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A note on the well-posedness of the compressible viscous fluid in the critical Besov space

By

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Abstract

We prove well-posedness of the compressible Navier-Stokes system in the Lagrangian formulation by the use of absolute temperature. The purpose of this article is to illustrate the difference between the use of the *total energy along the flow* in Chikami-Danchin [3] (*J. Difff. Eq*, 258 (2015), 3435-3467) and the absolute temperature.

§ 1. Introduction

We consider the Cauchy problem of the following compressible Navier-Stokes system with the full conservation law:

$$(1.1)\qquad \begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) + \nabla P = \mathrm{div}\,(\tau), & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ \partial_t(\rho\theta) + \mathrm{div}\,(u\rho\theta) + P\mathrm{div}\,u = k\Delta\theta + \tau:\nabla u, & (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ (\rho,u,\theta)|_{t=0} = (\rho_0,u_0,\theta_0), & x\in\mathbb{R}^n \end{cases}$$

where $n\geq 3$. In the above, $\rho=\rho(t,x)$, $u=u(t,x)$ and $\theta=\theta(t,x)$ are the unknown functions, representing the fluid density, the velocity vector and the absolute temperature, respectively. We assume that the stress tensor τ is given by

$$\tau = 2\mu D(u) + \lambda\mathrm{div}\,u\ \mathrm{Id},$$

where Id denote the identity matrix. The deformation tensor $D(u)$ is defined by

$$D(u) := \frac{1}{2}(Du+\nabla u)\ \text{with}\quad (Du)_{ij} := \partial_j u^i \quad\text{and}\quad (\nabla u)_{ij} := ({}^t(Du))_{ij} = \partial_i u^j$$

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where ${}^tA$ signifies the transpose of a matrix A. The operation $\tau : \nabla u$ signifies the trace product of the two matrices τ and ∇u, which is given by $\tau : \nabla u = \sum_{ij} \tau_{ij}\partial_j u_i$. The viscosity coefficients μ and λ are the Lamé constants satisfying $\mu > 0$ and $\lambda + 2\mu > 0$ and the heat conductive coefficient k is a positive constant. The given function P represents the pressure depending on ρ and θ. In this article, we restrict ourselves to the following ideal pressure law:

$$P(\rho, \theta) = \theta\pi(\rho),$$

where π is a smooth function of ρ. We assume that the initial density ρ_0 is bounded away from 0, i.e.,

$$0 < \underline{\rho} \leq \rho_0(x)$$

and tends to some positive constant ρ^* at infinity. We also assume that θ tends to a positive constant θ^* at spatial infinity. In the rest of the article, we assume $k = \rho^* = \theta^* = 1$ without loss of generality.

§ 1.1. Aim of this paper

Let us notice that system (1.1) considered is invariant under the scaling: $(\rho, u, \theta) \to (\rho_\nu, u_\nu, \theta_\nu)$ with

$$\rho_\nu(t, x) = \rho(\nu^2 t, \nu x), \quad u_\nu(t, x) = \nu u(\nu^2 t, \nu x) \quad \text{and} \quad \theta_\nu(t, x) = \nu^2\theta(\nu^2 t, \nu x). \tag{1.2}$$

This motivates us to introduce the idea of critical function spaces. The *critical spaces* for (1.1) are defined as the function spaces invariant under the scaling transformation (1.2). The idea of critical spaces was first adapted to the study of well-posedness for the (barotropic) compressible viscous fluid in [4], inspired by the work of [7] for the incompressible Navier-Stokes system. The work has given rise to a vast amount of results regarding the well-posedness issues of the compressible fluids. See [2, 5, 8], Chapter 10 of [1] and the references therein.

We employ homogeneous Besov spaces for defining the critical space for (1.1). Hereafter, we denote by L^p for $1 \leq p \leq \infty$ the usual Lebesgue space over $\mathbb{R}^n$ and ℓ^p as the sequence space over $\mathbb{Z}$. Let us introduce the Besov spaces as follows: Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity. Namely, let $\widehat{\phi} \in \mathcal{S}$ is a non-negative radially symmetric function that satisfies

$$\operatorname{supp}\widehat{\phi} \subset \{\xi \in \mathbb{R}^n; 2^{-1} < |\xi| < 2\},$$
$$\widehat{\phi_j}(\xi) := \widehat{\phi}(2^{-j}\xi) \text{ for all } j \in \mathbb{Z} \qquad \text{and} \qquad \sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi) = 1 \text{ for all } \xi \neq 0.$$

Definition 1.1 (Besov space with the interpolation index 1). Let $\mathcal{S}'$ be the space of tempered distributions on $\mathbb{R}^n$ and $\mathcal{P}$ be the space of all polynomials over $\mathbb{R}^n$. For

$s \in \mathbb{R}$ and $1 \le p \le \infty$ we define the homogeneous Besov space (with the interpolation index 1) $\dot{B}^s_{p,1}(\mathbb{R}^n) = \dot{B}^s_{p,1}$ as follows:

$$\dot{B}^s_{p,1} := \left\{ u \in \mathcal{S}' \;\Big|\; \sum_{j\in\mathbb{Z}} \phi_j * u = u \text{ in } \mathcal{S}',\ \|u\|_{\dot{B}^s_{p,1}} := \sum_{j\in\mathbb{Z}} 2^{js} \|\phi_j * u\|_{L^p} < \infty \right\}.$$

We denote by $\dot{S}_m u := \sum_{j<m} \phi_j * u$ the frequency cut-off of u.

