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A criterion for the linear independence of polylogarithms over a number field

By

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Abstract

Let $Li_s(z)$ be the s-th polylogarithmic function. Let $\alpha \in \overline{\mathbb{Q}}$, $0 < |\alpha| < 1$. In this article, we give a criterion for the linear independence of the $s+1$ numbers $Li_1(\alpha)$, $Li_2(\alpha)$, $\cdots$, $Li_s(\alpha)$ and 1 over the number field $\mathbb{Q}(\alpha)$. The new part is that we prove the linear independence of such polylogarithms over a number field of arbitrary degree. We also show examples and a linear independence measure.

§ 1. Introduction

Let $1 \le s \in \mathbb{Z}$. Denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Consider the s-th polylogarithmic function defined by

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad z \in \mathbb{C}, \ |z| < 1.$$

We obtain here a criterion for the linear independence of values $Li_s(\alpha)$ at $\alpha \in \overline{\mathbb{Q}}$ with $0 < |\alpha| < 1$, over the algebraic number field $\mathbb{Q}(\alpha)$ of *arbitrary degree.* Our result refines and generalizes the criterion due to E. M. Nikišin [9] which is limited to the rational case.

A lower bound for the dimension of a vector space spanned by polylogarithms over $\mathbb{Q}$ was given by T. Rivoal [10] in the rational case, and by R. Marcovecchio [6] in an algebraic case. However, these statements imply neither the irrationality nor the linear independence of chosen polylogarithms. M. Hata [2] showed a general linear

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independence criterion but still over $\mathbb{Q}$, by adapting Legendre polynomials. An analogy of Nikišin's work to the Lerch function was obtained by M. Kawashima [4].

§ 2. New results

Let α be an algebraic number of finite degree over $\mathbb{Q}$, with $0 < |\alpha| < 1$. Write $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. Let $\alpha = \alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(d)}$ be the d conjugates of α over $\mathbb{Q}$. We denote by $1 \leq b \in \mathbb{Z}$ the denominator of α^{-1}. Put $\delta = s \log s + (2s+1)\log 2$.

Theorem 2.1. *Suppose*

$$|\alpha| \times \prod_{i=2}^{d} \max\{1, \frac{1}{|\alpha^{(i)}|^s}\} < \frac{1}{b^{ds}} \exp\{s - (ds-1)\delta - \frac{ds(s+1)}{2}\} \tag{2.1}$$

Then the $s+1$ *numbers:* $Li_1(\alpha)$, $Li_2(\alpha)$, $\cdots, Li_s(\alpha)$ *and* 1 *are linearly independent over* $\mathbb{Q}(\alpha)$.

This result gives the first linear independence criterion at algebraic α for the function $Li_s(z)$. Write $v = v_1$ for the usual absolute value $|\cdot| = |\cdot|_{v_1}$, corresponding to the identity isomorphism, with the local degree denoted by $n_1 := n_{v_1}$ which is 1 or 2. The condition (2.1) is sufficient in the both cases where $n_1 = 1$ and $n_1 = 2$, although we may relax it when $n_1 = 2$. Our condition improves Nikišin's one [9] even in the rational case where $\alpha \in \mathbb{Q}$. Indeed, we have to compare (2.1) with (2.2) below, the corrected version in [4] of Nikišin's condition (not with the original one in [9] where an important factor is missing) in the rational case:

$$|\alpha| < \frac{1}{b^s} \exp\{-(s-1)(s+\delta)\}. \tag{2.2}$$

For given positive integers d, s we show here how to construct such α of degree exact d satisfying (2.1).

Theorem 2.2. *Let* $d, m \in \mathbb{Z}$ *with* $m \geq 1, d \geq 1$. *Define* $f_{m,d}(X) \in \mathbb{Q}[X]$ *by*

$$\begin{cases} f_{m,1}(X) = \left(2 + \frac{1}{m}\right) X - \frac{2}{m}, \\ f_{m,2}(X) = \left(2 + \frac{1}{m}\right) X^2 - 2X + \frac{2}{m}, \\ f_{m,d}(X) = \left(2 + \frac{1}{m}\right) X^d - \frac{2}{m} X^{d-1} - 2X + \frac{2}{m} \qquad (d \geq 3). \end{cases}$$

For each $s = 1, 2, \cdots$ *and* $d = 1, 2, \cdots$, *there exists a sufficiently large integer* $m_0 = m_0(d, s)$ *such that an algebraic number* α *of degree exact* d *over* $\mathbb{Q}$, *being given by the root of the smallest absolute value of the equation* $f_{m,d}(X) = 0$ *with* $m \geq m_0$, *satisfies the inequality* (2.1).

The polynomial $mf_{m,d}(X)$ is irreducible over $\mathbb{Z}$ by the Eisenstein criterion. An m_0 is calculated as follows.

Example 2.3. Let $d = 7$. For each s, we choose the index m_0 in Theorem 2.2 as in the table below, such that the following property holds. Let α be an algebraic number of degree 7 given by the root of the smallest absolute value of the equation $f_{m_0,7}(X) = 0$. Then the $s+1$ numbers: $Li_1(\alpha)$, $Li_2(\alpha)$, $\cdots, Li_s(\alpha)$ and 1 are linearly independent over $\mathbb{Q}(\alpha)$.

s	1	2	3	4	5	6	7	8	9 $\cdots$
m_0	10^{11}	10^{43}	10^{104}	10^{194}	10^{316}	10^{472}	10^{662}	10^{887}	$10^{1149}\cdots$

Indeed, for an s, the above m_0 gives the polynomial $f_{m_0,7}(X)$ whose roots are $\alpha_1, \ldots, \alpha_7$. Being $\alpha = \alpha_1$ the root of the smallest absolute value among them (we have $b \le 2$), the construction satisfies the inequality (2.1). For example, when $s = 5$ and $m_0 = 10^{316}$, by solving the equation, we have $|\alpha| < (1 + 10^{-317}) \times 10^{-316}$ and $\dfrac{1}{|\alpha^{(i)}|} < 1 + 10^{-317}$ $(i = 2, \ldots, 7)$, hence (2.1) holds.

