RIMS Kôkyûroku Bessatsu
B64 (2017), 67–76

On the Iwasawa λ-invariants of cyclotomic $\mathbb{Z}_2$-extensions of real abelian fields

By

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Abstract

Let p be a prime number and k a real abelian field. We denote by $\lambda_p(k)$ the Iwasawa λ-invariant associated to the ideal class group of the cyclotomic $\mathbb{Z}_p$-extension of k. It is conjectured that $\lambda_p(k) = 0$. In [9], [10] and [11], Ichimura and Sumida discovered a good method for verifying that $\lambda_p(k) = 0$ for an odd prime number p. In this paper, we give a criterion for $\lambda_2(k) = 0$, as a generalization of the preceding result [6]. Our criterion is considered as an even prime version of the theorem of Ichimura and Sumida.

§ 1. Introduction

Let p be a prime number. For a number field k, let k_∞ denote the cyclotomic $\mathbb{Z}_p$-extension. We denote respectively by $\lambda_p(k)$ and $\mu_p(k)$ the Iwasawa λ-invariant and the μ-invariant associated to the ideal class group of k_∞. If k is a totally real number field, it is conjectured that $\lambda_p(k) = \mu_p(k) = 0$ ([8], [13, page 316]), which is often called Greenberg's conjecture. For an abelian field k, we know $\mu_p(k) = 0$ by the Ferrero-Washington theorem [3].

When p is odd and k is a real abelian field whose degree is not divisible by p, Ichimura and Sumida ([9], [10], [11]) discovered a good method for verifying the conjecture. It is suitable for a practical computer calculation, and for example, using it they showed that $\lambda_3(\mathbb{Q}(\sqrt{m})) = 0$ with all positive integers m less than 10^4. In [17], the author removed the assumption that $p \nmid [k : \mathbb{Q}]$ in the theorem of Ichimura and Sumida and generalized their theorem to an arbitrary real abelian field k.

In this paper, we study the case where $p = 2$. When $p = 2$ and $k = \mathbb{Q}(\sqrt{m})$ is a real quadratic field with prime number m, a study of Greenberg's conjecture began

Received March 31, 2015. Revised October 14, 2015 and November 30, 2015.
2010 Mathematics Subject Classification(s): 11R23.
Key Words: *Iwasawa invariants, cyclotomic units.*
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from Ozaki and Taya [14] and further development has been made by Fukuda and Komatsu [4], [5]. They gave some criteria for $\lambda_2(\mathbb{Q}(\sqrt{m})) = 0$ with prime number m and showed that $\lambda_2(\mathbb{Q}(\sqrt{m})) = 0$ for all prime numbers m less than 10^5 except $m = 13841, 67073$. Furthermore, Fukuda, Komatsu, Ozaki and the author [6] gave a sufficient condition for $\lambda_2(\mathbb{Q}(\sqrt{m})) = 0$ with prime number m, which is considered a slight modification of the method of Ichimura and Sumida. By using this criterion, we showed that $\lambda_2(\mathbb{Q}(\sqrt{m})) = 0$ for $m = 13841, 67073$. In this paper, we study the case where k not only is a real quadratic field $\mathbb{Q}(\sqrt{m})$ but is an arbitrary real abelian field. We will give a sufficient and necessary condition for $\lambda_2(k) = 0$ (Theorem 2.1), which is the same form as the criterion of Ichimura and Sumida. For the case where p is odd, a key of the proof of the criterion is a structure theorem of semi-local units modulo cyclotomic units in the cyclotomic $\mathbb{Z}_p$-extension of k proved by Iwasawa [12], Gillard [7] and the author [15]. This structure theorem is invented by Iwasawa when p is odd and k is the p-cyclotomic field, and is generalized by Gillard to the case where p is an arbitrary prime number and k is an abelian field with $p \nmid [k : \mathbb{Q}]$. Furthermore, when p is odd, the author removed the assumption that $p \nmid [k : \mathbb{Q}]$ in the theorem of Iwasawa and Gillard. Recently, the author [18] determined the structure of semi-local units modulo cyclotomic units when $p = 2$ and k is an arbitrary abelian field. By using this result, we can prove our theorem in the same way as the proof of Ichimura and Sumida.

In [17, Theorem 2.2], for an odd prime number p, we not only generalized the criterion of Ichimura and Sumida but obtained a simple sufficient condition for $\lambda_p(k) = 0$ in the special case where the degree of k is divisible by p (for an application see [16]). When $p = 2$, a similar result does not hold since the structure of the semi-local units for $p = 2$ is different from that for an odd prime number p.

I would like to thank Professor Keiichi Komatsu and Professor Takashi Fukuda who recommended her writing this paper.

