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Large time behavior of solutions to 1D quasilinear wave equations

By

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Abstract

This note is a survey of [17]. We consider the large time behavior of solutions of the Cauchy problem of the quasilinear wave equation: $\partial_t^2 u = \partial_x((1+u)^{2a}\partial_x u)$, which describes shearing-motion in elastic-plastic rod. If $1+u(0,x)$ is bounded away from a positive constant, we can construct a local solution for smooth initial data. When $1+u(t,x)$ is going to 0 in finite time, the equation degenerates. We give a sufficient condition that the equation degenerates in finite time.

§ 1. Introduction

In this note, we consider the Cauchy problem of the following quasilinear wave equation:

$$
\begin{cases} \partial_t^2 u = \partial_x((1+u)^{2a}\partial_x u), & (t,x) \in (0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}
$$

where $u(t,x)$ is an unknown real valued function and $a > 0$. The equation in (1.1) describes shearing-motion in elastic-plastic rod (see Cristescu [4] and Ames, Lee and Vicario [1]).

Throughout this paper, we always assume that u_0 in (1.1) satisfies that there exists a constant $c_0 > 0$ such that

$$
1 + u_0(x) \geq c_0 \tag{1.2}
$$

for all $x \in \mathbb{R}$.

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The assumption (1.2) enable us to regard the equation in (1.1) as a strictly hyperbolic equation near $t = 0$. By the standard local existence theorem for strictly hyperbolic equations (e.g. Hughes, Kato and Marsden [7] or Majda [12]), the local solution of (1.1) with smooth initial data uniquely exists. If $1 + u(t, x)$ is going to 0 in finite time, the equation degenerates. When the equation degenerates, the standard local existence theorem does not work since the equation loses the strict hyperbolicity. In general, for non-strictly hyperbolic equations, the persistence of the regularity of solutions does not hold (see Remark 4). The main theorem in [17] gives a sufficient condition for the occurrence of the degeneracy.

Theorem 1.1 (S. [17])**.** *Let $(u_0, u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$. Suppose that the initial data (u_0, u_1) satisfy that* (1.2),

$$u_1(x) \pm (1 + u_0(x))^a \partial_x u_0(x) \leq 0 \quad \text{for all} \ \ x \in \mathbb{R} \tag{1.3}$$

and

$$\int_{\mathbb{R}} u_1(x) dx < \frac{-2}{a+1}. \tag{1.4}$$

Then there exists $T^* > 0$ such that a local unique solution $u \in C([0, T^*); H^2(\mathbb{R})) \cap C^1([0, T^*); H^1(\mathbb{R}))$ of (1.1) *exists and*

$$\lim_{t \nearrow T^*} 1 + u(t, x_0) = 0 \ \ \text{for some } x_0 \in \mathbb{R}. \tag{1.5}$$

Remark 1. In Theorem 1.1, we do not assume $u_1 \in L^1(\mathbb{R})$. The assumption (1.3) implies that u_1 is a non-positive function, from which, the left hand side of (1.4) is going to $-\infty$, if $u_1 \notin L^1(\mathbb{R})$. Hence the assumption (1.4) is satisfied.

The equation in (1.1) is formally equivalent to the following 2×2 conservation system:

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} - \partial_x \begin{pmatrix} V \\ \dfrac{(1+U)^{2a+1}}{2a+1} \end{pmatrix} = 0,$$

where $U(t, x) = u(t, x)$, $V(t, x) = \int_{-\infty}^{x} \partial_t u(t, y) dy$ and u is a solution to the equation in (1.1). The equation in (1.1) and this 2×2 conservation system have been studied by many authors (e.g. [10, 11, 13, 14, 23]). The global existence of solution to more general 2×2 conservation systems has been known (e.g. Johnson [8] and Yamaguchi and Nishida [20]). If (u_0, u_1) satisfies that (1.2), (1.3) and $\int_{\mathbb{R}} u_1(x) dx > -2/(a + 1)$, then (1.1) has a global smooth solution such that the equation does not degenerate. This global existence result and Theorem 1.1 say that $-2/(a+1)$ is a threshold of $\int_{\mathbb{R}} u_1(x) dx$

separating the global existence of solutions (such that the equation does not degenerate) and the degeneracy of the equation under the assumption (1.3). If (1.3) is not satisfied, then solutions can blow up in finite time (e.g. Klainerman and Majda [10], Manfrin [13] and Zabusky [23]).