Now we give a linear independence measure. For $\mathbf{x} \in \mathbb{P}_N(\overline{\mathbb{Q}})$ having coordinates $\mathbf{x} = (x_0, \ldots, x_N) \in \mathbb{P}_N(K)$ with an algebraic number field K, let us recall *the logarithmic height* of $\mathbf{x}$ defined by

$$h(\mathbf{x}) = \frac{1}{[K:\mathbb{Q}]} \sum_v n_v \log(\max\{|x_0|_v, \ldots, |x_N|_v\})$$

where the sum runs over all the normalized places of K and $n_v = [K_v : \mathbb{Q}_v]$ denote the local degree at v. We also define by $H(\mathbf{x}) = \exp(h(\mathbf{x}))$ *the exponential height* of $\mathbf{x}$. As is well-known, these definitions are independent of the choice of the projective coordinates. For α satisfying the condition (2.1) and $0 < |\alpha| < 1$, put

$$\rho = s + \delta - (s+1)\log|\alpha|,$$
$$\tau = -\log\left(b^{ds}|\alpha| \prod_{i=2}^{d} \max\{1, |\alpha^{(i)}|^{-s}\}\right) + s - (ds-1)\delta - ds(s+1)/2.$$

We have $\rho > s + \delta$ and $\tau > 0$ by (2.1).

The following theorem gives a new linear independence measure.

Theorem 2.4. *Fix $\alpha \in \overline{\mathbb{Q}}$ with $0 < |\alpha| < 1$ satisfying (2.1). For any $\varepsilon > 0$, there exists a constant $B_0 > 1$ depending on ε and the given data such that the following property holds. For any $\mathbf{x} = (x_0, x_1, \cdots, x_s) \in \mathbb{Q}(\alpha)^{s+1} - \{(0, 0, \cdots, 0)\}$ of exponential height $H(\mathbf{x}) = B \ge B_0$, we have*

$$|x_0 + x_1 Li_1(\alpha) + \cdots + x_s Li_s(\alpha)| > B^{-\frac{d\rho}{\tau} - \varepsilon}.$$

§ 3. Proof of Theorem 2.1

We construct sequences of polynomials in an algebraic setting of Padé approximation, however, to do this construction, we do not use the linear independence criterion of Nesterenko [8] which was an essential tool in previous works. Here, we introduce a modified matrix method (see §3.3) basically considered by Hermite and Nikišin to prove the linear independence of values of power series at algebraic numbers.

Fix $s \in \mathbb{Z}$. Let $q \in \mathbb{Z}$ with $0 \le q \le s$. Let $1 \le n \in \mathbb{Z}$ that we fix for the present and we will let grow sufficiently large later. Put $\sigma = ns + q$. Write $c, c_1, c_2, \cdots$, positive constants independent of n. Our construction of the sequences is carried out with respect to each index n, but we omit so often n in our notations.

Definition 3.1. For n and q, define the remainder function in t by

$$R(t) = R_q(t) = \frac{(t-1)(t-2)\cdots(t-\sigma+1)}{t^s(t+1)^s\cdots(t+n-1)^s(t+n)^q}.$$

Definition 3.2. Let $z \in \mathbb{C}$ with $|z| > 1$. For n and q, define a series in z by

$$N(z) = N_q(z) = \sum_{t=1}^{\infty} R(t) z^{-t}.$$

Let us construct sequences of polynomials. We show a proposition being valid for $z \in \mathbb{C}$, not only for $z \in \mathbb{R}$.

Proposition 3.3. *For each n, there exist polynomials $A_{kq}(z) \in \mathbb{Q}[z]$ $(k = 1, 2, \cdots, s)$ and $P_q(z) \in \mathbb{Q}[z]$ such that $A_{kq}(z)$ are not all identically zero, satisfying*

$$N_q(z) = \sum_{k=1}^{s} A_{kq}(z) Li_k(1/z) - P_q(z) = \frac{c_{0q}}{z^{\sigma}} + \frac{c_{1q}}{z^{\sigma+1}} + \cdots \tag{3.1}$$

with $c_{0q} \neq 0$ $(q = 0, \cdots, s)$, $\deg A_{kq}(z) \le n$, $(k = 1, \cdots, q)$, $\deg A_{kq}(z) \le n-1$ $(k = q+1, \cdots, s)$.

Proof. By

$$\int_0^1 x^{m-1} \left(\log \frac{1}{x}\right)^{k-1} dx = \frac{\Gamma(k)}{m^k} \qquad (m \in \mathbb{Z},\ m \ge 1), \tag{3.2}$$

we have

$$Li_k(1/z) = \sum_{m \ge 1} \frac{z^{-m}}{m^k} = \frac{1}{\Gamma(k)} \int_0^1 \left(\log \frac{1}{x}\right)^{k-1} \left(\sum_{m\ge 1} \frac{x^{m-1}}{z^m}\right) dx$$

$$= \frac{1}{\Gamma(k)} \int_0^1 \frac{\left(\log \frac{1}{x}\right)^{k-1}}{z - x} dx.$$

From the definition of $R(t)$, there exists $\gamma_{kj}^{(q)} \in \mathbb{Q}$ such that

$$R(t) = \sum_{j=0}^{n-1} \left(\sum_{k=1}^{s} \frac{\gamma_{kj}^{(q)}}{(t+j)^k} \right) + \sum_{k=1}^{q} \frac{\gamma_{kn}^{(q)}}{(t+n)^k}.$$

(3.3)

Put $A_{kq}(x)=\sum_{j=0}^{n-\varepsilon_k}\gamma_{kj}^{(q)}x^j\ (1\le k\le s,\ 0\le q\le s)$ with $\varepsilon_k=\begin{cases}0 & \text{for } 1\le k\le q,\\ 1 & \text{for } q+1\le k\le s.\end{cases}$

We then have

$$\begin{aligned}A_{kq}(z)Li_k(1/z)&=\frac{1}{\Gamma(k)}\int_0^1\frac{A_{kq}(z)}{z-x}\left(\log\frac{1}{x}\right)^{k-1}dx\\&=\frac{1}{\Gamma(k)}\int_0^1\frac{A_{kq}(z)-A_{kq}(x)}{z-x}\left(\log\frac{1}{x}\right)^{k-1}dx+\frac{1}{\Gamma(k)}\int_0^1\frac{A_{kq}(x)}{z-x}\left(\log\frac{1}{x}\right)^{k-1}dx.\end{aligned}$$

