RIMS Kôkyûroku Bessatsu
B63 (2017), 169–194

On the criteria for the stability of unduloids

By

Yoshihito KOHSAKA*

Abstract

The stability of steady states for the surface diffusion equation will be studied. In the axisymmetric setting, steady states of surface diffusion equation are the Delaunay surfaces, which are the axisymmetric constant mean curvature surfaces. Unduloid is one of the Delaunay surfaces. In this paper, We consider a linearized stability of unduloids and describe the criteria for the stability of them.

§ 1. Introduction

Let $\Gamma_t \subset \mathbb{R}^3$ be a moving surface with respect to time t governed by the geometric evolution law

$$V = -\Delta_{\Gamma_t} H \ \text{ on } \ \Gamma_t, \tag{1.1}$$

where V is the normal velocity of Γ_t, H is the mean curvature of Γ_t, and Δ_{Γ_t} is the Laplace-Beltrami operator on Γ_t. In our sign convention, the mean curvature H for spheres with outer unit normal is negative. (1.1) is called surface diffusion equation. The surface diffusion equation (1.1) is the H^{-1}-gradient flow of the area functional of Γ_t, so that this geometric evolution equation has a variational structure that the area of the surface decreases with respect to time t whereas the volume of the region enclosed by the surface is preserved.

In this paper, we consider the following problem. For $\phi_\pm : \mathbb{R}_+ \to \mathbb{R}$, set

$$\Pi_\pm = \{(\phi_\pm(|\boldsymbol{\eta}|), \boldsymbol{\eta})^T \mid \boldsymbol{\eta} \in \mathbb{R}^2\},$$
$$\Omega = \{(x, \boldsymbol{\eta})^T \mid \phi_-(|\boldsymbol{\eta}|) \le x \le \phi_+(|\boldsymbol{\eta}|),\ \boldsymbol{\eta} \in \mathbb{R}^2\}.$$

Received August 10, 2016. Revised October 25, 2016.
2010 Mathematics Subject Classification(s): 53C44, 53A10, 74G65
Key Words: *surface diffusion equation, unduloid, stability.*
This work was supported by JSPS KAKENHI Grant Numbers 24540200, 24244012, 25247008.
*Graduate School of Maritime Sciences, Kobe University, Kobe 658-0022, Japan.
e-mail: `kohsaka@maritime.kobe-u.ac.jp`

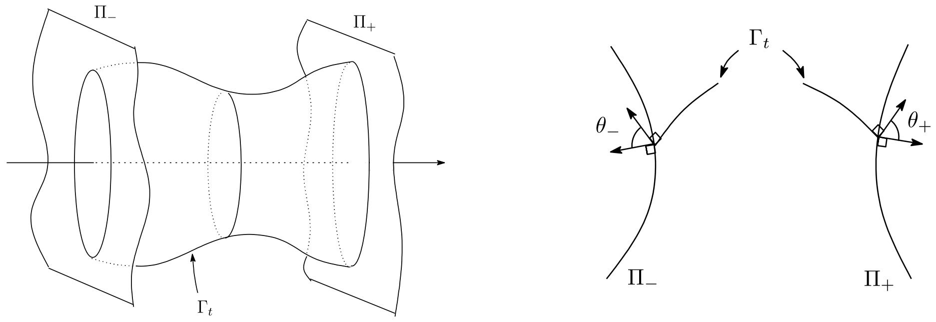


Figure 1. Setting of (1.2).

Note that $\partial\Omega = \Pi_-$ or Π_+. Let us assume that $\Gamma_t \subset \Omega$ and the motion of Γ_t is governed by

$$\begin{cases} V = -\Delta_{\Gamma_t} H \ \text{ on } \ \Gamma_t, \\ (N_{\Gamma_t}, N_{\Pi_\pm})_{\mathbb{R}^3} = \cos\theta_\pm \ \text{ on } \ \Gamma_t \cap \Pi_\pm, \\ (\nabla_{\Gamma_t} H, \nu_\pm)_{\mathbb{R}^3} = 0 \ \text{ on } \ \Gamma_t \cap \Pi_\pm, \\ \Gamma_t|_{t=0} = \Gamma_0. \end{cases} \tag{1.2}$$

Here, N_{Γ_t} and $N_{\Pi_\pm}$ are the outer unit normals to Γ_t and $\Pi_\pm$ $(= \partial\Omega)$, respectively, and $\nu_\pm$ are the outer unit co-normals to $\partial\Gamma_t$ on $\Gamma_t \cap \Pi_\pm$. The problem (1.2) are obtained as the H^{-1}-gradient flow of the capillary energy

$$\mathrm{Area}\,[\Gamma_t] + \mu_+ \mathrm{Area}\,[\Sigma_{t,+}] + \mu_- \mathrm{Area}\,[\Sigma_{t,-}],$$

where $\Sigma_{t,\pm}$ are the part of $\Pi_\pm$ with the boundary $\partial\Sigma_{t,\pm} = \Gamma_t \cap \Pi_\pm$. Note that contact angles $\theta_\pm$ are given by $\cos\theta_\pm = \mu_\pm$ (see [10]).

Let Γ_* be the steady states for (1.2) and H_* be the mean curvature of Γ_*. Then Γ_* satisfies

$$\begin{cases} \Delta_{\Gamma_*} H_* = 0 \ \text{ on } \ \Gamma_*, \\ (\nabla_{\Gamma_*} H_*, \nu_\pm)_{\mathbb{R}^3} = 0 \ \text{ on } \ \Gamma_* \cap \Pi_\pm. \end{cases}$$

This implies

$$\|\nabla_{\Gamma_*} H_*\|^2_{L^2(\Gamma_*)} = 0,$$

so that we see that the steady states of (1.2) are the constant mean curvature surfaces (CMC surfaces). In this paper, we consider the Delaunay surfaces, which are the axisymmetric CMC surfaces, as the steady states Γ_*. In Section 2 we analyze the eigenvaule problem corresponding to the linearized problem for (1.2) around the Delaunay surfaces Γ_*, and in Section 3 we focus on the unduloids, which is one of the Delaunay surfaces, and derive the criteria of the stability of them.

As regards the results on the stability of the Delaunay surfaces as the variational problem for the capillary energy, we refer to Athanassenas [2], Fel and Rubinstein [9, 16], and Vogel [17, 18, 19, 20, 21]. Concerning the results on the stability as steady states for the geometric flow, we refer to Abels, Garcke, and Müller [1], Athanassenas [3], Athanassenas and Kandanaarachchi [4], Bernoff, Bertozzi, and Witelski [5], Depner [7], and LeCrone and Simonett [14].

§ 2. Delaunay surfaces and the eigenvalue problem

Let Γ_* be a axisymmetric steady states of (1.2) and set

$$\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid s \in [0,d], \zeta \in [0,2\pi]\},$$

where s is the arc-length parameter of a generating curve $(x_*(s), y_*(s))^T$. We show the following theorem on the representation of the Delaunay surfaces with the non-zero constant mean curvature.

Theorem 2.1 ([12, 15])**.** *Let H_* be a constant satisfying $H_* \neq 0$ (assuming $H_* < 0$). Then a generating curve $(x_*(s), y_*(s))^T$ of the Delaunay surfaces with a constant mean curvature H_* is represented by*

$$x_*(s) = \int_0^s \frac{1 - B\sin(2H_*(\sigma-\tau))}{\sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}}\,d\sigma,$$
$$y_*(s) = -\frac{1}{2H_*}\sqrt{1+B^2-2B\sin(2H_*(s-\tau))},$$

where $B \geq 0$ and $\tau \in \mathbb{R}$ are constants.

Remark. In this paper, our main purpose is to analyze the stability of unduloids. Unduloid is a surface of revolution of an elliptic catenary, which is derived by tracing the focus of a rolling ellipse along a fixed line.

For $(x_*(s), y_*(s))$ given in Theorem 2.1, set

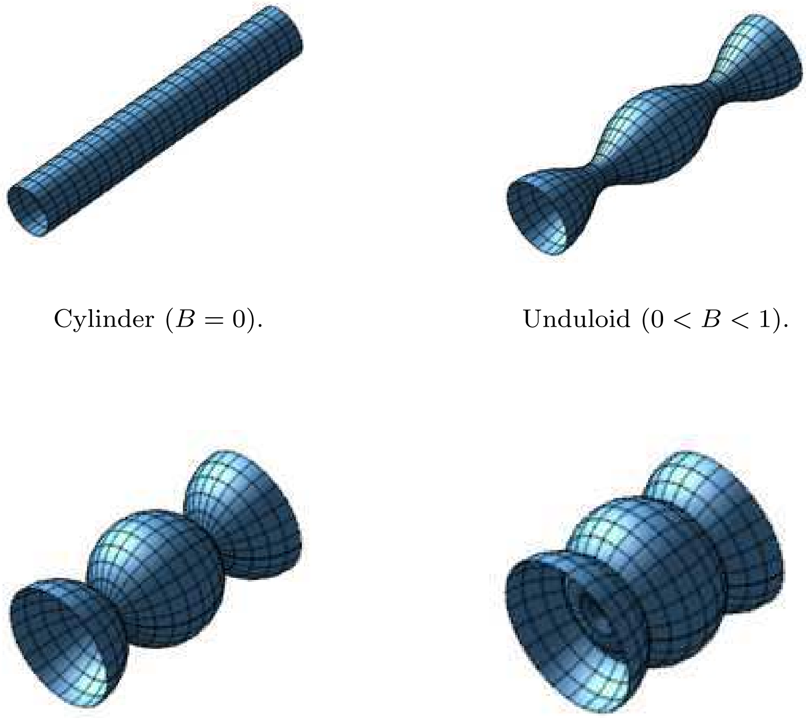
$$\Phi_*(s,\zeta) := (x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T$$

for $s \in [0,d]$ and $\zeta \in [0,2\pi]$. Then we define

$$\Psi(s,\zeta,\rho) := \Phi_*(\gamma(s,\rho),\zeta) + \rho N_*(\gamma(s,\rho),\zeta),$$

where N_* is the unit normal to Γ_* and

$$\gamma(s,\rho) := \gamma_-(\rho) + \frac{s}{d}\{\gamma_+(\rho) - \gamma_-(\rho)\}.$$



Cylinder ($B = 0$). Unduloid ($0 < B < 1$).

Sphere ($B = 1$). Nodoid ($B > 1$).

Figure 2. Delaunay surfaces ($H \neq 0$).