Given a Banach space X, we denote by $L^q(0,T;X)$ the Bochner space, i.e.,

$$L^q(0,T;X) := \left\{ u \in \mathcal{S}' \mid \|u\|_{L^q_T(X)} := \left(\int_0^T \|u(t)\|^q_X dt \right)^{\frac{1}{q}} < \infty \right\}$$

for $1 \le q < \infty$ (with obvious modification when $q = \infty$). We define the space $E_p(T)$ for some $T > 0$ by

$$E_p(T) := \left\{ (v,\psi) \;\middle|\; \begin{array}{l} v \in C([0,T] : \dot{B}^{\frac{n}{p}-1}_{p,1}),\ \partial_t v, \nabla^2 v \in L^1(0,T; \dot{B}^{\frac{n}{p}-1}_{p,1}), \\ \psi \in C([0,T] : \dot{B}^{\frac{n}{p}-2}_{p,1}), \partial_t \psi, \nabla^2 \psi \in L^1(0,T; \dot{B}^{\frac{n}{p}-2}_{p,1}) \end{array} \right\} \tag{1.3}$$

whose norm is given by

$$\begin{aligned} \|(v,\varphi)\|_{E_p(T)} :=& \|v\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \|\partial_t v, \nabla^2 v\|_{L^1_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \\ &+ \|\varphi\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-2}_{p,1})} + \|\partial_t \varphi, \nabla^2 \varphi\|_{L^1_T(\dot{B}^{\frac{n}{p}-2}_{p,1})}. \end{aligned}$$

We note that the energy conservation in system (1.1) can also be formulated by the use of the *total energy by unit volume* E that is interrelated with θ by

$$E = \rho(\frac{|u|^2}{2} + \theta). \tag{1.4}$$

Then system for (ρ, u, E) now writes:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div}(\tau), \\ \partial_t E + \operatorname{div}(uE) - k \operatorname{div}\big(\dfrac{1}{\rho}\nabla E\big) \\ \qquad\qquad = \operatorname{div}\Big[\tau \cdot u - \dfrac{k}{\rho^2} E \nabla\rho - k\nabla(\dfrac{|u|^2}{2}) - uP\Big(\rho, \dfrac{E}{\rho} - \dfrac{|u|^2}{2}\Big)\Big]. \end{cases} \tag{1.5}$$

In [3], the analysis of the above system is carried out by the Lagrangian method in the Besov spaces introduced in [5, 6] with a new physical quantity called *total energy along the flow*, which is defined in the Lagrangian coordinates and roughly equals to the total energy $E=\rho(\frac{|u|^2}{2}+\theta)$ multiplied by the Jacobian of the flow. The use of this quantity in particular enlarges the admissible range of Lebesgue exponent, as far as the "Lagrangian solution" is concerned. The result in [3] for system (1.5) in the Eulerian coordinates reads as follows:

Proposition 1.2 ([3]). *Let $n\geq 3$ and $1<p<n$. Let u_0 be a vector field in $\dot{B}^{\frac{n}{p}-1}_{p,1}$ and E_0, a real valued function with E_0-1 in $\dot{B}^{\frac{n}{p}-2}_{p,1}$. Assume that ρ_0 satisfies $\rho_0-1\in\dot{B}^{\frac{n}{p}}_{p,1}$ and*

$$\inf_x \rho_0(x)>0. \tag{1.6}$$

Then there exists some $T>0$ such that system (1.5) *has a unique local solution (ρ,u,E) with $(u,E-1)\in E_p(T)$, ρ bounded away from 0 and $\rho-1\in\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}}_{p,1})$.*

It is furthermore possible to rephrase the above theorem in terms of temperature through the relation in (1.4), under the same conditions on the initial data.

The purpose of this article is to present an alternative proof of Proposition 1.2 by the direct use of the absolute temperature θ in Lagrangian coordinates. The proof given here can not cover the optimal admissibility of the Lebesgue exponent in the Lagrangian coordinates, which is only possible with the use of the *total energy along the flow* in [3]. It seems inevitable to restrict the range of the Lebesgue exponent not just in the Eulerian coordinates but also in the Lagrangian coordinates, if one formulates the equation with the absolute temperature. However, our argument shows that if one aims only to prove the existence and uniqueness in the Eulerian coordinates, it suffices to work with the temperature formulation.

§ 1.2. Change of coordinates and main result

One characteristic of the compressible viscous fluid is that it is governed by a hyperbolic-parabolic composite system at the linear level. The effect of the composite system appears in the smoothing properties for different spectral regions, and the hyperbolic component in the high frequencies hinders the use of the contraction mapping principle due to the inevitable loss of derivative. To overcome this, it has been observed by many authors [9, 10, 11] that the use of *Lagrangian coordinates* has an important merit when constructing a solution for system (1.1). Namely, by the change of coordinates, system (1.1) can be treated as a quasi-linear parabolic system, locally-in-time. Following [3], we perform the Lagrangian change of variable to (1.1).

The flow X of u is (formally) defined by the solution of the integral equation

$$X(t,y)=y+\int_0^t u(\tau,X(\tau,y))d\tau. \tag{1.7}$$

We denote the functions after the change of variables by $\bar{\rho}(t,y) := \rho(t, X(t,y))$, $\bar{u}(t,y) := u(t, X(t,y))$ and so on. We also denote the Jacobian and the inverse matrix of DX by $J := \det(DX)$ and $A := (DX)^{-1}$, respectively. We refer to [5, 3] for the details on how one may perform the change of coordinates.