Hence

$$\sum_{k=1}^{s}A_{kq}(z)Li_k(1/z)=\sum_{k=1}^{s}I^{(k,q)}(z)+\int_0^1\sum_{k=1}^{s}\frac{A_{kq}(x)}{\Gamma(k)}\left(\log\frac{1}{x}\right)^{k-1}\frac{dx}{z-x}$$

where $I^{(k,q)}(z)=\frac{1}{\Gamma(k)}\int_0^1\frac{A_{kq}(z)-A_{kq}(x)}{z-x}\left(\log\frac{1}{x}\right)^{k-1}dx.$

Putting

$$P_q(z)=\sum_{k=1}^{s}I^{(k,q)}(z), \tag{3.4}$$

we have $P_q(z)\in\mathbb{Q}[z]$ and

$$\sum_{k=1}^{s}A_{kq}(z)Li_k(1/z)-P_q(z)=\int_0^1\sum_{k=1}^{s}\frac{A_{kq}(x)}{\Gamma(k)}\left(\log\frac{1}{x}\right)^{k-1}\frac{dx}{z-x}.$$

Expanding $\frac{1}{z-x}$, it is seen that (3.1) is nothing but

$$\int_0^1\sum_{k=1}^{s}\frac{A_{kq}(x)}{\Gamma(k)}\left(\log\frac{1}{x}\right)^{k-1}x^{\nu-1}dx=0\quad(\nu=1,\cdots,\sigma-1); \tag{3.5}$$

because we have

$$\begin{aligned}R(t)&=\sum_{k=1}^{q}\frac{\gamma_{kn}^{(q)}}{(t+n)^k}+\sum_{j=0}^{n-1}\left(\sum_{k=1}^{s}\frac{\gamma_{kj}^{(q)}}{(t+j)^k}\right)=\sum_{k=1}^{q}\left\{\sum_{j=0}^{n}\frac{\gamma_{kj}^{(q)}}{(t+j)^k}\right\}+\sum_{k=q+1}^{s}\left\{\sum_{j=0}^{n-1}\frac{\gamma_{kj}^{(q)}}{(t+j)^k}\right\}\\&=\sum_{k=1}^{s}\left\{\frac{1}{\Gamma(k)}\sum_{j=0}^{n-\varepsilon_k}\gamma_{kj}^{(q)}\frac{\Gamma(k)}{(t+j)^k}\right\}=\int_0^1\left\{\sum_{k=1}^{s}\frac{1}{\Gamma(k)}\sum_{j=0}^{n-\varepsilon_k}\gamma_{kj}^{(q)}x^j\left(\log\frac{1}{x}\right)^{k-1}\right\}x^{t-1}dx\\&=\int_0^1\sum_{k=1}^{s}\frac{A_{kq}(x)}{\Gamma(k)}\left(\log\frac{1}{x}\right)^{k-1}x^{t-1}dx.\end{aligned}$$

Consequently, the condition (3.5) is equivalent to $R(1) = R(2) = \cdots = R(\sigma - 1) = 0$, but this follows from the definition of $R(t)$. The property $c_{0q} \neq 0$ $(0 \leq q \leq s)$ follows from $R(\sigma) \neq 0$ by counting the number of zeros of $R(t)$. Thus the polynomials $A_{kq}(z)$, $P_q(z)$ are constructed as required. □

Lemma 3.4. *For $0 \leq q \leq s$ and $z \in \mathbb{C}$, $|z| > 1$, there exists a constant $c > 0$ such that, for sufficiently large n, we have*

$$|N_q(z)| \leq n^c \left(\frac{1}{|z|}\right)^{ns+q} \left(1 + \frac{1}{s}\right)^{-ns(s+1)}.$$

Proof. Recall

$$N_q(z) = \int_0^1 \sum_{k=1}^{s} \frac{A_{kq}(x)}{\Gamma(k)} \left(\log \frac{1}{x}\right)^{k-1} \frac{dx}{z - x}.$$

For $x \in \mathbb{R},\ 0 < x \leq 1$, we are going to show

$$\frac{1}{x - z} = \frac{1}{2iz} \int_{\Re(t)=\frac{1}{2}} \left(-\frac{x}{z}\right)^{t-1} \frac{dt}{\sin \pi(t-1)}. \tag{3.6}$$

For, the function $f(t) = \dfrac{1}{\sin \pi(t-1)}$ has poles at $t \in \mathbb{Z}$ of order 1, thus

$$\begin{aligned}
\mathrm{Res}_{t=n+1} \left(-\frac{x}{z}\right)^{t-1} \frac{1}{\sin \pi(t-1)} &= \lim_{t \to n+1} \frac{t-n-1}{\sin \pi(t-1)} \left(-\frac{x}{z}\right)^{t-1} \\
&= \lim_{h \to 0} \frac{h}{\sin \pi(n+h)} \left(-\frac{x}{z}\right)^{n+h} = \frac{1}{\pi} \left(\frac{x}{z}\right)^{n}.
\end{aligned}$$

Let $N \in \mathbb{Z}$ (we may reduce to the case $N \in \mathbb{Z}$). For sufficiently small $\varepsilon > 0$, consider

$$\begin{aligned}
C_- &: t = Ne^{i\theta} + \frac{1}{2} \quad (-\frac{\pi}{2} \leq \theta \leq -\varepsilon), \\
C_+ &: t = Ne^{i\theta} + \frac{1}{2} \quad (\varepsilon \leq \theta \leq \frac{\pi}{2}), \\
C_\varepsilon &: t = Ne^{i\theta} + \frac{1}{2} \quad (-\varepsilon \leq \theta \leq \varepsilon), \\
L &: \Re(t) = \frac{1}{2} \quad (|\Im(t)| \leq N,\ N \in \mathbb{Z}).
\end{aligned}$$

The residue formula gives $\displaystyle\int_{L+C_-+C_\varepsilon+C_+} \left(-\frac{x}{z}\right)^{t-1} \frac{dt}{\sin \pi(t-1)} = 2\pi i \cdot \frac{1}{\pi} \sum_{n=0}^{N-1} \left(\frac{x}{z}\right)^{n}$.