Here $\gamma_\pm(\rho)$ are given by

$$\gamma_-(\rho) := \min\{s \,|\, \Phi_*(s,\zeta) + \rho N_*(s,\zeta) \in \Omega\},$$
$$\gamma_+(\rho) := \max\{s \,|\, \Phi_*(s,\zeta) + \rho N_*(s,\zeta) \in \Omega\}.$$

For $v : [0,d] \times [0,T] \to [-\varepsilon,\varepsilon], (s,t) \mapsto v(s,t)$, setting

$$\Phi(s,\zeta,t) := \Psi(s,\zeta,v(s,t)),$$

an axisymmetric perturbation Γ_t from a Delaunay surface Γ_* is represented by

$$\Gamma_t = \{\Phi(s,\zeta,t) \,|\, s \in [0,d],\ \zeta \in [0,2\pi],\ t \in [0,T]\}.$$

This implies the nonlinear problem

$$\text{(2.1)} \qquad \begin{cases} V(v_t, v, \partial_s v) = -\Delta(v,\partial_s v) H(v, \partial_s v, \partial_s^2 v) \ \text{ for } \ (s,t) \in [0,d] \times [0,T], \\ (N(v,\partial_s v), N_{\Pi_\pm}(v))_{\mathbb{R}^3} = \cos\theta_\pm \ \text{ for } \ s = 0, d,\ t \in [0,T], \\ (\nabla_{\Gamma_t} H(v,\partial_s v, \partial_s^2 v), \nu_\pm(v,\partial_s v))_{\mathbb{R}^3} = 0 \ \text{ for } \ s = 0,d,\ t \in [0,T]. \end{cases}$$

Linearizing (2.1) (cf. [7]), we obtain

$$
\begin{cases}
v_t = -\dfrac{1}{2}\Delta_{\Gamma_*} L[v] \ \text{ for } \ (s,t) \in [0,d] \times [0,T], \\
\partial_s v \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) v = 0 \ \text{ for } \ s = 0, d,\ t \in [0,T], \\
\partial_s L[v] = 0 \ \text{ for } \ s = 0, d,\ t \in [0,T],
\end{cases}
\tag{2.2}
$$

where $L[v] = \Delta_{\Gamma_*} v + |A_*|^2 v$ with

$$
\Delta_{\Gamma_*} = \frac{1}{y_*}\left\{\partial_s(y_*\partial_s) + \frac{1}{y_*}\partial_\zeta^2\right\}, \quad |A_*|^2 = (-x_*'' y_*' + x_*' y_*'')^2 + \left(\frac{x_*'}{y_*}\right)^2,
$$

and

$$
\kappa_{\Pi_\pm} = \pm\frac{\ddot{\phi}_\pm(y_*)}{\{1 + (\dot{\phi}_\pm(y_*))^2\}^{3/2}}, \quad \kappa_{\Gamma_*} = -x_*'' y_*' + x_*' y_*''.
$$

Note that κ_{Π_-} and κ_{Π_+} are the curvature of $x = -\phi_-(y)$ at $y = y_*(0)$ and $x = \phi_+(y)$ at $y = y_*(d)$, respectively, and κ_{Γ_*} is the curvature of the generating curve $(x_*(s), y_*(s))^T$. Taking account of the fact that v is independent of ζ since v is an axisymmetric perturbation, we have

$$
\Delta_{\Gamma_*} v = \frac{1}{y_*}\left\{\partial_s(y_*\partial_s v)\right\}.
$$

Let us consider the eigenvalue problem

$$
\begin{cases}
-\Delta_{\Gamma_*} L[w] = \lambda w \ \text{ for } \ s \in [0,d], \\
\partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \ \text{ at } \ s = 0, d, \\
\partial_s L[w] = 0 \ \text{ at } \ s = 0, d.
\end{cases}
\tag{2.3}
$$

We say that the steady states Γ_* is linearly stable under an axisymmetric perturbation if and only if all of eigenvalues of (2.3) are negative. To analyze the eigenvalue problem (2.3), set

$$
\mathcal{E} = \left\{ w \in H^1(\Gamma_*) \,\middle|\, \int_0^d w\, y_*\, ds = 0 \right\},
$$

$$
\mathcal{X} = \left\{ w \in (H^1(\Gamma_*))^* \,\middle|\, \langle w, 1\rangle = 0 \rangle \right\},
$$

where $(H^1(\Gamma_*))^*$ is the duality space of $H^1(\Gamma_*)$ and $\langle\cdot,\cdot\rangle$ is the duality pairing $(H^1(\Gamma_*))^*$ and $H^1(\Gamma_*)$. In addition, define the symmetric bilinear form

$$
\begin{aligned}
I[w_1, w_2] = &\int_0^d \left\{\partial_s w_1 \partial_s w_2 - |A_*|^2 w_1 w_2\right\} y_*\, ds \\
&+ y_*(\kappa_{\Pi_+} \csc\theta_+ - \kappa_{\Gamma_*} \cot\theta_+) w_1 w_2\big|_{s=d} \\
&+ y_*(\kappa_{\Pi_-} \csc\theta_- - \kappa_{\Gamma_*} \cot\theta_-) w_1 w_2\big|_{s=0},
\end{aligned}
$$

and H^{-1}-inner product

$$(w_1, w_2)_{-1} = \int_0^d \partial_s u_{w_1} \partial_s u_{w_2}\, y_* ds,$$

where u_{w_i} is a weak solution of

$$\begin{cases} -\Delta_{\Gamma_*} u_{w_i} = w_i \quad \text{for} \quad s \in (0, d), \\ \partial_s u_{w_i} = 0 \quad \text{at} \quad s = 0, d \end{cases}$$

for $w_i \in \mathcal{X}$. Then we obtain the following theorem.

Theorem 2.2. *Let $\xi \in \mathcal{X}$ and $w \in \mathcal{E}$. Then the following (i) and (ii) are equivalent.*

(i) $w \in H^3(\Gamma_*)$ and w is a weak solution of

$$\begin{cases} -\Delta_{\Gamma_*} L[w] = \xi \quad \textit{for} \quad s \in [0, d], \\ \partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \quad \textit{at} \quad s = 0, d, \\ \partial_s L[w] = 0 \quad \textit{at} \quad s = 0, d. \end{cases}$$

(ii) w satisfies

$$-I[w, \psi] = (\xi, \psi)_{-1} \quad (\psi \in \mathcal{E}). \tag{2.4}$$

With regard to a proof, apply a similar argument to the proof in [7, 11].

Set

$$\mathcal{D}(\mathcal{A}) = \Big\{ w \in H^3(\Gamma_*) \,\Big|\, w \text{ satisfies}$$
$$\partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \text{ at } s = 0, d,$$
$$\text{and} \int_0^d w\, y_*\, ds = 0 \Big\}$$

and define the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ by

$$\langle \mathcal{A} w, \psi \rangle = \int_0^d \partial_s L[w]\, \partial_s \psi\, y_* ds \quad (w \in \mathcal{D}(\mathcal{A}),\ \psi \in \mathcal{E}).$$

Then, by the definition $\mathcal{A}$ and Theorem 2.2, we obtain

$$(\mathcal{A} w, \psi)_{-1} = -I[w, \psi] \quad (\psi \in \mathcal{E}).$$

This easily implies that $\mathcal{A}$ is symmetric with respect to the inner product $(\cdot, \cdot)_{-1}$. Then we have the following theorem and lemma.

Theorem 2.3. *$\mathcal{A}$ is self-adjoint with respect to $(\,\cdot\,,\,\cdot\,)_{-1}$.*

Lemma 2.4. *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be eigenvalues of $\mathcal{A}$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Then the following properties hold.*

(i) For $n \in \mathbb{N}$, $n \geq 2$,

$$\lambda_1 = -\inf_{w\in\mathcal{E}\setminus\{0\}} \frac{I[w,w]}{(w,w)_{-1}}, \qquad \lambda_n = -\sup_{\mathcal{W}\in\Sigma_{n-1}} \inf_{w\in\mathcal{W}^{\perp}\setminus\{0\}} \frac{I[w,w]}{(w,w)_{-1}}.$$

Here, Σ_n is the class of subspaces of $\mathcal{E}$ with n-dimension and $\mathcal{W}^{\perp}$ is the orthogonal subspace of $\mathcal{W}$ with respect to H^{-1}-inner product.

(ii) The eigenvalues of $\mathcal{A}$ depend continuously on $\kappa_{\Pi_\pm}\csc\theta_\pm$, $\kappa_{\Gamma_{\Gamma_*}}\cot\theta_\pm$, and d, and are monotone decreasing with respect to $\kappa_{\Pi_\pm}\csc\theta_\pm$.

Concerning proofs, see [7, 11] for Theorem 2.3 and [6, Chapter VI] with Theorem 2.2 above for Lemma 2.4.

If the maximal eigenvalue λ_1 for (2.3) is negative, the steady states Γ_* are linearly stable under an axisymmetric perturbation. In order to analyze the sign of eigenvalues for the eigenvalue problem (2.3), we first introduce the following lemma.

Lemma 2.5. *Set*

$$\Lambda_\pm := \kappa_{\Pi_\pm}\csc\theta_\pm - \kappa_{\Gamma_*}\cot\theta_\pm.$$

Then there exists $m > 0$ and $\delta > 0$ such that

$$I[w,w] > 0 \quad (w \in \mathcal{E}\setminus\{0\}),$$

provided that $\Lambda_-, \Lambda_+ > m$ and $d < \delta$.

With regard to a proof, see [13]. Roughly speaking, this lemma is proved by using a weighted Wirtinger inequality (cf. [8]) and applying a proof by contradiction.

