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# A report on the cohomological coprimality of Galois representations

By

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## Abstract

Let $K$ be a local or a global field and $G_K$ its absolute Galois group. In this note we announce some results on the vanishing of certain Galois cohomology groups associated with representations of $G_K$ coming from geometry. Motivated by our efforts to generalize some results of Coates, Sujatha and Wintenberger, we introduce the notion of "cohomological coprimality", which provides another notion of independence between such representations. We consider proper smooth varieties $X$ and $X'$ over a $p$-adic field $K$ with potential good reduction. Then it can be shown that in many cases where $X$ and $X'$ have "quite different" nature, their corresponding representations are cohomologically coprime. When $K$ is a number field, we can prove the cohomological coprimality of systems of $\ell$-adic representations of $G_K$ associated with elliptic curves which are non-isogenous over $\overline{K}$.

## § 1. Introduction

We announce some results on the vanishing of certain Galois cohomology groups associated with Galois representations coming from geometry which extends some results of Coates-Sujatha-Wintenberger [CSW01] and Sujatha [Su00]. Such vanishing results are useful in obtaining generalization of methods in Iwasawa theory to larger Galois extensions. For instance, it enables the computation of Euler characteristics for discrete modules associated to $p$-adic Galois representations coming from geometry (see [CSW01] and [CS99]) and for Selmer groups of elliptic curves over field extensions that contain all $p$-power roots of unity (see [CH01], [CSS03] and [Ze09]). An especially interesting case in generalizing the results of [CSW01] and [Su00] motivated us to introduce another

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notion of independence, called cohomological coprimality, between representations of a topological group.

Let $R$ be a topological commutative ring with unity and $V$ a topological $R$-module. Let $G$ be a closed subgroup of the group $\mathrm{Aut}_R(V)$ of topological $R$-automorphisms of $V$ endowed with the compact-open topology. We consider the continuous cohomology groups $H^m(G, V)$ of $G$ with coefficients in $V$ defined by continuous cochains. We say that $V$ has *vanishing $G$-cohomology* if the cohomology groups $H^m(G, V)$ are trivial for all $m = 0, 1, \ldots$.

Let $K$ be a finite extension of the field of $p$-adic rational numbers $\mathbb{Q}_p$ or of the field of rational numbers $\mathbb{Q}$. We fix a separable closure $\overline{K}$ of $K$. We put $G_L = \mathrm{Gal}(\overline{K}/L)$, for a subextension $L$ of $\overline{K}$. By a *variety* over $K$ we mean a separated scheme of finite type over $K$. Let $X$ be a proper smooth variety defined over $K$ and $i \geq 0$ an integer. Put $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, where $X_{\overline{K}} := X \otimes_K \overline{K}$, and consider the $p$-adic Galois representation $\rho \colon G_K \to \mathrm{GL}(V)$. Denote by $K(\mu_{p^\infty})$ the field extension of $K$ obtained by adjoining to $K$ all roots of unity whose order is a power of $p$. Let $G_V = \rho(G_K)$ and $H_V = \rho(G_{K(\mu_{p^\infty})})$.

**Theorem 1** ([CSW01], Theorems 1.1 and 1.5)**.** *Suppose that $[K : \mathbb{Q}_p] < \infty$ and assume that $X$ is a proper smooth variety defined over $K$ with potential good reduction. Then:*
(1) *if $i$ is nonzero, then $V$ has vanishing $G_V$-cohomology;*
(2) *if $i$ is odd, then $V$ has vanishing $H_V$-cohomology.*

Let $K(V)$ be the fixed subfield of $\overline{K}$ by the kernel of $\rho$. Observe that we may identify $G_V$ with the Galois group $\mathrm{Gal}(K(V)/K)$. Similarly, $H_V$ may be identified with $\mathrm{Gal}(K(V)/K(V) \cap K(\mu_{p^\infty}))$.

A result similar to Theorem 1 holds in the case where the variety $X$ is defined over a number field.