On $C_- + C_+$, we have

$$\begin{aligned}
\left|\sin \pi \left(Ne^{i\theta} - \frac{1}{2}\right)\right| &= |\cos(\pi N e^{i\theta})| = |\cos(\pi N \cos\theta + i\pi N \sin\theta)| \\
&= \frac{1}{2} \left|e^{i\pi N \cos\theta} e^{-\pi N \sin\theta} + e^{-i\pi N \cos\theta} e^{\pi N \sin\theta}\right| \\
&\geq \frac{1}{2} \left|e^{\pi N \sin\theta} - e^{-\pi N \sin\theta}\right|.
\end{aligned}$$

Choose conveniently $\arg\left(-\frac{x}{z}\right)$. Note $\left|-\frac{x}{z}\right| < 1$ by assumption. We obtain

$$\begin{aligned}
&\left|\int_{C_-+C_\varepsilon+C_+}\left(-\frac{x}{z}\right)^{t-1}\frac{dt}{\sin\pi(t-1)}\right| \\
&\quad = \left|2\int_{C_+}\left(-\frac{x}{z}\right)^{Ne^{i\theta}-\frac{1}{2}}\frac{1}{\sin\pi(Ne^{i\theta}-\frac{1}{2})}\cdot Nie^{i\theta}d\theta + \int_{C_\varepsilon}\left(-\frac{x}{z}\right)^{t-1}\frac{dt}{\sin\pi(t-1)}\right| \\
&\le c_1\cdot N\int_\varepsilon^{\frac{\pi}{2}}\frac{d\theta}{e^{\pi N\sin\theta}-e^{-\pi N\sin\theta}} + c_2\cdot N\left|\frac{x}{z}\right|^{\frac{N}{3}-\frac{1}{2}} \longrightarrow 0 \quad (N\to\infty),
\end{aligned}$$

thus we get (3.6).
Since $0 < x \le 1$ and $|z| > 1$, we obtain by the uniform convergence of the integral,

$$\begin{aligned}
N_q(z) &= \int_0^1\sum_{k=1}^s\frac{A_{kq}(x)}{\Gamma(k)}\left(\log\frac{1}{x}\right)^{k-1}\left\{-\frac{1}{2iz}\int_{\Re(t)=\frac{1}{2}}\left(-\frac{x}{z}\right)^{t-1}\frac{dt}{\sin\pi(t-1)}\right\}dx \\
&= \frac{1}{2iz}\int_{\Re(t)=\frac{1}{2}}\left\{\int_0^1\left(\sum_{k=1}^s\frac{A_{kq}(x)}{\Gamma(k)}\left(\log\frac{1}{x}\right)^{k-1}\right)\left(-\frac{x}{z}\right)^{t-1}dx\right\}\frac{dt}{\sin\pi t} \\
&= \frac{1}{2iz}\int_{\Re(t)=\frac{1}{2}}R(t)\left(-\frac{1}{z}\right)^{t-1}\frac{dt}{\sin\pi t}.
\end{aligned}$$

There is no pole of the integrand between $\Re(t) = \frac{1}{2}$ and $\Re(t) = \sigma - \frac{1}{2}$. When the imaginary part of t is large in the absolute value, there is no contribution of the integral. Then we have

$$|N_q(z)| = \frac{1}{|2z|}\left|\int_{\Re(t)=\sigma-\frac{1}{2}}R(t)\left(-\frac{1}{z}\right)^{t-1}\frac{1}{\sin\pi t}dt\right|.$$

For $t = \sigma - \frac{1}{2} + iw$ with w real, by using formulae for Γ-function and that of Stirling we have

$$\begin{aligned}
&\left|R(t)\left(-\frac{1}{z}\right)^{t-1}\frac{1}{\sin\pi t}\right| \\
&\le \frac{|z^{\frac{3}{2}}|}{|z^\sigma|e^{w\cdot arg(-1/z)}}\left|\frac{\Gamma(t)}{\Gamma(t-\sigma+1)\sin\pi t}\right|\left|\frac{1}{t^s(t+1)^s\cdots(t+n-1)^s(t+n)^q}\right| \\
&\le \frac{n^{c_3}}{|z^\sigma|e^{c_4 w}}\left(\frac{ns+s}{ns+n}\right)^{ns(s+1)} \le \frac{n^{c_3}}{|z^\sigma|e^{c_4 w}}\left(1+\frac{1}{s}\right)^{-ns(s+1)}.
\end{aligned}$$

This implies $$|N_q(z)| \le \frac{n^c}{|z|^\sigma}\left(1+\frac{1}{s}\right)^{-ns(s+1)}.$$ □

§ 3.1. Estimates for polynomials

Lemma 3.5. *Let d_n be the least common multiple of the n integers $1, 2, \cdots, n$. Then for $1 \leq k \leq s,\ 0 \leq j \leq n$, the number $s! d_n^{s-k} \gamma_{kj}^{(q)}$ is a rational integer $\in \mathbb{Z}$ (here we understand $\gamma_{kn}^{(q)} = 0$ for $k > q$).*

Proof. Fix j with $0 \leq j \leq n-1$. Then $\gamma_{kj}^{(q)} = \dfrac{1}{(s-k)!} \dfrac{d^{s-k}}{dt^{s-k}} R(t)(t+j)^s \Big|_{t=-j}$
The required property follows (confer Lemma 2 in [9]). □

Remark. We have noted that the coefficient c_{0q} for each q $(0 \leq q \leq s)$ in (3.1) does not vanish, from the definition of $R(t)$. Similarly the coefficient $\gamma_{qn}^{(q)}$ of the term of degree n in the polynomial $A_{qq}(z)$ is non-zero, because $t = -n$ is not the zero of $R(t)$, hence we have $\deg A_{qq}(z) = n$ for $q = 1, \cdots, s$. It is an very important fact that we will use later.

