in non-compact manifolds. Here we shall point out that our argument to prove Theorem 3.1 applies to some non-compact manifolds as well, and a typical example is the Schwarzschild space. Before proceeding further, we recall the definition of the Schwarzschild space. The Schwartzschild space is given by $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$ where g_{Sch} denotes the Schwarzschild metric of mass $m > 0$ and is defined as follows: $(g_{Sch})_{ij}(x) := (1 + \frac{m}{2r})^4 \delta_{ij}$ and $r = |x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. From the definition, it is easily seen that g_{Sch} is spherically symmetric, conformal to the Euclidean metric and asymptotically flat. Furthermore, the scalar curvature of $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$ is identically zero and the sphere at $\{r = m/2\}$ is totally geodesic. In fact, the Schwarzschild metric is symmetric under the mapping $r \mapsto \frac{m^2}{4r}$ and therefore it has two asymptotically flat ends.

Before stating the result, in the next remark we recall what is known about minimal and CMC surfaces in Schwarzschild metric (we thank Alessandro Carlotto for a discussion about this point).

Remark 3.5 (Minimal and CMC surfaces in Schwarzschild).

- In the Schwarzschild space there are no non-spherical closed minimal surfaces: indeed arguing by maximum principle using comparison with CMC slices (i.e. the spheres $\{r = const\}$), it is possible to show that the only immersed closed minimal hypersurface in $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$ is the horizon $\{r = m/2\}$, which in fact is totally geodesic.

- Regarding CMC surfaces in $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$, it was proved by Brendle [7] that the only embedded closed CMC surfaces in the outer Schwarzschild $(\mathbb{R}^3 \setminus B_{m/2}(0), g_{Sch})$ are the spherical slices $\{r = const\}$ (let us mention that the results of Brendle include a larger class of warped products metrics). The embeddedness assumption is crucial for this classification result, in view of possible phenomena analogous to the Wente tori (which are immersed and CMC) in $\mathbb{R}^3$. It is also essential that the closed surfaces do not intersect the horizon $\{r = m/2\}$. Indeed, solving the isoperimetric problem in $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$ for small volumes, it is expected (by perturbative arguments á la Pacard-Xu [41]) that the isoperimetric surfaces are spherical surfaces intersecting $\{r = m/2\}$.

Summarizing, it is known that in $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$ there are *no non-spherical embedded minimal* surfaces and it is expected there are *no non-spherical embedded CMC* surfaces.

Now we are in position to state our result in the Schwarzschild space. In sharp contrast to the aforementioned situation, our next theorem asserts the existence of *embedded tori* which are critical points of the Hawking mass under area constraint.

Theorem 3.6. *Let $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$, with $g_{ij}(x) = (1+\frac{m}{2r})^4\delta_{ij}$, be the Schwarzschild metric of mass $m > 0$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exist two smooth embedded Willmore tori in $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$ with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$, which are distinguished by the the values of W. By spherical symmetry, there are infinitely many Willmore tori, hence critical points of the Hawking mass with constrained area.*

In fact, by direct computations, we can check that both (3.1) and (3.2) hold for $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$. Hence, arguing as in [12], we can find a local maximum and a local minimum of the reduced function Φ_ε whose values are distinct. Therefore, there are at least two Willmore tori in $(\mathbb{R}^3 \setminus \{0\}, g_{Sch})$.

Remark 3.7. By analogous arguments to the proof of Theorem 3.6, one can prove the existence of at least two smooth embedded Willmore tori with small area constraint in *asymptotically locally Euclidean (ALE) scalar flat* 3-manifolds. As in Theorem 3.6, they are distinguished by the values of W and critical points of the Hawking mass under the area-constraint condition.
More precisely, the following three conditions are sufficient for our arguments.

1) (M, g) is a complete non-compact 3-manifold whose scalar curvature vanishes identically: $\mathrm{Sc} \equiv 0$.