For $n \times n$ matrices $A = (A_{ij})_{1\le i,j\le n}$ and $B = (B_{ij})_{1\le i,j\le n}$, we further define the trace product $A : B$ by

$$A : B = \sum_{ij} A_{ij} B_{ji}.$$

For a C^1 function $F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, we define $\operatorname{div} F : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(\operatorname{div} F)^j := \sum_i \partial_i F_{ij}, \quad 1 \le j \le m.$$

We denote by $\operatorname{adj}(A)$ the adjugate matrix of A i.e. the transpose of the cofactor matrix of A. If A is invertible then $\operatorname{adj}(A) = (\det A)A^{-1}$, where A^{-1} is the inverse matrix and $\det A$ is the determinant of A. We shall denote by ${}^tA$ the transpose of the matrix A. We define the "twisted" deformation tensor and divergence operator (acting on vector fields z) by the formulas

$$D_A(z) := \frac{1}{2}(DzA + {}^tA\nabla z) \quad \text{and} \quad \operatorname{div}_A z := {}^tA : \nabla z = Dz : A.$$

After changing the coordinates through (1.7), we have the following set of equations

$$(1.8) \qquad \begin{cases} \partial_t(J\bar{\rho}) = 0, \\ \rho_0 \partial_t \bar{u} - \operatorname{div}\Big(\operatorname{adj}(DX)(\tau_A(\bar{u}) - P(\bar{\rho}, \bar{\theta}))\Big) = 0, \\ \rho_0 \partial_t \bar{\theta} - \operatorname{div}(\operatorname{adj}(DX)({}^tA)\nabla\bar{\theta}) \\ \qquad + P(\bar{\rho},\bar{\theta})\operatorname{div}(\operatorname{adj}(DX)\bar{u}) - \tau_A(\bar{u}) : \operatorname{div}(\operatorname{adj}(DX)\bar{u}) = 0, \end{cases}$$

where $\tau_A(\bar{u}) := 2\mu D_A \bar{u} + \lambda \operatorname{div}_A \bar{u}$.

The following is our main theorem.

Theorem 1.3. *Let $n \ge 3$ and $1 < p < n$. Let u_0 be a vector field in $\dot{B}^{\frac{n}{p}-1}_{p,1}$ and θ_0, a real valued function with $\theta_0 - 1$ in $\dot{B}^{\frac{n}{p}-2}_{p,1}$. Assume that ρ_0 satisfies $\rho_0 - 1 \in \dot{B}^{\frac{n}{p}}_{p,1}$ and (1.6). Then there exists a small enough $T > 0$ such that system (1.8) admits a unique local solution $(\bar{\rho}, \bar{u}, \bar{\theta})$ with $\bar{\rho}$ bounded away from zero, $\bar{\rho} - 1$ in $\mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}}_{p,1})$ and $(\bar{u}, \bar{\theta} - 1)$ in $E_p(T)$.*

Furthermore, the flow map $(\rho_0 - 1, u_0, \theta_0 - 1) \mapsto (\bar{\rho} - 1, \bar{u}, \bar{\theta} - 1)$ is locally Lipschitz continuous from $\dot{B}^{\frac{n}{p}}_{p,1} \times \dot{B}^{\frac{n}{p}-1}_{p,1} \times \dot{B}^{\frac{n}{p}-2}_{p,1}$ to $\mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}}_{p,1}) \times E_p(T)$.

Once we obtain the Lagrangian solution $(\bar{\rho}, \bar{u}, \bar{\theta})$ for system (1.8), we may resort to Proposition 4.4 in Appendix to revert back to the Eulerian coordinates. The restriction

$n \geq 3$ and $1 < p < n$ ensure that the regularity of the "Eulerian solution" coincide with that of the "Lagrangian solution". Thus we may recover Proposition 1.2 through the relation in (1.4) without having to use to the *total energy along the flow*.

§ 1.3. Banach's Fixed point scheme

Our goal is to solve (1.8) by Banach's fixed point theorem. Here, we formulate our fixed point scheme. Note that having changed the coordinates we no longer have to retain any reference to the initial Eulerian vector-field u by defining directly the "flow" X of $\overline{u}$ by the formula

$$X(t,y) = y + \int_0^t \overline{u}(\tau,y)\,d\tau. \tag{1.9}$$

To simplify the notation, we drop the bars of the Lagrangian coordinates hereafter.

Setting $\psi := \theta - 1$ and recalling the pressure is given by $P(\rho,\theta) = \theta\pi(\rho)$, let us linearize system (1.8) around $(1,0,1)$:

$$\begin{cases} \rho = J^{-1}\rho_0, \\ \partial_t u - \rho_0^{-1}\mathrm{div}\,\tau + \rho_0^{-1}\nabla\big(\psi\pi(\rho_0)\big) \\ \ = \rho_0^{-1}\mathrm{div}\Big[\Big(\mathrm{adj}(DX)\tau_A(u) - \tau\Big) - \Big(\mathrm{adj}(DX)(\psi+1)\pi(\rho) - \psi\pi(\rho_0)\mathrm{Id}\Big)\Big], \\ \partial_t\psi - \rho_0^{-1}\Delta\psi = \rho_0^{-1}\Big(\mathrm{div}\,((\mathrm{adj}(DX)\,{}^tA - \mathrm{Id})\nabla\psi) \\ \quad - (\psi+1)\pi(\rho)\mathrm{div}\,(\mathrm{adj}(DX)u) + \tau : \mathrm{div}\,(\mathrm{adj}(DX)u)\Big). \end{cases}$$

We are no longer concerned with the mass conservation equation since it is explicitly solvable thanks to the change of coordinates. Hence, it suffices to construct a contraction map to the parabolic system for (u,ψ). With $a_0 := \rho_0 - 1 \in \dot{B}^{\frac{n}{p}}_{p,1}$, $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}$ and $\psi_0 := \theta_0 - 1 \in \dot{B}^{\frac{n}{p}-2}_{p,1}$ we aim to solve system (1.8) in the critical Besov space defined in (1.3). The system can now be written by

$$\begin{cases} \partial_t u - \rho_0^{-1}\mathrm{div}\,\tau + \rho_0^{-1}\nabla\big(\psi\pi(\rho_0)\big) = \rho_0^{-1}\mathrm{div}\,\big(I_1(u,u) + I_2(u,\psi)\big), \\ \qquad\qquad\qquad \partial_t\psi - \rho_0^{-1}\Delta\psi = \rho_0^{-1}\big(\mathrm{div}\,(I_3(u,\psi)) + I_4(u,\psi) + I_5(u,u)\big), \end{cases} \tag{1.10}$$

where

$$\begin{aligned} I_1(v,v) &:= \mathrm{adj}(DX_v)\tau_{A_v}(v) - \tau, \\ I_2(v,\varphi) &:= -\mathrm{adj}(DX_v)(\varphi+1)\pi(J_v^{-1}\rho_0) - \varphi\pi(\rho_0)\mathrm{Id}, \\ I_3(v,\varphi) &:= (\mathrm{adj}(DX_v)\,{}^tA_v - \mathrm{Id})\nabla\varphi, \\ I_4(v,\varphi) &:= -(\varphi+1)\pi(J_v^{-1}\rho_0)\mathrm{div}\,(\mathrm{adj}(DX_v)v) \\ \text{and}\quad I_5(v,v) &:= \tau : \mathrm{div}\,(\mathrm{adj}(DX_v)v). \end{aligned} \tag{1.11}$$