Remark 2. Many authors (e.g. [2, 5, 9, 16, 21, 22]) have studied the Cauchy problem of the following 1D quasilinear wave equation:

$$\partial_t^2 u = c(u)^2 \partial_x^2 u + \lambda c(u) c'(u)(\partial_x u)^2, \tag{1.6}$$

where $c'(\theta) = dc(\theta)/d\theta$ and $0 \le \lambda \le 2$. This parameterized equation has been introduced by Glassey, Hunter and Zheng [6] (see also Chen and Shen [3]). When $\lambda = 0, 1$ or 2, the equation (1.6) has some physical backgrounds (e.g. the second sound wave in superfluids, the nematic liquid crystal or long waves on a dipole chain in the continuum limit). (1.6) with $\lambda = 2$ and $c(\theta) = (1+\theta)^a$ is the equation in (1.1). If $\lambda \neq 2$, then the equation (1.6) does not have the structure of conservation system. In [9, 16], Kato and the author have given a sufficient condition the occurrence of the degeneracy of the equation with $\lambda = 0$ and 1 and $c(\theta) = (1+\theta)$. Using the same method as in [9, 16], one can show that the equation (1.6) with $0 \le \lambda < 2$ and $c(\theta) = (1+\theta)^a$ for $a > 0$ degenerates in finite time, if the initial data are smooth, compactly supported and satisfy (1.2) and (1.3). That is, the equation (1.6) with $0 \le \lambda < 2$ would degenerate regardless of $\int_{\mathbb{R}} u_1(x)dx$.

Remark 3. Theorem 1.1 can be generalized to the following 1D quasilinear wave equation under suitable assumptions on $c(\cdot)$:

$$\partial_t^2 u = \partial_x (c(u)^2 \partial_x u).$$

The details of this generalization is disused in Section 4.

Remark 4. It has been known that a loss of the regularity appears for solutions to the following non-strictly hyperbolic equation:

$$\partial_t^2 u - t^{2l} \partial_x^2 u - h t^{l-1} \partial_x u = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

where h is a constant and $l \in \mathbb{N}$. Namely, in general, $(u, \partial_t u)$ does not belong to $C^1([0,\infty), H^s(\mathbb{R})) \times C([0,\infty), H^{s-1})$ with initial data $(u(0,x), \partial_t u(0,x)) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ (see Taniguchi and Tozaki [18], Yagdjian [19] and Qi [15]). From this fact, we can expect that the solutions of (1.1) have a singularity when the equation degenerates.

In this note, we show Theorem 1.1 with $u_1 \in L^1(\mathbb{R})$ only. In the proof of Theorem 1.1 with $u_1 \in L^1(\mathbb{R})$, we use the following formula of solutions of (1.1) established by

Lax [11]:

$$\frac{2(1+u(t_0,x_0))^{a+1}}{a+1} = w_1(0,x_-(0)) - w_2(0,x_+(0)),$$

where $w_1 = \int_{-\infty}^{x} \partial_t u dx - (1+u)^{a+1}/(a+1)$ and $w_2 = \int_{-\infty}^{x} \partial_t u dx + (1+u)^{a+1}/(a+1)$ are Riemann invariants of (1.1) and $x_\pm(t)$ are the plus and minus characteristic curves through (t_0, x_0), which are defined in Section 3. Our proof of Theorem 1.1 is based on a contradiction argument for the function $F(t) = -\int_{\mathbb{R}} u(t,x) - u_0(x)dx$. The idea for the proof is to divide the domain of the integral $\mathbb{R}$ into three parts by using the plus and minus characteristic curves in the estimate of $F(t)$. From the integration by parts and the equation in (1.1), we easily see that $F(t)/t = -\int_{\mathbb{R}} u_1(x)dx$. While, by using the above idea, we can obtain $\liminf_{t\nearrow\infty} F(t)/t \le 2/(a+1)$. Hence we have $-\int_{\mathbb{R}} u_1(x)dx \le 2/(a+1)$, which contradicts to the assumption (1.4). In the case that u_1 does not belong to $L^1(\mathbb{R})$, the function $F(t)$ can not be defined in general. For this case, more careful analysis is needed (for the complete proof, see [17]).

