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Stark's conjecture over the rational number field and CM-periods of Fermat curves

By

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Abstract

This is an announcement of the results of the paper "Fermat curves and the reciprocity law on cyclotomic units" (arXiv:1502.04397). We will define a "period-ring-valued beta function" and give a reciprocity law on its special values, by using some results on Fermat curves due to Rohrlich and Coleman. Here the "period-ring" is in the sense of p-adic Hodge theory. As an application, we will discuss its relation with Stark's conjecture over the rational number field. In particular, we see that the reciprocity law given in this paper is a refinement of the reciprocity law on cyclotomic units.

§ 1. Introduction

This paper is an announcement of the results of [Ka]. However, we explain our motivation in more detail. We consider Yoshida's conjecture about CM-periods and a special case of Stark's conjecture. Let F be a totally real algebraic number field, K a finite abelian extension of F with $G := \mathrm{Gal}(K/F)$. The partial zeta function associated with $\sigma \in G$ is defined by

$$\zeta_F(s,\sigma) := \sum_{\mathfrak{a} \subset \mathcal{O}_F,\ (\frac{K/F}{\mathfrak{a}})=\sigma} N\mathfrak{a}^{-s},$$

where $\mathfrak{a}$ runs over all integral ideals of F whose image under the Artin map is equal to σ. This series converges when $\mathrm{Re}(s) > 1$, has a meromorphic continuation to the whole complex plane, and is analytic at $s = 0$. Then Stark's conjecture implies that

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when K has a real embedding, $\exp(\zeta_F'(0,\sigma))$ is an algebraic number satisfying a "reciprocity law", except for the case when $K=\mathbb{Q}$.

On the other hand, Yoshida's conjecture [Yo, Proposition 3.3, Chapter III] states that

the transcendental part of some product of Shimura's period symbols can be written as a monomial of $\exp(\zeta_F'(0,\sigma))$'s where K's are CM-fields.

We will give the precise statement of a version of Yoshida's conjecture (Conjecture 5.1) in §5, which expresses the transcendental part of each value of Shimura's period symbol, not only some product of them.

Stark's conjecture and Yoshida's conjecture are in contrast with each other: fields with real embeddings and CM-fields, the algebraicity and the transcendental part. My question is

(Q) whether there is any "relation" between these conjectures about $\exp(\zeta_F'(0,\sigma))$.

The results in [Ka] suggest a relation of them, at least when the base field F is $\mathbb{Q}$, as we will state in (Ans) below. First we recall the rank 1 abelian Stark conjecture when the splitting place is a real place. Assume that $K\neq\mathbb{Q}$ and that there exists a real embedding $\iota\colon K\hookrightarrow\mathbb{R}$. We denote the place of F corresponding to $\iota|_F$ by v. Then v splits in K. Put

$$u_F(\sigma):=\exp(-2\zeta_F'(0,\sigma)). \tag{1.1}$$

We can show that $u_F(\sigma)$ is a real number. Stark's conjecture states that $u_F(\sigma)$, which is called a Stark unit, satisfies the following four properties.

(SC1) For all $\sigma\in G$, $u_F(\sigma)\in K$. (Strictly speaking, $u_F(\sigma)$ is in the image of $\iota\colon K\hookrightarrow\mathbb{R}$.)

(SC2) For all $\sigma,\tau\in G$, $\tau(u_F(\sigma))=u_F(\tau\sigma)$.

(SC3) Let S_0 be the set of all infinite places of F and all ramifying places in K/F. There are two cases (Note that $|S_0|\geq 2$ by our assumption):

- (a) Assume that $|S_0|>2$. Then $u_F(\sigma)$ is a v-unit (that is, $|u_F(\sigma)|_{w'}=1$ for all places w' of K with $w'\nmid v$).
- (b) Assume that $|S_0|=2$, that is, $S_0=\{v,v'\}$ with another place v' of F. Then $u_F(\sigma)$ is an S_0-unit and $|u_F(\sigma)|_{w'}$ is a constant for all $w'\mid v'$.