Therefore our fixed point scheme is given by the following:

$$\left\{\begin{aligned}\partial_t u - \rho_0^{-1}\mathrm{div}\,\tau + \rho_0^{-1}\nabla\big(\psi\pi(\rho_0)\big) &= \rho_0^{-1}\mathrm{div}\,\big(I_1(v,v)+I_2(v,\varphi)\big),\\ \partial_t\psi - \rho_0^{-1}\Delta\psi &= \rho_0^{-1}\big(\mathrm{div}\,(I_3(v,\varphi)) + I_4(v,\varphi) + I_5(v,v)\big),\end{aligned}\right. \tag{1.12}$$

for a given $(v,\varphi)\in E_p(T)$. In order to solve (1.10) locally in time, it suffices to show that the map

$$\Phi : (v,\varphi)\mapsto (u,\psi) \tag{1.13}$$

with (u,ψ) the solution to (1.12) has a fixed point in $E_p(T)$ when $n\geq 3$ and $1<p<n$, for a small enough T.

The rest of the article unfolds as follows: in the second section, we show some a priori estimates for linear parabolic equations. In the third section, we prove Theorem 1.2. In Appendix we state some technical tools that are used in the second section without proof. We will denote by C harmless generic 'constants' that may change from line to line.

§ 2. A priori estimates for parabolic equations

We first recall a result for the following heat equation with non-smooth coefficients:

$$\partial_t u - a\mathrm{div}\,(b\nabla u) = f, \tag{2.1}$$

the study of which in the Besov spaces is summarized in the papers such as [3, 5] .

Proposition 2.1 ([5]). *Let a and b be bounded functions and assume that there exists a constant β such that $ab\geq\beta>0$. Suppose that $a\nabla b$ and $b\nabla a$ are in $L^\infty(0,T;\dot B^{\frac np-1}_{p,1})$ for some $1<p<\infty$. There exist two constants η and C such that if for some $m\in\mathbb{Z}$ we have*

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^n}\dot S_m(ab)(t,x)\geq\frac{\beta}{2},$$

$$\|(\mathrm{Id}-\dot S_m)(a\nabla b, b\nabla a)\|_{L^\infty_T(\dot B^{\frac np-1}_{p,1})}\leq\eta\beta,$$

then the solutions to (2.1) satisfy for all $t\in[0,T]$,

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot B^s_{p,1})}+\beta\|u\|_{L^1_t(\dot B^{s+2}_{p,1})}\\ &\qquad\leq C(\|u_0\|_{\dot B^s_{p,1}}+\|f\|_{L^1_t(\dot B^s_{p,1})})\exp\left(\frac{C}{\beta}\int_0^t(\|\dot S_m(a\nabla a, b\nabla a)\|^2_{\dot B^{\frac np}_{p,1}}\,d\tau\right)\end{aligned}$$

whenever $-\min(\frac np,\frac n{p'})<s\leq\frac np-1$, where p' is the Hölder conjugate exponent of p.

As a consequence of the above, we have the following a priori estimate obtained in [5], on which the analysis of the momentum equation is relied. We consider

$$\partial_t u - a\mathrm{div}\,(2\mu D(u) + \lambda \mathrm{div}\,u\mathrm{Id}) = f, \tag{2.2}$$

where both u and f are valued in $\mathbb{R}^n$ and a is some variable coefficient. We assume that the following uniform ellipticity condition is satisfied:

$$\alpha := \min\left(\inf_{(t,x)\in[0,T]\times\mathbb{R}^n}(a\mu)(t,x),\ \inf_{(t,x)\in[0,T]\times\mathbb{R}^n}(2a\mu + a\lambda)(t,x)\right) > 0. \tag{2.3}$$

Proposition 2.2 ([5]). *Let a, λ and μ be bounded functions satisfying* (2.3). *Assume that $a\nabla\mu$, $a\nabla\lambda$, $\mu\nabla a$ and $\lambda\nabla a$ are in $L^\infty(0,T;\dot{B}^{\frac{n}{p}-1}_{p,1})$ for some $1 < p < 2n$, and that there exist some constants $\bar{a}$, $\bar{\lambda}$ and $\bar{\mu}$ satisfying*

$$2\bar{a}\bar{\mu} + \bar{a}\bar{\lambda} > 0 \text{ and } \bar{a}\bar{\mu} > 0,$$

and $a-\bar{a}$, $\lambda-\bar{\lambda}$ and $\mu-\bar{\mu}$ are in $\mathcal{C}([0,T];\dot{B}^{\frac{n}{p}}_{p,1})$. Finally, suppose that

$$\lim_{m\to+\infty}\|(\mathrm{Id}-\dot{S}_m)(a\nabla\mu, a\nabla\lambda, \mu\nabla a, \lambda\nabla a)\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} = 0.$$

Then for any data $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}$ and $f \in L^1(0,T;\dot{B}^{\frac{n}{p}-1}_{p,1})$, system (2.2) *admits a unique solution $u \in \mathcal{C}([0,T];\dot{B}^{\frac{n}{p}-1}_{p,1})$ with $\nabla u \in L^1(0,T;\dot{B}^{\frac{n}{p}}_{p,1})$.*