Lemma 3.6. *There exists a constant $c > 0$ such that, for sufficiently large n, the polynomials $A_{kq}(z)$ $(1 \leq k \leq s,\ 0 \leq q \leq s)$ and $P_q(z)$ $(0 \leq q \leq s)$ satisfy the following estimates:*

$$
\begin{aligned}
|A_{kq}(z)| &\leq n^c \max\{1, |z|^n\} \exp(n\delta), \\
|P_q(z)| &\leq n^c \max\{1, |z|^n\} \exp(n\delta).
\end{aligned}
$$

Proof. Since $\gamma_{kj}^{(q)}$ is the coefficient of $(t+j)^{-k}$ by the residue formula at $t = -j$: we have

$$
\gamma_{kj}^{(q)} = \mathrm{Res}_{t=-j} R(t)(t+j)^{k-1} = \frac{1}{2\pi i} \int_{|t+j|=\frac{1}{2}} R(t)(t+j)^{k-1} dt.
$$

For t, j with $|t+j| = \dfrac{1}{2}$, we have $j - 1 < |t| < j + 1$ giving the three estimates:
i) $|t - \sigma + 1||t - \sigma + 2| \cdots |t - 1| \leq (j + \sigma) \cdots (j + 2)$
ii) $|t||t+1| \cdots |t+n-1| \geq (j-1)! \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot (n-1-j-1)! = \dfrac{1}{8}(j-1)! \cdot (n-j-2)!$
iii) $|t+n| \geq n - |t| \geq n - j - 1$.

Therefore,

$$\begin{aligned}
|\gamma_{kj}^{(q)}| &\leq \frac{1}{2\pi}\int_{|t+j|=\frac{1}{2}} |R(t)||t+j|^{k-1}dt \\
&\leq \frac{1}{2\pi}\int_{|t+j|=\frac{1}{2}} \frac{(j+2)\cdots(j+\sigma)}{\frac{1}{8^s}((j-1)!)^s((n-j-2)!)^s(n-j-1)^q}\cdot\frac{1}{2^{k-1}}dt \\
&= \frac{2^{3s}}{2^k}\cdot\frac{(j+2)\cdots(j+\sigma)}{((j-1)!)^s((n-j-2)!)^s(n-j-1)^q} \\
&\leq 2^{3s-k}\cdot {}_{j+\sigma}\mathrm{C}_j\cdot({}_n\mathrm{C}_j)^s\cdot\frac{\sigma!}{(n!)^s}\cdot\frac{j^s(n-j-1)^{s-q}(n-j)^s}{j+1} \\
&\leq n^{c_5}\cdot 2^{2ns+n}\cdot s^{ns} = n^{c_6}\exp(n\delta).
\end{aligned}$$

Consequently, from the definition (3.3) of $A_{kq}(z)$,

$$|A_{kq}(z)| \leq \sum_{j=0}^{n-\varepsilon_k} |\gamma_{kj}^{(q)}||z|^j \leq n^{c_7}\max\{1,|z|^n\}\exp(n\delta).$$

From the definition (3.4) of $P_q(z)$, we have

$$P_q(z) = \sum_{k=1}^{s}\sum_{j=0}^{n-\varepsilon_k}\gamma_{kj}^{(q)}\left(z^{j-1}+\frac{z^{j-2}}{2^k}+\cdots+\frac{1}{j^k}\right) \tag{3.7}$$

which implies $|P_q(z)| \leq n^{c_8}\cdot\max\{1,|z|\}^n\cdot\exp(n\delta)$. □

§ 3.2. An estimate for a determinant

We give an estimate for a determinant in our algebraic setting.

Lemma 3.7. *Consider $M(z)$ the following square matrix of size $s+1$:*

$$M(z) = \begin{pmatrix} A_{10}(z) & A_{20}(z) & \cdots & A_{s0}(z) & P_0(z) \\ A_{11}(z) & A_{21}(z) & \cdots & A_{s1}(z) & P_1(z) \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1s}(z) & A_{2s}(z) & \cdots & A_{ss}(z) & P_s(z) \end{pmatrix}.$$

Denote by $\Delta(z)$ the determinant of $M(z)$. Then $\Delta(z)\equiv \text{constant}\neq 0$.

Proof. For $q = 0, 1, \cdots, s$, we have

$$\Delta_q(z) = (-1)^{q+s} \begin{vmatrix} A_{10}(z) & A_{20}(z) & \cdots & A_{s0}(z) \\ A_{11}(z) & A_{21}(z) & \cdots & A_{s1}(z) \\ \vdots & \vdots & & \vdots \\ A_{1,q-1}(z) & A_{2,q-1}(z) & \cdots & A_{s,q-1}(z) \\ A_{1,q+1}(z) & A_{2,q+1}(z) & \cdots & A_{s,q+1}(z) \\ \vdots & \vdots & & \vdots \\ A_{1s}(z) & A_{2s}(z) & \cdots & A_{ss}(z) \end{vmatrix} \tag{3.8}$$

where $\Delta_q(z)$ is the co-factor corresponding to the $(q, s+1)$-th element.

Recall $\deg A_{kq}(z) \le n \, (1 \le k \le q)$, $\deg A_{kq}(z) \le n-1$ $(q+1 \le k \le s)$ by construction in Proposition 3.3, and $\deg A_{qq}(z) = n$ $(q = 1, \cdots, s)$ by Remark after Lemma 3.5. Let γ be the product of the leading coefficient of the term of degree exact n in $A_{qq}(z)$ over all $q = 1, \cdots, s$. Then $\gamma \ne 0$. When $q = 1, \cdots, s$, we then have $\deg \Delta_q(z) \le (n-1) + n + \cdots + n = ns - 1$, and $\deg \Delta_0(z) = ns$.

With suitable numbers $h_1, h_2, \cdots$, we are now going to show:

$$\Delta(z) = \sum_{q=0}^{s} P_q(z)\Delta_q(z) = -\left(\gamma c_{00} + \frac{h_1}{z} + \frac{h_2}{z^2} + \cdots\right). \tag{3.9}$$

For this, we sum up the $s+1$ identities below coming from (3.1) over $q = 0, 1, \cdots, s$:

$$\Delta_q(z)\left(\sum_{k=1}^{s} A_{kq}(z) Li_k(1/z) - P_q(z)\right) = \Delta_q(z)\left(\frac{c_{0q}}{z^{ns+q}} + \cdots\right). \tag{3.10}$$

