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Asymptotic limit of oscillatory integrals with certain smooth phases

By

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Abstract

Asymptotic limit of the oscillatory integral is explicitly computed in the case when its smooth phase contains some flat function.

§ 1. Introduction

Let us consider the oscillatory integral:

$$I_f(t;\varphi) = \int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)dx \qquad t > 0,$$

where f and φ are real-valued C^∞ smooth functions defined on an open neighborhood U of the origin in $\mathbb{R}^n$ and the support of φ is compact and is contained in U. Here, f and φ are called the *phase* and the *amplitude* respectively.

The oscillatory integral appears in many fields in mathematics and the information of its behavior as $t \to \infty$ often play important roles in the respective field (we only refer to [1] and [13]). Until now, many strong results about its behavior have been obtained. In particular, Varchenko [15] shows that the behavior can be described by using the geometry of the *Newton polyhedron* of the phase when the phase is real analytic and satisfies some conditions. Here, the Newton polyhedron is an important concept

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in singularity theory (see [1]). Later, his result has been improved and generalized in many kinds of cases. To be more specific, the following result about the asymptotic limit has been obtained in many cases ([12], [3], [5], [4], [11], [2], [7], [8], [9], [10], etc.):

$$\lim_{t\to\infty} t^{1/d(f)}(\log t)^{-m(f)+1} \cdot I_f(t;\varphi) = C_f(\varphi), \tag{1.1}$$

where $d(f)$ and $m(f)$ are simply defined through the geometry of the Newton polyhedron of f (see [1]): $d(f)$ is a positive number, called the *Newton distance* of f, and $m(f)$ is contained in the set $\{1,\dots,n\}$, called the *multiplicity* of $d(f)$. Moreover, $C_f(\varphi)$ is a constant, which is nonzero when $\varphi(0)$ is positive and φ is nonnegative on U. We remark that the constant $C_f(\varphi)$ has been exactly computed in many cases ([12], [3], [11], [2], [7], [8], [9], [10], etc.).

But, unfortunately, the above result (1.1) cannot be extended to the general C^∞ smooth case. The purpose of this note is to show the following theorem:

Theorem 1.1. *When $f(x_1,x_2) = x_2^q + e^{-1/|x_1|^p}$, where p is a positive real number and q is an integer not less than 2, and the support of φ is compact, we have*

$$\lim_{t\to\infty} t^{1/q}(\log t)^{1/p} \cdot \int_{\mathbb{R}^2} e^{itf(x_1,x_2)}\varphi(x_1,x_2)dx_1dx_2 = C_q\varphi(0,0),$$

where C_q is a nonzero constant defined by

$$C_q = \begin{cases} 4\Gamma(1/q+1)\cdot e^{\frac{\pi}{2q}i} & (q \text{ is even}); \\ 4\Gamma(1/q+1)\cdot \cos\frac{\pi}{2q} & (q \text{ is odd}). \end{cases}$$

When $q=2$, Iosevich and Sawyer [6] have given an estimate from the above: $|I_f(t;\varphi)| \le Ct^{-1/2}(\log t)^{-1/p}$, with $C>0$.

The above theorem implies that equality (1.1) does not hold in the above case (note that $d(f)=q$ and $m(f)=1$) and, moreover, the behavior of the oscillatory integral cannot always be determined by the information of only Newton polyhedron of the phase when the phase is smooth. On the other hand, the above theorem shows that for any positive number α, there exists a phase whose oscillatory integral satisfies $\lim_{t\to\infty} t^{1/d(f)}(\log t)^{\alpha}\cdot I_f(t;\varphi) = C\varphi(0,0)$ with $C\neq 0$ in the two-dimensional case.

Throughout this article, we sometimes use the symbol: $X := \log t$ for brief description. Moreover, we often use the same symbols t_0 and C to express various constants which are independent of t.