It follows from Lemma 2.5 that there exists $m > 0$ and $\delta > 0$ such that the maximal eigenvalue λ_1 is non-positive, provided that $\kappa_{\Pi_-}, \kappa_{\Pi_+} > m$ and $d < \delta$. That is, all of eigenvalues are non-positive in such case. According to Lemma 2.4(ii), the eigenvalues depend continuously on the parameters. Thus we investigate the condition that the zero is an eigenvalue for the eigenvalue problem (2.3). To do it, we should solve

$$\Delta_{\Gamma_*} L[w] = 0 \ \text{ for } \ s \in [0,d], \tag{2.5}$$

$$\partial_s w \pm (\kappa_{\Pi_\pm}\csc\theta_\pm - \kappa_{\Gamma_*}\cot\theta_\pm) w = 0 \ \text{ at } \ s = 0, d, \tag{2.6}$$

$$\partial_s L[w] = 0 \ \text{ at } \ s = 0, d. \tag{2.7}$$

By (2.5) and (2.7), we have

$$\|\partial_s L[w]\|^2_{L^2(\Gamma_*)} = 0.$$

This implies that $L[w]$ is equal to constants. Thus we can get the fundamental solutions of the boundary value problem (2.5) and (2.7) if we solve

$$L[w] = 0, \quad L[w] = \beta\,(\neq 0). \tag{2.8}$$

Let w_1, w_2 be fundamental solutions of $L[v] = 0$ and let w_3 be a solution of $L[v] = \beta$. Then a solution of the boundary value problem (2.5) and (2.7) is represented by

$$w(s) = c_1 w_1(s) + c_2 w_2(s) + c_3 w_3(s).$$

Deriving the condition of parameters that this w is a non-trivial solution under (2.6) and

$$\int_0^d w\, y_* ds = 0,$$

it gives the condition that the zero is an eigenvalue for (2.3). That is, the zero is an eigenvalue if and only if the parameters satisfy

$$\begin{vmatrix} w_1'(0) - \Lambda_- w_1(0) & w_2'(0) - \Lambda_- w_2(0) & w_3'(0) - \Lambda_- w_3(0) \\ w_1'(d) + \Lambda_+ w_1(d) & w_2'(d) + \Lambda_+ w_2(d) & w_3'(d) + \Lambda_+ w_3(d) \\ \displaystyle\int_0^d w_1\, y_* ds & \displaystyle\int_0^d w_2\, y_* ds & \displaystyle\int_0^d w_3\, y_* ds \end{vmatrix} = 0, \tag{2.9}$$

where $\Lambda_\pm = \kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm$. Then, setting

$$\boldsymbol{w}(s) = (w_1(s), w_2(s), w_3(s))^T, \quad \boldsymbol{I}(d) = \left(\int_0^d w_1\, y_* ds, \int_0^d w_2\, y_* ds, \int_0^d w_3\, y_* ds\right)^T,$$

(2.9) is equivalent to

$$\begin{aligned} &- (\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \Lambda_- \Lambda_+ - (\boldsymbol{w}(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \Lambda_- \\ &\quad + (\boldsymbol{w}'(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \Lambda_+ + (\boldsymbol{w}'(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} = 0. \end{aligned}$$

Moreover, we can rewrite it as

$$A^w \kappa_{\Pi_-} \kappa_{\Pi_+} + B_-^w \kappa_{\Pi_-} + B_+^w \kappa_{\Pi_+} + C^w = 0, \tag{2.10}$$

where

$$\begin{cases} A^w = -(\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3}, \\ B_-^w = \big\{-(\boldsymbol{w}(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \\ \qquad\quad +(\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(d) \cot\theta_+ \big\} \sin\theta_+, \\ B_+^w = \big\{(\boldsymbol{w}'(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \\ \qquad\quad +(\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(0) \cot\theta_- \big\} \sin\theta_-, \\ C^w = \big\{(\boldsymbol{w}'(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \\ \qquad\quad -(\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(d) \kappa_{\Gamma_*}(0) \cot\theta_+ \cot\theta_- \big\} \sin\theta_+ \sin\theta_-. \end{cases} \tag{2.11}$$

Then we obtain the following three representations of (2.10).

Case I: $A^w \neq 0$ and $B_-^w B_+^w - A^w C^w \neq 0$.

$$(2.10) \quad \Leftrightarrow \quad \kappa_{\Pi_+} = -\frac{B_-^w}{A^w} + \frac{\dfrac{B_-^w B_+^w - A^w C^w}{(A^w)^2}}{\kappa_{\Pi_-} - \left(-\dfrac{B_+^w}{A^w}\right)}.$$

Case II : $A^w \neq 0$ and $B_-^w B_+^w - A^w C^w = 0$.

$$(2.10) \quad \Leftrightarrow \quad \left\{\kappa_{\Pi_-} - \left(-\frac{B_+^w}{A^w}\right)\right\}\left\{\kappa_{\Pi_+} - \left(-\frac{B_-^w}{A^w}\right)\right\} = 0.$$

Case III : $A^w = 0$.

$$(2.10) \quad \Leftrightarrow \quad B_-^w \kappa_{\Pi_-} + B_+^w \kappa_{\Pi_+} + C^w = 0.$$

The details of the condition (2.10) depend on the configuration of Γ_*. In Section 3, we focus on unduloids as the steady states Γ_*.

§ 3. Stability analysis for unduloids

§ 3.1. Zero-eigenvalue condition for unduloids

In this subsection, we derive the precise form of the condition (2.10) for the case that Γ_* is a unduloid.

Let us consider the case that Γ_* is a unduloid with a constant mean curvature H_* (<0). Remember that for $\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid s \in [0,d],\ \zeta \in [0,2\pi]\}$ the generating curves $(x_*(s), y_*(s))^T$ of unduloids are given by

$$x_*(s) = \int_0^s \frac{1 - B\sin(2H_*(s-\tau))}{\sqrt{1 + B^2 - 2B\sin(2H_*(s-\tau))}}\, d\sigma,$$
$$y_*(s) = -\frac{1}{2H_*}\sqrt{1 + B^2 - 2B\sin(2H_*(s-\tau))},$$

where $B \in (0,1)$. Then $|A_*|^2$ in the operator $L[w]$ and κ_{Γ_*} in the boundary condition (2.6) are given by

$$|A_*|^2 = \frac{4H_*^2\{B^2(B - \sin(2H_*(s-\tau))^2 + (1 - B\sin(2H_*(s-\tau))^2\}}{(1 + B^2 - 2B\sin(2H_*(s-\tau)))^2},$$
$$\kappa_{\Gamma_*} = \frac{2BH_*(B - \sin(2H_*(s-\tau))}{1 + B^2 - 2B\sin(2H_*(s-\tau))},$$

so that $L[w] = 0$ and $L[w] = 1$ (we choose 1 as β in (2.8)) are linear second order ordinary differential equation with variable coefficients. Solving them, we obtain

$$\begin{cases} w_1(s) = \dfrac{\cos(2H_*(s-\tau))}{\sqrt{1+B^2-2B\sin(2H_*(s-\tau))}}, \\ w_2(s) = \sin(2H_*(s-\tau)) + 2H_* \left\{ \dfrac{1+B^2}{2} I_1(s) - \dfrac{1}{2} I_2(s) \right\}, \\ w_3(s) = \dfrac{1}{4H_*^2} + \dfrac{B}{2H_*} I_1(s)\, w_1(s), \end{cases} \tag{3.1}$$

where

$$I_1(s) = I_1(s; H_*, B, \tau) := \int_0^s \frac{1}{\sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}}\, d\sigma,$$
$$I_2(s) = I_2(s; H_*, B, \tau) := \int_0^s \sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}\, d\sigma.$$

Set

$$H_*^+ = -H_* \,(>0), \quad \alpha = H_*^+\tau + \frac{\pi}{4}$$

and let $\alpha \in (-\pi/2, \pi/2]$. For $-\pi/2 + m\pi < H_*^+ s - \alpha < -\pi/2 + (m+1)\pi$ ($m \in \mathbb{N} \cup \{0\}$), $I_1(s; -H_*^+, B, \tau)$ and $I_2(s; -H_*^+, B, \tau)$ are given by

$$\begin{aligned} &I_1(s; -H_*^+, B, \tau) \\ &= \frac{1}{H_*^+(1+B)} \big\{ 2mK(k) + (-1)^m F(\sin(H_*^+ s - \alpha); k) - F(\sin(-\alpha); k) \big\}, \\ &I_2(s; -H_*^+, B, \tau) \\ &= \frac{1+B}{H_*^+} \big\{ 2mE(k) + (-1)^m E(\sin(H_*^+ s - \alpha); k) - E(\sin(-\alpha); k) \big\}, \end{aligned}$$

where

$$k = \frac{2\sqrt{B}}{1+B}. \tag{3.2}$$

Note that $k \in (0,1)$ because of

$$\begin{aligned} &\frac{dk}{dB} = \frac{2(1-B)}{\sqrt{B}(1+B)^2} > 0 \quad (B \in (0,1)), \\ &k|_{B=0} = 0, \quad k|_{B=1} = 1. \end{aligned} \tag{3.3}$$

Also, $K(k)$ and $E(k)$ are the complete elliptic integrals of the 1st and 2nd kind, and $F(s;k)$ and $E(s;k)$ are the incomplete elliptic integrals of the 1st and 2nd kind. In this paper, the elliptic integrals are given by

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}}\,d\xi, \quad E(k) = \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\,d\xi,$$
$$F(\eta;k) = \int_0^\eta \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}}\,d\xi, \quad E(\eta;k) = \int_0^\eta \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\,d\xi.$$

Then we are led to the following lemma.

Lemma 3.1. *The zero is an eigenvalue of the eigenvalue problem* (2.3) *if and only if parameters* $\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d, \tau, \theta_+, \theta_-$ *satisfy*

$$\begin{aligned}(3.4)\quad & A^u(H_*^+, B, d, \tau)\kappa_{\Pi_-}\kappa_{\Pi_+} + B_-^u(H_*^+, B, d, \tau, \theta_+)\kappa_{\Pi_-} + B_+^u(H_*^+, B, d, \tau, \theta_-)\kappa_{\Pi_+} \\ & + C^u(H_*^+, B, d, \tau, \theta_+, \theta_-) = 0.\end{aligned}$$

Here $A^u, B_\pm^u, C^u$ *denote the coefficents* (2.11) *for the case that* w_i $(i = 1, 2, 3)$ *are* (3.1).

Proof. Substituting (3.1) for (2.10) and calculating it, we get (3.4). □

The precise forms of A^u, $B_\pm^u$, and C^u are obtained by Maple 17. Here, we show only the form of A^u:

$$\begin{aligned}
& A^u(H_*^+, B, d, \tau) \\
& = \frac{1}{8(H_*^+)^3 PQ}\Big[(H_*^+)^2(1-B^2)^2 I_1^2 \cos(2H_*^+\tau)\cos(2H_*^+(d-\tau)) \\
& \quad - 4(H_*^+)^2(1+B^2)I_1 I_2 \cos(2H_*^+\tau)\cos(2H_*^+(d-\tau)) \\
& \quad + 3(H_*^+)^2 I_2^2 \cos(2H_*^+\tau)\cos(2H_*^+(d-\tau)) \\
& \quad + 2H_*^+(1+B^2)I_1\big\{P\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau)) + Q\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau))\big\} \\
& \quad - 4H_*^+ B I_1\big\{P\cos(2H_*^+(d-\tau)) - Q\cos(2H_*^+\tau)\big\} \\
& \quad - 4H_*^+ I_2\big\{P\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau)) + Q\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau))\big\} \\
& \quad + 2PQ\big\{1 + \sin(2H_*^+\tau)\sin(2H_*^+(d-\tau))\big\} \\
& \quad - (P^2+Q^2)\cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\Big],
\end{aligned}$$

where

$$P(H_*^+, B, \tau) = \sqrt{1 + B^2 - 2B\sin(2H_*^+\tau)},$$
$$Q(H_*^+, B, d, \tau) = \sqrt{1 + B^2 + 2B\sin(2H_*^+(d-\tau))}.$$

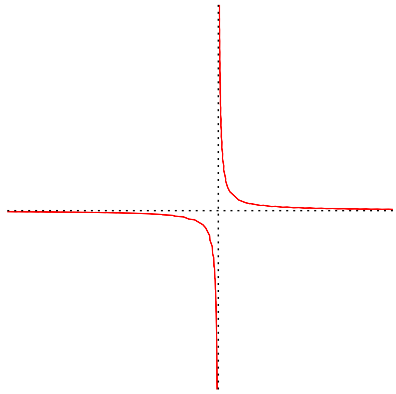


Figure 3. The hyperbola given by $\kappa_{\Pi_-}\kappa_{\Pi_+} = c\,(c > 0)$.