The sum of the right hand side of the identity (3.10) over all $q = 0, 1, \cdots, s$ has the constant term, with respect to z, arising only from the leading coefficient of $\Delta_0(z)$ (the coefficient of z^{ns}, namely the product of the coefficients of z^n in $A_{qq}(z)$ for $q = 1, \cdots s$), multiplied by c_{00} (the coefficient of $\frac{1}{z^{ns}}$). We obtain from the definition of γ:

$$D - \sum_{q=0}^{s} P_q(z)\Delta_q(z) = \gamma c_{00} + \frac{h_1}{z} + \frac{h_2}{z^2} + \cdots, \tag{3.11}$$

$$\text{with} \quad D = \sum_{k=1}^{s} Li_k(1/z)\left(\sum_{q=0}^{s} A_{kq}(z)\Delta_q(z)\right).$$

However, for each $1 \le k \le s$, the sum $\sum_{q=0}^{s} A_{kq}(z)\Delta_q(z)$ in D is an expansion of the determinant

$$\begin{vmatrix} A_{10}(z) & A_{20}(z) & \cdots & A_{s0}(z) & A_{k0}(z) \\ A_{11}(z) & A_{21}(z) & \cdots & A_{s1}(z) & A_{k1}(z) \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1s}(z) & A_{2s}(z) & \cdots & A_{ss}(z) & A_{ks}(z) \end{vmatrix}$$

which has two same columns. Consequently $D = 0$, then by (3.11) we have (3.9).
Since $\Delta(z)$ is a polynomial with respect to z, by combining with (3.9) and the fact $c_{00} \neq 0$ in Proposition 3.3, the lemma is achieved. □

§ 3.3. Conclusion

Let us take $\alpha \in \overline{\mathbb{Q}}$ $(0 < |\alpha| < 1)$ satisfying (2.1). Denote by $\mathfrak{o}_K$ the ring of algebraic integers of K.

Now suppose that $Li_1(\alpha), Li_2(\alpha), \cdots, Li_s(\alpha)$ and 1 are linearly dependent over K. Then there exist $x_1, x_2, \cdots, x_s, -x_0 \in \mathfrak{o}_K$, not all zero, such that

$$\ell := x_1 Li_1(\alpha) + \cdots + x_s Li_s(\alpha) - x_0 = 0.$$

Lemma 3.7 guarantees that the $s+1$ lines of the matrix $M(\alpha^{-1})$ are linearly independent over K. Then along the hyperplane of K^{s+1}, defined by $x_1X_1 + \cdots x_sX_s - x_0X_0 = 0$, we may choose s linearly independent vectors form the $s+1$ lines of the matrix $M(\alpha^{-1})$, such that together with the new vector $(x_1, x_2, \cdots, x_s, -x_0)$, the $s+1$ vectors are linearly independent over K. We assume that these chosen s lines consist of the j-th lines with $j \neq q$ of $M(\alpha^{-1})$.
Define

$$\Delta := \det \begin{pmatrix} A_{10}(\alpha^{-1}) & A_{20}(\alpha^{-1}) & \cdots & A_{s0}(\alpha^{-1}) & P_0(\alpha^{-1}) \\ A_{11}(\alpha^{-1}) & A_{21}(\alpha^{-1}) & \cdots & A_{s1}(\alpha^{-1}) & P_1(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1,q-1}(\alpha^{-1}) & A_{2,q-1}(\alpha^{-1}) & \cdots & A_{s,q-1}(\alpha^{-1}) & P_{q-1}(\alpha^{-1}) \\ x_1 & x_2 & \cdots & x_s & x_0 \\ A_{1,q+1}(\alpha^{-1}) & A_{2,q+1}(\alpha^{-1}) & \cdots & A_{s,q+1}(\alpha^{-1}) & P_{q+1}(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1s}(\alpha^{-1}) & A_{2s}(\alpha^{-1}) & \cdots & A_{ss}(\alpha^{-1}) & P_s(\alpha^{-1}) \end{pmatrix}.$$

Then we have $\Delta \neq 0$ by the argument above. On the other hand, Lemma 3.5 shows that when we multiply the determinant Δ by

$$\partial := (s!b^n)^s d_n^{s(s+1)/2},$$

the number $\Delta^* := \partial\Delta$ belongs to $\mathfrak{o}_K$.
By linear algebra, we have

$$\Delta = -\det\begin{pmatrix} A_{10}(\alpha^{-1}) & A_{20}(\alpha^{-1}) & \cdots & A_{s0}(\alpha^{-1}) & N_0(\alpha^{-1}) \\ A_{11}(\alpha^{-1}) & A_{21}(\alpha^{-1}) & \cdots & A_{s1}(\alpha^{-1}) & N_1(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1,q-1}(\alpha^{-1}) & A_{2,q-1}(\alpha^{-1}) & \cdots & A_{s,q-1}(\alpha^{-1}) & N_{q-1}(\alpha^{-1}) \\ x_1 & x_2 & \cdots & x_s & \ell \\ A_{1,q+1}(\alpha^{-1}) & A_{2,q+1}(\alpha^{-1}) & \cdots & A_{s,q+1}(\alpha^{-1}) & N_{q+1}(\alpha^{-1}) \\ \vdots & \vdots & & \vdots & \vdots \\ A_{1s}(\alpha^{-1}) & A_{2s}(\alpha^{-1}) & \cdots & A_{ss}(\alpha^{-1}) & N_s(\alpha^{-1}) \end{pmatrix}.$$

Since $0 \neq \Delta^* \in \mathfrak{o}_K$, by the product formula we have $\prod_{v|\infty} |\Delta^*|_v^{n_v} \geq 1$ with n_v the local degree corresponding to each Archimedean place v of K.
We distinguish the usual absolute value $|\cdot|$ from the others as is done in [6], namely pick up the usual absolute value $|\cdot| = |\cdot|_{v_1}$, with the local degree denoted by $n_1 = n_{v_1}$.
We obtain

(3.12)

$$1 \leq \prod_{v|\infty} |\Delta^*|_v^{n_v} = |\Delta^*|_{v_1}^{n_1} \cdot \prod_{v|\infty, v\neq v_1} |\Delta^*|_v^{n_v}$$

(3.13)

$$\leq \left\{ \partial \sum_{j\neq q} |N_j(\alpha^{-1})| \left(\max_{i,j} |A_{ij}(\alpha^{-1})| \right)^{s-1} \max_{1\leq\mu\leq s} |x_\mu| + \partial|\ell| \left(\max_{i,j} |A_{ij}(\alpha^{-1})| \right)^s \right\}^{n_1}$$