§ 2. Behavior of an associated one-dimensional integral

To prove Theorem 1.1, we prepare some auxiliary lemma concerning about an associated one-dimensional integral. Let ψ be a smooth function defined on $\mathbb{R}$ whose support is compact. Let $L(t;\psi)$ be the integral defined by

$$L(t;\psi) = \int_0^\infty e^{ite^{-1/x^p}} \psi(x)dx,$$

where p is a positive real number. Moreover, $L(t;\psi)$ can be written as

$$L(t;\psi) = L^{(1)}(t;\psi) + L^{(2)}(t;\psi),$$

with

$$L^{(1)}(t;\psi) = \int_0^{\frac{1}{(\log t)^{1/p}}} e^{ite^{-1/x^p}} \psi(x)dx,$$
$$L^{(2)}(t;\psi) = \int_{\frac{1}{(\log t)^{1/p}}}^\infty e^{ite^{-1/x^p}} \psi(x)dx.$$

The asymptotic behaviors of the integrals $L(t;\psi)$, $L^{(1)}(t;\psi)$ and $L^{(2)}(t;\psi)$ as $t \to \infty$ are seen as follows.

Lemma 2.1.

(i)

$$\lim_{t\to\infty} (\log t)^{1/p} \cdot L^{(1)}(t;\psi) = \psi(0).$$

(ii)

$$\lim_{t\to\infty} (\log t)^{1/p+1} \cdot L^{(2)}(t;\psi) = \psi(0) \cdot \int_1^\infty \frac{e^{iw}}{w} dw.$$

In particular, we have

$$\lim_{t\to\infty} (\log t)^{1/p} \cdot L(t;\psi) = \psi(0).$$

Proof. We may assume that the support of ψ is contained in $(\frac{-1}{\log 2}, \frac{1}{\log 2})$ from the principle of stationary phase (see [1], [13]).

(i). By exchanging the integral variable x by u as

$$x = \frac{1}{[X(u+1)]^{1/p}} \Longleftrightarrow u = \frac{1}{x^p X} - 1 \qquad (X := \log t),$$

the integral $L^{(1)}(t;\psi)$ can be written as

$$L^{(1)}(t;\psi) = \frac{1}{p(\log t)^{1/p}} \cdot \int_0^\infty e^{it^{-u}} \frac{\psi\left(\frac{1}{[X(u+1)]^{1/p}}\right)}{(u+1)^{1+1/p}} du.$$

Therefore, the Lebesgue convergence theorem implies

$$\lim_{t\to\infty} (\log t)^{1/p} \cdot L^{(1)}(t;\psi) = \frac{1}{p}\psi(0) \cdot \int_0^\infty \frac{du}{(u+1)^{1+1/p}} = \psi(0).$$

(ii). By exchanging the integral variable x by u as

$$u = e^{-1/x^p} \Longleftrightarrow x = \left(\frac{-1}{\log u}\right)^{1/p},$$

the integral $L^{(2)}(t;\psi)$ can be written as

$$L^{(2)}(t;\psi) = \int_{1/t}^{1/2} e^{itu} \frac{1}{u} \left(\frac{-1}{\log u}\right)^{1/p+1} \tilde{\psi}(u) du. \tag{2.1}$$

Here, let $\tilde{\psi}$ be the function defined on $[0,1)$ satisfying that $\tilde{\psi}(u) := \psi(\left(\frac{-1}{\log u}\right)^{1/p})$ for $u \in (0,1)$ and $\tilde{\psi}(0) := \psi(0)$. Note that $\tilde{\psi}$ is continuous on $[0,1)$, smooth in $(0,1)$ and its support is contained in $[0,1/2)$. Applying integration by parts to (2.1), we have

$$L^{(2)}(t;\psi) = M^{(1)}(t) + M^{(2)}(t), \tag{2.2}$$

with

$$M^{(1)}(t) = \left[\frac{1}{it} e^{itu} \frac{1}{u} \left(\frac{-1}{\log u}\right)^{1/p+1} \tilde{\psi}(u)\right]_{1/t}^{1/2},$$

$$M^{(2)}(t) = -\frac{1}{it} \int_{1/t}^{1/2} e^{itu} \frac{d}{du} \left\{\frac{1}{u} \left(\frac{-1}{\log u}\right)^{1/p+1} \tilde{\psi}(u)\right\} du.$$

The behaviors of $M^{(1)}(t)$ and $M^{(2)}(t)$ as $t \to \infty$ can be seen as follows.