2) Fixed some base point $x_0 \in M$, there exists $r > 0$ with the following property: for every $\epsilon > 0$ there exists $R_\epsilon > 0$ such that for any $x \in M \setminus B^M_{R_\epsilon}(x_0)$ there exists a diffeomorphism $\Psi : B^{\mathbb{R}^3}_r(0) \to B^M_r(x)$ satisfying $\|\delta_{ij} - (\Psi^* g)_{ij}\|_{C^2(B^{\mathbb{R}^3}_r(0))} \leq \epsilon$.

3) At some point $x_1 \in M$, the Ricci tensor is not zero.

Notice that condition 1) is equivalent to the constrained Einstein equations in the vacuum case, and 2) is a mild uniform control of the local geometry of M together with a mild asymptotic condition. Regarding 3), if $\mathrm{Ric} \equiv 0$ on M, (M, g) becomes flat since (M, g) is 3-dimensional and the Riemann curvature tensor determined by the Ricci tensor (see, for example, Lee-Parker [25]). Moreover, from 3) and $\mathrm{Sc} \equiv 0$, we find that the Ricci tensor at x_1 has both positive and negative eigenvalues, which implies that both (3.1) and (3.2) hold.

§ 3.2. Outline of the strategy for a proof of Theorem 3.1

As in subsection 2.2, the construction relies on a Lyapunov-Schmidt reduction. Here, I_ε is the Willmore functional W and X a set of smooth immersions from the Clifford torus $\mathbb{T}$ into the Riemannian manifold (M, g) whose area is equal to $4\sqrt{2}\pi^2\varepsilon^2$.

As we also treat small scale objects, a candidate of Z_ε is a set of Willmore tori in the Euclidean space whose areas are equal to $4\sqrt{2}\pi^2$ as in subsection 2.2. Since the Euclidean Willmore functional $W_{\mathbb{R}^3}$ is conformally invariant, the Clifford torus $\mathbb{T}$ and its images under the Möbius transformations with area equal to $4\sqrt{2}\pi^2$ form a non-compact critical manifold of $W_{\mathbb{R}^3}$. Moreover, by the result of Weiner [50], the second variation of W is non-degenerate in the sense of subsection 2.2 and by the recent gap-theorem proved by Nguyen and the third author [34], this critical manifold is isolated in energy from the next Willmore torus. As expected, we can prove that the images of small Clifford tori via exponential map form a manifold of *almost critical points* of W.

Next, the finite-dimensional reduction of the problem is carried out. Through the implicit function theorem, for every exponentiated torus, we shall find a perturbation which solve our problem up to Lagrange multipliers and define the reduced function Φ_ε. In this case, the Lagrange multiplies are given by the Jacobi fields of translations, rotations and Möbius inversions (in other words, spherical inversions). The main difficulty here is the non-compactness of the critical manifold Z_ε derived from the Möbius inversions. In fact, we can construct the Möbius inversions which preserve the area of the Clifford torus $\mathbb{T}$ and make $\mathbb{T}$ degenerate into a round sphere with the area equal to $4\sqrt{2}\pi^2$.

To overcome this issue, we employ the variational structure of the problem and compare the Willmore energy of the exponentiated symmetric torus $\mathbb{T}$ and degenerating tori via the Möbius inversions. For the energy expansion at $R\mathbb{T}$, $R \in SO(3)$ and $P \in M$, we prove the following expansion in which a combination of the scalar curvature and the sectional curvature of the plane of symmetry of $R\mathbb{T}$ plays a role (cf Remark 3.2 (ii)):

$$\Phi_\varepsilon = 8\pi^2 - 4\sqrt{2}\,\{\mathrm{Sc}_P - \mathrm{Ric}_P(R\nu, R\nu)\}\,\varepsilon^2 + \text{higher order term} \tag{3.3}$$

where ν represents the axial vector of $\mathbb{T}$. Here $8\pi^2 = W_{\mathbb{R}^3}(\mathbb{T})$ and the same quantity to (3.1) appears in the second term of the right hand side in (3.3).