This paper is organized as follows: In Section 2, we recall some properties of solutions of (1.1). In Section 3, we show Theorem 1.1. In Section 4, we discus the generalization in Remark 3.

Notation

We denote Lebesgue space for $1 \le p \le \infty$ and L^2 Sobolev space with the order $m \in \mathbb{N}$ on $\mathbb{R}$ by $L^p(\mathbb{R})$ and $H^m(\mathbb{R})$. For a Banach space X, $C^j([0,T];X)$ denotes the set of functions $f : [0,T] \to X$ such that $f(t)$ and its k times derivatives for $k = 1, 2, \ldots, j$ are continuous. Various positive constants are simply denoted by C.

§ 2. Preliminary

We set $R(t,x)$ and $S(t,x)$ as follows

$$\begin{cases} R = \partial_t u + (1+u)^a \partial_x u, \\ S = \partial_t u - (1+u)^a \partial_x u. \end{cases}$$

The functions R and S have been used in Glassey, Hunter and Zheng [5, 6] and Zhang and Zheng [21]. We recall some properties of R and S established by Lax [11].

We denote the time when the blow-up or the degeneracy occur by T^*. That is,

$$T^* := \sup\{\ T > 0 \mid \sup_{[0,T]} \{\|\partial_t u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty}\} < \infty,\ \inf_{[0,T]\times\mathbb{R}} 1 + u(t,x) > 0\ \}.$$

We note that if $(u_0, u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and (1.2) is satisfied, then (1.1) has a unique solution u satisfying

$$u \in \bigcap_{j=0,1,2} C^j([0,T]; H^{2-j}(\mathbb{R}))$$

and

$$1+u(t,x)\geq c_1 \quad \text{for} \quad (t,x)\in[0,T]\times\mathbb{R}$$

for some positive constants T and c_1 (see Remark 5).

By (1.1), R and S are solutions to the system of the following first order equations:

$$\begin{cases} \partial_t R-(1+u)^a\partial_x R=\dfrac{a}{2(1+u)}(R^2-RS),\\ \partial_t u=\dfrac{1}{2}(R+S),\\ \partial_t S+(1+u)^a\partial_x S=\dfrac{a}{2(1+u)}(S^2-SR). \end{cases} \tag{2.1}$$

We note that the assumption (1.3) means $R(0,x),S(0,x)\leq 0$ for all $x\in\mathbb{R}$. By the method of characteristic, we can obtain the following lemmas:

Lemma 2.1. *Let $(u_0,u_1)\in H^2(\mathbb{R})\times H^1(\mathbb{R})$. Suppose that* (1.2) *and* (1.3) *are satisfied. Then we have*

$$R(t,x),\ S(t,x)\leq 0 \text{ for } (t,x)\in[0,T^*)\times\mathbb{R}. \tag{2.2}$$

Lemma 2.2. *Let $(u_0,u_1)\in H^2(\mathbb{R})\times H^1(\mathbb{R})$. Suppose that* (1.2) *and* (1.3) *are satisfied. Then $\|R(t)\|_{L^\infty}$ and $\|S(t)\|_{L^\infty}$ are uniformly bounded with $t\in[0.T^*)$.*

The proofs of these lemmas are given in [17].

Remark 5. One can check that the definition of T^* is suitable from the well-known theorem for blow-up criterion (e.g. Majda [12]). Namely, if $\|\partial_t u(t)\|_{L^\infty}+\|\partial_x u(t)\|_{L^\infty}$ is bounded on $[0,T]$ and $1+u(t,x)$ is bounded away from a positive constant on $[0,T]\times\mathbb{R}$, then we have the boundedness of $\|\partial_t u(t)\|_{H^1}+\|u(t)\|_{H^2}$ with the initial condition $(u_0,u_1)\in H^2(\mathbb{R})\times H^1(\mathbb{R})$.