(Estimate for $M^{(1)}(t)$.)
Noticing that the support of $\tilde{\psi}$ is contained in $[0,1/2)$, we have

$$M^{(1)}(t) = \frac{2\tilde{\psi}(1/2)}{(\log 2)^{1/p+1}} \cdot \frac{e^{it/2}}{it} + ie^i \frac{\tilde{\psi}(1/t)}{(\log t)^{1/p+1}} = ie^i \frac{\tilde{\psi}(1/t)}{(\log t)^{1/p+1}}.$$

Therefore

$$\lim_{t \to \infty} (\log t)^{1/p+1} M^{(1)}(t) = ie^i \tilde{\psi}(0) = ie^i \psi(0). \tag{2.3}$$

(Estimate for $M^{(2)}(t)$.)
By a simple computation, the integral $M^{(2)}(t)$ can be written as

$$M^{(2)}(t) = \frac{-1}{it} \int_{1/t}^{1/2} e^{itu} \frac{1}{u^2} \left(\frac{-1}{\log u}\right)^{1/p+1} a(u) du, \tag{2.4}$$

where a is a smooth function defined on $(0,1)$ defined by

$$a(u) :=$$
$$\left[-1 + \left(\frac{1}{p}+1\right)\left(\frac{-1}{\log u}\right)\right] \psi\left(\left(\frac{-1}{\log u}\right)^{1/p}\right) + \frac{1}{p}\psi'\left(\left(\frac{-1}{\log u}\right)^{1/p}\right)\left(\frac{-1}{\log u}\right)^{1/p+1}.$$

Note that a can be naturally extended to be continuous on $[0,1)$ and its support is contained in $[0,1/2)$. Moreover, by exchanging the integral variable u by v as

$$u = \frac{e^v}{t} \Longleftrightarrow v = \log(ut),$$

(2.4) can be rewritten as

$$\begin{aligned} M^{(2)}(t) &= \frac{-1}{it}\int_0^{\log t - \log 2} e^{ie^v}\left(\frac{t}{e^v}\right)^2\left(\frac{-1}{v-\log t}\right)^{1/p+1} a\left(\frac{e^v}{t}\right)\frac{e^v}{t}dv \\ &= \frac{i}{X^{1/p+1}}\int_0^{X-\log 2} e^{ie^v}e^{-v}\left(\frac{1}{1-v/X}\right)^{1/p+1} a\left(\frac{e^v}{t}\right)dv. \end{aligned} \tag{2.5}$$

Since the following inequality always holds:

$$\frac{1}{1-v/X} \le \frac{v}{\log 2} + 1 \qquad \text{for } v \in [0, X-\log 2],$$

the integrand in (2.5) can be estimated as follows. There exists a positive number C such that

$$\begin{aligned} &\left| e^{ie^v}e^{-v}\left(\frac{1}{1-v/X}\right)^{1/p+1} a\left(\frac{e^v}{t}\right)\right| \\ &\qquad \le Ce^{-v}\left(\frac{v}{\log 2}+1\right)^{1/p+1} \qquad \text{for } v \in (0, X-\log 2). \end{aligned} \tag{2.6}$$

Since the right hand side of (2.6) is integrable on $[0,\infty)$, the Lebesgue convergence theorem implies that

$$\begin{aligned} \lim_{t\to\infty}(\log t)^{1/p+1}\cdot M^{(2)}(t) &= i\int_0^\infty e^{ie^v}e^{-v}a(0)dv \\ &= -i\psi(0)\int_1^\infty \frac{e^{iw}}{w^2}dw. \end{aligned} \tag{2.7}$$

by exchanging the integral variable v by w: $w = e^v$.

Putting (2.2), (2.3), (2.7) together, we obtain (ii) in Lemma 2.1. Note that integration by parts implies

$$i\left(e^i - \int_1^\infty \frac{e^{iw}}{w^2}dw\right) = \int_1^\infty \frac{e^{iw}}{w}dw.$$

$\square$

Remarks.

1. In the above proof of (ii), integration by parts for $L^{(2)}(t;\psi)$ is crusial. Indeed, the behavior of $M^{(2)}(t)$ can be more easily understood. The essential difference between $L^{(2)}(t;\psi)$ and $M^{(2)}(t)$ is seen in the powers of u (i.e., $1/u$ in (2.1) and $1/u^2$ in (2.4) respectively) and it plays useful roles in the above computation.
2. The integral in (ii) in Lemma 2.1 seems difficult to express its value in more clear form. But, by using the integrals:
$$\mathrm{si}(z) = -\int_z^\infty \frac{\sin x}{x}dx, \quad \mathrm{Ci}(z) = -\int_z^\infty \frac{\cos x}{x}dx, \quad E_n(z) = \int_n^\infty \frac{e^{-zx}}{x}dx,$$
the value of the integral can be expressed as $-\mathrm{Ci}(1) - i\mathrm{si}(1) = E_1(-i)$. The above integrals are the so-called *sine integral*, *cosine integral*, *exponetial integral*, respectively, which are some kinds of *error functions*. (See, for example, [14], p.6, p.60.)