§ 2. Main result

Let χ be a non-trivial $\overline{\mathbb{Q}_2}$-valued *even* Dirichlet character of the first kind with respect to the cyclotomic $\mathbb{Z}_2$-extension, i.e. the conductor of χ is not divisible by 8, and $k = k_\chi$ the fixed field of the kernel of χ. We denote by k_∞ the cyclotomic $\mathbb{Z}_2$-extension of k with its n-th layer k_n ($n \geq 0$). We write $A_n = A_{k_n}$ for the 2-Sylow subgroup of the ideal class group of k_n and put $X := \varprojlim A_n$, the projective limit being taken with respect to the relative norms. We put $\Delta := \mathrm{Gal}(k/\mathbb{Q})$ and $\Gamma := \mathrm{Gal}(k_\infty/k)$, so $\mathrm{Gal}(k_\infty/\mathbb{Q}) = \Delta \times \Gamma$ since χ is of the first kind. Then we regard X as a module over the completed group ring $\mathbb{Z}_2[\Delta][\![\Gamma]\!]$. It is well-known that X is finitely generated and torsion over $\mathbb{Z}_2[\Delta][\![\Gamma]\!]$ ([13, Theorem 5]). Let $\mathcal{O}$ denote the ring generated by the values of χ over $\mathbb{Z}_2$. For any $\mathbb{Z}_2[\Delta]$-module M, we define an $\mathcal{O}$-module M^χ, the χ-part of M, by

$$M^{\chi} := \{m \in M \otimes_{\mathbb{Z}_2} \mathcal{O} \mid \delta m = \chi(\delta) m \ \forall \delta \in \Delta\}.$$

If χ' is an $\mathbb{Q}_2$-conjugate of χ, then $M^{\chi'} \cong M^{\chi}$. Then $\mathcal{O}$-modules A_n^{χ} and X^{χ} are defined, and $X^{\chi} = \varprojlim A_n^{\chi}$ becomes an $\mathcal{O}[\![\Gamma]\!]$-module. Denote by $\boldsymbol{\mu}_{2^n}$ the group of 2^n-th roots of unity for $n \geq 0$ and $\boldsymbol{\mu}_{2^\infty} := \bigcup_n \boldsymbol{\mu}_{2^n}$. We fix a topological generator γ of Γ and define $q \in 4\mathbb{Z}_2$ by $(\zeta + \zeta^{-1})^{\gamma} = \zeta^{1+q} + \zeta^{-1-q}$ for all $\zeta \in \boldsymbol{\mu}_{2^\infty}$. We identify, as usual, the completed group ring $\mathcal{O}[\![\Gamma]\!]$ with the power series ring $\Lambda := \mathcal{O}[\![T]\!]$ by $\gamma = 1 + T$. Thus, X^{χ} is regarded as a module over Λ, and is finitely generated and torsion over Λ. For a finitely generated torsion Λ-module M, denote by $\mathrm{char}_{\Lambda} M$ the characteristic polynomial of M, which is a uniquely determined distinguished polynomial times a power of a fixed prime element of $\mathcal{O}$. We denote by $\lambda_2(\chi)$ (resp. $\mu_2(\chi)$) the λ-invariant (resp. μ-invariant) of X^{χ}. We know $\mu_2(\chi) = 0$ by the Ferrero-Washington theorem [3]. Greenberg's conjecture for χ is as follows:

Conjecture. Let χ be an even Dirichlet character of the first kind. It is conjectured X^{χ} to be finite, that is, $\mathrm{char}_{\Lambda} X^{\chi} = 1$ or equivalently $\lambda_2(\chi) = 0$.

For the trivial character χ_0, we know $\lambda_2(\chi_0) = 0$. One can see that Greenberg's conjecture for a real abelian field K, i.e. $\lambda_2(K) = 0$ is equivalent to saying that $\lambda_2(\chi) = 0$ for all characters χ of $\mathrm{Gal}(K/\mathbb{Q})$ of the first kind by using the Ferrero-Washington theorem [3] (see [17, Lemma 2.1]).

We will give a criterion for $\lambda_2(\chi) = 0$. To state this, we need to recall the relation between $\mathrm{char}_{\Lambda} X^{\chi}$ and the Kubota-Leopoldt 2-adic L-function $L_2(s, \chi)$ associated to χ. By Iwasawa, there exists a unique power series $g_{\chi}(T)$ in $\mathcal{O}[\![T]\!]$ such that

$$g_{\chi}((1+q)^s - 1) = \frac{1}{2} L_2(1-s, \chi)$$

(cf. [19, Theorem 7.10]). Using the 2-adic Weierstrass preparation theorem and the Ferrero-Washington theorem [3], one can uniquely write

$$g_{\chi}(T) = u_{\chi}(T) P_{\chi}(T) \tag{2.1}$$

where $P_{\chi}(T)$ is a distinguished polynomial in $\mathcal{O}[T]$ and $u_{\chi}(T)$ is a unit of Λ. Put $\lambda_2(\chi)^* = \deg P_{\chi}(T)$. It follows from the Iwasawa main conjecture proved in [20] that

$$\mathrm{char}_{\Lambda} X^{\chi} \mid P_{\chi}(T), \tag{2.2}$$

and hence $\lambda_2(\chi) \leq \lambda_2(\chi)^*$ (see (3.2) and (3.3)). Therefore we have $\mathrm{char}_{\Lambda} X^{\chi} = 1$ if and only if $P(T) \nmid \mathrm{char}_{\Lambda} X^{\chi}$ for all distinguished irreducible factors $P(T)$ of $P_{\chi}(T)$.