§ 3. Proof of Theorem 1.1 with $u_1\in L^1(\mathbb{R})$

Proof. We show that $T^*<\infty$ and $\lim_{t\nearrow T^*}\inf_{(s,x)\in[0,t]\times\mathbb{R}}1+u(s,x)=0$ only. From Lemma 2.1 (see also Remark 5), the degeneracy occurs under the assumption of Theorem 1.1, if $T^*<\infty$. Hence it is enough to show that $T^*<\infty$. Now we suppose that $T^*=\infty$.

We set $(w_1(t,x),w_2(t,x))$ and $(v_1(t,x),v_2(t,x))$ as follows:

$$w_1=\int_{-\infty}^x\partial_t u dx+\frac{(1+u)^{a+1}}{a+1},$$
$$w_2=\int_{-\infty}^x\partial_t u dx-\frac{(1+u)^{a+1}}{a+1}$$

and

$$v_1 = \int_x^\infty \partial_t u dx - \frac{(1+u)^{a+1}}{a+1},$$
$$v_2 = \int_x^\infty \partial_t u dx + \frac{(1+u)^{a+1}}{a+1}.$$

Integrating the first and third equations of (2.1) on $(-\infty, x]$ and $[x, \infty)$ respectively, from the second equation of (2.1), we have the following systems:

$$\begin{cases} \partial_t w_1 - (1+u)^a \partial_x w_1 = 0, \\ \partial_t w_2 + (1+u)^a \partial_x w_2 = 0 \end{cases} \tag{3.1}$$

and

$$\begin{cases} \partial_t v_1 - (1+u)^a \partial_x v_1 = 0, \\ \partial_t v_2 + (1+u)^a \partial_x v_2 = 0. \end{cases}$$

Let $x_\pm(t)$ be characteristic curves of the equations of (3.1) respectively. That is, $x_\pm(t)$ are solutions to the following differential equations:

$$\frac{d}{dt} x_\pm(t) = \pm(1 + u(t, x_\pm(t)))^a. \tag{3.2}$$

w_1 and w_2 are invariants on the minus and plus characteristic curves defined in (3.2) respectively. Namely we have

$$w_1(t, x_-(t)) = w_1(0, x_-(0)) \tag{3.3}$$

and

$$w_2(t, x_+(t)) = w_2(0, x_+(0)). \tag{3.4}$$

From (3.3) and (3.4), if the plus and minus characteristic curves cross at some point (t_0, x_0), then we have

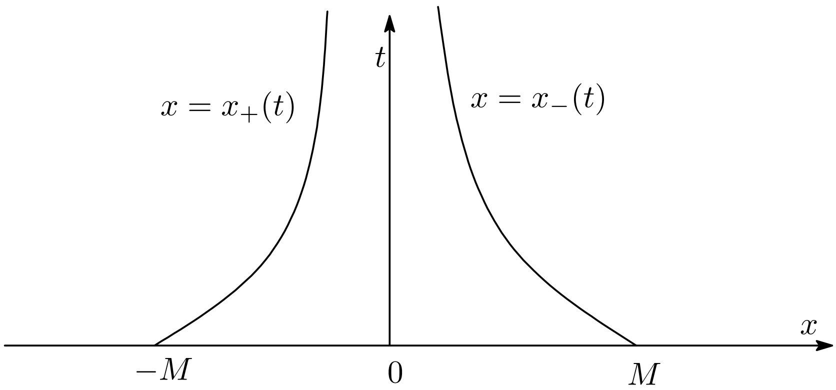
$$\begin{aligned} \frac{2(1+u(t_0,x_0))^{a+1}}{a+1} =& w_1(0, x_-(0)) - w_2(0, x_+(0)) \\ =& \int_{x_+(0)}^{x_-(0)} u_1(y) dy + \frac{(1+u_0(x_+(0)))^{a+1} + (1+u_0(x_-(0)))^{a+1}}{a+1}. \end{aligned} \tag{3.5}$$

