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Adelically summable normalized weights and adelic equidistribution of effective divisors having small diagonals and small heights on the Berkovich projective lines

By

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Abstract

We introduce the notion of an adelically summable normalized weight g, which is a family of normalized weights on the Berkovich projective lines satisfying a summability condition. We then establish an adelic equidistribution of effective k-divisors on the projective line over the separable closure k_s in $\overline{k}$ of a product formula field k having small g-heights and small diagonals. This equidistribution result generalizes Ye's for the Galois conjugacy classes of algebraic numbers with respect to quasi-adelic measures.

§ 1. Introduction

Equidistribution of small points is quite classical ([15], [5], [13], [6], [1], [2], [3], [7], [10], and, most recently, [18]) but, to extend it to effective divisors, one would need the further assumption of *small diagonals* ([12]). The *adelic* condition on weights (or metrics of line bundles) is quite natural from the arithmetic point of view, but what is really necessary in the proof of the equidistribution is a weaker *summability* ([17]).

Our aim in this article is to contribute to the study of adelic equidistribution of a sequence of *effective divisors* defined over a product formula field k on the projective line $\mathbb{P}^1(k_s)$ over the separable closure k_s of k in an algebraic closure $\overline{k}$ having *small*

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diagonals and *small g-heights* with respect to an *adelically summable normalized weight* g. This equidistribution result generalizes Ye [17, Theorem 1.1], which is a generalization of Baker–Rumely [3, Theorem 2.3], Chambert-Loir [7, Théorème 4.2], Favre–Rivera-Letelier [10, Théorème 2].

In Sections 2 and 3, we recall background from arithmetic and potential theory on the Berkovich projective line, introduce the notion of an adelically summable normalized weight g, and state the main result (Theorem 1). In Section 4, we show Theorem 1. A part of the proof of Theorem 1 is an adaption of the proof of [12, Theorem 2]. In Section 5, we give an example of an adelically summable normalized weight g which is not an adelic normalized weight.

§ 2. Background

For the details including references of this section, see [12].

Definition 2.1. An *effective k-divisor* (or an effective divisor defined over k) on $\mathbb{P}^1(\overline{k})$ is the scheme theoretic vanishing of a non-constant homogeneous polynomial in two variables with coefficients in k. An effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$ is said to be *on* $\mathbb{P}^1(k_s)$ if $\operatorname{supp}\mathcal{Z} \subset \mathbb{P}^1(k_s)$.

Effective divisors include Galois conjugacy classes of *algebraic numbers*, and are also called *Galois stable multisets*.

Definition 2.2. A field k is a *product formula field* if k is equipped with (i) the (possibly uncountable) family M_k of all places of k, (ii) a family $(|\cdot|_v)_{v\in M_k}$, where for each $v \in M_k$, $|\cdot|_v$ is a non-trivial absolute value of k representing v, and (iii) a family $(N_v)_{v\in M_k}$ in $\mathbb{N}$ such that the following *product formula property* holds: for every $z \in k \setminus \{0\}$, $|z|_v = 1$ for all but finitely many $v \in M_k$ and (PF) $\prod_{v\in M_k} |z|_v^{N_v} = 1$.

Product formula fields include number fields and function fields over curves. A product formula field k is a number field if and only if k has an *infinite* place v, i.e., $|\cdot|_v$ is archimedean (see, e.g., the paragraph after [4, Definition 7.51]).

Notation 2.3. For each $v \in M_k$, let k_v be the completion of k with respect to $|\cdot|_v$ and $\mathbb{C}_v$ the completion of an algebraic closure of k_v with respect to (the extended) $|\cdot|_v$, and we fix an embedding of $\overline{k}$ to $\mathbb{C}_v$ which extends that of k to k_v.

By convention, the dependence of a local quantity induced by $|\cdot|_v$ on each $v \in M_k$ is emphasized by adding the suffix v to it.

Let K be an algebraically closed field that is complete with respect to a non-trivial absolute value $|\cdot|$ (e.g., $\mathbb{C}_v$ for a product formula field k and each $v \in M_k$), which is either non-archimedean or archimedean.

Notation 2.4 (the normalized chordal metric on $\mathbb{P}^1$). On K^2, let $\|(p_0, p_1)\|$ be either the maximal norm $\max\{|p_0|, |p_1|\}$ (for non-archimedean K) or the Euclidean norm $\sqrt{|p_0|^2 + |p_1|^2}$ (for archimedean K), and let $\pi = \pi_K : K^2 \setminus \{(0,0)\} \to \mathbb{P}^1 = \mathbb{P}^1(K)$ be the canonical projection such that $\pi(p_0, p_1) = p_1/p_0 \in K$ if $p_0 \neq 0$ and that $\pi(0,1) = \infty$. With the wedge product $(z_0, z_1) \wedge (w_0, w_1) := z_0 w_1 - z_1 w_0$ on K^2, the *normalized chordal metric* $[z, w]$ on $\mathbb{P}^1$ is defined by

$$(z, w) \mapsto [z, w] = |p \wedge q| / (\|p\| \cdot \|q\|) \leq 1$$

on $\mathbb{P}^1 \times \mathbb{P}^1$, where $p \in \pi^{-1}(z)$ and $q \in \pi^{-1}(w)$.