We state our main theorem of this paper, which is an even prime version of the theorems of Ichimura and Sumida [11] and the author [17]. We have to prepare some notation. We fix a distinguished polynomial $P(T)$ in $\mathcal{O}[T]$ such that $P(T) \mid P_\chi(T)$. Put $\omega_n = \omega_n(T) = (1+T)^{2^n} - 1$ and $\nu_n = \nu_n(T) = \omega_n/T$ for $n \geq 0$. By the Leopoldt conjecture for $p = 2$ and k_n proved in [1], $\Lambda/(P, \omega_n)$ and $\Lambda/(P, \nu_n)$ are finite abelian groups for any $n \geq 0$. We denote by $m_{P,n}$ the exponent of $\Lambda/(P, \omega_n)$ (resp. $\Lambda/(P, \nu_n)$) if $\chi(2) \neq 1$ (resp. $\chi(2) = 1$). Then we take a polynomial $X_{P,n}(T)$ in $\mathcal{O}[T]$ satisfying

$$X_{P,n}(T)P(T) \equiv m_{P,n} \mod \begin{cases} \omega_n & \text{if } \chi(2) \neq 1, \\ \nu_n & \text{if } \chi(2) = 1. \end{cases} \tag{2.3}$$

This polynomial $X_{P,n}$ is uniquely determined modulo ω_n (resp. ν_n) since ω_n and $P(T)$ are relatively prime. Choose an element $\widetilde{Y}_{P,n}$ in $\mathbb{Z}[\Delta][T]$ such that

$$\widetilde{Y}_{P,n} \equiv \widetilde{X}_{P,n} \bmod m_{P,n},$$

where $\widetilde{X}_{P,n}$ is an element of $\mathbb{Z}_2[\Delta][T]$ satisfying $\chi(\widetilde{X}_{P,n}) = X_{P,n}$. Here we regard χ as a $\mathbb{Z}_2[T]$-linear homomorphism $\mathbb{Z}_2[\Delta][T] \to \mathcal{O}[T]$ induced by χ. For any $m \geq 1$, we fix a primitive m-th root ζ_m of unity with the property that $\zeta_{mm'}^{m'} = \zeta_m$ for all $m' \geq 1$. Let f be the odd part of the conductor of χ, so the conductor of χ is f or $4f$. Since χ is a non-trivial even character of the first kind, we have $f \neq 1$. Put $k' = \mathbb{Q}(\zeta_f) \cap k(\zeta_4)$, then we have $k'(\zeta_4) = k(\zeta_4)$. We put $G = \mathrm{Gal}(k'/\mathbb{Q})$ and identify G with $\mathrm{Gal}(k(\zeta_4)/\mathbb{Q}(\zeta_4))$. We have $\mathrm{Gal}(k(\zeta_4)/\mathbb{Q}) \cong G \times \mathrm{Gal}(\mathbb{Q}(\zeta_4)/\mathbb{Q})$. We regard χ as a character of $\mathrm{Gal}(k(\zeta_4)/\mathbb{Q})$ and $\psi = \chi|_G$. Let ω be the Teichmüller character mod 4, i.e. the non-trivial character of $\mathrm{Gal}(\mathbb{Q}(\zeta_4)/\mathbb{Q})$. Then χ is ψ or $\psi\omega$ according to the conductor of χ is f or $4f$. Let G_2 (resp. G') denote the 2-Sylow subgroup (resp. the odd part) of G:

$$G = G_2 \times G'.$$

We put $\psi' = \psi|_{G'}$. Let

$$e_{\psi'} := \frac{1}{\#G'} \sum_{g \in G'} \mathrm{Tr}(\psi'(g))g^{-1}$$

be the idempotent of $\mathbb{Z}_2[G']$ corresponding to ψ', where Tr is the trace map from the field generated by the values of ψ' over $\mathbb{Q}_2$ to $\mathbb{Q}_2$. Let $\alpha \in \mathbb{Z}[G]$ denote an element of G_2 of order 2 (resp. $\alpha = 0$) if G_2 is non-trivial (resp. trivial), and $\mathbf{e}_\psi = \mathbf{e}_{\psi,P,n}$ an element of $\mathbb{Z}[G]$ such that

$$\mathbf{e}_\psi \equiv e_{\psi'}(1-\alpha) \bmod m_{P,n}.$$

We use cyclotomic units defined as follows

$$c_n = N_{\mathbb{Q}(\zeta_{f2^{n+2}})/k(\zeta_{2^{n+2}})}(1 - \zeta_{f2^{n+2}}).$$