§ 3. The proof of Theorem 1.1

We respectively define the integrals:
$$\tilde{I}^{(\pm)}(t) = \int_0^\infty \int_0^\infty e^{it[\pm x_2^q + e^{-1/|x_1|^p}]}\varphi(x_1,x_2)dx_1dx_2.$$
The integral $I_f(t;\varphi)$ can be written as
$$I_f(t;\varphi) = \sum_{(\theta_1,\theta_2)\in\{\pm1,\pm1\}} \int_0^\infty \int_0^\infty e^{it[\theta_2^q x_2^q + e^{-1/|x_1|^p}]}\varphi(\theta_1 x_1,\theta_2 x_2)dx_1dx_2.$$
Therefore, in order to prove the theorem, it suffices to show
$$\lim_{t\to\infty} t^{1/q}(\log t)^{1/p}\cdot \tilde{I}^{(\pm)}(t) = \Gamma(1/q+1)\cdot e^{\pm\frac{\pi}{2q}i}\cdot\varphi(0,0). \tag{3.1}$$
Since the form of $\tilde{I}^{(-)}(t)$ is similar to that of $\tilde{I}^{(+)}(t)$, we only consider the case of the integral $\tilde{I}^{(+)}(t)$.

Now, the integral $\tilde{I}^{(+)}(t)$ can be devided as follows.
$$\tilde{I}^{(+)}(t) = J^{(1)}(t) + J^{(2)}(t), \tag{3.2}$$
with
$$\begin{aligned} J^{(1)}(t) &= \int_0^\infty \int_0^{\frac{1}{(\log t)^{1/p}}} e^{it[x_2^q + e^{-1/|x_1|^p}]}\varphi(x_1,x_2)dx_1dx_2,\\ J^{(2)}(t) &= \int_0^\infty \int_{\frac{1}{(\log t)^{1/p}}}^\infty e^{it[x_2^q + e^{-1/|x_1|^p}]}\varphi(x_1,x_2)dx_1dx_2. \end{aligned} \tag{3.3}$$
The behaviors of $J^{(1)}(t)$ and $J^{(2)}(t)$ as $t\to\infty$ are seen as follows.

Lemma 3.1.

(i)

$$\lim_{t\to\infty} t^{1/q}(\log t)^{1/p}\cdot J^{(1)}(t)=\Gamma(1/q+1)\cdot e^{\frac{\pi}{2q}i}\cdot\varphi(0,0).$$

(ii) *There exist positive numbers C and t_0 independent of t such that*

$$|J^{(2)}(t)|\le\frac{C}{t^{1/q}(\log t)^{1/p+1}}\qquad \textit{for } t\ge t_0.$$

From (3.2), the above lemma easily implies the equation (3.1).

§ 4. The proof of Lemma 3.1

Let us prove Lemma 3.1. Let α be a smooth function defined on $\mathbb{R}$ satisfying that $\alpha(x)=1$ for $|x|\le 1$ and $\alpha(x)=0$ for $|x|\ge 2$, and let $\beta(x):=1-\alpha(x)$.

(i). Let $P(x_1,x_2):=e^{x_2^q}\varphi(x_1,x_2)$. It is easy to see

$$P(x_1,x_2)=P(x_1,0)+x_2\int_0^1\frac{\partial P}{\partial x_2}(x_1,sx_2)ds.$$

By using the functions α and β, we have

$$\begin{aligned}P(x_1,x_2)&=P(x_1,0)-\beta(x_2)P(x_1,0)+x_2e^{x_2^q}R(x_1,x_2)\\&=\alpha(x_2)P(x_1,0)+x_2e^{x_2^q}R(x_1,x_2),\end{aligned}$$

where

$$R(x_1,x_2)=e^{-x_2^q}\left(\frac{1}{x_2}\beta(x_2)P(x_1,0)+\int_0^1\frac{\partial P}{\partial x_2}(x_1,sx_2)ds\right).$$