The *Berkovich* projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ is a compact augmentation of $\mathbb{P}^1$. Letting $\delta_{\mathcal{S}}$ be the Dirac measure on P^1 at a point $\mathcal{S} \in \mathsf{P}^1$, we set

$$\Omega_{\mathrm{can}} := \begin{cases} \delta_{\mathcal{S}_{\mathrm{can}}} & \text{for non-archimedean } K, \\ \omega & \text{for archimedean } K, \end{cases}$$

where $\mathcal{S}_{\mathrm{can}}$ is the canonical (or Gauss) point in P^1 (represented by the ring $\mathcal{O}_K = \{z \in K : |z| \leq 1\}$ of K-integers) for non-archimedean K and ω is the Fubini-Study area element on $\mathbb{P}^1$ normalized as $\omega(\mathbb{P}^1) = 1$ for archimedean K. For non-archimedean K, the *generalized Hsia kernel* $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ on P^1 with respect to $\mathcal{S}_{\mathrm{can}}$ is the unique (jointly) upper semicontinuous and separately continuous extension of the normalized chordal metric $[z, w]$ on $\mathbb{P}^1 (\times \mathbb{P}^1)$ to $\mathsf{P}^1 \times \mathsf{P}^1$. In particular,

$$[\mathcal{S}, \mathcal{S}']_{\mathrm{can}} \leq 1 \quad \text{on } \mathsf{P}^1 \times \mathsf{P}^1$$

and $[\mathcal{S}_{\mathrm{can}}, \mathcal{S}_{\mathrm{can}}]_{\mathrm{can}} = 1$. By convention, for archimedean K, the kernel function $[\mathcal{S}, \mathcal{S}']_{\mathrm{can}}$ is defined by the $[z, w]$ itself.

Let $\Delta = \Delta_{\mathsf{P}^1}$ be the Laplacian on P^1 normalized so that for each $\mathcal{S}' \in \mathsf{P}^1$,

$$\Delta \log[\cdot, \mathcal{S}']_{\mathrm{can}} = \delta_{\mathcal{S}'} - \Omega_{\mathrm{can}}$$

on P^1. For a construction of Δ in non-archimedean case, see [4, §5], [9, §7.7], [16, §3] and also [11, §2.5]; in [4] the opposite sign convention on Δ is adopted.

Definition 2.5. A *continuous weight g on* P^1 is a continuous function on P^1 such that

$$\mu^g := \Delta g + \Omega_{\mathrm{can}}$$

is a probability Radon measure on P^1. We also call g a (continuous Ω_{can}-)potential on P^1 of μ^g.

For a continuous weight g on P^1, the *g-potential kernel* on P^1 (or the negative of an Arakelov Green kernel function on P^1 relative to μ^g [4, §8.10]) is the function

$$\Phi_g(\mathcal{S},\mathcal{S}') := \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} - g(\mathcal{S}) - g(\mathcal{S}') \quad \text{on } \mathsf{P}^1 \times \mathsf{P}^1$$

and the *g-equilibrium energy* $V_g \in (-\infty,+\infty)$ *of* P^1 is the supremum of the *g-energy* functional

$$\nu \mapsto \int_{\mathsf{P}^1\times\mathsf{P}^1} \Phi_g \mathrm{d}(\nu\times\nu) \tag{2.1}$$

on the space of all probability Radon measures ν on P^1; indeed, $V_g \le 2\cdot\sup_{\mathsf{P}^1}|g| < \infty$ and

$$\int_{\mathsf{P}^1\times\mathsf{P}^1} \log[\mathcal{S},\mathcal{S}']_{\mathrm{can}} \mathrm{d}(\Omega_{\mathrm{can}}\times\Omega_{\mathrm{can}})(\mathcal{S},\mathcal{S}') = \begin{cases} -\frac{1}{2} & \text{for archimedean } K \\ 0 & \text{otherwise} \end{cases} > -\infty \tag{2.2}$$

(for archimedean $K \cong \mathbb{C}$, the left hand side in (2.2) equals $\int_{\mathbb{C}\setminus\{0\}} \log[\cdot,\infty]\mathrm{d}\omega$, which is computed by $[z,\infty] = 1/\sqrt{1+r^2}$ and $\omega = r\mathrm{d}r\mathrm{d}\theta/(\pi(1+r^2)^2)$ ($z = re^{i\theta}$, $r>0, \theta\in\mathbb{R}$)). A probability Radon measure ν on P^1 at which the g-energy functional (2.1) attains the supremum V_g is called a *g-equilibrium mass distribution on* P^1; indeed, μ^g is the unique g-equilibrium mass distribution on P^1 (for non-archimedean K, see [4, Theorem 8.67, Proposition 8.70]).

A continuous weight g on P^1 is a *normalized weight* on P^1 if $V_g = 0$. For every continuous weight g on P^1, $\overline{g} := g + V_g/2$ is the unique normalized weight on P^1 such that $\mu^{\overline{g}} = \mu^g$ on P^1.

Example 2.6. The function $g_0 :\equiv 0$ on P^1 is a continuous weight on P^1 since (it is a continuous function on P^1 and) $\mu^{g_0} = \Delta g_0 + \Omega_{\mathrm{can}} = \Omega_{\mathrm{can}}$ on P^1. By (2.2), g_0 is itself a normalized weight on P^1 for non-archimedean K, and $\overline{g_0} = g_0 + V_{g_0}/2 \equiv -1/4$ on P^1 is a normalized weight on P^1 for archimedean K.