Since $f \neq 1$, it is well-known that c_n is a unit. Furthermore, we can see that $c_n^{\mathbf{e}_\psi}$ is an element of k_n ([18]). We remark that our cyclotomic unit c_n is slightly different from the cyclotomic unit c_n which appeared in the criteria in [9], [11] and [17]. By the identification $\gamma = 1+T$, the element $\widetilde{Y}_{P,n}$ of $\mathbb{Z}[\Delta][T]$ can act on the group $k_n^\times$. For each $n \geq 0$ consider the following condition:

$$(H_{P,n}) \qquad c_n^{\mathbf{e}_\psi \widetilde{Y}_{P,n}} \notin (k_n^\times)^{m_{P,n}}.$$

We can see that the condition $(H_{P,n})$ does not depend on the choices of $X_{P,n}$, $\widetilde{X}_{P,n}$ and $\widetilde{Y}_{P,n}$. We remark that, in the case where $\chi(2) = 1$, the condition $(H_{P,0})$ does not hold since $c_0 = 1$ and $m_{P,0} = 1$ for any $P(T)$. We can show that the condition $(H_{P,n})$ implies $(H_{P,n+1})$ in a way similar to [11, Lemma 1] by using Lemma 4.1.

Our main theorem is stated as follows:

Theorem 2.1. *Let $P(T)$ be a distinguished polynomial in $\mathcal{O}[T]$ such that $P(T) \mid P_\chi(T)$. Then we have $P(T) \nmid \mathrm{char}_\Lambda X^\chi$ if and only if the condition $(H_{P,n})$ holds for some $n \geq 0$.*

In the case where $\chi\omega^{-1}(2) = 1$, we know that $T - q \mid P_\chi(T)$ and $T - q \nmid \mathrm{char}_\Lambda X^\chi$ ([2, Proposition 2]). This theorem is an even prime version of [11, Theorem] and [17, Theorem 2.6]. We note that we can verify the condition in the theorem by a congruence relation. That is, the condition $(H_{P,n})$ is equivalent to saying that there exists a prime ideal $\mathfrak{l}$ of k_n of degree one for which the condition

$$c_n^{\mathbf{e}_\psi \widetilde{Y}_{P,n}} \bmod \mathfrak{l} \notin ((\mathbb{Z}/l\mathbb{Z})^\times)^{m_{P,n}}$$

holds where $l\mathbb{Z} = \mathfrak{l} \cap \mathbb{Q}$ by the Chebotarev density theorem.

By using (2.2), we obtain the following:

Corollary 2.2. *We have $\lambda_2(\chi) = 0$ if and only if for any distinguished irreducible factor $P(T)$ of $P_\chi(T)$, the condition $(H_{P,n})$ holds for some $n \geq 0$.*

§ 3. Preliminaries

We first recall the Iwasawa main conjecture and see its consequence (2.2). Let M/k_∞ be the maximal abelian 2-extension unramified outside 2 and L/k_∞ the maximal unramified abelian 2-extension. As usual, we consider $\mathrm{Gal}(M/k_\infty)$, $\mathrm{Gal}(L/k_\infty)$ and $\mathrm{Gal}(M/L)$ as $\mathbb{Z}_2[\Delta][\![\Gamma]\!]$-modules. Let $\wp$ be a prime ideal of k over 2. There exists a unique prime ideal $\wp_n$ of k_n over $\wp$ since k is of the first kind. We denote by $U_{n,\wp}$ the group of principal units in the completion $k_{n,\wp}$ of k_n at $\wp_n$. Put

$$\mathcal{U}_n := \prod_{\wp \mid 2} U_{n,\wp},$$

where $\wp$ runs over all prime ideals of k over 2. Let E'_n be the group of units ϵ of k_n such that $\epsilon \equiv 1 \bmod \wp_n$ for all $\wp_n \mid 2$. Let $\mathcal{E}_n$ be the closure of the image of E'_n under the diagonal map $E'_n \to \mathcal{U}_n$. Put

$$\mathcal{U} := \varprojlim \mathcal{U}_n, \qquad \mathcal{E} := \varprojlim \mathcal{E}_n,$$

where the projective limits are taken with respect to the relative norms. We regard $\mathcal{U}$ and $\mathcal{E}$ as modules over $\mathbb{Z}_2[\Delta][\![\Gamma]\!]$. By class field theory, we have the following isomorphisms of $\mathbb{Z}_2[\Delta][\![\Gamma]\!]$-modules:

$$X \cong \mathrm{Gal}(L/k_\infty), \qquad \mathcal{U}/\mathcal{E} \cong \mathrm{Gal}(M/L). \tag{3.1}$$