(SC4) $K\left(\sqrt{u_F(\sigma)}\right)/F$ is an abelian extension.

When $F=\mathbb{Q}$, we can derive this conjecture from the following (L), (E), (CU):

(L) Lerch's formula: $\exp(\zeta'(0,v,z)) = \frac{\Gamma(\frac{z}{v})}{\sqrt{2\pi}v^{\frac{1}{2}-\frac{z}{v}}}$. Here $\zeta(s,v,z)$ is the Hurwitz zeta function $\zeta(s,v,z) := \sum_{n=0}^{\infty}(z+vn)^{-s}$ $(v,z>0)$.

(E) Euler's reflection formula: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$.

(CU) Cyclotomic units: Let ζ_m be a primitive mth root of unity. Then $1-\zeta_m$ is a p-unit if m is a power of a prime p, is a unit otherwise.

We provide a sketch of the proof in the case $K/F = \mathbb{Q}(\zeta_m)^+/\mathbb{Q}$ with $\zeta_m := \exp(\frac{2\pi\sqrt{-1}}{m})$, $\mathbb{Q}(\zeta_m)^+ := \mathbb{Q}(\zeta_m+\zeta_m^{-1})$, $m \geq 3$. We define an element $\sigma_{\pm\frac{a}{m}}$ in $G = \mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ by $\sigma_{\pm\frac{a}{m}}(\zeta_m+\zeta_m^{-1}) := \zeta_m^a + \zeta_m^{-a}$ for $0<a<\frac{m}{2}$, $(a,m)=1$. Then $\zeta_{\mathbb{Q}}(s,\sigma_{\pm\frac{a}{m}}) = \sum_{n\in\mathbb{N},\ n\equiv\pm a \bmod m} n^{-s}$. Here we denote the set of natural numbers by $\mathbb{N} := \{1,2,3,\dots\}$. We see that

$$u_{\mathbb{Q}}(\sigma_{\pm\frac{a}{m}}) = \left(\frac{2\pi}{\Gamma(\frac{a}{m})\Gamma(\frac{m-a}{m})}\right)^2 \tag{1.2}$$

by (L). In addition, using (E), we obtain

$$u_{\mathbb{Q}}(\sigma_{\pm\frac{a}{m}}) = \left(2\sin(\tfrac{a}{m}\pi)\right)^2 = 2-(\zeta_m^a+\zeta_m^{-a}).$$

Then (SC1), (SC2), (SC4) for $u_{\mathbb{Q}}(\sigma_{\pm\frac{a}{m}})$ follow immediately, (SC3) follows from (CU). We note that we can rewrite (SC1), (SC2) in the case $K/F = \mathbb{Q}(\zeta_m)^+/\mathbb{Q}$ as a reciprocity law on special values of the form $\Gamma(\alpha)\Gamma(1-\alpha)$ $(\alpha\in\mathbb{Q}\cap(0,1))$, as follows: For $a,m\in\mathbb{N}$ with $a<m$, we have

$$\begin{gathered}\frac{\Gamma(\frac{a}{m})\Gamma(\frac{m-a}{m})}{\pi} \in \overline{\mathbb{Q}},\\ \tau\left(\frac{\Gamma(\frac{a}{m})\Gamma(\frac{m-a}{m})}{\pi}\right) = \pm\frac{\Gamma(\tau(\frac{a}{m}))\Gamma(\tau(\frac{m-a}{m}))}{\pi} \qquad (\tau\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})).\end{gathered} \tag{1.3}$$

Here for $\tau\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we define $\tau(\frac{a}{m}) := \frac{a'}{m}$ where a' is the integer satisfying $\tau(\zeta_m^a) = \zeta_m^{a'}$, $1\leq a'<m$.

A brief sketch of the main result in [Ka] is as follows: First we define a "period-ring-valued beta function" (Definition 4.1) by using

(R) Rohrlich's formula (2.1) for CM-periods of Fermat curves.