Noticing that the supports of $P(x_1,x_2)$ and $\alpha(x_2)P(x_1,0)$ are compact, we see that R is a smooth function on $\mathbb{R}^2$ with a compact support. Since $\varphi(x_1,x_2)=e^{-x_2^q}P(x_1,x_2)$, φ can be expressed as

$$\varphi(x_1,x_2)=e^{-x_2^q}\varphi(x_1,0)-e^{-x_2^q}\beta(x_2)\varphi(x_1,0)+x_2R(x_1,x_2). \tag{4.1}$$

By substituting (4.1) into (3.3) and applying Fubini's theorem, the integral $J^{(1)}(t)$ can be expressed as

$$J^{(1)}(t)=K^{(1)}(t)-K^{(2)}(t)+K^{(3)}(t),$$

with

$$\begin{aligned}K^{(1)}(t)&=L^{(1)}(t;\varphi(\cdot,0))\cdot\int_0^\infty e^{-[1-it]x_2^q}dx_2,\\K^{(2)}(t)&=L^{(1)}(t;\varphi(\cdot,0))\cdot\int_0^\infty e^{-[1-it]x_2^q}\beta(x_2)dx_2,\\K^{(3)}(t)&=\int_0^\infty e^{itx_2^q}L^{(1)}(t;R(\cdot,x_2))x_2dx_2.\end{aligned} \tag{4.2}$$

Now, let us investigate the behaviors of the above three functions as $t \to \infty$.

(Behavior of $K^{(1)}(t)$.)

First, let us consider the integral $K^{(1)}(t)$. Setting $z = [1-it]^{1/q}x_2$ and noting that the rapid decay of e^{-z^q} allows us to replace the contour $[1-it]^{1/q} \cdot [0,\infty)$ by $[0,\infty)$, we see that

$$\int_0^\infty e^{-[1-it]x_2^q} dx_2 = \frac{1}{(1-it)^{1/q}} \cdot \int_0^\infty e^{-z^q} dz = \frac{\Gamma(1/q+1)}{(1-it)^{1/q}} = \frac{1}{t^{1/q}} \frac{\Gamma(1/q+1)}{(1/t-i)^{1/q}}.$$

On the other hand, Lemma 2.1 (i) implies

$$\lim_{t\to\infty} (\log t)^{1/p} \cdot L^{(1)}(t; \varphi(\cdot, 0)) = \varphi(0,0).$$

Applying the above equalities to (4.2), we have

$$\lim_{t\to\infty} t^{1/q} (\log t)^{1/p} \cdot K^{(1)}(t) = \Gamma(1/q+1) e^{\frac{\pi}{2q} i} \cdot \varphi(0,0).$$

(Estimate of $K^{(2)}(t)$.)

Let N be an arbitrary natural number. Applying N-times integrations by parts, we have

$$\int_0^\infty e^{-[1-it]x_2^q} \beta(x_2) dx_2 = \frac{1}{q^N [1-it]^N} \int_0^\infty e^{-[1-it]x_2^q} \left(\frac{\partial}{\partial x_2} \cdot \frac{1}{x_2^{q-1}} \right)^N \beta(x_2) dx_2.$$

A simple computation implies that there exist positive numbers t_N and C_N such that

$$\left| \int_0^\infty e^{-[1-it]x_2^q} \beta(x_2) dx_2 \right| \leq \frac{C_N}{t^N} \qquad \text{for } t \geq t_N.$$

Therefore, from (4.2) and Lemma 2.1 (i), there exist positive numbers $\tilde{t}_N$ and $\tilde{C}_N$ such that

$$|K^{(2)}(t)| \leq \frac{\tilde{C}_N}{t^N (\log t)^{1/p}} \qquad \text{for } t \geq \tilde{t}_N.$$

(Estimate of $K^{(3)}(t)$.)

For the proof of (i), it suffices to show that the integral $K^{(3)}(t)$ is dominated by $Ct^{-2/q}(\log t)^{-1/p}$ for large t.