Put

$$\mathfrak{X} := \mathrm{Gal}(M/k_\infty).$$

It is known that $\mathfrak{X}$ is finitely generated and torsion over $\mathbb{Z}_2[\Delta][\![\Gamma]\!]$ ([13, Theorem 17]), and we further see that this is finitely generated as a $\mathbb{Z}_2$-module by [3], so X and $\mathcal{U}/\mathcal{E}$ are also finitely generated over $\mathbb{Z}_2$. Hence we have

$$\mathrm{char}_\Lambda X^\chi \cdot \mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi) = \mathrm{char}_\Lambda \mathfrak{X}^\chi. \tag{3.2}$$

The Iwasawa main conjecture proved in [20] asserts that the torsion Λ-module $\mathfrak{X}^\chi$ has the characteristic polynomial $P_\chi(T)$:

$$\mathrm{char}_\Lambda \mathfrak{X}^\chi = P_\chi(T). \tag{3.3}$$

Hence the relation (2.2), $\mathrm{char}_\Lambda X^\chi \mid P_\chi(T)$, holds. Furthermore, $\lambda_2(\chi) = 0$ is equivalent to the following:

$$\mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi) = P_\chi(T).$$

Next we recall results on the structures of the Λ-modules $\mathcal{U}^\chi$ and $\mathcal{U}^\chi/\mathcal{C}^\chi$ in [18] (Theorem 3.1), which are essentially used in the proof of our main theorem. ($\mathcal{C}^\chi$ is a group of cyclotomic units defined below.) Since $f \neq 1$, $c_n = N_{\mathbb{Q}(\zeta_{f2^{n+2}})/k(\zeta_{2^{n+2}})}(1 - \zeta_{f2^{n+2}})$ is a unit of $k_n(\zeta_4) = k(\zeta_{2^{n+2}})$, and $c_n \equiv 1 \mod \wp'_n$ where $\wp'_n$ is the unique prime ideal of $k(\zeta_4)$ above $\wp_n$. We regard c_n as an element of $\widehat{k_n(\zeta_4)}^\times \otimes \mathcal{O}$, where $\widehat{k_n(\zeta_4)}^\times$ is the 2-adic completion of $k_n(\zeta_4)^\times$. We define $\xi_\psi \in \mathcal{O}[G]$ by

$$\xi_\psi := \sum_{\delta \in G} \psi(\delta)^{-1}\delta.$$

Then we see that $c_n^{\xi_\psi} = \sum_{\delta \in G} c_n^\delta \otimes \psi(\delta)^{-1}$ is an element of $\mathcal{E}_n^\chi$ in [18]. We can see that $N_{m,n}(c_m) = c_n$ for all $m \geq n \geq 0$. Then we put

$$c_\infty^{\xi_\psi} := (c_n^{\xi_\psi})_{n \geq 0} \in \mathcal{U}^\chi = \varprojlim \mathcal{U}_n^\chi$$

and denote by $\mathcal{C}^\chi$ the submodule of $\mathcal{U}^\chi$ generated by $c_\infty^{\xi_\psi}$ over Λ.

Let $\mathbb{T}_n$ denote the $\mathbb{Z}_2$-torsion of $\mathcal{U}_n$ and put $\mathbb{T} := \varprojlim \mathbb{T}_n$, where the projective limit is taken with respect to the relative norms. Then $\mathbb{T}_n$ is $\boldsymbol{\mu}_{2^{n+2}} \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\Delta/D]$ or $\{\pm 1\} \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\Delta/D]$ according to $\zeta_4 \in U_{n,\wp}$ or not, where D denotes the decomposition group of 2 in Δ. The χ-part $\mathbb{T}_n^\chi$ of $\mathbb{T}_n$ is the $\mathbb{Z}_2$-torsion of $\mathcal{U}_n^\chi$ and $\mathbb{T}^\chi := \varprojlim \mathbb{T}_n^\chi$. Then we have

$$\mathbb{T}^\chi \cong \begin{cases} \{1\} & \text{if } \chi\omega^{-1}(2) \neq 1, \\ \Lambda/(T-q) & \text{if } \chi\omega^{-1}(2) = 1. \end{cases} \tag{3.4}$$

The following fact plays an important role to prove Theorem 2.1. When the order of χ is odd (and $p = 2$), this is the theorem of Gillard ([7]).