Then, by the help with Maple 17, we obtain

$$\begin{aligned}
&B^u_-(H^+_*, B, d, \tau, \theta_+)B^u_+(H^+_*, B, d, \tau, \theta_-) - A^u(H^+_*, B, d, \tau)C^u(H^+_*, B, d, \tau, \theta_+, \theta_-)\\
&= \frac{1}{16(H^+_*)^4 PQ}\Big[H^+_*\big\{(1 + B^2)(1 + \sin(2H^+_*\tau)\sin(2H^+_*(d - \tau))) - (P^2 + Q^2)\big\}I_1\\
&\qquad + H^+_*\big(3 - \sin(2H^+_*(d - \tau))\sin(2H^+_*\tau)\big)I_2\\
&\qquad - P\cos(2H^+_*\tau)\sin(2H^+_*(d - \tau)) - Q\sin(2H^+_*\tau)\cos(2H^+_*(d - \tau))\Big]^2 \geq 0.
\end{aligned}$$

Thus, when $A^u(H^+_*, B, d, \tau) \neq 0$, the zero-eigenvalue condition (3.4) becomes the hyperbola which is derived from $\kappa_{\Pi_-}\kappa_{\Pi_+} = c\,(c > 0)$ by the translation (see Fig. 3).

§ 3.2. Criteria of stability for unduloids with $\tau = \pi/(4H^+_*)$ and $\theta_\pm = \pi/2$

In this subsection, our goal is to obtain criteria of stability for unduloids when the translating parameter τ and the contact angles $\theta_\pm$ between Γ_* and $\Pi_\pm$ are given by

$$\tau = \frac{\pi}{4H^+_*}, \quad \theta_\pm = \frac{\pi}{2}$$

for each $H^+_* > 0$. In [13], we gave criteria for $H^+_* = 1$, $\tau = \pi/4$, and $\theta_\pm = \pi/2$. In this paper, we generalize them for $H^+_* > 0$.

Hereafter, $\hat{A}^u, \hat{B}^u_\pm, \hat{C}^u$ denote $A^u, B^u_\pm, C^u$ with $\tau = \pi/(4H^+_*)$ and $\theta_\pm = \pi/2$, respectively. Also, for $k \in (0, 1)$ and $p \in \mathbb{N}$, set

$$\begin{aligned}
&\mathcal{G}_{1,p}(k) := (1 - k^2)K(k) - pE(k), \quad \mathcal{G}_{2,p}(k) := K(k) - pE(k),\\
&\mathcal{G}_3(k) := (1 - k^2)\{K(k)\}^2 - 2(2 - k^2)K(k)E(k) + 3\{E(k)\}^2.
\end{aligned}$$

First, let us consider the zero points of $\hat{A}^u(H^+_*, B, d)$. Since $H^+_* d = \pi/2 + m\pi$ ($m \in \mathbb{N} \cup \{0\}$) does not satisfy $\hat{A}^u(H^+_*, B, d) = 0$, we assume $H^+_* d \neq \pi/2 + m\pi$. Then we have

$$\begin{aligned}&\hat{A}^u(H_*^+, B, d)\\&= \frac{\sin(H_*^+ d)\cos(H_*^+ d)}{2(H_*^+)^3 \hat{Q}(H_*^+, B, d)} \Big\{ H_*^+(1-B)^2 \hat{I}_1(d; H_*^+, B) - 2H_*^+ \hat{I}_2(d; H_*^+, B)\\&\qquad + \hat{Q}(H_*^+, B, d)\tan(H_*^+ d)\Big\},\end{aligned}$$

where $\hat{Q}(H_*^+, B, d) := Q(H_*^+, B, d, \pi/(4H_*^+)) = \sqrt{1 + B^2 - 2B\cos(2H_*^+ d)}$ and

$$\hat{I}_1(d; H_*^+, B) := I_1\Big(d; -H_*^+, B, \frac{\pi}{4H_*^+}\Big) = \int_0^d \frac{1}{\sqrt{1 + B^2 - 2B\cos(2H_*^+ \sigma)}}\, d\sigma,$$

$$\hat{I}_2(d; H_*^+, B) := I_2\Big(d; -H_*^+, B, \frac{\pi}{4H_*^+}\Big) = \int_0^d \sqrt{1 + B^2 - 2B\cos(2H_*^+ \sigma)}\, d\sigma.$$

We easily see that $H_*^+ d = m\pi$ ($m \in \mathbb{N}$) are the zero points of $\hat{A}^u(H_*^+, B, d)$ for $B \in (0, 1)$. Set

$$\begin{aligned}f(d; H_*^+, B) &:= H_*^+(1-B)^2 \hat{I}_1(d; H_*^+, B) - 2H_*^+ \hat{I}_2(d; H_*^+, B)\\&\quad + \hat{Q}(H_*^+, B, d)\tan(H_*^+ d).\end{aligned}$$

Applying a similar calculation to [13], we get

$$\partial_d f(d; H^+, B) = H_*^+ \sqrt{(1+B)^2 - 4B\cos^2(H_*^+ d)}\, \tan^2(H_*^+ d) > 0$$

for $H_*^+ d \in (m\pi, (m+1)\pi)$ and $H_*^+ d \neq m\pi + \pi/2$ ($m \in \mathbb{N} \cup \{0\}$). Thus $f(d; H_*^+, B)$ is strictly monotone increasing in d. Moreover, it follows that for $m \in \mathbb{N}$

$$f\Big(\frac{m\pi}{H_*^+}; H_*^+, B\Big) = m(1+B)\mathcal{G}_{1,2}(k),$$

where k is given by (3.2). By virtue of Lemma 4.1 with $p = 2$, we obtain $\mathcal{G}_{1,2}(k) < 0$, so that

$$f\Big(\frac{m\pi}{H_*^+}; H_*^+, B\Big) < 0.$$

Also we have

$$\lim_{H_*^+ d \to m\pi + \frac{\pi}{2} \mp 0} f(d; H_*^+, B) = \pm\infty$$

for $m \in \mathbb{N} \cup \{0\}$. Consequently, it follows that for each $H_*^+ > 0$ and $B \in (0, 1)$ there exists a unique $d_m = d_m(H_*^+, B)$ with $H_*^+ d_m \in \big(m\pi, m\pi + \pi/2\big)$ ($m \in \mathbb{N}$) such that $f(d_m; H_*^+, B) = 0$. Thus we see that $\hat{A}^u(H_*^+, B, d) = 0$ at $d = m\pi/H_*^+$ and $d = d_m(H_*^+, B)$ for each $H_*^+ > 0$ and $B \in (0, 1)$. Set

$$q_m(H_*^+, B) := \begin{cases} \dfrac{\ell\pi}{H_*^+} & (m = 2\ell - 1),\\ d_\ell(H_*^+, B) & (m = 2\ell) \end{cases} \tag{3.5}$$

for $\ell \in \mathbb{N}$. Note that $\hat{A}^u(H_*^+, B, d) > 0$ for $d \in (0, q_1)$.

Second, let us consider the asymptotic behavior of $-\hat{B}_-^u/\hat{A}^u$ and $-\hat{B}_+^u/\hat{A}^u$ around the zero points of $\hat{A}^u(H_*^+, B, d)$. (We remark that in [13] the figures of $-\hat{B}_-^u/\hat{A}^u$ and $-\hat{B}_+^u/\hat{A}^u$ with $H_*^+ = 1$ and $B = 0.6$ by Maple 17 are only shown without the proof.) It follows from the above argument that

$$\hat{A}^u(H_*^+, B, d) \begin{cases} < 0 \; (d \in (q_{2\ell-1}, q_{2\ell})), \\ > 0 \; (d \in (q_{2\ell}, q_{2\ell+1})) \end{cases} \tag{3.6}$$

for $\ell \in \mathbb{N}$. Since

$$\hat{B}_-^u(H_*^+, B, q_{2\ell-1}) = \frac{\ell}{(H_*^+)^2\sqrt{1-k^2}}\, \mathcal{G}_{1,2}(k) < 0,$$
$$\hat{B}_+^u(H_*^+, B, q_{2\ell-1}) = \frac{\ell}{(H_*^+)^2\sqrt{1-k^2}}\, \mathcal{G}_{1,2}(k) < 0,$$

where k is given by (3.2), we are led to

$$\lim_{d \to q_{2\ell-1} \mp 0} \left(-\frac{\hat{B}_-^u}{\hat{A}^u} \right) = \pm\infty, \quad \lim_{d \to q_{2\ell-1} \mp 0} \left(-\frac{\hat{B}_+^u}{\hat{A}^u} \right) = \pm\infty. \tag{3.7}$$

Also, taking account of $H_*^+ q_{2\ell} = H_*^+ d_\ell \in \big(\ell\pi, \ell\pi + \pi/2\big)$ $(\ell \in \mathbb{N})$, we obtain

$$\hat{B}_-^u(H_*^+, B, q_{2\ell}) = \frac{\tan(H_*^+ q_{2\ell}) \sin^2(H_*^+ q_{2\ell})}{2(H_*^+)^2} > 0,$$
$$\hat{B}_+^u(H_*^+, B, q_{2\ell}) = \frac{\big\{\hat{I}_2(q_{2\ell}; H_*^+, B)\big\}^2 \sin(H_*^+ q_{2\ell}) \cos(H_*^+ q_{2\ell})}{2(1-B)\hat{Q}(H_*^+, B, q_{2\ell})} > 0.$$