(3.14)

$$\times \prod_{v|\infty, v\neq v_1} \left\{ \partial \sum_{j\neq q} |P_j(\alpha^{-1})|_v \left(\max_{i,j} |A_{ij}(\alpha^{-1})|_v \right)^{s-1} \max_{1\leq\mu\leq s} |x_\mu|_v + \partial|x_0|_v \left(\max_{i,j} |A_{ij}(\alpha^{-1})|_v \right)^s \right\}^{n_v}.$$

Note that in the expression $|\Delta^*|_{v_1}$, we do not need the algebraicity of the element.
We start the final step of the proof of our theorem. For the term (3.11), we have

$$\left\{ \partial \sum_{j\neq q} |N_j(\alpha^{-1})| \left(\max_{i,j} |A_{ij}(\alpha^{-1})| \right)^{s-1} \max_{1\leq\mu\leq s} |x_\mu| + \partial|\ell| \left(\max_{i,j} |A_{ij}(\alpha^{-1})| \right)^s \right\}^{n_1}$$
$$\leq n^{c_9} \left\{ b^{ns} d_n^{s(s+1)/2} \left(|\alpha|^n e^{-ns} e^{n(s-1)\delta} + |\ell||\alpha|^{-ns} e^{ns\delta} \right) \right\}^{n_1} \cdot\cdot\, (\mathrm{I}).$$

Consider the term (3.12). We have

$$\prod_{v|\infty,v\neq v_1}\left\{\partial\sum_{j\neq q}|P_j(\alpha^{-1})|_v\left(\max_{i,j}|A_{ij}(\alpha^{-1})|_v\right)^{s-1}\max_{1\leq\mu\leq s}|x_\mu|_v+\partial|x_0|_v\left(\max_{i,j}|A_{ij}(\alpha^{-1})|_v\right)^s\right\}^{n_v}$$
$$\leq n^{c_{10}}\prod_{i=n_1+1}^{d}\left\{b^{ns}d_n^{s(s+1)/2}\max\{1,|\alpha^{(i)}|^{-1}\}^{ns}e^{ns\delta}\left(s\max_{1\leq\mu\leq s}|x_\mu^{(i)}|+|x_0^{(i)}|\right)\right\}\cdots\text{(II)}.$$

Let $\varepsilon(n)=(\log n)^{1/2}\exp\{-\sqrt{(\log n)/R}\}$ with $R=\frac{515}{(\sqrt{546}-\sqrt{322})^2}$. We see $\varepsilon(n)\to 0$ when $n\to\infty$. The Rosser-Schoenfeld theorem [11], an explicit version of the prime number theorem, implies an estimate of the least common multiple d_n of the form $n\{1-\varepsilon(n)\}\leq\log d_n\leq n\{1+\varepsilon(n)\}$, then combining (I) and (II) together with this estimate and the assumption $\ell=0$, we have (here c_{11} depends on $|x_\mu^{(i)}|$):

$$1\leq n^{c_{11}}\left\{b^{ns}|\alpha|^n e^{-ns}e^{n(1+\varepsilon(n))(s(s+1)/2)}e^{n(s-1)\delta}\right\}^{n_1}$$
$$\times\prod_{i=n_1+1}^{d}\left\{b^{ns}\max\{1,|\alpha^{(i)}|^{-ns}\}\,e^{n(1+\varepsilon(n))(s(s+1)/2)}e^{ns\delta}\right\}.$$

By taking the $1/n$-th power, we get when $n_1=1$:

$$1\leq(n^{c_{12}})^{\frac{1}{n}}b^{ds}e^{-s}e^{d(1+\varepsilon(n))(s(s+1)/2)}e^{(ds-1)\delta}\cdot|\alpha|\prod_{i=2}^{d}\max\{1,|\alpha^{(i)}|^{-s}\}.$$

When $n_1=2$:

$$1\leq(n^{c_{13}})^{\frac{1}{n}}b^{ds}e^{-2s}e^{d(1+\varepsilon(n))(s(s+1)/2)}e^{(ds-2)\delta}\cdot|\alpha|^2\prod_{i=3}^{d}\max\{1,|\alpha^{(i)}|^{-s}\}.$$

By our condition (2.1), we obtain a contradiction, because the right-hand side <1 for sufficiently large n, in the both cases $n_1=1$ and $n_1=2$ (by $0<|\alpha|<1$ and $s+\delta>0$). Thus $\ell\neq 0$, which again gives us a contradiction. The proof of Theorem 2.1 is achieved. □

§ 4. Proof of Theorem 2.2

Recall
$$\begin{cases} f_{m,1}(X)= & \left(2+\frac{1}{m}\right)X-\frac{2}{m},\\ f_{m,2}(X)= & \left(2+\frac{1}{m}\right)X^2-2X+\frac{2}{m},\\ f_{m,d}(X)= & \left(2+\frac{1}{m}\right)X^d-\frac{2}{m}X^{d-1}-2X+\frac{2}{m}\quad(d\geq 3).\end{cases}$$

The polynomial $m\times f_{m,d}(X)$ is irreducible in $\mathbb{Z}[X]$ thanks to the Eisenstein criterion,

then the roots of the polynomial are of degree exact d over $\mathbb{Q}$. We also see that the inverse of the roots of $m \times f_{m,d}(X)$ have the denominator 1 or 2.
Let α be the root, of the smallest absolute value, of $f_{m,d}(X) = 0$. The roots of $2X^d - 2X = 2X(X^{d-1} - 1)$ are 0 and the $d-1$ roots of the unity. We recall the classical result of A. M. Ostrowski stating the continuity of the complex roots of a polynomial with respect to its coefficients (this fact is described as Theorem 1.4 of [7], and in a more general situation due to A. Hurwitz, as Theorem 1.5 in [7], see also Theorem 6.2 of [1]). Then among the roots of $m \times f_{m,d}(X)$, there is one root α whose absolute value is sufficiently small, and the other $d-1$ roots are of absolute value very close to 1. Although $1/|\alpha^{(i)}|$ is not always less than 1, it is possible to choose sufficiently large m_0 such that the algebraic number α satisfies the assumption of Theorem 2.1. $\square$

§ 5. Proof of Theorem 2.4

We show a linear independence measure by a variant of the method due to M. Hata in [3]. Write again the usual absolute value $|\cdot| = |\cdot|_{v_1}$, with the local degree denoted by $n_1 = n_{v_1}$.