The integral $K^{(3)}(t)$ can be written as follows:

$$K^{(3)}(t) = H^{(1)}(t) + H^{(2)}(t), \tag{4.3}$$

with

$$H^{(1)}(t) = \int_0^\infty e^{itx_2^q} L^{(1)}(t; R(\cdot, x_2)) \alpha(t^{1/q} x_2) x_2 dx_2,$$
$$H^{(2)}(t) = \int_0^\infty e^{itx_2^q} L^{(1)}(t; R(\cdot, x_2)) \beta(t^{1/q} x_2) x_2 dx_2,$$

where the functions α and β are as in the beginning of this subsection.

Let us investigate the behaviors of the functions $H^{(1)}(t)$ and $H^{(2)}(t)$ as $t \to \infty$.

(Behavior of $H^{(1)}(t)$.)

Exchanging the integral variable x_2 by u_2: $u_2 = t^{1/q}x_2$, we have

$$H^{(1)}(t) = \frac{1}{t^{2/q}} \int_0^2 e^{iu_2^q} L^{(1)}(t; R(\cdot, \frac{u_2}{t^{1/q}}))\alpha(u_2)u_2 du_2. \tag{4.4}$$

In order to investigate the behavior of $L^{(1)}(t; R(\cdot, \frac{u_2}{t^{1/q}}))$ as $t \to \infty$, consider the following inequality:

$$\begin{aligned} &\left|(\log t)^{1/p} \cdot L^{(1)}(t; R(\cdot, \frac{u_2}{t^{1/q}})) - R(0,0)\right| \\ &\leq (\log t)^{1/p} \left|L^{(1)}(t; R(\cdot, \frac{u_2}{t^{1/q}})) - L^{(1)}(t; R(\cdot, 0))\right| \\ &\quad + \left|(\log t)^{1/p} L^{(1)}(t; R(\cdot, 0)) - R(0,0)\right|. \end{aligned} \tag{4.5}$$

The first term in the right hand side of (4.5) is dominated by

$$(\log t)^{1/p} \int_0^{\frac{1}{(\log t)^{1/p}}} \left|R(x_1, \frac{u_2}{t^{1/q}}) - R(x_1, 0)\right| dx_1 \quad \text{for } u_2 \in [0,2].$$

The uniform of the continuity of the function R implies that the above integral tends to zero as $t \to \infty$. Moreover, from Lemma 2.1 (i), the second term in the right hand side of (4.5) tends to zero as $t \to \infty$. Therefore, we have

$$\lim_{t\to\infty} (\log t)^{1/p} L^{(1)}(t; R(\cdot, \frac{u_2}{t^{1/q}})) = R(0,0) \quad \text{for } u_2 \in [0,2]. \tag{4.6}$$

Note that the limit in (4.6) is uniform with respect to $u_2 \in [0,2]$. Applying the equality (4.6) to (4.4), we can easily get

$$\lim_{t\to\infty} t^{2/q}(\log t)^{1/p} \cdot H^{(1)}(t) = R(0,0) \cdot \int_0^2 e^{iu_2^q} \alpha(u_2)u_2 du_2. \tag{4.7}$$

(Estimate for $H^{(2)}(t)$.)

By applying two-times integrations by parts to the integral $H^{(2)}(t)$, $H^{(2)}(t)$ can be written as

$$H^{(2)}(t) = \left(\frac{-1}{qit}\right)^2 \int_0^\infty e^{itx_2^q} L^{(1)}(t; F(\cdot, x_2; t))dx_2,$$

where

$$F(x_1, x_2; t) = \frac{\partial}{\partial x_2}\left(\frac{1}{x_2^{q-1}} \cdot \frac{\partial}{\partial x_2}\left(\frac{1}{x_2^{q-1}} \cdot x_2 R(x_1, x_2)\beta(t^{1/q}x_2)\right)\right).$$

A simple computation shows that there is a positive constant C independent of x_1 and t such that

$$|F(x_1, x_2; t)| \leq \frac{C}{x_2^{2q-1}} \quad \text{for } x_2 > 0. \tag{4.8}$$

Note that $t^{1/q}$ is dominated by $2/x_2$ when $t^{1/q}x_2$ is contained in the support of β'. Moreover, there exist positive numbers t_0, C such that

$$\begin{aligned} |L^{(1)}(t; F(\cdot, x_2; t))| &\leq \left| \int_0^{\frac{1}{(\log t)^{1/p}}} e^{ite^{-1/x_1^p}} F(x_1, x_2; t) dx_1 \right| \\ &\leq \int_0^{\frac{1}{(\log t)^{1/p}}} |F(x_1, x_2; t)| dx_1 \leq \frac{C}{x_2^{2q-1}(\log t)^{1/p}} \quad \text{for } x_2 > 0,\ t \geq t_0. \end{aligned} \tag{4.9}$$