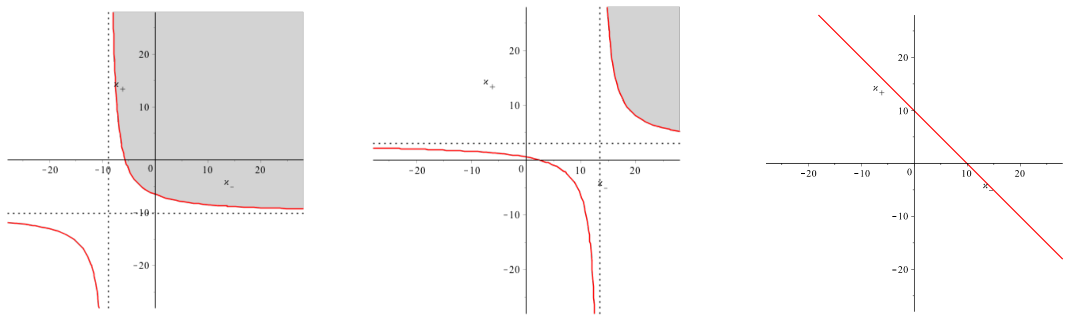
These imply

$$\lim_{d \to q_{2\ell} \mp 0} \left(-\frac{\hat{B}_-^u}{\hat{A}^u} \right) = \pm\infty, \quad \lim_{d \to q_{2\ell} \mp 0} \left(-\frac{\hat{B}_+^u}{\hat{A}^u} \right) = \pm\infty. \tag{3.8}$$

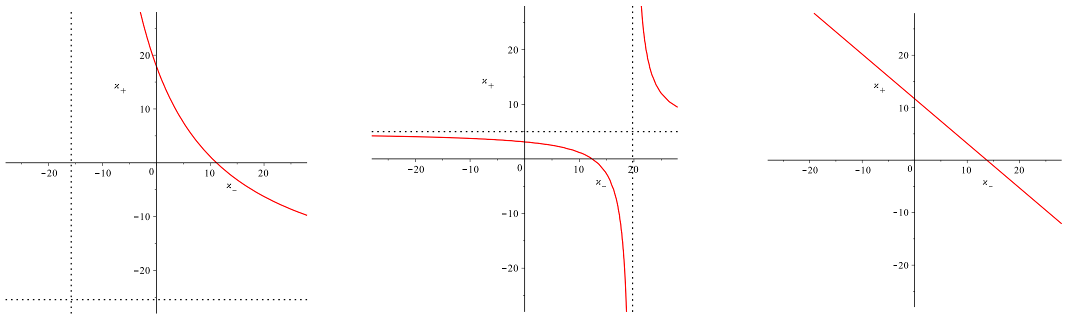
Finally, applying a similar calculation to [13], we see that there are no $d > 0$ such that $\hat{B}_-^u \hat{B}_+^u - \hat{A}^u \hat{C}^u = 0$ for each $H_*^+ > 0$ and $B \in (0, 1)$. Thus, in the case that $\tau = \pi/(4H_*^+)$ and $\theta_\pm = \pi/2$, only Case I and Case III appear, so that for each $d > 0$ the zero-eigenvalue condition (3.4) with $\tau = \pi/(4H_*^+)$ and $\theta_\pm = \pi/2$ draws the hyperbolas or the straight lines, which are not parallel to $\kappa_{\Pi_\pm}$-axes, in the $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$-coordinate plane (see Fig. 4). Combining these hyperbolas and straight lines, (3.4) with $\tau = \pi/(4H_*^+)$ and $\theta_\pm = \pi/2$ forms the surface of Fig. 5 for each $H_*^+ > 0$ and $B \in (0, 1)$ in the $(\kappa_{\Pi_-}, d, \kappa_{\Pi_+})$-coordinate space. Consequently, the following theorem is derived.

Theorem 3.2. *Set*

$$\begin{aligned} &\hat{D}(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d) \\ &:= \hat{A}^u(H_*^+, B, d)\kappa_{\Pi_-}\kappa_{\Pi_+} + \hat{B}_-^u(H_*^+, B, d)\kappa_{\Pi_-} + \hat{B}_+^u(H_*^+, B, d)\kappa_{\Pi_+} + \hat{C}^u(H_*^+, B, d). \end{aligned}$$



$0 < d < q_1$. $\qquad\qquad$ $d = q_1$.



$q_1 < d < q_2$. $\qquad\qquad$ $d = q_2$.

Figure 4. The hyperbolas and the straight lines given by (3.4) with $\tau = \pi/(4H_*^+)$ and $\theta_\pm = \pi/2$ in the $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$-coordinate plane. In these figures, $\kappa_\pm := \kappa_{\Pi_\pm}$.

If the parameters $\kappa_{\Pi_\pm}, H_*^+, B, d$ satisfy

$$\hat{D}(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d) > 0, \quad \kappa_{\Pi_-} > -\frac{\hat{B}_+^u(H_*^+, B, d)}{\hat{A}^u(H_*^+, B, d)}, \quad \text{and} \quad d < q_1, \tag{3.9}$$

then unduloids are linearly stable under an axisymmetric perturbation.

Proof. By Lemma 2.5, we see that there exist $m > 0$ and $\delta > 0$ such that $\lambda_1 \leq 0$ provided that $\kappa_{\Pi_-}, \kappa_{\Pi_+} > m$ and $d < \delta$. On the other hand, By Lemma 3.1, zero is eigenvalue if and only if the parameters $\kappa_{\Pi_\pm}, H_*^+, B, d$ satisfy

$$\hat{D}(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d) = 0. \tag{3.10}$$

For each $H_*^+ > 0$ and $B \in (0, 1)$, set

$$\mathcal{R}_S^3(H_*^+, B) := \Big\{ (\kappa_{\Pi_-}, d, \kappa_{\Pi_+}) \,\Big|\, \hat{D}(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d) > 0,$$
$$\kappa_{\Pi_-} > -\frac{\hat{B}_+^u(H_*^+, B, d)}{\hat{A}^u(H_*^+, B, d)}, \ d < q_1 \Big\},$$

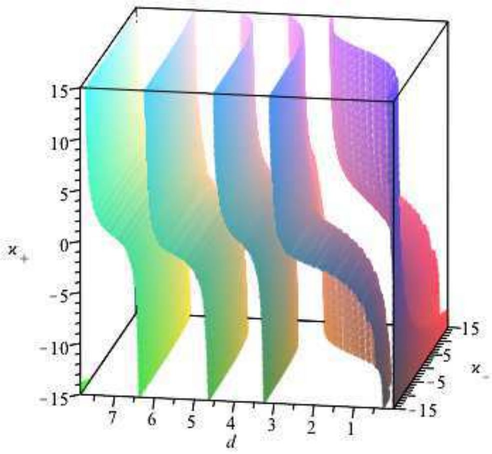


Figure 5. The surface given by (3.4) with $\tau = \pi/(4H_*^+)$ and $\theta_\pm = \pi/2$ in the $(\kappa_{\Pi_-}, d, \kappa_{\Pi_+})$-coordinate space. In this figure, $\kappa_\pm := \kappa_{\Pi_\pm}$.

$$\begin{aligned}\mathcal{R}_U^3(H_*^+, B) := &\{(\kappa_{\Pi_-}, d, \kappa_{\Pi_+}) \,|\, \hat{D}(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d) < 0,\ d < q_1\} \\ &\cup \left\{(\kappa_{\Pi_-}, d, \kappa_{\Pi_+}) \,\middle|\, \hat{D}(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d) > 0, \right. \\ &\qquad\qquad \left. \kappa_{\Pi_-} < -\frac{\hat{B}_+^u(H_*^+, B, d)}{\hat{A}^u(H_*^+, B, d)},\ d < q_1\right\}, \\ &\cup \{(\kappa_{\Pi_-}, d, \kappa_{\Pi_+}) \,|\, d \geq q_1\}.\end{aligned}$$

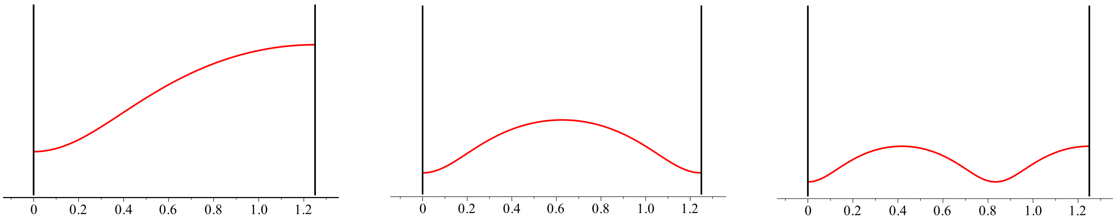
Note that $\overline{\mathcal{R}_U^3(H_*^+, B)} = \{\mathcal{R}_S^3(H_*^+, B)\}^c$. Then, for each $H_*^+ > 0$ and $B \in (0,1)$ the surface (3.10) with $\kappa_{\Pi_-} > -\hat{B}_+^u(H_*^+, B, d)/\hat{A}^u(H_*^+, B, d)$ and $d < q_1$ divide the $(\kappa_{\Pi_-}, d, \kappa_{\Pi_+})$-coordinate space into two connected sets $\mathcal{R}_S^3(H_*^+, B)$ and $\mathcal{R}_U^3(H_*^+, B)$. Thus it follows from the continuity with respect to the parameters and the monotonicity with respect to $\kappa_{\Pi_\pm}$ that $\lambda_1 < 0$ if $(\kappa_{\Pi_-}, d, \kappa_{\Pi_+}) \in \mathcal{R}_S^3(H_*^+, B)$ for each $H_*^+ > 0$ and $B \in (0,1)$. That is, in this case, unduloids are linearly stable under an axisymmetric perturbation. □

Remark. We remark that in [13] the statements of Theorem 5.1 for cylinders and Theorem 5.2, 5.3 for unduloids are something strange and there are lack of conditions. For example, the correct statement corresponding to Theorem 5.2 in [13] is Theorem 3.2 with $H^+ = 1$ and $B = 0.6$ mentioned above.

Theorem 3.3. *If $d \geq q_1$, then there are no pairs of $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$ such that unduloids are stable.*

Proof. For each $H_*^+ > 0$, $B \in (0,1)$, and $d > 0$, set

$$\mathcal{R}_S^2(H_*^+, B, d)$$



$d = \dfrac{\pi}{2H_*^+}$ and $\tau = \dfrac{\pi}{4H_*^+}$. $\quad$ $d = \dfrac{\pi}{H_*^+}$ and $\tau = \dfrac{\pi}{4H_*^+}$. $\quad$ $d = \dfrac{3\pi}{2H_*^+}$ and $\tau = \dfrac{\pi}{4H_*^+}$.