Since $\lim_{|x|\to\infty} u_0(x) = 0$ and $u_1 \in L^1(\mathbb{R})$, from (1.4), there exists a number $M_1 > 0$ such that if $M \geq M_1$, then

$$\int_{-M}^{M} u_1 dx < -\frac{(1+u_0(M))^{a+1} + (1+u_0(-M))^{a+1}}{a+1}. \tag{3.6}$$

Suppose that the plus and minus characteristic curves $x_\pm(t)$ defined in (3.2) pass through $(0, \mp M)$ respectively. The characteristic curves $x_\pm(t)$ are drawn on (x, t) plane as follows:

Figure 1: characteristic curves $x_{\pm}(t)$ on (x,t) plane



From (3.6) and (3.5), these characteristic curves $x_+(t)$ and $x_-(t)$ do not cross for all $t \geq 0$. Hence it follows that $x_{\pm}(t)$ are uniformly bounded with $t \geq 0$. From (2.2), $1 + u(t, x_{\pm}(t))$ are monotone decreasing functions with $t \geq 0$, since $\frac{d}{dt}u(t, x_+(t)) = R(t, x_+(t))$ and $\frac{d}{dt}u(t, x_-(t)) = S(t, x_-(t))$. Therefore we have

$$\lim_{t\to\infty} 1 + u(t, x_{\pm}(t)) = 0. \tag{3.7}$$

Now we estimate $-\int_{-\infty}^{x_+(t)} u(t,x) - u_0(x)dx$. Since $\frac{dx_{\pm}(t)}{dt} = \pm(1 + u(t, x_{\pm}(t)))^a$, we have

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{x_+(t)} u(t,x) - u_0(x)dx = &\int_{-\infty}^{x_+(t)} \partial_t u(t,x)dx \\ &+ (u(t, x_+(t)) - u_0(x_+(t)))(1 + u(t, x_+(t)))^a. \end{aligned} \tag{3.8}$$

The equality (3.4) implies that

$$\begin{aligned} \int_{-\infty}^{x_+(t)} \partial_t u(t,x)dx = &-\frac{(1 + u_0(-M))^{a+1}}{a+1} + \frac{(1 + u(t, x_+(t)))^{a+1}}{a+1} \\ &+ \int_{-\infty}^{-M} u_1(x)dx. \end{aligned}$$

Since $\lim_{|x|\to\infty} u_0(x) = 0$ and $u_1 \in L^1(\mathbb{R})$, for any $\varepsilon > 0$, if M is sufficiently large, then we have

$$\int_{-\infty}^{x_+(t)} \partial_t u(t,x)dx \geq -\frac{1}{a+1} - C\varepsilon. \tag{3.9}$$

From (3.7), (3.8) and (3.9), we have

$$\begin{aligned} -\int_{-\infty}^{x_+(t)} u(t,x) - u_0(x)dx \leq &\left(\frac{1}{a+1} + C\varepsilon\right)t \\ &+ C\int_0^t (1 + u(s, x_+(s)))^a ds. \end{aligned} \tag{3.10}$$

Using the Riemann invariant (v_1, v_2) instead of (w_1, w_2), in the same way as in the derivation of (3.10), we have

$$
\begin{aligned}
-\int_{x_-(t)}^{\infty} u(t,x) - u_0(x)dx \leq \left(\frac{1}{a+1} + C\varepsilon\right) t \\
+ C\int_0^t (1 + u(s, x_-(s)))^a ds.
\end{aligned}
\tag{3.11}
$$