By noticing that $2q - 1 \geq 3(> 1)$ and that the support of $F(u_1, \cdot; t)$ is contained in $(t^{-1/q}, \infty)$, the inequalities in (4.9) imply that

$$\begin{aligned} |H^{(2)}(t)| \leq &\frac{C}{t^2(\log t)^{1/p}} \cdot \int_{t^{-1/q}}^{\infty} \frac{1}{x_2^{2q-1}} dx_2 \\ \leq &\frac{C}{t^2(\log t)^{1/p} \cdot t^{-2+2/q}} \leq \frac{C}{t^{2/q}(\log t)^{1/p}} \quad \text{for } t \geq t_0. \end{aligned} \tag{4.10}$$

We remark that the function F with the estimate (4.8) was obtained by applying *integration by parts* and it played an important role in the estimate (4.10) of the integral $H^{(2)}(t)$ (see also the first remark in the end of Section 2).

Putting (4.3), (4.7), (4.10) together, we can get the desired estimate:

$$|K^{(3)}(t)| \leq \frac{C}{t^{2/q}(\log t)^{1/p}} \quad \text{for } t \geq t_0.$$

(ii). By using the functions α and β in the beginning of Section 4, the integral $J^{(2)}(t)$ can be devided as

$$J^{(2)}(t) = N^{(1)}(t) + N^{(2)}(t), \tag{4.11}$$

with

$$\begin{aligned} N^{(1)}(t) &= \int_0^{\infty} e^{itx_2^q} L^{(2)}(t; \varphi(\cdot, x_2)) \alpha(t^{1/q}x_2) dx_2, \\ N^{(2)}(t) &= \int_0^{\infty} e^{itx_2^q} L^{(2)}(t; \varphi(\cdot, x_2)) \beta(t^{1/q}x_2) dx_2. \end{aligned} \tag{4.12}$$

To obtain the estimate in (ii) in Lemma 3.1, we give appropriate estimates for $N^{(1)}(t)$ and $N^{(2)}(t)$.

(Estimate for $N^{(1)}(t)$.)

Exchanging the integral variable x_2 by u_2: $x_2 = u_2/t^{1/q}$, we have

$$N^{(1)}(t) = \frac{1}{t^{1/q}} \int_0^2 e^{iu_2^q} L^{(2)}(t; \varphi(\cdot, \frac{u_2}{t^{1/q}}))\alpha(u_2)du_2. \tag{4.13}$$

From Lemma 2.1 (ii), there exist positive numbers t_0, C independent of t and x_2 such that

$$|L^{(2)}(t; \varphi(\cdot, x_2))| \leq \frac{|\varphi(0, x_2)| + C}{(\log t)^{1/p+1}} \leq \frac{C}{(\log t)^{1/p+1}} \quad \text{for } t \geq t_0,\ x_2 > 0.$$

Therefore, applying the above estimate to (4.13), we have

$$|N^{(1)}(t)| \leq \frac{C}{t^{1/q}(\log t)^{1/p+1}} \quad \text{for } t \geq t_0. \tag{4.14}$$

(Estimate for $N^{(2)}(t)$.)

By appying integration by parts to (4.12), the integral $N^{(2)}(t)$ can be written as

$$N^{(2)}(t) = \frac{-1}{qit} \cdot \int_0^\infty e^{itx_2^q} L^{(2)}(t; G(\cdot, x_2; t))dx_2,$$

where

$$G(x_1, x_2; t) = \frac{\partial}{\partial x_2} \left(\frac{1}{x_2^{q-1}} \varphi(x_1, x_2)\, \beta(t^{1/q} x_2) \right).$$

Let $\tilde{G}(x_1, x_2; t) = x_2^q G(x_1, x_2; t)$. A simple computation shows that $\tilde{G}$ is bounded on $[0, \infty)^3$. Note that $t^{1/q}$ is dominated by $2/x_2$ when $t^{1/q}x_2$ is contained in the support of β'. From Lemma 2.1 (ii),

$$\lim_{t \to \infty} (\log t)^{1/p+1} \cdot L^{(2)}(t; \tilde{G}(\cdot, x_2; t)) = ie^i \cdot \tilde{G}(0, x_2; t).$$