Theorem 3.1 ([18])**.** *There is a natural Λ-homomorphism*

$$\Psi : \mathcal{U}^\chi \longrightarrow \Lambda$$

for which the kernel is $\mathbb{T}^\chi$ and the image is Λ (resp. $(T-q)\Lambda$) if $\chi\omega^{-1}(2) \neq 1$ (resp. if $\chi\omega^{-1}(2) = 1$). Furthermore, we have

$$\Psi(c_\infty^{\xi_\psi}) = g_\chi(T).$$

Put $\mathcal{V}_n^\chi := \bigcap_{m \geq n} N_{m,n}(\mathcal{U}_m^\chi)$, with the norm maps $N_{m,n}$ from k_m to k_n. In the following lemma, we can determine the structure of the Λ-modules $\mathcal{V}_n^\chi$ in the same way as the proof of [17, Lemma 3.1] (see also [7, Proposition 2]).

Lemma 3.2.

(i) *The projection $\mathcal{U}^\chi \to \mathcal{V}_n^\chi$ induces the following isomorphisms:*

$$\mathcal{V}_n^\chi \cong \begin{cases} \mathcal{U}^\chi/(\mathcal{U}^\chi)^{\omega_n} & \text{if } \chi(2) \neq 1, \\ \mathcal{U}^\chi/((\mathcal{U}^\chi)^{2\nu_n} \cdot (\mathcal{U}^\chi)^{\omega_n}) & \text{if } \chi(2) = 1. \end{cases}$$

(ii) *If $\chi(2) = 1$ (resp. $\chi(2) \neq 1$) then $\mathcal{U}_n^\chi/\mathcal{V}_n^\chi$ is isomorphic to $\Lambda/(T)$ (resp. is killed by 2).*

Corollary 3.3. *Let v_n be an element of $\mathcal{V}_n^\chi$ and $X(T)$ a polynomial in $\mathcal{O}[T]$ relatively prime to $\omega_n(T)$ (resp. $\nu_n(T)$) if $\chi(2) \neq 1$ (resp. $\chi(2) = 1$). If $v_n^{X(T)} = 1$ holds, then we have $v_n \in \mathbb{T}_n^\chi$.*

Proof. By Lemma 3.2 (i), the map Ψ in Theorem 3.1 induces a Λ-homomorphism

$$\Psi_n : \mathcal{V}_n^\chi \longrightarrow \Lambda/(\vartheta_n^\chi)$$

where $\vartheta_n^\chi = \omega_n$ (resp. ν_n) if $\chi(2) \neq 1$ (resp. $\chi(2) = 1$). We further see that the kernel of Ψ_n is contained in $\mathbb{T}_n^\chi$. This proves the corollary. $\square$

Finally, we shall show the freeness of $\mathcal{E}^\chi$ (Lemma 3.6). We need the following lemma.

Lemma 3.4.

(i) *The inclusion* $E'_n \to \mathcal{E}_n$ *induces an isomorphism*

$$E'_n/E'^{2^a}_n \cong \mathcal{E}_n/\mathcal{E}_n^{2^a}$$

for any $a \geq 0$.

(ii) $\mathcal{E} \cap \mathbb{T} = \{1\}$.

Proof. (i) This follows from the Leopoldt conjecture for $p = 2$ and k_n proved in [1](cf. [19, §5-5]).

(ii) The $\mathbb{Z}_2$-torsion of $\mathcal{E}_n$ is $\mathcal{E}_n \cap \mathbb{T}_n = \{\pm 1\}$. Therefore $\mathcal{E} \cap \mathbb{T} = \varprojlim \mathcal{E}_n \cap \mathbb{T}_n = \{1\}$. $\square$

By this lemma, we can regard $\mathbb{T}$ as a submodule of $\mathcal{U}/\mathcal{E}$ and also of $\mathfrak{X}$. We can show the following lemma similarly to the proof of [17, Lemma 3.4] by using the fact that $\mathfrak{X}$ has no non-trivial finite $\mathbb{Z}_2[\Delta][[\Gamma]]$-submodule ([13, Theorem 18]).

Lemma 3.5. $\mathfrak{X}/\mathbb{T}$ *has no non-trivial finite* $\mathbb{Z}_2[\Delta][[\Gamma]]$*-submodule.*

This lemma produces the following lemma in the same way as in the odd prime case ([17, Lemma 3.5]):

Lemma 3.6. $\Psi(\mathcal{E}^\chi)$ *is a principal ideal generated by* $\mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi)$*. In particular* $\mathcal{E}^\chi$ *is a free* Λ*-module of rank one.*

Remark. As in the odd prime case, $\lambda_2(\chi) = 0$ is equivalent to the assertion that $\mathcal{E}^\chi = \mathcal{C}^\chi$.