On the other hand, degenerating tori look like geodesic spheres with small handles. We show that the handle parts are negligible and check the following expansion in which the scalar curvature plays a role as in (2.1):

$$\Phi_\varepsilon = 8\pi^2 - \frac{8\sqrt{2}}{3}\pi^2 \mathrm{Sc}_P \varepsilon^2 + \text{higher order term.} \tag{3.4}$$

Now combining the expansions (3.3) and (3.4) with our assumption (3.1), the Möbius degenerations cost more in the sense of the Willmore energy and we can rule out the degenerations by the minimization procedure. Thus Φ_ε achieves a minimum and one finds a smooth embedded Willmore torus satisfying the area constraint. Similarly, if (3.2) holds, then we use the maximizing procedure instead of the minimization to find a critical point of Φ_ε.

Finally, we comment that the expansions (3.3) and (3.4) (as well as those in the next subsection) are probably the main contribution of the work. We also believe that they might play a role in further developments of the topic, especially in ruling-out possible degeneracy phenomena under global (non-perturbative) variational approaches to the problem. This has already happened for the case of Willmore spheres.

§ 3.3. Existence and multiplicity via Morse theory

In our second paper [13] we construct smooth embedded Willmore tori with small area constraint in Riemannian 3-manifolds, under some curvature/topological condition different from the ones in Theorem 3.1. More precisely we obtain the following existence result.

Theorem 3.8 (Existence). *Assume that*

(M1) (M, g) *is closed, connected and orientable three-manifold.*

(M2) *The scalar curvature* Sc *of* (M, g) *is a Morse function.*

(M3) *If P is a critical point of* Sc*, then the Ricci tensor* Ric_P *has three distinct eigenvalues.*
Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there exists a smooth embedded Willmore torus in (M, g) with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$. More precisely, the above surfaces are obtained as normal graphs over exponentiated (Möbius transformations of) Clifford tori and the corresponding graph functions (dilated by a factor $1/\varepsilon$) converge to 0 in $C^{4,\alpha}$-norm as $\varepsilon \to 0$ with decay rate $O(\varepsilon^2)$.

Remark 3.9. (i) The assumptions in Theorem 3.8 are generic in the metric g.

(ii) If the Ricci tensor is not a multiple of the identity at all points of global maximum and minimum of the scalar curvature, then we have at least two critical tori. For the details, we refer to [13, Remark 5.4] and its argument there.

Next we turn to a multiplicity result. Before stating the result, we need some preparations. First, we suppose (M1)–(M3) in Theorem 3.8. Next, we introduce some numbers depending on the scalar curvature as follows. For $q = 0, \dots, 3$, set

$$C_q := \sharp\{P_i \in M \ : \ \nabla\mathrm{Sc}(P_i) = 0, \quad \mathrm{index}\,(-\nabla^2\mathrm{Sc}(P_i)) = q\}$$

and define

$$\tilde{C}_0 = \tilde{C}_1 := 0; \quad \tilde{C}_2 := 4\,C_0; \quad \tilde{C}_q := 4\,C_{q-2} + 2\,C_{q-3} \text{ for } q = 3, 4, 5; \quad \tilde{C}_6 := 2\,C_3. \tag{3.5}$$

The meaning of $\tilde{C}_q$ is explained from the Morse theory on a manifold with boundary (see Morse-Van Schaack [39]) and the expansion of the Willmore energy and its derivative in the Möbius inversions. See the end of subsection 3.4.

Next, let us consider the Betti numbers of M with $\mathbb{Z}_2$ coefficients

$$\beta_q := \mathrm{rank}_{\mathbb{Z}_2}(H_q(M; \mathbb{Z}_2)); \qquad q \geq 0,$$

and define

$$\tilde{\beta}_0 = 1; \quad \tilde{\beta}_1 = \beta_1 + 1; \quad \tilde{\beta}_2 = \tilde{\beta}_3 = \beta_1 + \beta_2 + 1; \quad \tilde{\beta}_4 = \beta_2 + 1; \quad \tilde{\beta}_5 = 1; \quad \tilde{\beta}_k = 0 \quad \text{for } k \geq 6. \tag{3.6}$$

These numbers are the Betti numbers with $\mathbb{Z}_2$ coefficients of the finite dimensional manifold Z_ε consisting of the exponentiated Clifford tori with the area equal to $4\sqrt{2}\pi^2\varepsilon^2$. For details, we refer to Remark 3.12 and subsection 3.4.