The boundedness of $\tilde{G}$ implies that there exist positive numbers t_0 and C independent of t, x_2 such that

$$|L^{(2)}(t; \tilde{G}(\cdot, x_2; t))| \leq \frac{C}{(\log t)^{1/p+1}} \quad \text{for } t \geq t_0,\ x_2 > 0. \tag{4.15}$$

Therefore, by noticing that the support of $\tilde{G}(x_1, \cdot; t)$ is contained in $(t^{-1/q}, \infty)$, (4.15) implies that there exist positive numbers t_0 and C independent of t such that

$$\begin{aligned} |N^{(2)}(t)| &= \frac{1}{qt} \left| \int_0^\infty e^{itx_2^q} L^{(2)}(t; G(\cdot, x_2; t))dx_2 \right| \\ &= \frac{1}{qt} \left| \int_0^\infty e^{itx_2^q} \frac{1}{x_2^q} L^{(2)}(t; \tilde{G}(\cdot, x_2; t))dx_2 \right| \\ &\leq \frac{C}{t} \cdot \int_{t^{-1/q}}^\infty \frac{1}{x_2^q} \frac{1}{(\log t)^{1/p+1}} dx_2 \\ &\leq \frac{C}{t} \cdot \frac{1}{t^{1/q-1}} \cdot \frac{1}{(\log t)^{1/q+1}} = \frac{C}{t^{1/q}(\log t)^{1/p+1}} \quad \text{for } t \geq t_0. \end{aligned} \tag{4.16}$$

Putting (4.11), (4.14), (4.16) together, we can get (ii) in Lemma 3.1.

References

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko: *Singularities of Differentiable Maps II*, Birkhäuser, 1988.

[2] K. Cho, J. Kamimoto and T. Nose: Asymptotic analysis of oscillatory integrals via the Newton polyhedra of the phase and the amplitude, J. Math. Soc. Japan, **65** (2013), 521–562.

[3] J. Denef, J. Nicaise and P. Sargos: Oscillating integrals and Newton polyhedra, J. Anal. Math. **95** (2005), 147–172.

[4] I. A. Ikromov and D. Müller: Uniform estimates for the Fourier transform of surface carried measures in $\mathbb{R}^3$ and an application to Fourier restriction, J. Fourier Anal. Appl. **17** (2011), 1292–1332.

[5] I. A. Ikromov, M. Kempe and D. Müller: Estimates for maximal functions associated with hypersurfaces in $\mathbb{R}^3$ and related problems of harmonic analysis, Acta Math. **204** (2010), 151–271.

[6] A. Iosevich and E. Sawyer: Maximal averages over surfaces, Adv. Math. **132** (1997), 46–119.

[7] J. Kamimoto and T. Nose: Toric resolution of singularities in a certain class of C^∞ functions and asymptotic analysis of oscillatory integrals, J. Math. Sci. Univ. Tokyo **23** (2016), 425–485. (Preprint arXiv:1208.3924.)

[8] ______: Newton polyhedra and weighted oscillatory integrals with smooth phases, Trans. Amer. Math. Soc. **368** (2016), 5301–5361. (Preprint arXiv:1406.4325.)

[9] ______: On the asymptotic expansion of oscillatory integrals with smooth phases in two dimensions, RIMS Kôkyûroku Bessatsu, **B57**, (2016), 141–157.

[10] ______: On local zeta functions in two dimensions, in preparation.

[11] T. Okada and K. Takeuchi: Meromorphic continuations of local zeta functions and their applications to oscillating integrals, Tohoku Math. J. (2), **65**, no. 2 (2013), 159–178.

[12] H. Schulz: Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms, Indiana Univ. Math. J., **40** (1991), 1267–1275.

[13] E. M. Stein: *Harmonic Analysis. Real-variable methods, orthogonality and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.

[14] N. M. Temme: *Asymptotic methods for integrals.* Series in Analysis, **6**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[15] A. N. Varchenko: Newton polyhedra and estimation of oscillating integrals, Functional Anal. Appl., **10-3** (1976), 175–196.