Definition 2.7. We say a sequence (ν_n) of positive and discrete Radon measures on P^1 satisfying $\lim_{n\to\infty}\nu_n(\mathsf{P}^1) = \infty$ has *small diagonals* if

$$\lim_{n\to\infty} \frac{(\nu_n\times\nu_n)(\mathrm{diag}_{\mathbb{P}^1(K)})}{\nu_n(\mathsf{P}^1)^2} = 0,$$

where $\mathrm{diag}_{\mathbb{P}^1}$ is the diagonal of $\mathbb{P}^1\times\mathbb{P}^1$. For a continuous weight g on P^1 and a Radon measure ν on P^1, the *g-Fekete sum* with respect to ν is defined by

$$(\nu,\nu)_g := \int_{\mathsf{P}^1\times\mathsf{P}^1\setminus\mathrm{diag}_{\mathbb{P}^1}} \Phi_g \mathrm{d}(\nu\times\nu),$$

and we say a sequence (ν_n) of positive and discrete Radon measures on P^1 satisfying $\lim_{n\to\infty}\nu_n(\mathsf{P}^1)=\infty$ is an *asymptotically g-Fekete configuration on* P^1 if (ν_n) not only has small diagonals but also satisfies

$$\lim_{n\to\infty}\frac{(\nu_n,\nu_n)_g}{(\nu_n(\mathsf{P}^1))^2}=V_g.$$

Remark 2.8. In the definition of an asymptotically g-Fekete configuration (ν_n) on P^1, under the former small diagonal assumption, the latter one is equivalent to the weaker $\liminf_{n\to\infty}(\nu_n,\nu_n)_g/(\nu_n(\mathsf{P}^1))^2\ge V_g$ since we always have

$$\limsup_{n\to\infty}\frac{(\nu_n,\nu_n)_g}{(\nu_n(\mathsf{P}^1))^2}\le V_g \tag{2.3}$$

(see, e.g., [4, Lemma 7.54]). By a classical argument (cf. [14, Theorem 1.3 in Chapter III]), if (ν_n) is an asymptotically g-Fekete configuration on P^1, then the weak convergence $\lim_{n\to\infty}\nu_n/(\nu_n(\mathsf{P}^1))=\mu^g$ on P^1 holds.

Let k be a field, and K be an *algebraic and metric completion* of k in that K is an algebraically closed field that is complete with respect to a non-trivial absolute value $|\cdot|$ and is a field extension of k (e.g., a product formula field k and $K=\mathbb{C}_v$ for each $v\in M_k$). An effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$ is regarded as a positive and discrete Radon measure $\sum_{w\in\operatorname{supp}\mathcal{Z}}(\operatorname{ord}_w\mathcal{Z})\delta_w$ on $\mathsf{P}^1(K)$, which is still denoted by $\mathcal{Z}$ and whose support is in $\mathbb{P}^1(\overline{k})$, and then the *diagonal*

$$(\mathcal{Z}\times\mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(\overline{k})})=\sum_{w\in\operatorname{supp}\mathcal{Z}}(\operatorname{ord}_w\mathcal{Z})^2 \tag{2.4}$$

of $\mathcal{Z}$ is independent of the choice of K.

Definition 2.9. For every continuous weight g on P^1, the *logarithmic g-Mahler measure* of an effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$ is defined by

$$M_g(\mathcal{Z}):=\int_{\mathsf{P}^1}g\mathrm{d}\mathcal{Z}+M^{\#}(\mathcal{Z}),$$

where we set $M^{\#}(\mathcal{Z}):=-\sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{\infty\}}(\operatorname{ord}_w\mathcal{Z})\log[w,\infty]\ge 0$.

§ 3. Main result

Let k be a product formula field. In the following, a sum over M_k will be indeed a sum over an at most countable subset in M_k, the convergence of which will be understood in the absolute sense.

Definition 3.1. A family $g = (g_v)_{v\in M_k}$ is an *adelically summable normalized weight* if (i) for every $v \in M_k$, g_v is a normalized weight on $\mathsf{P}^1(\mathbb{C}_v)$, i.e., $V_{g_v} = 0$, and (ii) the following summability condition holds: $g_v \equiv 0$ on $\mathbb{P}^1(k_s)$ for every $v \in M_k$ but some *countable* subset E_g in M_k, and

$$\sum_{v\in M_k} N_v \cdot \sup_{\mathbb{P}^1(k_s)} |g_v| < \infty. \tag{3.1}$$

In particular, $\sum_{v\in M_k} N_v \cdot g_v$ is absolutely convergent pointwise on $\mathbb{P}^1(k_s)$.