Furthermore, there exist two constants η and C such that if m is so large as to satisfy

$$\min\left(\inf_{(t,x)\in[0,T]\times\mathbb{R}^n}\dot{S}_m(a\mu)(t,x),\ \inf_{(t,x)\in[0,T]\times\mathbb{R}^n}\dot{S}_m(2a\mu+a\lambda)(t,x)\right) \geq \frac{\alpha}{2},$$

$$\|(\mathrm{Id}-\dot{S}_m)(a\nabla\mu, a\nabla\lambda, \mu\nabla a, \lambda\nabla a)\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \leq \eta\alpha,$$

then we have for all $t \in [0,T]$,

$$\begin{aligned}&\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \alpha\|\nabla u\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})}\\ &\quad\leq C(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|f\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})})\exp\left(\frac{C}{\alpha}\int_0^t \|\dot{S}_m(a\nabla\mu, a\nabla\lambda, \mu\nabla a, \lambda\nabla a)\|^2_{\dot{B}^{\frac{n}{p}}_{p,1}}\,d\tau\right).\end{aligned}$$

Now we derive the a priori estimate for the linearized system

$$\begin{cases}\partial_t u - a\mathrm{div}\,(2\mu D(u) + \lambda\mathrm{div}\,u\mathrm{Id} + \psi\pi\mathrm{Id}) = f,\\ \partial_t\psi - a\mathrm{div}\,(b\nabla\psi) = g.\end{cases} \tag{2.4}$$

In fact, the following statement is merely a corollary of Proposition 2.1 and 2.2.

Corollary 2.3. *Let* $n \geq 3$, $1 < p < n$, $u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}$, $\psi_0 \in \dot{B}^{\frac{n}{p}-2}_{p,1}$, $f \in L^1(0,T;\dot{B}^{\frac{n}{p}-1}_{p,1})$ *and* $g \in L^1(0,T;\dot{B}^{\frac{n}{p}-2}_{p,1})$. *Assume that* a, b, λ *and* μ *satisfy the assumptions of Proposition* 2.2. *Let* a *and* b *further satisfy assumptions of Proposition* 2.1 *with* $s = \frac{n}{p} - 2$. *Assume that* π *belongs to the multiplier space* [1] $\mathcal{M}(\dot{B}^{\frac{n}{p}}_{p,1})$. *Then system* (2.4) *admits a unique solution* (u, ψ) *with*

$$u \in \mathcal{C}([0,T];\dot{B}^{\frac{n}{p}-1}_{p,1}) \cap L^1(0,T;\dot{B}^{\frac{n}{p}+1}_{p,1}) \quad \textit{and} \quad \psi \in \mathcal{C}([0,T];\dot{B}^{\frac{n}{p}-2}_{p,1}) \cap L^1(0,T;\dot{B}^{\frac{n}{p}}_{p,1}).$$

Besides, if m *is large enough (as in Propositions* 2.2 *and* 2.1*) then* (u,ψ) *fulfills for all* $t \in [0,T]$,

$$\|\psi\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-2}_{p,1})} + \beta\|\psi\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})} \leq C\Big(\|\psi_0\|_{\dot{B}^{\frac{n}{p}-2}_{p,1}} + \|g\|_{L^1_t(\dot{B}^{\frac{n}{p}-2}_{p,1})}\Big) \times \exp\Big(\frac{C}{\beta}\int_0^t \|\dot{S}_m(a\nabla b, b\nabla a)\|^2_{\dot{B}^{\frac{n}{p}}_{p,1}}\,d\tau\Big),$$

$$\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \alpha\|u\|_{L^1_t(\dot{B}^{\frac{n}{p}+1}_{p,1})} \leq C\Big(\|u_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}} + \|f\|_{L^1_t(\dot{B}^{\frac{n}{p}-1}_{p,1})} + \|a\|_{\mathcal{M}(\dot{B}^{\frac{n}{p}-1}_{p,1})}\|\pi\|_{\mathcal{M}(\dot{B}^{\frac{n}{p}}_{p,1})}\|\psi\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})}\Big)\exp\Big(\frac{C}{\alpha}\int_0^t \|\dot{S}_m(a\nabla\mu, a\nabla\lambda, \mu\nabla a, \lambda\nabla a)\|^2_{\dot{B}^{\frac{n}{p}}_{p,1}}\,d\tau\Big).$$

Proof. It suffices to solve the second equation of (2.4) according to Proposition 2.1. Note that in order to apply Proposition 2.1 with $s = \frac{n}{p} - 2$, we need $n \geq 3$ and $1 < p < n$. Then we look at u as the solution to

$$\partial_t u - a\mathrm{div}\,(2\mu D(u) + \lambda \mathrm{div}\, u \mathrm{Id}) = f - a\nabla(\pi\psi).$$

The assumptions on a and π, and the control of ψ in $L^1(0,T;\dot{B}^{\frac{n}{p}}_{p,1})$ imply that u may be constructed according to Proposition 2.2 and satisfies the desired a priori bound. □

We remark that unlike in [3] where the "total energy along the flow" is used, here we do not need to derive a new estimate for the absolute temperature ψ. For the "total energy along the flow", one needs to establish some new parabolic estimate for equations that differ from (2.1) or (2.2).