Figure 6. Unduloids between parallel planes.

$$:= \left\{ (\kappa_{\Pi_-}, \kappa_{\Pi_+}) \,\middle|\, \hat{D}(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B, d) > 0,\ \kappa_{\Pi_-} > -\frac{\hat{B}_+^u(H_*^+, B, d)}{\hat{A}^u(H_*^+, B, d)} \right\}.$$

(In Fig. 4, the set $\mathcal{R}_S^2(H_*^+, B, d)$ is the region painted by gray.) If $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) \in \mathcal{R}_S^2(H_*^+, B, d)$ for each $H_*^+ > 0$, $B \in (0,1)$, and $d \in (0, q_1)$, unduloids are linearly stable. By virtue of (3.7) and (3.8), the set $\mathcal{R}_S^2(H_*^+, B, d)$ vanishes from the $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$-coordinate plane when $d \to q_1 - 0$ and does not appear for $d \geq q_1$ (see Fig. 4). Thus, for $d \geq q_1$, any pairs of (K_-, K_+) does not imply that unduloids are stable. □

§ 3.3. Stabilization of unduloids with $\theta_\pm = \pi/2$

According to Athanassenas [2] and Vogel [17], unduloids between the the parallel planes are unstable. In this subsection, we make sure of their result by using our criteria and give some conditions of parameters which stabilize unduloids.

Let $\Pi_\pm$ be the parallel planes. When $H_*^+ d = m\pi/2\,(m \in \mathbb{N})$, we can put unduloids between $\Pi_\pm$ with the angles $\theta_\pm = \pi/2$ (see Fig. 3.3).

It follows from Theorem 3.3 and $q_1 = \pi/H_*^+$ that if $H_*^+ d \geq \pi$, there are no pairs of $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$ such that unduloids are stable. This means that if $H_*^+ d \geq \pi$, unduloids between parallel planes $\Pi_\pm$ are unstable. Let us consider the case that $H_*^+ d = \pi/2$. Then we obtain the fact that the zero is an eigenvalue if and only if $\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B$ satisfy

$$\hat{A}^0(H_*^+)\kappa_{\Pi_-}\kappa_{\Pi_+} + \hat{B}_-^0(H_*^+, B)\kappa_{\Pi_-} + \hat{B}_+^0(H_*^+, B)\kappa_{\Pi_+} + \hat{C}^0(H_*^+, B) = 0 \tag{3.11}$$

with

$$\hat{A}^0(H_*^+) := \frac{1}{(H_*^+)^2}, \quad \hat{B}_-^0(H_*^+, B) := -\frac{1}{H_*^+}\,\mathcal{G}_{1,2}(k),$$
$$\hat{B}_+^0(H_*^+, B) := -\frac{1}{H_*^+\sqrt{1-k^2}}\,\mathcal{G}_{2,2}(k), \quad \hat{C}^0(H_*^+, B) := \frac{1}{1-k^2}\,\mathcal{G}_3(k),$$

where k is given by (3.2). Since $\pi/2 < \pi = H_*^+ q_1$, (3.11) gives a criterion of the stability for unduloids which exist between the parallel planes $\Pi_\pm$ and intersect $\Pi_\pm$ with the angle

$\pi/2$. Set

$$\begin{aligned}&\hat{D}^0(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B)\\&:= \hat{A}^0(H_*^+)\kappa_{\Pi_-}\kappa_{\Pi_+} + \hat{B}_-^0(H_*^+, B)\kappa_{\Pi_-} + \hat{B}_+^0(H_*^+, B)\kappa_{\Pi_+} + \hat{C}^0(H_*^+, B).\end{aligned}$$

According to Lemma 4.3, we see that at $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0,0)$

$$\hat{D}^0(0,0,H_*^+,B) = \hat{C}^0(H_*^+,B) < 0$$

for each $H_*^+ > 0$ and $B \in (0,1)$. This means that $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0,0)$ is included in the region of parameters which derives instability of unduloids .

Remark. When $B = 0$, the steady state Γ_* is cylinder. For $B = 0$, we have $y_*(s) = 1/(2H_*^+)$ which gives a radius of a cylinder. Note that $H_*^+ d = \pi/2$ means $d/(1/2H_*^+) = \pi$. Let r be radii of cylinders. According to [13], if $d/r < \pi$, $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0,0)$ is included in the region of parameters which derives stability of cylinders, whereas if $d/r > \pi$, $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0,0)$ is included in that which derives instability of cylinders (also see Athanassenas [2] and Vogel [17]). The restriction $d/r = \pi$ implies that $(\kappa_{\Pi_-}, \kappa_{\Pi_+}) = (0,0)$ is a neutral (0-eigenvalue) state for cylinders with the mean curvature $1/(2r)$ (see Fig. 7 (a) below).

By changing the parameters $(\kappa_{\Pi_-}, \kappa_{\Pi_+}, H_*^+, B)$, let us stabilize unduloids under the conditions $H_*^+ d = \pi/2$ and $x_*(d)\,(= x_*(d) - x_*(0)) = 1.25$. To do it, we analyze (3.11) in details. Since $\hat{A}^0(H_*^+) > 0$ and it follows from Lemma 4.3 that

$$\hat{B}_-^0\hat{B}_+^0 - \hat{A}^0\hat{C}^0 = \frac{1}{\sqrt{1-k^2}}\left[\left(1 - \frac{1}{\sqrt{1-k^2}}\right)\mathcal{G}_3(k) + \{E(k)\}^2\right] > 0,$$

where k is given by (3.2), the condtion (3.11) implies a hyperbola

$$\kappa_{\Pi_+} = -\frac{\hat{B}_-^0}{\hat{A}^0} + \frac{\dfrac{\hat{B}_-^0\hat{B}_+^0 - \hat{A}^0\hat{C}^0}{(\hat{A}^0)^2}}{\kappa_{\Pi_-} - \left(-\dfrac{\hat{B}_+^0}{\hat{A}^0}\right)}.$$

Then the axes of a hyperbola are

$$\kappa_{\Pi_-} = -\frac{\hat{B}_+^0(H_*^+,B)}{\hat{A}^0(H_*^+)} = \frac{H_*^+\mathcal{G}_{2,2}(k)}{\sqrt{1-k^2}}, \quad \kappa_{\Pi_+} = -\frac{\hat{B}_-^0(H_*^+,B)}{\hat{A}^0(H_*^+)} = H_*^+\mathcal{G}_{1,2}(k),$$

where k is given by (3.2). Differentiating $-\hat{B}_+^0/\hat{A}^0$ and $-\hat{B}_-^0/\hat{A}^0$ with respect to k and using (3.3), Lemma 4.1(i), and Lemma 4.2(i), we see that for $H_*^+ > 0$ and $B \in (0,1)$

$$
\begin{aligned}
\frac{\partial}{\partial B}\left(-\frac{\hat{B}_+^0(H_*^+,B)}{\hat{A}^0(H_*^+)}\right) &= H_*^+\frac{dk}{dB}\left\{\frac{k}{(1-k^2)^{\frac{3}{2}}}\,\mathcal{G}_{2,2}(k)+\frac{1}{\sqrt{1-k^2}}\frac{\partial}{\partial k}\mathcal{G}_{2,2}(k)\right\}\\
&= H_*^+\frac{dk}{dB}\cdot\frac{1}{k(1-k^2)^{\frac{3}{2}}}\,\mathcal{G}_{2,1}(k)>0,\\
\frac{\partial}{\partial B}\left(-\frac{\hat{B}_-^0(H_*^+,B)}{\hat{A}^0(H_*^+)}\right) &= H_*^+\frac{dk}{dB}\frac{\partial}{\partial k}\mathcal{G}_{1,2}(k)<0.
\end{aligned}
$$

Thus $-\hat{B}_+^0/\hat{A}^0$ is monotone increasing in B and $-\hat{B}_-^0/\hat{A}^0$ is monotone decreasing in B. In addition, it follows from Lemma 4.2(ii), (3.2), and (3.3) that there exists a unique $B_c \in (0,1)$ such that

$$
-\frac{\hat{B}_+^0(H_*^+,B)}{\hat{A}^0(H_*^+)}\begin{cases} <0\,(B\in(0,B_c)),\\ =0\,(B=B_c),\\ >0\,(B\in(B_c,1)),\end{cases}
$$

and from Lemma 4.1(ii) that

$$
-\frac{\hat{B}_-^0(H_*^+,B)}{\hat{A}^0(H_*^+)}<0 \quad (B\in(0,1)).
$$

Note that B_c are derived from k_c which satisfies $K(k_c)-2E(k_c)=0$. By using Maple 17 we see $k_c \approx 0.9089$ which gives $B_c \approx 0.4114$. Also, κ_{Π_-}-intercept and κ_{Π_+}-intercept of hyperbola are

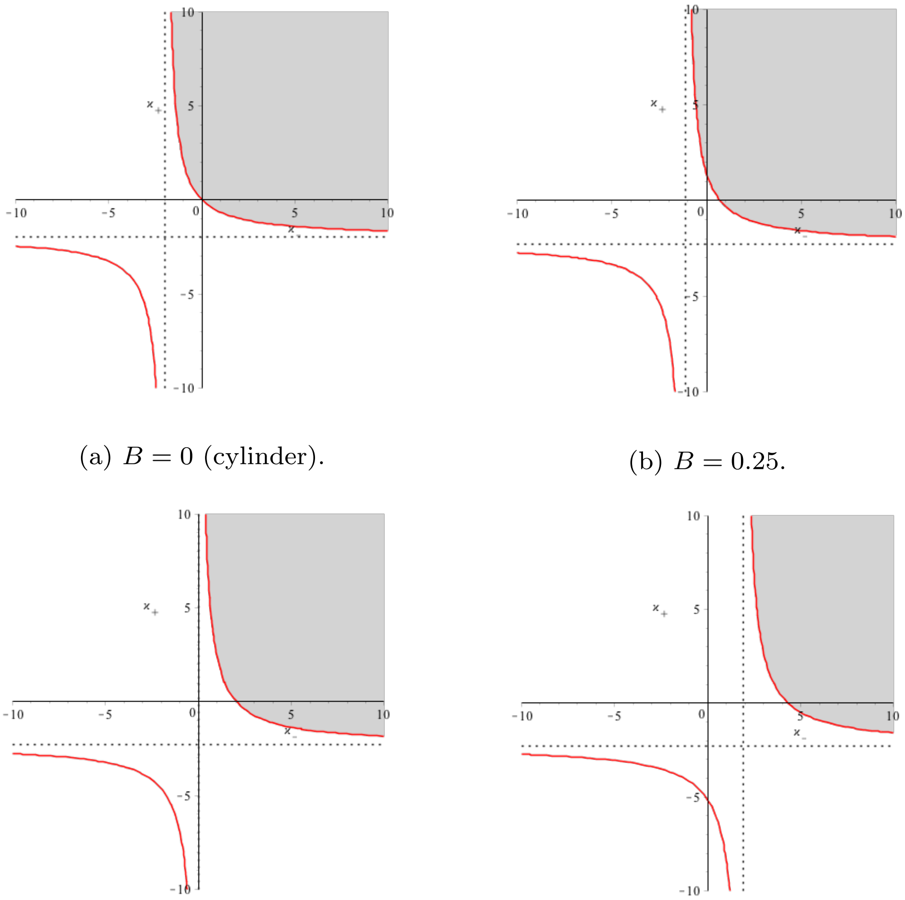
$$
\begin{aligned}
\kappa_{\Pi_-} &= -\frac{\hat{C}^0(H_*^+,B)}{\hat{B}_-^0(H_*^+,B)} = \frac{H_*^+\mathcal{G}_3(k)}{(1-k^2)\mathcal{G}_{1,2}(k)},\\
\kappa_{\Pi_+} &= -\frac{\hat{C}^0(H_*^+,B)}{\hat{B}_+^0(H_*^+,B)} = \frac{H_*^+\mathcal{G}_3(k)}{\sqrt{1-k^2}\,\mathcal{G}_{2,2}(k)}.
\end{aligned}
$$