Remark 3.2. In Ye [17, §2.2], the family $(\mu^{g_v})_{v\in M_k}$ of the probability Radon measures $\mu^{g_v} = \Delta g_v + \Omega_{\mathrm{can},v}$ on $\mathsf{P}^1(\mathbb{C}_v)$ associated with a family $g = (g_v)_{v\in M_k}$ of normalized weights g_v on $\mathsf{P}^1(\mathbb{C}_v)$ is called a *quasi-adelic probability measure* if the following multiplicativity condition holds: setting

$$\begin{aligned}
G^v &:= g_v \circ \pi_{\mathbb{C}_v} + \log\|\cdot\|_v \quad \text{on } \mathbb{C}_v^2 \setminus \{(0,0)\},\\
K^v &:= \{p \in \mathbb{C}_v^2 \setminus \{(0,0)\} : G^v(p) \le 0\} \cup \{(0,0)\}\\
&= \{p \in \mathbb{C}_v^2 \setminus \{(0,0)\} : \|p\|_v \le e^{-g_v(\pi_{\mathbb{C}_v}(p))}\} \cup \{(0,0)\},\\
r^{\#}_{\mathrm{outer}}(K^v) &:= \sup\{\|p\|_v : p \in K^v\}, \quad \text{and}\\
r^{\#}_{\mathrm{inner}}(K^v) &:= \inf\left\{\|p\|_v : p \in \mathbb{C}_v^2 \setminus K^v\right\}
\end{aligned}$$

for each $v \in M_k$ (the $r^{\#}_{\mathrm{outer}}(K^v)$ and $r^{\#}_{\mathrm{inner}}(K^v)$ are called the *outer* and *inner* radii of K_v in $(\mathbb{C}_v^2, \|\cdot\|_v)$, respectively, and the normalization $\mathrm{Cap}(K^v)_v = 1$ in [17, §2.1] is equivalent to $V_{g_v} = 0$), we not only have $G^v \equiv \log\|\cdot\|_v$ on $\mathbb{C}_v^2$, i.e., $g_v \equiv 0$ on $\mathbb{P}^1(\mathbb{C}_v)$ for all but countably many $v \in M_k$ but also

$$\sum_{v\in M_k} N_v \cdot \log(r^{\#}_{\mathrm{outer}}(K^v)) \in \mathbb{R} \quad \text{and} \quad \sum_{v\in M_k} N_v \cdot \log(r^{\#}_{\mathrm{inner}}(K^v)) \in \mathbb{R}. \tag{3.1$'$}$$

For every $v \in M_k$, we have

$$r^{\#}_{\mathrm{outer}}(K^v) \ge e^{-\inf_{\mathbb{P}^1(k_s)} g_v} \quad \text{and} \quad r^{\#}_{\mathrm{inner}}(K^v) \le e^{-\sup_{\mathbb{P}^1(k_s)} g_v}; \tag{3.2}$$

indeed, for every $\epsilon > 0$, by the continuity of g_v on $\mathbb{P}^1(\mathbb{C}_v)$ and the surjectivity of $\pi_{\mathbb{C}_v} : \mathbb{C}_v^2 \setminus \{(0,0)\} \to \mathbb{P}^1(\mathbb{C}_v)$, there is $p \in \mathbb{C}_v^2 \setminus \{(0,0)\}$ such that $e^{-\inf_{\mathbb{P}^1(\mathbb{C}_v)} g_v} - \epsilon < e^{-g_v(\pi_{\mathbb{C}_v}(p))}$, and by the density of $|\mathbb{C}_v^*|_v$ in $\mathbb{R}_{\ge 0}$, there is $c \in \mathbb{C}_v^*$ such that $e^{-\inf_{\mathbb{P}^1(\mathbb{C}_v)} g_v} - \epsilon < \|c\cdot p\|_v < e^{-g_v(\pi_{\mathbb{C}_v}(p))} = e^{-g_v(\pi_{\mathbb{C}_v}(c\cdot p))}$. Hence $e^{-\inf_{\mathbb{P}^1(\mathbb{C}_v)} g_v} \le r^{\#}_{\mathrm{outer}}(K^v)$, so that $e^{-\inf_{\mathbb{P}^1(k_s)} g_v} \le r^{\#}_{\mathrm{outer}}(K^v)$. A similar argument also yields $e^{-\sup_{\mathbb{P}^1(k_s)} g_v} \ge (e^{-\sup_{\mathbb{P}^1(\mathbb{C}_v)} g_v} \ge) r^{\#}_{\mathrm{inner}}(K^v)$.

In particular, (3.1′) is stronger than (3.1); indeed, by (3.2), the condition (3.1′) implies

$$\sum_{v\in M_k} N_v \cdot \inf_{\mathbb{P}^1(k_s)} g_v \in \mathbb{R} \quad \text{and} \quad \sum_{v\in M_k} N_v \cdot \sup_{\mathbb{P}^1(k_s)} g_v \in \mathbb{R},$$

which is equivalent to (3.1).

Example 3.3. A family $g = (g_v)_{v \in M_k}$ is an *adelic normalized weight* if g satisfies the condition (i) in Definition 3.5 and the *at most finitely many non-triviality* condition that $g_v \equiv 0$ on $\mathsf{P}^1(\mathbb{C}_v)$ for all but finitely many $v \in M_k$.

An adelic normalized weight g is an adelically summable normalized weight.

Definition 3.4. The *g-height* of an effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(k_s)$ with respect to an adelically summable normalized weight $g = (g_v)_{v \in M_k}$ is

$$h_g(\mathcal{Z}) := \sum_{v \in M_k} N_v \frac{M_{g_v}(\mathcal{Z})}{\deg \mathcal{Z}}.$$

Remark 3.5. In Definition 3.4, using the product formula property of k (and by a standard argument involving the ramification theory of valuation), for every $v \in M_k$ but some *finite* subset $E_{\mathcal{Z}}$ in M_k, we have $M^{\#}(\mathcal{Z})_v = 0$. In particular, $h_g(\mathcal{Z}) \in \mathbb{R}$.