Now we are ready to state our second main theorem.

Theorem 3.10 (Generic multiplicity). *Assume* (M1)*. Then for generic metrics g, if $\tilde{\beta}_q - \tilde{C}_q > 0$ holds for some $q \in \{0, \dots, 4\}$, then there exists $\varepsilon_0 > 0$ such that for* *every $\varepsilon \in (0, \varepsilon_0]$ there are at least $\tilde{\beta}_q - \tilde{C}_q$ smooth embedded Willmore tori in (M, g) with constrained area equal to $4\sqrt{2}\pi^2\varepsilon^2$ and with index q. In particular there are at least $\sum_{q=0}^{4}(\tilde{\beta}_q - \tilde{C}_q)^+$ area-constrained Willmore tori.*

Remark 3.11. Notice that we always have $\tilde{\beta}_q - \tilde{C}_q > 0$, for $q = 0, 1$, so the above result implies in particular that for generic metrics there exist at least two area-constrained Willmore tori, one with index zero and the other with index one, the index being intended for critical points of the Willmore functional under area constraint. Also, as the Morse inequalities on M imply $C_q \geq \beta_q$ for generic metrics, the condition $\tilde{\beta}_q - \tilde{C}_q > 0$ is not satisfied for $q = 5$ or $q = 6$.

Remark 3.12.

(i) The numbers $\tilde{\beta}_q$ are the Betti numbers (with $\mathbb{Z}_2$ coefficients) of the projective tangent bundle over M. By a classical result of differential topology due to Stiefel (see for instance [31, page 148]), three-dimensional oriented manifolds are parallelizable, i.e., the tangent bundle is trivial: $TM \simeq M \times \mathbb{R}^3$. As a consequence, the projective tangent bundle is homeomorphic to $M \times \mathbb{RP}^2$. Since $H_k(\mathbb{RP}^2, \mathbb{Z}_2) = \mathbb{Z}_2$ for $0 \leq k \leq 2$ and zero otherwise, the $\tilde{\beta}$'s in (3.6) can be computed as a direct application of Künneth's formula.

(ii) Using the homology of M with $\mathbb{Z}_2$ coefficients is more convenient than using standard $\mathbb{Z}$ coefficients for a number of reasons: first of all Künneth's formula turns out to be easier. Secondly, the Betti numbers with $\mathbb{Z}_2$ coefficients of a compact manifold X are always bounded below by the Betti numbers with $\mathbb{Z}$ coefficients, this because they also keep track of the $\mathbb{Z}_2$-torsion part. The precise relation between the two is given by the Universal Coefficients Theorem (see for instance [15, Chapter 3.A]), which implies that $H_k(X, \mathbb{Z}_2)$ consists of

- a $\mathbb{Z}_2$ summand for each $\mathbb{Z}$ summand of $H_k(X, \mathbb{Z})$,
- a $\mathbb{Z}_2$ summand for each $\mathbb{Z}_{2^n}$ summand in $H_k(X, \mathbb{Z})$, $n \geq 1$,
- a $\mathbb{Z}_2$ summand for each $\mathbb{Z}_{2^n}$ summand in $H_{k-1}(X, \mathbb{Z})$, $n \geq 1$.

In particular, in our case of $X = M \times \mathbb{RP}^2$, the $\mathbb{Z}$-Betti numbers vanish in dimension larger than three while the $\mathbb{Z}_2$-Betti numbers do not vanish in dimension 4 and 5. Clearly this permits stronger conclusions in terms of existence and multiplicity of critical points via Morse-theoretic arguments.