§ 3. Proof of Theorem 1.2

In the rest of the paper, we shall show that the fixed point scheme (1.10) does indeed work. We denote the linear part of the solution (u,ψ) by (u_L,ψ_L), respectively,

[1] The multiplier space $\mathcal{M}(\dot{B}^s_{p,1})$ is the set of all distributions $f \in \mathcal{S}'$ satisfying $hf \in \dot{B}^s_{p,1}$ for all $h \in \dot{B}^s_{p,1}$, and $\|f\|_{\mathcal{M}(\dot{B}^s_{p,1})} := \sup_{\|h\|_{\dot{B}^s_{p,1}}=1} \|hf\|_{\dot{B}^s_{p,1}} < \infty$.

i.e.,

$$\partial_t u_L - \rho_0^{-1}\mathrm{div}\,(2\mu D u_L + \lambda \mathrm{div}\, u_L \mathrm{Id}) = 0,\ u_L|_{t=0} = u_0,$$
$$\partial_t \psi_L - \rho_0^{-1}\Delta\psi_L = 0,\ \psi_L|_{t=0} = \psi_0.$$

Let $\widetilde{u} := u - u_L$ and $\widetilde{\psi} := \psi - \psi_L$ then $(\widetilde{u}, \widetilde{\theta})$ has to satisfy the following equations

$$(3.1)\quad \begin{cases} \partial_t \widetilde{u} - \rho_0^{-1}\mathrm{div}\,(2\mu D\widetilde{u} + \lambda \mathrm{div}\,\widetilde{u}\mathrm{Id}) = \rho_0^{-1}\mathrm{div}\,(I_1(v,v) + I_4(v,\varphi)), \\ \partial_t \widetilde{\psi} - \rho_0^{-1}\Delta\widetilde{\psi} = \rho_0^{-1}\big(\mathrm{div}\,(I_3(v,\varphi) + I_4(v,\varphi)) + I_5(v,v)\big), \end{cases}$$

with $(v,\varphi) \in E_p(T)$, where the terms on the right hand-side are defined in (1.11). We claim that the Banach fixed point theorem applies to the map Φ defined in (1.13) in a closed ball $\bar{B}_{E_p(T)}((u_L,\theta_L),R)$ centered at the linear solutions (u_L,ψ_L) with a radius R, provided T and R are suitably small.

If the right-hand side of the first equation is in $L^1(0,T;\dot{B}^{\frac{n}{p}-1}_{p,1})$ and if there exists some $m \in \mathbb{Z}$ so that the conditions of Proposition 2.2 are satisfied then $u \in E_p(T)$. The existence of m so that we have, for some α,

$$\inf_{x\in\mathbb{R}^n} \dot{S}_m\Big(\frac{1}{\rho_0}\Big) \geq \frac{\alpha}{2} \quad \text{and} \quad \big\|(\mathrm{Id} - \dot{S}_m)\Big(\frac{\nabla\rho_0}{\rho_0^2}\Big)\big\|_{L^\infty_T(\dot{B}^{\frac{n}{p}-1}_{p,1})} \leq \eta\alpha$$

is ensured by the fact that $\rho_0 - 1$ belong to the space $\dot{B}^{\frac{n}{p}}_{p,1}$ which is defined in terms of a convergent series and embeds continuously in the set of bounded continuous functions decaying at infinity. Note that the choice of m only depends on ρ_0

As the proof of Theorem is carried out by the Banach fixed point theorem: the procedure is divided into 4 steps: We shall show

1. that the map Φ in (1.13) is a map from the ball $\bar{B}_{E_p(T)}((u_L,\theta_L),R)$ into itself for suitably small T and R;

2. that the map Φ is a contraction;

3. the time-continuity of the solutions;

4. the Lipschitz stability of the solution with respect to the initial data.

We do not give the full proof of the above but the process is quite standard and easy (see [3, 5]). Here we shall only show the stability of the ball.

First step: Stability of the ball $\bar{B}_{E_p(T)}((u_L,\psi_L),R)$ for suitably small T and R.
From now on, we assume that for a small enough $\widetilde{c}$, we have

$$(3.2)\qquad \|Dv\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})} \leq \widetilde{c}.$$

Proposition 2.3 and the definition of the multiplier space $\mathcal{M}(\dot{B}_{p,1}^{\frac{n}{p}-1})$ ensure that

$$\begin{aligned}\|(\widetilde{u},\widetilde{\psi})\|_{E_p(T)} \leq Ce^{C_{\rho_0,m}T}\Big(&\|\rho_0^{-1}\|_{\mathcal{M}(\dot{B}_{p,1}^{\frac{n}{p}-1})}\|I_1(v,v)+I_2(v,\varphi)\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}\\ &+\|\rho_0^{-1}\|_{\mathcal{M}(\dot{B}_{p,1}^{\frac{n}{p}-2})}\|\operatorname{div} I_3(v,\varphi)+I_4(v,\varphi)+I_5(v,v)\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}-2})}\Big).\end{aligned} \tag{3.3}$$

The coefficient ρ_0^{-1} indeed belongs to $\mathcal{M}(\dot{B}^s_{p,1})$ for $s=\frac{n}{p}-1$ and $s=\frac{n}{p}-2$ by the product estimate:

$$\|\rho_0^{-1}h\|_{\dot{B}^s_{p,1}} \leq \Big\|(\frac{a_0}{1+a_0}-1)h\Big\|_{\dot{B}^s_{p,1}} \leq C(\|a_0\|_{\dot{B}_{p,1}^{\frac{n}{p}}}+1)\|h\|_{\dot{B}^s_{p,1}}$$

under our assumptions on n and p (recall that $a_0 := \rho_0 - 1$).