By virtue of Lemma 4.1(ii), Lemma 4.2(ii), and Lemma 4.3, we have

$$
\begin{aligned}
&-\frac{\hat{C}^0(H_*^+,B)}{\hat{B}_-^0(H_*^+,B)}>0 \quad (B\in(0,1)),\\
&-\frac{\hat{C}^0(H_*^+,B)}{\hat{B}_+^0(H_*^+,B)}\begin{cases} >0\,(B\in(0,B_c)),\\ <0\,(B\in(B_c,1))\end{cases}
\end{aligned}
$$

for $H_*^+>0$. Note that $\hat{C}^0(H_*^+,0)=0$. These observations lead us to Fig. 7.

Let us keep $\kappa_{\Pi_+}=0$. Then, since we have κ_{Π_-}-intercept of hyperbola $-\hat{C}^0/\hat{B}_-^0$ bigger than $-\hat{B}_+^0/\hat{A}^0$, we can stabilize unduloids if we choose κ_{Π_-} bigger than $-\hat{C}^0/\hat{B}_-^0$ (see Fig. 7 (b), (c), (d) and Fig. 8 (a), (b)). On the other hand, let us keep $\kappa_{\Pi_-}=0$. If



(a) $B = 0$ (cylinder).

(b) $B = 0.25$.

(c) $B = B_c \, (\approx 0.4114)$.

(d) $B = 0.55$.

Figure 7. The hyperbolas under the conditions $d = \dfrac{\pi}{2H_*^+}$ and $\tau = \dfrac{\pi}{4H_*^+}$. In these figures, $\kappa_\pm := \kappa_{\Pi_\pm}$.

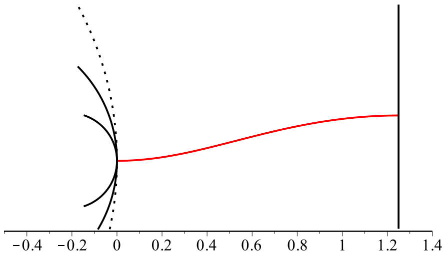
$B \in (0, B_c)$, we have κ_{Π_+}-intercept of hyperbola $-\hat{C}^0/\hat{B}_+^0$ bigger than $-\hat{B}_-^0/\hat{A}^0$, so that we can stabilize unduloids if we choose κ_{Π_+} bigger than $-\hat{C}^0/\hat{B}_+^0$ (see Fig. 7 (b) and Fig. 8 (c)). But, if $B \in (B_c, 1)$, we do not have κ_{Π_+}-intercept of hyperbola $-\hat{C}^0/\hat{B}_+^0$ bigger than $-\hat{B}_-^0/\hat{A}^0$ (see Fig. 7 (c), (d)). This means that we can not stabilize unduloids even if we choose sufficiently large κ_{Π_+}.

Remark. Let $\tau = \pi/(4H_*^+)$ and $\theta_\pm = \pi/2$. Under the relation

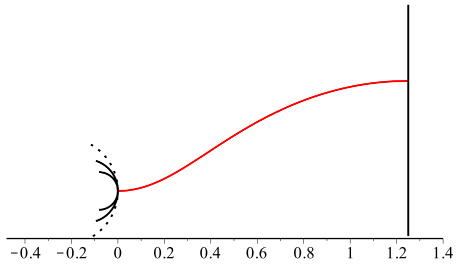
$$\kappa_{\Pi_+} = \frac{J_1(H_*^+, B, d)\kappa_{\Pi_-} - J_3(H_*^+, B, d)}{(1 - B^2)\kappa_{\Pi_-} - J_2(H_*^+, B, d)},$$

where

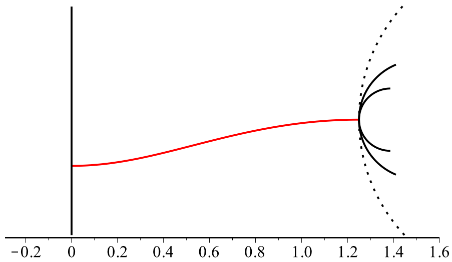
$$J_1(H_*^+, B, d) = (H_*^+)^2(1 - B)\big\{(1 - B)^2\hat{I}_1(d; H_*^+, B) - 2\hat{I}_2(d; H_*^+, B)\big\},$$



(a) $B = 0.25$. If $\kappa_{\Pi_-} > \kappa^0_{\Pi_-} \approx 0.6816$, unduloids with $H^+_* \approx 1.2367$ are linearly stable.



(b) $B = 0.55$. If $\kappa_{\Pi_-} > \kappa^0_{\Pi_-} \approx 4.3131$, unduloids with $H^+_* \approx 1.1553$ are linearly stable.



(c) $B = 0.25$. If $\kappa_{\Pi_+} > \kappa^0_{\Pi_+} \approx 1.3459$, unduloids with $H^+_* \approx 1.2367$ are linearly stable.

Figure 8. States of stable unduloids. If the generation curve of Π_- or Π_+ is the curve with dots, it is a neutral (0-eigenvalue) state.

$$
\begin{aligned}
J_2(H^+_*, B, d) &= (H^+_*)^2(1+B)\big\{(1+B)^2\hat{I}_1(d; H^+_*, B) - 2\hat{I}_2(d; H^+_*, B)\big\},\\
J_3(H^+_*, B, d) &= (H^+_*)^4\big[\,(1-B^2)^2\big\{\hat{I}_1(d; H^+_*, B)\big\}^2\\
&\qquad - 4(1+B^2)\hat{I}_1(d; H^+_*, B)\hat{I}_2(d; H^+_*, B) + 3\big\{\hat{I}_2(d; H^+_*, B)\big\}^2\,\big],
\end{aligned}
$$

set

$$
\begin{aligned}
\hat{c}_1 &:= -\frac{H^+_*(1-B)\hat{I}_2(d; H^+_*, B)\kappa_{\Pi_-}}{(H^+_*)^2\big\{(1+B)^2\hat{I}_1(d; H^+_*, B) - 3\hat{I}_2(d; H^+_*, B)\big\} - (1-B)\kappa_{\Pi_-}},\\
\hat{c}_2 &:= 1,\\
\hat{c}_3 &:= -\frac{4(H^+_*)^2\big[\,(H^+_*)^2\big\{(1+B)^2\hat{I}_1(d; H^+_*, B) - \hat{I}_2(d; H^+_*, B)\big\} - (1-B)\kappa_{\Pi_-}\big]}{(H^+_*)^2\big\{(1+B)^2\hat{I}_1(d; H^+_*, B) - 3\hat{I}_2(d; H^+_*, B)\big\} - (1-B)\kappa_{\Pi_-}}.
\end{aligned}
$$

Then the eigenspace for the zero eigenvalue is given by

$$
\big\{\eta\big(\hat{c}_1\hat{w}_1(s) + \hat{c}_2\hat{w}_2(s) + \hat{c}_3\hat{w}_3(s)\big)\,\big|\,\eta \in \mathbb{R}\big\},
$$

where $\hat{w}_i(s) := w_i(s)|_{\tau=\frac{\pi}{4H_*^+}}$ $(i = 1, 2, 3)$. Thus the dimension of the eigenspace for the zero eigenvalue is equal to 1, so that we see that the multiplicity of the zero eigenvalue is 1.

§ 4. Appendix : Properties of the elliptic integrals

Let $K(k)$ and $E(k)$ be the complete elliptic integrals of the 1st and 2nd kind, respectively. For $k \in (0,1)$ and $p \in \mathbb{N}$, set

$$\begin{aligned}
&\mathcal{G}_{1,p}(k) := (1-k^2)K(k) - pE(k),\\
&\mathcal{G}_{2,p}(k) := K(k) - pE(k),\\
&\mathcal{G}_3(k) := (1-k^2)\{K(k)\}^2 - 2(2-k^2)K(k)E(k) + 3\{E(k)\}^2.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{d}{dk}K(k) &= \frac{1}{k(1-k^2)}\{E(k) - (1-k^2)K(k)\},\\
\frac{d}{dk}E(k) &= \frac{1}{k}\{E(k) - K(k)\}.
\end{aligned}$$

Lemma 4.1. *Let $p \in \mathbb{N}$. Then the following properties hold.*

(i) For $p = 1, 2$,

$$\frac{d}{dk}\mathcal{G}_{1,p}(k) < 0 \quad (k \in (0,1)).$$

(ii) For $p \in \mathbb{N}$

$$\mathcal{G}_{1,p}(k) < 0 \quad (k \in (0,1)).$$

Proof. Differentiating $\mathcal{G}_{1,p}$ with respect to k, we have

$$\begin{aligned}
\frac{d}{dk}\mathcal{G}_{1,p}(k) =& -2kK(k) + (1-k^2)\cdot\frac{1}{k(1-k^2)}\{E(k) - (1-k^2)K(k)\}\\
&- \frac{p}{k}\{E(k) - K(k)\}\\
=&\frac{1}{k}\left\{(p-1-k^2)K(k) - (p-1)E(k)\right\}.
\end{aligned}$$

Then we see

$$\begin{aligned}
&(p-1-k^2)K(k) - (p-1)E(k)\\
&= \int_0^{\frac{\pi}{2}} \left\{\frac{p-1-k^2}{\sqrt{1-k^2\sin^2\eta}} - (p-1)\sqrt{1-k^2\sin^2\eta}\right\} d\eta\\
&= -k^2\int_0^{\frac{\pi}{2}} \frac{1-(p-1)\sin^2\eta}{\sqrt{1-k^2\sin^2\eta}}\, d\eta,
\end{aligned}$$

so that for $p = 1, 2$

$$\frac{d}{dk}\mathcal{G}_{1,p}(k) < 0 \quad (k \in (0,1)).$$

Let us prove (ii). For $p = 1, 2$, (i) implies that $\mathcal{G}_{1,p}$ is monotone decreasing in k. Moreover, we have

$$\mathcal{G}_{1,p}(0) = K(0) - pE(0) = -\frac{(p-1)\pi}{2} \leq 0.$$

Thus it follows that for $p = 1, 2$

$$\mathcal{G}_{1,p}(k) < 0 \quad (k \in (0,1)).$$

For $p \geq 3$, we obtain

$$\mathcal{G}_{1,p}(k) = (1-k^2)K(k) - 2E(k) - (p-2)E(k) = \mathcal{G}_{1,2}(k) - (p-2)E(k) < 0.$$

This completes the proof. □

Lemma 4.2. *Let $p \in \mathbb{N}$. Then*

$$\frac{d}{dk}\mathcal{G}_{2,p}(k) > 0 \quad (k \in (0,1)).$$

In addition, the following properties hold.