Since $\prod_{1\le i\le d} \max\{1, |x_0^{(i)}|, \cdots, |x_s^{(i)}|\} \le H(\mathbf{x})^d$, we have when $n_1 = 1$:

$$\begin{aligned}
1 \le \prod_{v|\infty} |\Delta^*|_v^{n_v} &= |\Delta^*|_{v_1} \cdot \prod_{v|\infty, v\ne v_1} |\Delta^*|_v^{n_v} \\
&\le n^{c_{14}} H(\mathbf{x})^d \left(\prod_{i=2}^{d} \max\{1, |\alpha^{(i)}|^{-1}\}^s b^{ds} e^{(1+\varepsilon(n))(ds(s+1)/2)} e^{s\delta(d-1)}\right)^n \\
&\times \left\{ \left(e^{-s}|\alpha| e^{(s-1)\delta}\right)^n + |\ell| \cdot \left(|\alpha|^{-s} e^{s\delta}\right)^n \right\} \cdot\cdot \text{(III)}.
\end{aligned}$$

When $n_1 = 2$, we have:

$$\begin{aligned}
1 \le \prod_{v|\infty} |\Delta^*|_v^{n_v} &= |\Delta^*|_{v_1}^2 \cdot \prod_{v|\infty, v\ne v_1} |\Delta^*|_v^{n_v} \\
&\le n^{c_{15}} H(\mathbf{x})^d \left(\prod_{i=3}^{d} \max\{1, |\alpha^{(i)}|^{-1}\}^s b^{ds} e^{(1+\varepsilon(n))(ds(s+1)/2)} e^{s\delta(d-2)}\right)^n \\
&\times \left\{ \left(e^{-s}|\alpha| e^{(s-1)\delta}\right)^n + |\ell| \cdot \left(|\alpha|^{-s} e^{s\delta}\right)^n \right\}^2 \cdot\cdot \text{(IV)}.
\end{aligned}$$

Taking the 1/2-th power of the inequality (IV), we get by $0 < |\alpha| < 1$ and $H(\mathbf{x}) \ge 1$:

$$\begin{aligned}
1 \le\, & n^{c_{16}} H(\mathbf{x})^d \left\{\prod_{i=2}^{d} \max\{1, |\alpha^{(i)}|^{-1}\}^s b^{ds} e^{(1+\varepsilon(n))(ds(s+1)/2)} e^{s\delta(d-1)}\right\}^n \\
&\times \left\{ \left(e^{-s}|\alpha| e^{(s-1)\delta}\right)^n + |\ell| \cdot \left(|\alpha|^{-s} e^{s\delta}\right)^n \right\} \quad \text{as (III), even if } n_1 = 2.
\end{aligned}$$

Recall $e^{-\tau} = |\alpha| e^{-s} b^{ds} e^{ds(s+1)/2} e^{(ds-1)\delta} \prod_{i=2}^{d} \max\{1, |\alpha^{(i)}|^{-1}\}^{s}$ and $e^{\rho} = \frac{e^{s+\delta}}{|\alpha|^{s+1}}$.
By our assumption (2.1), we have $\tau > 0$ and $\rho > s+\delta$. Let $\varepsilon > 0$. Take a suitable small $0 < \varepsilon' < \tau$ satisfying $\frac{\rho}{\tau} + \frac{\varepsilon}{2} > \frac{\rho}{\tau - \varepsilon'}$. Write $\tau' = \tau - \varepsilon'$. There exists a positive integer n^* such that for any $n \geq n^*$ we have:

$$n^{c_{17}} \Big(e^{-s} |\alpha| e^{(ds-1)\delta} \cdot b^{ds}\ e^{(1+\varepsilon(n))(ds(s+1)/2)}\Big)^{n} \prod_{i=2}^{d} \max\{1, |\alpha^{(i)}|^{-1}\}^{ns} \leq e^{-\tau' n} \quad \text{and}$$

$$n^{c_{18}} \Big(|\alpha|^{-s} e^{ds\delta} \cdot b^{ds}\ e^{(1+\varepsilon(n))(ds(s+1)/2)}\Big)^{n} \prod_{i=2}^{d} \max\{1, |\alpha^{(i)}|^{-1}\}^{ns} \leq e^{-\tau' n} e^{\rho n} = e^{(\rho - \tau')n}.$$

Consequently,

$$|\ell| > \frac{1 - e^{-\tau' n} \cdot H(\mathbf{x})^{d}}{e^{(\rho-\tau')n} \cdot H(\mathbf{x})^{d}}.$$

Now for this fixed n^*, we consider $B_0 > 1$ such that for all $B := H(\mathbf{x}) \geq B_0$ we have $e^{-\tau' n^*} \cdot B^{d} \geq \frac{1}{2}$, namely $1 - e^{-\tau' n^*} \cdot B^{d} \leq 1 - e^{-\tau' n^*} \cdot B_0^{d} \leq \frac{1}{2}$.
Let $\tilde{n}$ be the least positive integer such that we have $e^{-\tau' \tilde{n}} \cdot B^{d} < \frac{1}{2}$, that is $1 - e^{-\tau' \tilde{n}} \cdot B^{d} > \frac{1}{2}$. Since $\tilde{n} > n^*$, we get

$$|\ell| > \frac{\frac{1}{2}}{e^{(\rho-\tau')\tilde{n}} \cdot B^{d}}.$$

From the definition of $\tilde{n}$, we have $e^{-(\tilde{n}-1)\tau'} \cdot B^{d} \geq \frac{1}{2}$, then $e^{\tilde{n}} \leq (2B^{d})^{\frac{1}{\tau'}} \cdot e$. Finally we obtain

$$|\ell| > \frac{1}{2^{\frac{\rho}{\tau'}} \cdot e^{(\rho-\tau')} \cdot B^{\frac{d\rho}{\tau'}}} \geq \frac{1}{B^{\frac{d\rho}{\tau} + \frac{d\varepsilon}{2}}}.$$

This completes the proof of Theorem 2.4. □

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