§ 4. Proof of the main result

We can rewrite, in the same way as the proof of [17, Lemma 4.1], the condition $(H_{P,n})$ as follows:

Lemma 4.1. *For each* $n \geq 0$*, the condition* $(H_{P,n})$ *is equivalent to the following condition*:

$$(\mathcal{H}_{P,n}) \qquad c_n^{\xi_\psi X_{P,n}} \notin (\mathcal{E}_n^\chi)^{m_{P,n}}.$$

Proof of Theorem 2.1. Our proof is the same as in [11] by using Theorem 3.1 and Corollary 3.3. We shall show that $P(T) \mid \mathrm{char}_\Lambda X^\chi$ holds if and only if the opposite

$$(\neg\mathcal{H}_{P,n}) \qquad c_n^{\xi_\psi X_{P,n}} \in (\mathcal{E}_n^\chi)^{m_{P,n}}$$

of $(\mathcal{H}_{P,n})$ holds for all $n \geq 0$. We put $Q(T) = P_\chi(T)/P(T)$. By the Iwasawa main conjecture (3.3) and (3.2), we have

$$\mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi) \cdot \mathrm{char}_\Lambda X^\chi = P_\chi(T).$$

Then $P(T) \mid \mathrm{char}_\Lambda X^\chi \iff \mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi) \mid Q(T)$. By Lemma 3.6, the latter condition is equivalent to saying that $Q(T) \in \Psi(\mathcal{E}^\chi)$. Since $\Psi(\mathcal{E}^\chi) \subset \Lambda$, we have $Q(T) \in \Psi(\mathcal{E}^\chi) \iff P(T)Q(T) \in \Psi((\mathcal{E}^\chi)^{P(T)})$. Furthermore, we have $P(T)Q(T) \in \Psi((\mathcal{E}^\chi)^{P(T)}) \iff \Psi(c_\infty^{\xi_\psi}) \in \Psi((\mathcal{E}^\chi)^{P(T)}) \iff c_\infty^{\xi_\psi} \in \mathbb{T}^\chi(\mathcal{E}^\chi)^{P(T)}$ by using Theorem 3.1 and $g_\chi(T) = u_\chi(T)P(T)Q(T)$. By Lemma 3.4 (ii), we have $c_\infty^{\xi_\psi} \in \mathbb{T}^\chi(\mathcal{E}^\chi)^{P(T)} \iff c_\infty^{\xi_\psi} \in (\mathcal{E}^\chi)^{P(T)}$. If we assume that

$$(4.1) \qquad c_\infty^{\xi_\psi} \in (\mathcal{E}^\chi)^{P(T)}$$

holds then $c_n^{\xi_\psi} \in (\mathcal{E}_n^\chi \cap \mathcal{V}_n^\chi)^{P(T)}$ holds for all $n \geq 0$. This implies that

$$(4.2) \qquad c_n^{\xi_\psi X_{P,n}} \in (\mathcal{E}_n^\chi \cap \mathcal{V}_n^\chi)^{X_{P,n}P(T)}.$$

By the definition (2.3) and Lemma 3.2 (i), we have $(\mathcal{E}_n^\chi \cap \mathcal{V}_n^\chi)^{X_{P,n}P(T)} = (\mathcal{E}_n^\chi \cap \mathcal{V}_n^\chi)^{m_{P,n}}$. Therefore the condition (4.2) implies the condition $(\neg\mathcal{H}_{P,n})$. Conversely, we assume that the condition $(\neg\mathcal{H}_{P,n})$ holds for all $n \geq 0$. By Lemma 3.2 (ii), $\mathcal{E}_n^\chi/(\mathcal{E}_n^\chi \cap \mathcal{V}_n^\chi)$ is killed by $2T$. Hence $c_n^{\xi_\psi 2T X_{P,n}} \in (\mathcal{E}_n^\chi \cap \mathcal{V}_n^\chi)^{X_{P,n}P(T)}$, so there exists $\epsilon_n \in \mathcal{E}_n^\chi \cap \mathcal{V}_n^\chi$ such that $c_n^{\xi_\psi 2T X_{P,n}} = \epsilon_n^{X_{P,n}P(T)}$ for all $n \geq 0$. Since $X_{P,n}(T)$ is relatively prime to $\omega_n(T)$ (resp. $\nu_n(T)$) if $\chi(2) \neq 1$ (resp. $\chi(2) = 1$), by Corollary 3.3, we have $c_n^{\xi_\psi 2T}/\epsilon_n^{P(T)} \in \mathbb{T}_n \cap \mathcal{E}_n = \{\pm 1\}$, that is $c_n^{\xi_\psi 2T} = \pm\epsilon_n^{P(T)}$. Then we have $c_{n+1}^{\xi_\psi 2T} = N_{n+2,n+1}(c_{n+2}^{\xi_\psi 2T}) = N_{n+2,n+1}(\pm\epsilon_{n+2}^{P(T)}) = N_{n+2,n+1}(\epsilon_{n+2})^{P(T)}$ for all $n \geq 0$. Therefore we have $N_{m+2,n+1}(\epsilon_{m+2})^{P(T)} = N_{m+1,n+1}(N_{m+2,m+1}(\epsilon_{m+2})^{P(T)}) = N_{m+1,n+1}(c_{m+1}^{\xi_\psi 2T}) = c_{n+1}^{\xi_\psi 2T} = N_{n+2,n+1}(\epsilon_{n+2})^{P(T)}$ for all $m \geq n \geq 0$. By using Corollary 3.3 and $\mathbb{T}_n \cap \mathcal{E}_n = \{\pm 1\}$, we have $N_{m+2,n+1}(\epsilon_{m+2}) = \pm N_{n+2,n+1}(\epsilon_{n+2})$. Taking $N_{n+1,n}$ of this equation, we obtain $N_{m,n}(N_{m+2,m}(\epsilon_{m+2})) = N_{n+2,n}(\epsilon_{n+2})$. Therefore we have $\epsilon = (N_{n+2,n}(\epsilon_{n+2}))_{n\geq 0} \in \varprojlim \mathcal{E}_n^\chi = \mathcal{E}^\chi$ and $N_{n+2,n}(\epsilon_{n+2})^{P(T)} = c_n^{\xi_\psi 2T}$, that is $c_\infty^{\xi_\psi 2T} \in (\mathcal{E}^\chi)^{P(T)}$. By the same argument as in the above, this is equivalent to saying that $\mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi) \mid 2TQ(T)$. Since $\mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi)$ is prime to $2T$, we have $\mathrm{char}_\Lambda(\mathcal{U}^\chi/\mathcal{E}^\chi) \mid Q(T)$ which is equivalent to the condition (4.1). This completes the proof. $\square$