Definition 3.6. For an adelically summable normalized weight g, we say a sequence $(\mathcal{Z}_n)$ of effective k-divisors on $\mathbb{P}^1(k_s)$ has *small g-heights* if

$$\limsup_{n \to \infty} h_g(\mathcal{Z}_n) \leq 0.$$

Our principal result is the following *adelic asymptotically Fekete configuration* theorem, which is stronger than an *adelic equidistribution* theorem and generalizes Ye [17, Theorem 1.1].

Theorem 1. *Let k be a product formula field and k_s the separable closure of k in $\overline{k}$. Let $g = (g_v)_{v \in M_k}$ be an adelically summable normalized weight. If a sequence $(\mathcal{Z}_n)$ of effective k-divisors on $\mathbb{P}^1(k_s)$ satisfying $\lim_{n \to \infty} \deg \mathcal{Z}_n = \infty$ has both small diagonals and small g-heights, then the uniform convergence*

$$\lim_{n \to \infty} \sup_{v \in M_k} N_v \left| \frac{(\mathcal{Z}_n, \mathcal{Z}_n)_{g_v}}{(\deg \mathcal{Z}_n)^2} \right| = 0$$

holds. In particular, for every $v \in M_k$, $(\mathcal{Z}_n)$ is an asymptotically g_v-Fekete configuration on $\mathsf{P}^1(\mathbb{C}_v)$, so that $\lim_{n \to \infty} \mathcal{Z}_n / \deg \mathcal{Z}_n = \mu^{g_v}$ weakly on $\mathsf{P}^1(\mathbb{C}_v)$.

In Theorem 1, if all the divisors $\mathcal{Z}_n$ are the Galois conjugacy classes of k-algebraic numbers, then the small diagonal assumption always holds.

§ 4. Proof of Theorem 1

Notation 4.1. Let k be a field. For an effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$, set

$$D^*(\mathcal{Z}|\overline{k}) := \prod_{w \in \operatorname{supp} \mathcal{Z} \setminus \{\infty\}} \prod_{w' \in \operatorname{supp} \mathcal{Z} \setminus \{w, \infty\}} (w - w')^{(\operatorname{ord}_w \mathcal{Z})(\operatorname{ord}_{w'} \mathcal{Z})} \in \overline{k} \setminus \{0\},$$

which is indeed in $k \setminus \{0\}$ if $\mathcal{Z}$ is on $\mathbb{P}^1(k_s)$ (cf. [12, Theorem 7]).

Recall the following local computation from [12].

Lemma 4.2 ([12, Lemma 5.2]). *Let k be a field and K an algebraic and metric augmentation of k. Then for every continuous weight g on $\mathsf{P}^1(K)$ and every effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(\overline{k})$,*

$$(\mathcal{Z},\mathcal{Z})_g + 2\cdot\sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{\infty\}}(\operatorname{ord}_w\mathcal{Z})^2\log[w,\infty] = \log|D^*(\mathcal{Z}|\overline{k})| - 2(\deg\mathcal{Z})^2\frac{M_g(\mathcal{Z})}{\deg\mathcal{Z}} + 2\cdot\sum_{w\in\operatorname{supp}\mathcal{Z}}(\operatorname{ord}_w\mathcal{Z})^2 g(w). \tag{4.1}$$

Let k be a product formula field and k_s the separable closure of k in $\overline{k}$, and let $g=(g_v)_{v\in M_k}$ be an adelically summable normalized weight. For every $v\in M_k$ and every effective k-divisor $\mathcal{Z}$ on $\mathbb{P}^1(k_s)$, recalling $[\mathcal{S},\mathcal{S}']_{\mathrm{can},v}\le 1$ and (2.4), we have

$$\begin{aligned}
\frac{(\mathcal{Z},\mathcal{Z})_{g_v}}{(\deg\mathcal{Z})^2} &= \int_{(\mathsf{P}^1(\mathbb{C}_v)\times\mathsf{P}^1(\mathbb{C}_v))\setminus\operatorname{diag}_{\mathbb{P}^1(k_s)}}\log[\mathcal{S},\mathcal{S}']_{\mathrm{can},v}\mathrm{d}\frac{(\mathcal{Z}\times\mathcal{Z})}{(\deg\mathcal{Z})^2}(\mathcal{S},\mathcal{S}')\\
&\quad -2\cdot\int_{(\mathsf{P}^1(\mathbb{C}_v)\times\mathsf{P}^1(\mathbb{C}_v))\setminus\operatorname{diag}_{\mathbb{P}^1(k_s)}} g_v(\mathcal{S})\mathrm{d}\frac{(\mathcal{Z}\times\mathcal{Z})}{(\deg\mathcal{Z})^2}(\mathcal{S},\mathcal{S}')\\
&\le -2\cdot\int_{\mathsf{P}^1(\mathbb{C}_v)} g_v\mathrm{d}\frac{\mathcal{Z}}{\deg\mathcal{Z}} + 2\cdot\frac{\sum_{w\in\operatorname{supp}\mathcal{Z}}(\operatorname{ord}_w\mathcal{Z})^2 g_v(w)}{(\deg\mathcal{Z})^2}\\
&\le -2\inf_{\mathbb{P}^1(k_s)} g_v + 2\cdot\frac{(\mathcal{Z}\times\mathcal{Z})(\operatorname{diag}_{\mathbb{P}^1(k_s)})}{(\deg\mathcal{Z})^2}\cdot\sup_{\mathbb{P}^1(k_s)} g_v\\
&\le 4\cdot\sup_{\mathbb{P}^1(k_s)}|g_v|. \qquad (4.2)
\end{aligned}$$