Example 3.13. If M is homeomorphic to S^3, $S^2 \times S^1$ or $S^1 \times S^1 \times S^1$, we get the following values for $\tilde{\beta}_k$.

$$M = S^3 : \tilde{\beta}_k = 1 \ \text{ for } k = 0, \dots, 5, \ \tilde{\beta}_k = 0 \text{ for } k \geq 6.$$
$$M = S^2 \times S^1 : \tilde{\beta}_0 = \tilde{\beta}_5 = 1, \ \tilde{\beta}_1 = \tilde{\beta}_4 = 2, \ \tilde{\beta}_2 = \tilde{\beta}_3 = 3, \ \tilde{\beta}_k = 0 \text{ for } k \geq 6.$$
$$M = (S^1)^3 : \tilde{\beta}_0 = \tilde{\beta}_5 = 1, \ \tilde{\beta}_1 = \tilde{\beta}_4 = 4, \ \tilde{\beta}_2 = \tilde{\beta}_3 = 7, \ \tilde{\beta}_k = 0 \text{ for } k \geq 6.$$

§ 3.4. Outline of the strategy for proofs of Theorems 3.8 and 3.10

We use a similar approach to that of subsection 3.2. The difference from the previous one is that we shall use a Morse theoretical approach instead of the minimizing (or maximizing) to find critical points of the reduced functional $\Phi_\varepsilon : Z_\varepsilon \to \mathbb{R}$. A difficulty here is same to the previous one, that is the degeneration of the Clifford tori. To avoid it and apply Morse theory on a manifold with boundary due to Morse-Van Schaack [39], for which it is important to observe behaviors of the derivative of Φ_ε with respect to Möbius inversions. This corresponds to understanding the behaviors of the normal derivative of Φ_ε on the boundary.

To apply Morse theory, we first consider the topology (the Betti numbers) of the finite dimensional manifold Z_ε. For this purpose, since M is parallelizable, we observe that Z_ε is diffeomorphic to $M \times \mathbb{BRP}^2$ where $\mathbb{BRP}^2 \subset T\mathbb{RP}^2$ consists of the couples $(P, v) \in \mathbb{RP}^2 \times T_P\mathbb{RR}^2$ satisfying $|v| < 1$. The degeneration of the Clifford tori is represented by $|v| \nearrow 1$. Since $\mathbb{BRP}^2$ can be deformed into $\mathbb{RP}^2 \times \{0\}$ continuously, by Remark 3.12 (i), one can check that the $\tilde{\beta}_q$'s are the Betti numbers of Z_ε.

Next, we need to compute the derivative of Φ_ε in the parameter of the Möbius inversions at the boundary where the Clifford tori are close to a round sphere with a small handle. Since the tori are degenerating, this computation is delicate and contains singularities. After a careful analysis, we find that the following function $\mathcal{F}(P, R)$ plays a crucial role to detect the behaviors of the normal derivative of Φ_ε: for $P \in M$ and $R \in SO(3)$,

$$\mathcal{F}(P, R) := \mathrm{Ric}_P(R\mathbf{e}_{P,2}, R\mathbf{e}_{P,2}) - \mathrm{Ric}_P(R\mathbf{e}_{P,3}, R\mathbf{e}_{P,3})$$

where $\{\mathbf{e}_{P,1}, \mathbf{e}_{P,2}, \mathbf{e}_{P,3}\}_{P \in M}$ is an orthonormal frame at P, $\mathbf{e}_{P,3}$ the axial vector of $\mathbb{T}$ and $R\mathbf{e}_{P,1}$ denotes the direction of the shrinking handle. Here we remark that we use another parametrization $(P, R, r) \in M \times SO(3) \times (0, 1)$ of Z_ε instead of $M \times \mathbb{BRP}^2$, r represents the degeneracy of the torus and when $r \to 1$, the torus degenerates. Then the assumptions of Theorem 3.8 or Theorem 3.10 give the following non-degeneracy condition for Sc_P and $\mathcal{F}$:

$(ND1)$ The function $P \mapsto \mathrm{Sc}_P : M \to \mathbb{R}$ is a Morse function. In particular, Sc has finitely many critical points $P_1, \dots, P_k$.