**Theorem 2** ([Su00], Theorem 2.7)**.** *Suppose that $[K : \mathbb{Q}] < \infty$. Then:*
(1) *if $i$ is nonzero, then $V$ has vanishing $G_V$-cohomology;*
(2) *if $i$ is odd, then $V$ has vanishing $H_V$-cohomology.*

Our aim is to generalize Theorems 1 and 2 to computation of cohomology of groups that correspond to field extensions other than $K$ and $K(\mu_{p^\infty})$. More precisely, we seek the answer to the following

*Problem 1.* Consider a Galois extension $L$ of $K$ and put $J_V = \rho(G_L)$. When does the representation $V$ have vanishing $J_V$-cohomology?

We may consider the above problem with $L$ taken to be a field extension of $K$ which corresponds to the kernel of another representation $V'$ of $G_K$. The problem in this scenario becomes symmetric as we may ask the same question to the representation $V'$ and the field extension $L'$ corresponding to the kernel of $V$. As such, the problem in this case becomes a problem of comparison, or "independence", between the representations $V$ and $V'$ from which we can derive some cohomological results. There can be several notions of "independence", the simplest being non-isomorphism. Another notion is the "independence" among representations in a given system of representations of a profinite group which was studied by Serre [Se13]. In view of the above discussion we propose another notion of "independence" between representations.

**Definition 3.** Let $G$ be a compact topological group and $R$ and $R'$ topological rings. Let $\rho\colon G \to \mathrm{GL}_R(V)$ and $\rho'\colon G \to \mathrm{GL}_{R'}(V')$ be two continuous linear representations of $G$ on a Hausdorff topological $R$-module $V$ and a Hausdorff topological $R'$-module $V'$, respectively. Put $\mathcal{G} = \rho(\mathrm{Ker}\rho')$ and $\mathcal{G}' = \rho'(\mathrm{Ker}\rho)$. We say that $V$ and $V'$ are *cohomologically coprime* if $V$ has vanishing $\mathcal{G}$-cohomology and $V'$ has vanishing $\mathcal{G}'$-cohomology.

Note that when $G = \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of a field $K$, then $\mathcal{G} \simeq \mathrm{Gal}(K(V)/K(V) \cap K(V'))$ and $\mathcal{G}' \simeq \mathrm{Gal}(K(V')/K(V) \cap K(V'))$.

## § 2. Result I - Local setting

We now present our main results in the case where the base field $K$ is a $p$-adic field. Following Bloch-Kato (see [BK86], Definition 7.2) we say that a proper smooth variety $X$ over $K$ has *good ordinary reduction* over $K$ if there exists a proper smooth model $\mathfrak{X}$ over $\mathcal{O}_K$ with special fiber $\mathcal{Y}$ such that the cohomology groups $H^r(\mathcal{Y}, d\Omega^s_{\mathcal{Y}})$ are trivial for all $r$ and all $s$. Here $d\Omega^s_{\mathcal{Y}}$ denotes the sheaf of locally exact differential $(s+1)$-forms on $\mathcal{Y}$. We say that $X$ has *potential good ordinary reduction* over $K$ if it has good ordinary reduction after a finite extension $K'/K$. The reader is advised to see Proposition 7.3 of [BK86] for equivalent formulations of the above definition.

**Theorem 4** ([Di14-2], Theorem 4.3, Remark 4.7 and Theorem 4.8)**.** *Let $X$ be a proper smooth variety with potential good reduction over $K$, $i$ a positive odd integer and $A$ an abelian variety with potential good reduction over $K$. Put $V = H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$ and $V' = V_p(A)$, the representation of $G_K$ given by the $p$-adic Tate module of $A$. Then $V$ and $V'$ are cohomologically coprime in either of the following situations* (a), (b)*:*
(a) *$A$ has potential good ordinary reduction over $K$ and moreover the following conditions are satisfied:*
(i) *the residue field of $K(V)$ is a potential prime-to-$p$ extension* (*in the sense of* [Oz09], *Definition* 2.1) *of the residue field of $K$;*
(ii) *$V^{G_L} = 0$ for every finite extension $L$ of $K(V')$;*

(b) *$X$ has potential good ordinary reduction over $K$ and $A$ is an elliptic curve with potential good supersingular reduction over $K$.*