Let E_g and $E_{\mathcal{Z}}$ be as in Definition 3.1 and Remark 3.5, respectively. In Notation 4.1, we already mentioned that $D^*(\mathcal{Z}|\overline{k})\in k\setminus\{0\}$ when $\operatorname{supp}\mathcal{Z}\subset\mathbb{P}^1(k_s)$. Hence by the product formula property of k, $\tilde{E}_{\mathcal{Z}}:=E_{\mathcal{Z}}\cup\{v\in M_k:|D^*(\mathcal{Z}|\overline{k})|_v\neq 1\}$ is still a finite subset in M_k. For every $v\in M_k\setminus E_{\mathcal{Z}}$, we have $\sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{\infty\}}(\operatorname{ord}_w\mathcal{Z})^2\log[w,\infty]_v=0$ since

$$0=-(\deg\mathcal{Z})M^{\#}(\mathcal{Z})_v\le\sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{\infty\}}(\operatorname{ord}_w\mathcal{Z})^2\log[w,\infty]_v\le 0,$$

which with (4.1) applied to $g=g_v$ yields $(\mathcal{Z},\mathcal{Z})_{g_v}=0$ for every $v\in M_k\setminus(E_g\cup\tilde{E}_{\mathcal{Z}})$. Summing up $N_v\times$(4.1) applied to $g=g_v$ over all $v\in M_k$ and applying (PF) to $D^*(\mathcal{Z}|\overline{k})$, we have

$$\begin{aligned}\sum_{v\in M_k} N_v(\mathcal{Z},\mathcal{Z})_{g_v} =& -2(\deg\mathcal{Z})^2 h_g(\mathcal{Z}) + 2\cdot\sum_{w\in\operatorname{supp}\mathcal{Z}}(\operatorname{ord}_w\mathcal{Z})^2\sum_{v\in M_k}N_v\cdot g_v(w)\\ &-2\cdot\sum_{v\in M_k}N_v\sum_{w\in\operatorname{supp}\mathcal{Z}\setminus\{\infty\}}(\operatorname{ord}_w\mathcal{Z})^2\log[w,\infty]_v\\ \ge& -2(\deg\mathcal{Z})^2 h_g(\mathcal{Z}) + 2\cdot\sum_{w\in\operatorname{supp}\mathcal{Z}}(\operatorname{ord}_w\mathcal{Z})^2\sum_{v\in M_k}N_v\cdot g_v(w).\end{aligned}\tag{4.3}$$

Let $(\mathcal{Z}_n)$ be a sequence of effective k-divisors on $\mathbb{P}^1(k_s)$ satisfying $\lim_{n\to\infty}\deg\mathcal{Z}_n=\infty$ and having both small diagonals and small g-heights. In the following, sums over M_k are indeed sums over $E_g\cup\bigcup_{n\in\mathbb{N}}\tilde{E}_{\mathcal{Z}_n}$. For every sequence (n_j) in $\mathbb{N}$ tending to ∞ as $j\to\infty$ and every $v_0\in M_k$, we have

$$\begin{aligned}0 \ge& \limsup_{j\to\infty} N_{v_0}\frac{(\mathcal{Z}_{n_j},\mathcal{Z}_{n_j})_{g_{v_0}}}{(\deg\mathcal{Z}_{n_j})^2} \ge \sum_{v\in M_k}\limsup_{j\to\infty}N_v\frac{(\mathcal{Z}_{n_j},\mathcal{Z}_{n_j})_{g_v}}{(\deg\mathcal{Z}_{n_j})^2}\\ \ge& \limsup_{j\to\infty}\sum_{v\in M_k}N_v\frac{(\mathcal{Z}_{n_j},\mathcal{Z}_{n_j})_{g_v}}{(\deg\mathcal{Z}_{n_j})^2} \ge \liminf_{j\to\infty}\sum_{v\in M_k}N_v\frac{(\mathcal{Z}_{n_j},\mathcal{Z}_{n_j})_{g_v}}{(\deg\mathcal{Z}_{n_j})^2}\\ \ge& -2\cdot\limsup_{j\to\infty}h_g(\mathcal{Z}_{n_j}) + 2\cdot\liminf_{j\to\infty}\frac{(\mathcal{Z}_{n_j}\times\mathcal{Z}_{n_j})(\operatorname{diag}_{\mathbb{P}^1(k_s)})}{(\deg\mathcal{Z}_{n_j})^2}\times\sum_{v\in M_k}N_v\cdot\inf_{\mathbb{P}^1(k_s)}g_v\\ \ge& 0(=V_{g_{v_0}}),\end{aligned}$$

where the first and second inequalities are by (2.3) applied to g_v and $V_{g_v}=0$ for every v, the third one holds by Fatou's lemma, which can be used by (4.2) and the absolute summability condition (3.1), the fifth one is by (4.3) and (2.4), and the final one holds under the assumption that $(\mathcal{Z}_n)$ has both small diagonals and small g-heights (and $\sum_{v\in M_k}N_v\cdot\inf_{\mathbb{P}^1(k_s)}g_v\in\mathbb{R}$ by (3.1)). In particular, we have not only

$$\lim_{n\to\infty}\sum_{v\in M_k}N_v\frac{(\mathcal{Z}_n,\mathcal{Z}_n)_{g_v}}{(\deg\mathcal{Z}_n)^2}=0\tag{4.4}$$

but also, for every $v\in M_k$,

$$\lim_{n\to\infty}N_v\frac{(\mathcal{Z}_n,\mathcal{Z}_n)_{g_v}}{(\deg\mathcal{Z}_n)^2}=0.\tag{4.5}$$

The second assertion in Theorem 1 already follows from (4.5), and the final one is a consequence of the second (see Remark 2.8).