<u>*Estimate of I_1*</u> : The first term I_1 of (3.1) have been estimated in [5, 3] as follows :

$$\|I_1(v,v)\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})} \leq C(\|a_0\|_{\dot{B}_{p,1}^{\frac{n}{p}}}+1)\|Dv\|^2_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}$$

<u>*Estimate of I_2*</u> : Since

$$\begin{aligned}I_2(v,\varphi) &= -\operatorname{adj}(DX_v)(\varphi+1)\pi(J_v^{-1}\rho_0)-\varphi\pi(\rho_0)\mathrm{Id}\\ &= -(\operatorname{adj}(DX_v)\pi(J_v^{-1}\rho_0)-\pi(\rho_0)\mathrm{Id})\varphi-\operatorname{adj}(DX_v)\pi(J_v^{-1}\rho_0),\end{aligned}$$

we have thanks to Propositions 4.1 and 4.3 of Appendix

$$\|I_2(v,\varphi)\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})} \leq C(\|a_0\|_{\dot{B}_{p,1}^{\frac{n}{p}}}+1)\big(\|Dv\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}\|\varphi\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}+T\|Dv\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}\big).$$

<u>*Estimate of I_3*</u> : Owing to (3.2), the application of Propositions 4.1 and 4.3 gives us

$$\begin{aligned}\|I_3(v,\varphi)\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}-1})} &\leq C\|\operatorname{adj}(DX_v)\,{}^tA_v-\mathrm{Id}\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}\|\nabla\varphi\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}-1})}\\ &\leq C\|Dv\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}\|\varphi\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}.\end{aligned}$$

<u>*Estimate of I_4*</u> : We decompose the term I_4 as follows:

$$\begin{aligned}I_4(v,\varphi) &= -(\varphi+1)\pi(J_v^{-1}\rho_0)\operatorname{div}(\operatorname{adj}(DX_v)v)\\ &= -\pi(J_v^{-1}\rho_0)\varphi\operatorname{div}(\operatorname{adj}(DX_v)v)-\pi(J_v^{-1}\rho_0)\operatorname{div}(\operatorname{adj}(DX_v)v).\end{aligned}$$

Then thanks to (3.2) and Proposition 4.1 we have

$$\begin{aligned}&\|I_4(v,\varphi)\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}-2})}\\ &\leq C(\|a_0\|_{\dot{B}_{p,1}^{\frac{n}{p}}}+1)(\|\varphi\operatorname{div}(\operatorname{adj}(DX_v)v)\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}-2})}+\|\operatorname{adj}(DX_v)v\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}-1})})\\ &\leq C(\|a_0\|_{\dot{B}_{p,1}^{\frac{n}{p}}}+1)(\|\varphi\|_{L^1_T(\dot{B}_{p,1}^{\frac{n}{p}})}+1)T\|v\|_{L^\infty_T(\dot{B}_{p,1}^{\frac{n}{p}-1})}.\end{aligned}$$

Estimate of I_5 :

$$I_5(v,v) = \tau : \operatorname{div}(\operatorname{adj}(DX_v)v).$$

Similarly to the previous computations, we may easily obtain by Proposition 4.1

$$\begin{aligned}\|I_5(v,v)\|_{L^1_T(\dot B^{\frac{n}{p}-2}_{p,1})} &= \|\tau : \operatorname{div}(\operatorname{adj}(DX_v)v)\|_{L^1_T(\dot B^{\frac{n}{p}-2}_{p,1})} \\ &\le CT\|v\|^2_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})}.\end{aligned}$$

In summary, we obtain the following estimates:

$$\begin{aligned}&\|\rho_0^{-1}\|_{\mathcal{M}(\dot B^{\frac{n}{p}-1}_{p,1})}\|I_1(v,v)+I_2(v,\varphi)\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})} \\ &\qquad + \|\rho_0^{-1}\|_{\mathcal{M}(\dot B^{\frac{n}{p}-2}_{p,1})}\|\operatorname{div} I_3(v,\varphi)+I_4(v,\varphi)+I_5(v,v)\|_{L^1_T(\dot B^{\frac{n}{p}-2}_{p,1})} \\ &\qquad \le C(\|a_0\|_{\dot B^{\frac{n}{p}}_{p,1}}+1)^2\Big((\|\varphi\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+\|Dv\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+T)\|Dv\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})} \\ &\qquad\qquad + (\|\varphi\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+\|v\|_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})}+1)T\|v\|_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})}\Big).\end{aligned}$$

Plugging all the inequalities above to (3.3), we obtain

$$\begin{aligned}&\|(\widetilde u,\widetilde\psi)\|_{E_p(T)} \\ &\le Ce^{C_{\rho_0,m}T}(\|a_0\|_{\dot B^{\frac{n}{p}}_{p,1}}+1)^2\Big((\|\varphi\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+\|Dv\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+T)\|Dv\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})} \\ &\qquad + (\|\varphi\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+\|v\|_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})}+1)T\|v\|_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})}\Big).\end{aligned}$$

Since (v,φ) belongs to the ball $\bar B_{E_p(T)}((u_L,\psi_L),R)$, decomposing v into $\widetilde v+u_L$ and φ into $\widetilde\varphi+\psi_L$ gives us

$$\begin{aligned}&\|(\widetilde u,\widetilde\psi)\|_{E_p(T)} \\ &\le C_{\rho_0}e^{C_{\rho_0,m}T}\Big((\|\psi_L\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+\|Du_L\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+R+T)(\|Du_L\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+R) \\ &\qquad + (\|\psi_L\|_{L^1_T(\dot B^{\frac{n}{p}}_{p,1})}+\|u_L\|_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})}+R+1)T(\|u_L\|_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})}+R)\Big)\end{aligned}$$

where C_{ρ_0} is some constant depending only on ρ_0 and the dimension. Note that we have

$$\|u_L\|_{L^\infty_T(\dot B^{\frac{n}{p}-1}_{p,1})} \le C\|u_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}}$$

by Proposition 2.2.

Choosing R and T sufficiently small we may see that Φ is indeed a self-map on the ball $\bar B_{E_p(T)}((u_L,K_L),R)$. The contraction estimate is also carried out in a similar fashion to the proofs given in [5, 6, 3]. We omit the details.

§ 4. Appendix

The purpose of this section is to present some technical results that have been used in the paper. In the first paragraph, we recall common product estimates regarding homogeneous Besov norms. Next, we state estimates for the flow and finally some necessary tools for the Lagrangian transformation.

§ 4.1. Estimate for product, composition and flows

For the proofs of the following propositions, see Chapter 2 of [1] and Appendix of [5, 6].