References

[1] A. Brumer, On the units of algebraic number fields, *Mathematatika*, **14** (1967), 121–124.
[2] B. Ferrero and R. Greenberg, On the behavior of p-adic L-functions at $s = 0$, *Invent. math.*, **50** (1978), 91–102.
[3] B. Ferrero and L. C. Washington, The Iwasawa invariant μ_p vanishes for abelian number fields, *Ann. of Math. (2)*, **109** (1979), 377–395.
[4] T. Fukuda and K. Komatsu, On the Iwasawa λ-invariant of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}(\sqrt{p})$, *Math. Comp.*, **78** (2009), 1797–1808.
[5] T. Fukuda and K. Komatsu, On the Iwasawa λ-invariant of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}(\sqrt{p})$ II, *Funct. Approx. Comment. Math.* **51** (2014), 167–179.
[6] T. Fukuda, K. Komatsu, M. Ozaki and T. Tsuji, On the Iwasawa λ-invariant of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}(\sqrt{p})$ III, *Funct. Approx. Comment. Math.* **54** (2016), no.1, 7–17.
[7] R. Gillard, Unités cyclotomiques, unités semi-locales et $\mathbb{Z}_\ell$-extensions II, *Ann. Inst. Fourier (Grenoble)*, **29** (1979), 1–15.
[8] R. Greenberg, On the Iwasawa invariants of totally real number fields, *Amer. J. Math.*, **98** (1976), 263–284.
[9] H. Ichimura and H. Sumida, On the Iwasawa invariants of certain real abelian fields, *Tôhoku Math. J.*, **49** (1997), 203–215.
[10] H. Ichimura and H. Sumida, On the Iwasawa λ-invariant of the p-cyclotomic field, *J. Math. Sci. Univ. Tokyo.*, **3** (1996), 457–470.
[11] H. Ichimura and H. Sumida, On the Iwasawa invariants of certain real abelian fields II, *Internat. J. Math.*, **7** (1996), 721–744.
[12] K. Iwasawa, On some modules in the theory of cyclotomic fields, *J. Math. Soc. Japan*, **16** (1964), 42–82.
[13] K. Iwasawa, On $\mathbb{Z}_\ell$-extension of algebraic number fields, *Ann. of Math.* (2), **98** (1973), 246–326.
[14] M. Ozaki and H. Taya, On the Iwasawa λ_2-invariants of certain families of real quadratic fields, *Manuscripta Math.*, **94** (1997), no.4, 437–444.
[15] T. Tsuji, Semi-local units modulo cyclotomic units, *J. Number Theory*, **78** (1999), 1–26.
[16] T. Tsuji, Greenberg's conjecture for Dirichlet characters of order divisible by p, *Proc. Japan Acad. Ser. A*, **77** (2001), 52–54.
[17] T. Tsuji, On the Iwasawa λ-invariants of real abelian fields, *Trans. of AMS*, **355-9** (2003), 3699–3714.
[18] T. Tsuji, Semi-local units modulo cyclotomic units in the cyclotomic $\mathbb{Z}_2$-extensions, in preparation.
[19] L. C. Washington, "Introduction to Cyclotomic Fields", Grad. Texts in Math. 83, Springer-Verlag, 1982.
[20] A. Wiles, Iwasawa conjecture of totally real fields, *Ann. of Math (2)*, **131** (1990), 493–540.