$(ND2)$ For each $i = 1, \dots, k$, $\mathcal{F}_i(R) := \mathcal{F}(P_i, R) : SO(3) \to \mathbb{R}$ is a Morse function for every $1 \leq i \leq k$, and $\mathcal{F}_i(R) \neq 0$ if $\nabla\mathcal{F}_i(R) = 0$.

By $(ND2)$, every $\mathcal{F}_i$ has finitely many critical points and we call them $R_{i,1}, \dots, R_{i,\ell_i}$. Then by our energy expansions, if r is close to 1, it turns out that the $\tilde{C}_q$'s in (3.5) satisfy (for details, see [13])

(3.7)

$$\begin{aligned}\tilde{C}_q &= \frac{1}{2}\sharp\{(P_i, R_{i,\ell}) \in M \times SO(3) \ : \ \text{index}\,(-\nabla^2 \text{Sc}(P_i)) + \text{index}\,(-\nabla^2 \mathcal{F}_i(R_{i,\ell})) = q \\ &\qquad\qquad \text{and } \mathcal{F}_i(R_{i,\ell}) < 0\} \\ &= \frac{1}{2}\sharp\{(P,R) \in M \times SO(3) \ : \ \nabla_{P,R}\Psi_{\varepsilon,r}(P,R) = 0, \ \nabla\Phi_\varepsilon(P,R,r) \text{ points inward,} \\ &\qquad\qquad \text{and index}\,(\nabla^2_{P,R}\Psi_{\varepsilon,r}(P,R)) = q\}\end{aligned}$$

where $\Psi_{\varepsilon,r}(P,R) := \Phi_\varepsilon(P,R,r)$. The factor $\frac{1}{2}$ in (3.7) is due to the symmetry of the degenerate Clifford torus. Indeed for every degenerate Clifford torus, there exists a nontrivial rotation $R \in SO(3), R \neq Id$ leaving the surface invariant. Then applying the results of [39] for Φ_ε, we obtain the existence and generic multiplicity in Theorems 3.8 and 3.10.

References

[1] A. Ambrosetti, M. Badiale, *Homoclinics: Poincaré-Melnikov type results via a variational approach,* Ann. Inst. Henri Poincaré Analyse Non Linèaire 15, (1998), 233–252.

[2] A. Ambrosetti, M. Badiale, *Variational perturbative methods and bifurcation of bound states from the essential spectrum* Proc. R. Soc. Edinb. **18**, (1998), 1131–1161.

[3] A. Ambrosetti, A. Malchiodi, *Perturbation methods and semilinear elliptic problems in* $\mathbb{R}^n$*,* Progress in mathematics, Birkhauser, (2006).

[4] M. Barros, A. Ferrández, P. Lucas, M. A. Merono, *Willmore tori and Willmore-Chen submanifolds in pseudo-Riemannian spaces*, J. Geom. Phys., Vol. 28, (1998), 45–66.

[5] M. Bauer, E. Kuwert, *Existence of minimizing Willmore surfaces of prescribed genus,* Int. Math. Res. Not., Vol.10, (2003), 553–576.

[6] Y. Bernard, T. Rivière , *Energy quantization for Willmore surfaces and applications,* Annals of Math., Vol. 180, Num. 1, (2014). 87–136.

[7] S. Brendle, *Constant mean curvature surfaces in warped product manifolds,* Publ. Math. de l'IHÉS 117, 247–269, (2013).

[8] A. Carlotto, A. Mondino, *Existence of Generalized Totally Umbilic* 2*-Spheres in Perturbed* 3*-Spheres,* Int. Math. Res. Not. IMRN 2014, no. 21, 6020-6052, doi:10.1093/imrn/rnt155.