Suppose that $E$ and $E'$ are elliptic curves over $K$. We can prove the cohomological coprimality of $V_p(E)$ and $V_p(E')$ by distinguishing the reduction types of $E$ and $E'$. As such, it provides an extension of Ozeki's results [Oz09] in this setting. This is summarized in the following

**Theorem 5** ([Di14-1], Theorem 1.5)**.** *Let $E$ and $E'$ be elliptic curves over $K$. The cohomological coprimality of $V_p(E)$ and $V_p(E')$ is given by the following table:*

| $E$ | $E'$ | *Cohomologically coprime* |
|---|---|---|
| *ordinary* | *ordinary* | *No* |
| | *supersingular* | *Yes* |
| | *multiplicative* | *"No"* |
| *supersingular with FCM* | *supersingular with FCM* | *Yes** |
| | *supersingular without FCM* | *Yes* |
| | *multiplicative* | *"No"* |
| *supersingular without FCM* | *supersingular without FCM* | *Yes** |
| | *multiplicative* | *"No"* |
| *multiplicative* | *multiplicative* | *"No"* |

Here FCM means formal complex multiplication. The symbol * means conditional cohomological coprimality. The cohomological coprimality in this case holds under the additional assumption that the group $E(L')[p^\infty]$ of $L'$-rational points of $E$ of $p$-power order is finite for every finite extension $L'$ of $K(E'_{p^\infty})$. This property is in fact equivalent to the statement that the $p$-divisible groups associated with $E$ and $E'$ are non-isogenous over $\mathcal{O}_{\overline{K}}$ (*op. cit.*, Corollary 5.5). For the entries where at least one of $E$ and $E'$ has multiplicative reduction over $K$, the 'No' is written in quotes to mean that cohomological coprimality is not attained in many cases. We refer the reader to ([Di14-1], Remark 5.7) for a brief discussion on this.

## § 3. Result II - Global setting

In this section we consider the setting where the base field $K$ is a number field. We let $S$ and $S'$ be sets of prime numbers and suppose that $E$ and $E'$ are elliptic curves over $K$. We consider the systems of $\ell$-adic representations associated with $E$ and $E'$ indexed by $S$ and $S'$, respectively:

$$(\rho_\ell \colon G_K \to \mathrm{GL}(V_\ell(E)))_{\ell \in S}$$

and

$$(\rho'_\ell\colon G_K \to \mathrm{GL}(V_\ell(E')))_{\ell\in S'}.$$

Let $V_S(E) = \bigoplus_{\ell\in S} V_\ell(E)$ and $V_{S'}(E') = \bigoplus_{\ell\in S'} V_\ell(E')$.

**Theorem 6** ([Di14-2], Theorem 1.3). *Let $S$ and $S'$ be sets of prime numbers. Let $E$ and $E'$ be elliptic curves over $K$.*
(i) *Assume that $E$ and $E'$ are not isogenous over $\overline{K}$. Then $V_S(E)$ and $V_{S'}(E')$ are cohomologically coprime.*
(ii) *If $S \cap S' = \emptyset$, then $V_S(E)$ and $V_{S'}(E')$ are cohomologically coprime.*

From the above theorem we obtain the following result which is reminiscent of the Isogeny Theorem for elliptic curves (cf. [Fa83], §5 Korollar 2).

**Corollary 7** ([Di14-2], Corollary 1.4). *Let $E$ and $E'$ be elliptic curves over $K$. The following statements are equivalent:*
(i) *$E$ and $E'$ are not isogenous over $\overline{K}$;*
(ii) *$V_S(E)|_{G_{K'}}$ and $V_{S'}(E')|_{G_{K'}}$ are cohomologically coprime for any $S$ and $S'$ and for every finite extension $K'$ of $K$;*
(iii) *$V_\ell(E)|_{G_{K'}}$ $(= V_{\{\ell\}}(E)|_{G_{K'}})$ and $V_\ell(E')|_{G_{K'}}$ $(= V_{\{\ell\}}(E')|_{G_{K'}})$ are cohomologically coprime for some prime number $\ell$ and for every finite extension $K'$ of $K$.*

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