For completeness, we include a proof of the following.

Lemma 4.3. *Let $(a_{n,m})_{n\in\mathbb{N},m\in\mathbb{N}}$ be a doubly indexed sequence in $\mathbb{R}$ and (b_m) be a sequence in $\mathbb{R}_{\ge0}$ such that for every $m\in\mathbb{N}$, $\sup_{n\in\mathbb{N}}a_{n,m}\le b_m$, that $\sum_{m\in\mathbb{N}}b_m<\infty$,* *and that for every* $n \in \mathbb{N}$, $\sum_{m\in\mathbb{N}} a_{n,m}$ *converges* (*absolutely*). *If* $\lim_{n\to\infty} \sum_{m\in\mathbb{N}} a_{n,m} = 0$ *and for every* $m \in \mathbb{N}$, $\lim_{n\to\infty} a_{n,m} = 0$, *then we have* $\lim_{n\to\infty} \sup_{m\in\mathbb{N}} |a_{n,m}| = 0$.

Proof. For every $\epsilon > 0$, there is $M \in \mathbb{N}$ such that $\sum_{m>M} b_m < \epsilon/4$, and then there is $N \in \mathbb{N}$ such that for every $n > N$, $|\sum_{m\in\mathbb{N}} a_{n,m}| < \epsilon/4$ and $\sup_{m\le M} |a_{n,m}| < \epsilon/(4M)$. We claim that for every $n > N$, $\sup_{m\in\mathbb{N}} |a_{n,m}| < \epsilon$; indeed, by the choice of M and N, for every $n > N$, $\sup_{m\le M} |a_{n,m}| < \epsilon/(4M) \le \epsilon/4$. Moreover, for every $m_0 > M$ and every $n > N$, we have not only $a_{n,m_0} \le b_{m_0} \le \sum_{m>M} b_m < \epsilon/4$ but also

$$\begin{aligned} -\frac{3}{4}\epsilon < -\frac{\epsilon}{4} + \sum_{m\in\mathbb{N}} a_{n,m} - \frac{\epsilon}{4} &= \left(\sum_{m\le M} a_{n,m} - \frac{\epsilon}{4}\right) + a_{n,m_0} + \left(\sum_{m>M, m\ne m_0} a_{n,m} - \frac{\epsilon}{4}\right) \\ &\le \left(M \cdot \sup_{m\le M} |a_{n,m}| - \frac{\epsilon}{4}\right) + a_{n,m_0} + \left(\sum_{m>M} b_m - \frac{\epsilon}{4}\right) < a_{n,m_0}, \end{aligned}$$

so that for every $n > N$, $\sup_{m>M} |a_{n,m}| \le 3\epsilon/4$.

Hence the claim holds, and the proof of Lemma 4.3 is complete. □

Once Lemma 4.3 is at our disposal, the first assertion in Theorem 1 follows from (4.2), (3.1), ((4.3),) (4.4), and (4.5). Now the proof of Theorem 1 is complete. □

§ 5. An Example

Let us focus on the product formula field $(\mathbb{Q}, M_{\mathbb{Q}}, (N_v \equiv 1)_{v\in M_{\mathbb{Q}}})$. Recall that $M_{\mathbb{Q}} \cong \{\text{prime numbers}\} \cup \{\infty\}$, where $|\cdot|_\infty$ is the Euclidean norm on $\mathbb{Q}$ and, for every $p \in M_{\mathbb{Q}}$, $|\cdot|_p$ is the normalized p-adic norm on $\mathbb{Q}$.

For every $p \in M_{\mathbb{Q}} \setminus \{\infty\}$, since $|\overline{\mathbb{Q}}^*|_p$ accumulates to 1 in $\mathbb{R}$, we can fix an $a_p \in \overline{\mathbb{Q}}^*$ such that $|a_p|_p \in (1, \exp(p^{-2})]$, and let us define the function

$$g_p(\mathcal{S}) := \begin{cases} -\log[a_p(\mathcal{S}), \infty]_{\mathrm{can},p} + \log[\mathcal{S}, \infty]_{\mathrm{can},p} - \frac{\log|a_p|_p}{2} & \text{if } \mathcal{S} \in \mathsf{P}^1(\mathbb{C}_p) \setminus \{\infty\}, \\ \frac{\log|a_p|_p}{2} & \text{if } \mathcal{S} = \infty \end{cases}$$

on $\mathsf{P}^1(\mathbb{C}_p)$, where the linear function $z \mapsto a_p(z) := a_p \cdot z$ on $\mathbb{C}_p$ uniquely extends to a continuous automorphism on $\mathsf{P}^1(\mathbb{C}_p)$ (see, e.g., [4, §2.3]). For every $p \in M_{\mathbb{Q}} \setminus \{\infty\}$, since $[\cdot, \infty]_p = 1/\max\{1, |\cdot|_p\}$ on $\mathbb{C}_p$, we have