Next we estimate $-\int_{x_+(t)}^{x_-(t)} u(t,x) - u_0(x)dx$. By Lemma 2.1, $u(t,x)$ is a monotone decreasing function with $t \geq 0$ for all $x \in \mathbb{R}$, from which, we have $-1 \leq u(t,x) \leq C$ on $(t,x) \in [0,\infty) \times \mathbb{R}$. Since the length of the interval $[x_+(t), x_-(t)]$ is uniformly bounded with $t \geq 0$, we have

$$
-\int_{x_+(t)}^{x_-(t)} u(t,x) - u_0(x)dx \leq C. \tag{3.12}
$$

We set $F(t) = -\int_{\mathbb{R}} u(t,x) - u_0(x)dx$. From the integration by parts and (1.1), we have $F''(t) = 0$. Integrating this equality twice on $[0,t]$ and dividing by t, we have

$$
\frac{F(t)}{t} = F'(0).
$$

The inequalities (3.10), (3.11) and (3.12) imply that

$$
\begin{aligned}
F'(0) = \frac{F(t)}{t} &= \frac{-1}{t}\left(\int_{-\infty}^{x_+(t)} + \int_{x_+(t)}^{x_-(t)} + \int_{x_-(t)}^{\infty}\right) u(t,x) - u_0(x)dx \\
&\leq \left(\frac{2}{a+1} + C\varepsilon\right) + \frac{C}{t} \\
&\quad + \frac{C}{t}\int_0^t (1 + u(s, x_+(s)))^a + (1 + u(s, x_-(s)))^a ds.
\end{aligned}
\tag{3.13}
$$

Since $\lim_{t\to\infty} 1 + u(t, x_\pm(t)) = 0$, it follows that

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t (1 + u(s, x_\pm(s)))^a ds = 0.
$$

Hence by taking $t \to \infty$ in the both sides of (3.13), we have

$$
-\int_{-\infty}^{\infty} u_1(x)dx \leq \frac{2}{a+1} + C\varepsilon,
$$

which contradicts to (1.4) for sufficiently small $\varepsilon > 0$. Hence we have $T^* < \infty$. Therefore the degeneracy occurs in finite time. Here we omit the proof of (1.5) (see [17] for the proof of (1.5)).

$\square$

§ 4. Generalization in Remark 3

In this section, we give a detail of the small generalization pointed out in Remark 3. We consider the Cauchy problem of the following quasilinear wave equation:

$$\begin{cases} \partial_t^2 u = \partial_x(c(u)^2\partial_x u), & (t,x) \in (0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}. \end{cases} \tag{4.1}$$

We assume that $c \in C^\infty((-1,\infty)) \cap C([-1,\infty))$ satisfies that

$$c(\theta) > 0 \quad \text{for all } \theta > -1, \tag{4.2}$$

$$c(-1) = 0, \tag{4.3}$$

$$c'(\theta) \geq 0 \quad \text{for all } \theta > -1. \tag{4.4}$$

We note that the assumptions (1.2) and (4.2) ensure the strictly hyperbolicity of the equation in (4.1). Under the above assumptions on c, Theorem 1.1 is generalized as follows:

Theorem 4.1. *Let $(u_0,u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$. Suppose that the initial data (u_0,u_1) and c satisfy that* (4.2)-(4.4), (1.2),

$$u_1(x) \pm c(u_0(x))\partial_x u_0(x) \leq 0 \quad \textit{for all} \ \ x \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} u_1(x)dx < -2\int_{-1}^{0} c(\theta)d\theta.$$

Then there exists $T^* > 0$ such that a solution $u \in C([0,T^*);H^2(\mathbb{R})) \cap C^1([0,T^*);H^1(\mathbb{R}))$ of (1.1) *exists uniquely and that*

$$\lim_{t \nearrow T^*} c(u(t,x_0)) = 0 \quad \textit{for some } x_0 \in \mathbb{R}.$$

The proof of Theorem 4.1 is almost same as of Theorem 1.1. For the equation (4.1), we define the functions (R,S) and the Riemann invariants (w_1,w_2) as follows:

$$R = \partial_t u + c(u)\partial_x u,$$
$$S = \partial_t u - c(u)\partial_x u$$

and

$$w_1 = \int_{-\infty}^{x} \partial_t u dx + G(u),$$
$$w_2 = \int_{-\infty}^{x} \partial_t u dx - G(u),$$

where $G(u) = \int_{-1}^{u} c(\theta)d\theta$. (R, S) has the same properties as in Lemmas 2.1 and 2.2 and we can apply the same method as in Theorem 1.1 to the proof of Theorem 4.1.

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