[9] J. Chen, Y. Li, *Bubble tree of branched conformal immersions and applications to the Willmore functional,* Amer. J. Math. 136 (2014), no. 4, 1107-1154.

[10] D. Christodoulou, S.-T. Yau, *Some remarks on the quasi-local mass, In: Mathematics and general relativity*, Contemporary Math., Vol. 71, 9–14, (1986).

[11] O. Druet, *Sharp local Isoperimetric inequalities involving the scalar curvature,* Proc. Am. Math. Soc., Vol. 130, Num. 8, 2351–2361, (2002).

[12] N. Ikoma, A. Malchiodi, A. Mondino, *Embedded area-constrained Willmore tori of small area in Riemannian 3-manifolds I: Minimization*, preprint arXiv:1411.4396, (2014).

[13] N. Ikoma, A. Malchiodi, A. Mondino, *Embedded area-constrained Willmore tori of small area in Riemannian three-manifolds II: Morse Theory*, to appear in American Journal of Math.

[14] N. Ikoma, A. Malchiodi, A. Mondino, *Multiplicity and foliation properties of embedded area-constrained Willmore spheres of small area in Riemannian three-manifolds*, in preparation.

[15] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, (2002).

[16] W. Helfrich, *Elastic properties of lipid bilayers - theory and possible experiments,* Zeitschrift Für Naturforschung C - A Journal Of Biosciences. 28. (1973), 693–703

[17] L. Keller, A. Mondino, T. Rivière, *Embedded surfaces of arbitrary genus minimizing the Willmore energy under isoperimetric constraint*, Arch. Rational Mech. Anal., Vol. 212, (2014), 645–682.

[18] R. Kusner, *Estimates for the biharmonic energy on unbounded planar domains, and the existence of surfaces of every genus that minimize the squared-mean-curvature integral*, Elliptic and Parabolic Methods in Geometry (Minneapolis, Minn, 1994), A. K. Peters, Massachusetts, (1996), 67-72.

[19] E. Kuwert, A. Mondino, J. Schygulla, *Existence of immersed spheres minimizing curvature functionals in compact 3-manifolds,* Math. Ann. 359 (2014), no. 1-2, 379–425.

[20] E. Kuwert, R. Schätzle, *Removability of isolated singularities of Willmore surfaces,* Annals of Math. Vol. 160, Num. 1, (2004), 315–357.

[21] T. Lamm, J. Metzger, *Small surfaces of Willmore type in Riemannian manifolds,* Int. Math. Res. Not. 19 (2010), 3786–3813.

[22] T. Lamm, J. Metzger, *Minimizers of the Willmore functional with a small area constraint,* Annales IHP-Anal. Non Lin., Vol. 30, (2013), 497–518.

[23] T. Lamm, J. Metzger, F. Schulze, *Foliations of asymptotically flat manifolds by surfaces of Willmore type,* Math. Ann., Vol. 350, Num. 1, (2011), 1–78.

[24] P. Laurain, A. Mondino, *Concentration of small Willmore spheres in Riemannian 3-manifolds,* Analysis & PDE, Vol. 7, Num. 8, (2014), 1901–1921.

[25] J. M. Lee, T. H. Parker, *The Yamabe problem,* Bull. Amer. Math. Soc. (N.S.), Vol. 17, Num. 1, (1987), 37–91.

[26] P. Li, S. T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces,* Invent. Math., Vol. 69, Num. 2, (1982), 269–291.

[27] J. M. Manzano, R. Souam *The classification of totally umbilical surfaces in homogeneous 3-manifolds*, Math. Zeit., Vol. 279, Num.1-2, (2015), 557–576.

[28] F. Marques, A. Neves, *Min-max theory and the Willmore conjecture,* Ann. Math., (2) 179 (2014), no. 2, 683–782.

[29] W. H. Meeks III, J. Perez, *Costant mean curvature surfaces in metric Lie groups*, "Geometric Analysis: Partial Differential Equations and Surfaces", Contemporary Mathematics (AMS), Vol. 570, (2012), 25–110.