$$\begin{aligned} &-\log[a_p(z), \infty]_p + \log[z, \infty]_p \\ &\quad= \begin{cases} \log\max\{1, |a_p \cdot z|_p\} - \log\max\{1, |z|_p\} \equiv 0 & \text{if } |z|_p < |a_p|_p^{-1} (< 1), \\ \log|a_p|_p + \log\min\{1, |z|_p\} (\in [0, \log|a_p|_p]) & \text{if } |z|_p \ge |a_p|_p^{-1} \end{cases} \end{aligned}$$

on $\mathbb{C}_p$, which with the density of $\mathbb{C}_p$ in $\mathsf{P}^1(\mathbb{C}_p)$ implies that g_p is in fact a continuous function on $\mathsf{P}^1(\mathbb{C}_p)$ and satisfies

$$\sup_{\mathsf{P}^1(\mathbb{C}_p)} |g_p| \le \frac{\log|a_p|_p}{2} \le \frac{1}{2p^2}. \tag{5.1}$$

For $v = \infty \in M_{\mathbb{Q}}$, set $g_v \equiv -1/4$ on $\mathsf{P}^1(\mathbb{C}_v)$.

Let us see that *the family* $g := (g_v)_{v\in M_{\mathbb{Q}}}$ *is not an adelic normalized weight but an adelically summable normalized weight* (recall Example 3.3 and Definition 3.1, respectively).

For $v = \infty \in M_{\mathbb{Q}}$, $g_\infty \equiv -1/4$ is a normalized weight on $\mathsf{P}^1(\mathbb{C}_v)$ (see Example 2.6). For every $p \in M_{\mathbb{Q}} \setminus \{\infty\}$, g_p is a continuous weight on $\mathsf{P}^1(\mathbb{C}_p)$ by

$$\mu^{g_p} := \Delta g_p + \Omega_{\mathrm{can},p} = -a_p^*(\delta_\infty - \delta_{\mathcal{S}_{\mathrm{can},p}}) + (\delta_\infty - \delta_{\mathcal{S}_{\mathrm{can},p}}) + \delta_{\mathcal{S}_{\mathrm{can},p}} = \delta_{a_p^{-1}(\mathcal{S}_{\mathrm{can},p})}$$

on $\mathsf{P}^1(\mathbb{C}_p)$ (for the functoriality $\Delta a_p^* = a_p^*\Delta$, see e.g. [4, §9.5]), and since $[z,w]_p = |z-w|_p \cdot [z,\infty]_p \cdot [w,\infty]_p$ on $\mathbb{C}_p \times \mathbb{C}_p$, we also have

$$\Phi_{g_p}(\mathcal{S},\mathcal{S}') = \log[\mathcal{S},\mathcal{S}']_{\mathrm{can},p} - g_p(\mathcal{S}) - g_p(\mathcal{S}') = \log[a_p(\mathcal{S}), a_p(\mathcal{S}')]_{\mathrm{can},p}$$

on $\mathbb{C}_p \times \mathbb{C}_p$, and in turn on $\mathsf{P}^1(\mathbb{C}_p) \times \mathsf{P}^1(\mathbb{C}_p)$ by the density of $\mathbb{C}_p$ in $\mathsf{P}^1(\mathbb{C}_p)$ and the separate continuity of (the exp of) Φ_{g_p} on $\mathsf{P}^1(\mathbb{C}_p) \times \mathsf{P}^1(\mathbb{C}_p)$. Hence for every $p \in M_{\mathbb{Q}} \setminus \{\infty\}$,

$$V_{g_p} = \int_{\mathsf{P}^1(\mathbb{C}_p)\times\mathsf{P}^1(\mathbb{C}_p)} \Phi_{g_p} \mathrm{d}(\mu^{g_p} \times \mu^{g_p}) = \log[\mathcal{S}_{\mathrm{can},p}, \mathcal{S}_{\mathrm{can},p}]_{\mathrm{can},p} = 0,$$

which implies that g_p is still a normalized weight on $\mathsf{P}^1(\mathbb{C}_v)$.

For every $p \in M_{\mathbb{Q}} \setminus \{\infty\}$, $g_p \not\equiv 0$ on $\mathsf{P}^1(\mathbb{C}_p)$; for, if $g_p \equiv 0$ on P^1, then we have $\delta_{\mathcal{S}_{\mathrm{can},p}} = \mu^{g_p} = \delta_{a_p^{-1}(\mathcal{S}_{\mathrm{can},p})}$ on $\mathsf{P}^1(\mathbb{C}_p)$, which contradicts $|a_p|_p > 1$. Hence g is not an adelic normalized weight. On the other hand, by (5.1), we have

$$\sum_{v\in M_{\mathbb{Q}}\setminus\{\infty\}} N_v \cdot \sup_{\mathbb{P}^1(\overline{\mathbb{Q}})} |g_v| \le \sum_{p\in M_{\mathbb{Q}}\setminus\{\infty\}} \sup_{\mathsf{P}^1(\mathbb{C}_p)} |g_p| \le \frac{1}{2} \sum_{p\in M_{\mathbb{Q}}\setminus\{\infty\}} \frac{1}{p^2} < \infty,$$

which shows that g is an adelically summable normalized weight.

Remark 5.1. For a dynamical and highly non-trivial example, see DeMarco–Wang–Ye [8, §7.2] and Ye [17, §4].

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