Proposition 4.1 ([5, 6]). *Let $\nu \geq 0$ and $-\min(\frac{n}{p}, \frac{n}{p'}) < \sigma \leq \frac{n}{p} - \nu$. The following product law holds:*

$$\|uv\|_{\dot{B}^{\sigma}_{p,1}} \leq C\|u\|_{\dot{B}^{\frac{N}{p}-\nu}_{p,1}} \|v\|_{\dot{B}^{\sigma+\nu}_{p,1}}.$$

Proposition 4.2. *Let $F : I \to \mathbb{R}$ be a smooth function (with I an open interval of $\mathbb{R}$ containing 0) vanishing at 0. Then for any $s > 0$, $1 \leq p \leq \infty$ and interval I' compactly supported in I there exists a constant C such that*

$$\|F(a)\|_{\dot{B}^{s}_{p,1}} \leq C\|a\|_{\dot{B}^{s}_{p,1}}$$

for any $a \in \dot{B}^{s}_{p,1}$ with values in I'.

The following flow estimates are found in Appendix of [5, 6].

Proposition 4.3 ([5, 6]). *Let $1 \leq p < \infty$ and $v \in E_p(T)$. Assume that for small enough $\widetilde{c}$,*

$$\int_0^T \|Dv\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \, dt \leq \widetilde{c}.$$

We define X as the flow of v as in (1.9)*. Then for all $t \in [0, T]$, we have*

$$\|\mathrm{Id} - \mathrm{adj}(DX_v(t))\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq C\|D\overline{v}\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})},$$

$$\|\mathrm{Id} - A_v(t)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq C\|D\overline{v}\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})},$$

$$\|J_v^{\pm 1}(t) - 1\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq C\|D\overline{v}\|_{L^1_T(\dot{B}^{\frac{n}{p}}_{p,1})}.$$

§ 4.2. Lagrangian coordinate

The necessary tools for the Lagrangian transformations are given below. For more details, we refer to [3, 5, 6].

Let X be a C^1-diffeomorphism over $\mathbb{R}^n$. For a vector-valued function $H : \mathbb{R}^n \to \mathbb{R}^m$, denote $\bar{H}(y) := H(x)$ with $x = X(y)$. The chain rule states that

$$D_y \bar{H}(y) = D_x H(X(y)) \cdot D_y X(y)$$

with $(D_y X)_{ij} = \partial_{y_j} X^i$, and

$$\nabla_y \bar{H}(y) = \nabla_y X(y) \cdot \nabla_x H(X(y)).$$

Hence, we have

$$D_x H(X(y)) = D_y \bar{H}(y) \cdot A(y)$$

with $A(y) = (D_y X(y))^{-1} = D_x X^{-1}$.

See Appendix of [5, 6] for the proofs of the following propositions.

Proposition 4.4 ([5, 6]). *Let X be a globally bi-Lipschitz diffeomorphism of $\mathbb{R}^n$ and (s, p, q) with $1 \le p < \infty$ and $-\frac{n}{p'} < s < \frac{n}{p}$ (or just $-\frac{n}{p'} < s \le \frac{n}{p}$ if $q = 1$ and just $-\frac{n}{p'} \le s < \frac{n}{p}$ if $q = \infty$). Then $a \mapsto a \circ X$ is a self-map over $\dot{B}^s_{p,q}$ if one of following cases holds:*

1. *$s \in (0, 1)$,*
2. *$s \in (-1, 0]$ and $J_{X^{-1}}$ is in the multiplier space $\mathcal{M}(\dot{B}^s_{p,q})$,*
3. *$s \ge 1$ and $(DX - \mathrm{Id}) \in \dot{B}^s_{p,q}$.*

Proposition 4.5 ([5, 6]). *Let K be a C^1-scalar function over $\mathbb{R}^n$ and H be a C^1-vector field. If X is a C^1-diffeomorphism such that $J := \det(D_y X) > 0$, then*

$$\overline{\nabla_x K} = J^{-1} \mathrm{div}_y (\mathrm{adj}(D_y X) \overline{K}),$$

$$\overline{\mathrm{div}_x H} = J^{-1} \mathrm{div}_y (\mathrm{adj}(D_y X) \overline{H}),$$

where $\mathrm{adj}(D_y X)$ is the adjugate of $D_y X$.

From the above proposition, we have the following set of change of coordinates:

$$\begin{aligned} \overline{\Delta_x u} &= J^{-1} \mathrm{div}_y (\mathrm{adj}(D_y X) \overline{\nabla_x u}) \\ &= J^{-1} \mathrm{div}_y (\mathrm{adj}(D_y X)({}^t A) \nabla_y \overline{u}), \end{aligned}$$

$$\begin{aligned} \overline{\nabla_x \mathrm{div}_x u} &= J^{-1} \mathrm{div}_y (\mathrm{adj}(D_y X) \overline{\mathrm{div}_x u}) \\ &= J^{-1} \mathrm{div}_y (\mathrm{adj}(D_y X)({}^t A) : \nabla_y \overline{u}), \\ \overline{\nabla_x P} &= J^{-1} \mathrm{div}_y (\mathrm{adj}(D_y X) \overline{P}), \end{aligned}$$

where $A = (D_y X(y))^{-1}$ and $J = \det(D_y X)$. The following is stated as a lemma in [3].

Lemma 4.6 ([3]). *Let* $z : [0, T] \times \mathbb{R}^n \to \mathbb{R}^m$ *and* $X : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *be differentiable functions with, in addition,* $X(t) : \mathbb{R}^n \to \mathbb{R}^n$ *being a* $\mathcal{C}^1$ *diffeomorphism given by* (1.9) *for all* $t \in \mathbb{R}$. *Then the following relation holds:*

$$\partial_t(J\bar{z}) = J\,\overline{(\partial_t z + \mathrm{div}_x\,(zu))},$$

where $J := \det(DX)$.

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