[30] W. H. Meeks III, P. Mira, J. Perez, A. Ros, *Constant mean curvature spheres in homogeneous three-spheres*, Preprint arXiv:1308.2612, (2013).

[31] J. Milnor, J. D. Stasheff, *Characteristic classes*, Annals of Math. Studies, Num. 76., Princeton University Press, (1974).

[32] A. Mondino, *Some results about the existence of critical points for the Willmore functional,* Math. Zeit., Vol. 266, Num. 3, (2010), 583–622.

[33] A. Mondino, *The conformal Willmore Functional: a perturbative approach,* J. Geom.

Anal., Vol. 23, Num. 2, (2013), 764–811.
[34] A. Mondino, H. T. Nguyen, *A Gap Theorem for Willmore Tori and an application to the Willmore Flow,* Nonlinear Analysis TMA, Vol. 102, (2014), 220-225.
[35] A. Mondino, T. Rivière, *Immersed Spheres of Finite Total Curvature into Manifolds,* Advances Calc. Var., Vol. 7, Num. 4, (2014), 493–538.
[36] A. Mondino, T.Rivière, *Willmore spheres in compact Riemannian manifolds,* Advances Math., Vol. 232, Num. 1, (2013), 608–676.
[37] A. Mondino, J. Schygulla, *Existence of immersed spheres minimizing curvature functionals in non-compact 3-manifolds,* Annales IHP-Anal. Non Lin., Vol. 31, (2014), 707–724.
[38] S. Montiel, A. Ros, *Minimal immersions of surfaces by the first Eigenfunctions and conformal area,* Invent. Math., Vol. 83, (1986), 153–166.
[39] M. Morse, G. B. Van Schaack, *The Critical Point Theory Under General Boundary Conditions*, Annals of Math., Vol. 35. Num. 3, (1934), 545–571.
[40] S. Nardulli, *The isoperimetric profile of a smooth Riemannian manifold for small volumes,* Ann. Glob. Anal. Geom., Vol. 36, Num. 2, (2009), 111–131.
[41] F. Pacard, X. Xu, *Constant mean curvature spheres in Riemannian manifolds,* Manuscripta Math. Vol.128, 275–295 (2008).
[42] T. Rivière, *Analysis aspects of Willmore surfaces,* Invent. Math., Vol. 174, Num. 1, (2008), 1–45.
[43] T. Rivière, *Variational Principles for immersed Surfaces with L^2-bounded Second Fundamental Form,* J. Reine. Angew. Math., Vol. 695, (2014), 41–98.
[44] A. Ros, *The Willmore conjecture in the real projective space,* Math. Res. Lett., Vol. 6, (1999), 487–493.
[45] J. Schygulla, *Willmore minimizers with prescribed isoperimetric ratio*, Arch. Ration. Mech. Anal. 203 (2012), no. 3, 901–941
[46] L. Simon, *Existence of surfaces minimizing the Willmore functional,* Comm. Anal. Geom., Vol. 1, Num. 2, (1993), 281–325.
[47] P. Topping, *Towards the Willmore conjecture.* Calc. Var. and PDE, Vol. 11, (2000), 361–393.
[48] F. Urbano, *Minimal surfaces with low index in the three-dimensional sphere*, Proc. Amer. Math. Soc., Vol. 108, (1990), 989–992.
[49] P. Wang, *On the Willmore functional of 2-tori in some product Riemannian manifolds*, Glasgow Math. J., Vol. 54, (2012), 517–528.
[50] J. L. Weiner, *On a problem of Chen, Willmore, et al.,* Indiana Univ. Math. J., Vol 27, Num. 1, (1978), 19–35.
[51] T.J. Willmore, *Riemannian Geometry,* Oxford Science Publications, Oxford University Press (1993).
[52] R. Ye, *Foliation by constant mean curvature spheres,* Pacific J. Math. Vol. 147, Num. 2, (1991), 381–396.