(i) For $p = 1$,

$$\mathcal{G}_{2,1}(k) > 0 \quad (k \in (0,1)).$$

(ii) For $p \geq 2$, there exists $k_2^c = k_2^c(p) \in (0,1)$ such that

$$\mathcal{G}_{2,p}(k) \begin{cases} < 0 \, (k \in (0, k_2^c)), \\ = 0 \, (k = k_2^c), \\ > 0 \, (k \in (k_2^c, 1)). \end{cases}$$

Proof. Differentiating $\mathcal{G}_{2,p}$ with respect to k, we have

$$\begin{aligned} \frac{d}{dk}\mathcal{G}_{2,p}(k) &= \frac{1}{k(1-k^2)}\{E(k) - (1-k^2)K(k)\} - \frac{p}{k}\{E(k) - K(k)\} \\ &= \frac{1}{k(1-k^2)}\left[(p-1)\{(1-k^2)K(k) - E(k)\} + pk^2E(k)\right]. \end{aligned}$$

Here we observe

$$\begin{aligned} (1-k^2)K(k) - E(k) &= \int_0^{\frac{\pi}{2}} \left(\frac{1-k^2}{\sqrt{1-k^2\sin^2\eta}} - \sqrt{1-k^2\sin^2\eta} \right) d\eta \\ &= \int_0^{\frac{\pi}{2}} \frac{-k^2(1-\sin^2\eta)}{\sqrt{1-k^2\sin^2\eta}}\, d\eta \end{aligned}$$

and

$$\begin{aligned}&(p-1)\{-k^2(1-\sin^2\eta)\}+pk^2(1-k^2\sin^2\eta)\\&=pk^2(1-k^2)\sin^2\eta+k^2(1-\sin^2\eta)>0.\end{aligned}$$

Thus we obtain for $p\in\mathbb{N}$

$$\frac{d}{dk}\mathcal{G}_{2,p}(k)>0\quad(k\in(0,1)),$$

so that $\mathcal{G}_{2,p}$ is monotone increasing in k. Since

$$\begin{aligned}&\mathcal{G}_{2,p}(0)=K(0)-pE(0)=-\frac{(p-1)\pi}{2}\begin{cases}=0\,(p=1),\\<0\,(p\geq 2),\end{cases}\\&\lim_{k\to 1-0}\mathcal{G}_{2,p}(k)=\infty,\end{aligned}$$

it follows that if $p=1$,

$$\mathcal{G}_{2,1}(k)>0\quad(k\in(0,1)),$$

and if $p\geq 2$, there exists $k_2^c=k_2^c(p)\in(0,1)$ such that

$$\mathcal{G}_{2,p}(k)\begin{cases}<0\,(k\in(0,k_2^c)),\\=0\,(k=k_2^c),\\>0\,(k\in(k_2^c,1)).\end{cases}$$

This completes the proof. □

Lemma 4.3. $\mathcal{G}_3(k)<0$ *for* $k\in(0,1)$.

A proof of this lemma is found in the Japanese book by Yotsutani and Murai [22]. For readers' convenience, we give a proof.

Proof. Differentiating $\mathcal{G}_3$ with respect to k, we obtain

$$\begin{aligned}&\frac{d}{dk}\mathcal{G}_3(k)\\&=-2k\{K(k)\}^2+(1-k^2)\cdot 2K(k)\cdot\frac{1}{k(1-k^2)}\big\{E(k)-(1-k^2)K(k)\big\}\\&\quad-2\bigg[-2kK(k)E(k)+(2-k^2)\cdot\frac{1}{k(1-k^2)}\big\{E(k)-(1-k^2)K(k)\big\}E(k)\\&\qquad\quad+(2-k^2)K(k)\cdot\frac{1}{k}\big\{E(k)-K(k)\big\}\bigg]\\&\quad+3\cdot 2E(k)\cdot\frac{1}{k}\big\{E(k)-K(k)\big\}\\&=\frac{2}{k(1-k^2)}\big[(1-k^2)^2\{K(k)\}^2-2(1-k^2)^2K(k)E(k)+(1-2k^2)\{E(k)\}^2\big]\\&=\frac{2}{k(1-k^2)}\big\{(1-k^2)K(k)-E(k)\big\}\big\{(1-k^2)K(k)-(1-2k^2)E(k)\big\}.\end{aligned}$$

It follows from Lemma 4.1 and Lemma 4.2 that for $k \in (0,1)$

$$(1-k^2)K(k) - E(k) = \mathcal{G}_{1,1}(k) < 0$$

and

$$\begin{aligned}(1-k^2)K(k) - (1-2k^2)E(k) &= (1-k^2)\bigl\{K(k) - E(k)\bigr\} + k^2 E(k) \\ &= (1-k^2)\mathcal{G}_{2,1}(k) + k^2 E(k) > 0.\end{aligned}$$

Thus we have

$$\frac{d}{dk}\mathcal{G}_3(k) < 0,$$

so that $\mathcal{G}_3$ is monotone decreasing in k. Since

$$\mathcal{G}_3(0) = \{K(0)\}^2 - 4K(0)E(0) + 3\{E(0)\}^2 = 0,$$

we are led to

$$\mathcal{G}_3(k) < 0 \quad (k \in (0,1)).$$

This completes the proof. □

Acknowledgements I would like to express my gratitude to Professor Miyuki Koiso and Professor Shoji Yotsutani for the fruitful discussion.

References

[1] H. Abels, H. Garcke, and L. Müller, *Stability of spherical caps under the volume-preserving mean curvature flow with line tension*, Nonlinear Anal., 117 (2015), 8–37.
[2] M. Athanassenas, *A variational problem for constant mean curvature surfaces with free boundary*, J. Reine Angew. Math., 377 (1987), 97–107.
[3] M. Athanassenas, *Volume-preserving mean curvature flow of rotationally symmetric surfaces*, Comment. Math. Helv., 72 (1997), no. 1, 52–66.
[4] M. Athanassenas, S. Kandanaarachchi, *Convergence of axially symmetric volume-preserving mean curvature flow*, Pacific J. Math., 259 (2012), no. 1, 41–54.
[5] A. J. Bernoff, A. L. Bertozzi, T. P. Witelski, *Axisymmetric surface diffusion: dynamics and stability of self-similar pinchoff*, J. Statist. Phys., 93 (1998), no. 3-4, 725–776.
[6] R. Courant and D. Hilbert, *Methods of mathematical physics*, vol.I, Interscience, New York, 1953.
[7] D. Depner, *Linearized stability analysis of surface diffusion for hypersurfaces with boundary contact*, Math. Nachr., 285 (2012), no. 11-12, 1385–1403.
[8] F. Farroni, R. Giova, and T. Ricciardi, *Best constants and extremals for a vector Poincaré inequality with weights*, Scientiae Math. Japonicae, 71 (2010), 111–126.
[9] L. G. Fel and B. Y. Rubinstein, *Stability of axisymmetric liquid bridges*, Z. Angew. Math. Phys., 66 (2015), Issue 6, 3447–3471.

[10] R. Finn, *Equilibrium Capillary Surfaces*, Grundlehren der mathematischen Wissenschaften 284, Springer New York, 1986.

[11] H. Garcke, K. Ito, and Y. Kohsaka, *Surface diffusion with triple junctions: a stability criterion for stationary solutions*, Adv. Differential Equations, 15 (2010), no. 5-6, 437–472.

[12] Katsuei Kenmotsu, *Surfaces with Constant Mean Curvature*, Translations of Mathematical Monographs, AMS, 2003.

[13] Yoshihito Kohsaka, *Stability analysis of Delaunay surfaces as steady states for the surface diffusion equation*, Geometric Properties for Parabolic and Elliptic PDE's, Springer Proceedings in Mathematics & Statistics 176, Springer International Publishing Switzerland, 2016.

[14] J. LeCrone, G. Simonett, *On well-posedness, stability, and bifurcation for the axisymmetric surface diffusion flow*, SIAM J. Math. Anal., 45 (2013), no. 5, 2834–2869.

[15] A. D. Myshkis, V. G. Babskii, N. D. Kopachevskii, L. A. Slobozhanin, and A. D. Tyuptsov, *Low-Gravity Fluid Mechanics*, Springer-Verlag Berlin Heidelberg, 1987.

[16] B. Y. Rubinstein and L. G. Fel, *Stability of unduloidal and nodoidal menisci between two solid spheres*, J. Geom. Symmetry Phys., 39 (2015), 77–98.

[17] T. I. Vogel, *Stability of a liquid drop trapped between two parallel planes*, SIAM J. Appl. Math., 47 (1987), no. 3, 516–525.

[18] T. I. Vogel, *Stability of a liquid drop trapped between two parallel planes. II. General contact angles*, SIAM J. Appl. Math., 49 (1989), no. 4, 1009–1028.

[19] T. I. Vogel, *Convex, rotationally symmetric liquid bridges between spheres*, Pacific J. Math., 224 (2006), no. 2, 367–377.

[20] T. I. Vogel, *Liquid Bridges Between Balls: The Small Volume Instability*, J. Math. Fluid Mech., 15 (2013), no. 2, 397–413.

[21] T. I. Vogel, *Liquid Bridges between Contacting Balls*, J. Math. Fluid Mech., 16 (2014), 737–744.

[22] S. Yotsutani and M. Murai, Daenkansuu to nakayoku narou (in Japanese), Nippon Hyoron Sha, 2013.