Here we also use the comparison isomorphisms of p-adic Hodge theory, and the "period-ring" means Fontaine's p-adic period ring. Then we derive a reciprocity law on its special values (Theorem 4.2), which is the main result in [Ka], from

(C) Coleman's formula (Theorem 2.3) for the absolute Frobenius action on Fermat curves.

On the other hand, by using (L), we can write a Stark unit $u_{\mathbb{Q}}(\sigma)$ over $\mathbb{Q}$ as a product of special values of our beta function, up to multiplication by a root of unity (Lemma 4.3-(4)). Therefore we see that the main result (Theorem 4.2) is a refinement of the following reciprocity law on Stark units in the case $K/F = \mathbb{Q}(\zeta_m)^+/\mathbb{Q}$ (Corollary 4.4):

$$
\begin{aligned}
&u_{\mathbb{Q}}(\sigma) \in \overline{\mathbb{Q}} \quad (\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})), \\
&\tau(u_{\mathbb{Q}}(\sigma)) \equiv u_{\mathbb{Q}}(\tau\sigma) \bmod \mu_\infty \quad (\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q}),\ \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})).
\end{aligned} \tag{1.4}
$$

Here we denote the group of roots of unity by μ_∞. Note that (1.4) is weaker than (SC1), (SC2) because of the root of unity ambiguity.

Now we provide an answer to the question (Q): Yoshida's conjecture is a generalization of (R), from $\mathbb{Q}$ to general totally real fields. Besides, Yoshida and the author formulated a p-adic analogue of Yoshida's conjecture in [KY1, KY2], which is a generalization of (C). Therefore, very very roughly speaking,

(Ans) Lerch's formula (L), a special case (R) of Yoshida's conjecture and its p-adic analogue (C) imply a refinement (Theorem 4.2) of a part of Stark's conjecture when $F = \mathbb{Q}$.

The results in [Ka] also have the following importance: There is also a generalization (5.1) of (L) by Shintani and Yoshida, which we will recall in §5. We also note that, as far as the author knows, there does not exist a generalization of (E) which is useful for Stark's conjecture. Therefore the author believes that it is meaningful to provide an alternative proof for Stark's conjecture without using (E), even in the well-known case $F = \mathbb{Q}$.

The outline of this paper is as follows. In §2, we introduce some results by Rohrlich (2.1) and Coleman (Theorem 2.3) concerning Fermat curves. In §3, we formulate a reciprocity law on p-adic periods of Fermat curves (Theorem 3.1). In §4, we define a period-ring-valued beta function $\mathfrak{B}$ (Definition 4.1) and study its properties. Our main result is a reciprocity law on its special values (Theorem 4.2). By Lemma 4.3-(4), special values of the form $\Gamma(\alpha)\Gamma(1-\alpha)$ $(\alpha \in \mathbb{Q} \cap (0,1))$ can be decomposed into a product of special values of $\mathfrak{B}$. Therefore (1.3) follows from Theorem 4.2 as we will see in Corollary 4.4, up to multiplication by a root of unity. We note that our main result is not only an alternative proof (Corollary 4.4) but also a refinement (Theorem 4.2) of (1.3), (1.4).

In §5, for future reference, we will give a brief sketch of a generalization of Lerch's formula (L) by Shintani and Yoshida, and conjectural generalizations of Rohrlich's formula (R) and Coleman's formula (C) by Yoshida and the author. The author hopes that the results in [Ka] will be generalized by using those works.

§ 2. Some results on Fermat curves

In this section, we recall some results on Fermat curves.

§ 2.1. CM-periods

Let F_m be the mth projective Fermat curve defined by the affine equation $x^m+y^m = 1$ over $\mathbb{Q}$. Then $\eta_{\frac{i}{m},\frac{j}{m}} := x^{i-1}y^{j-m}dx$ ($i,j \in \mathbb{N}$, $i,j < m$, $i+j \neq m$) are differentials of the second kind and their classes form a basis of $H^1_{\mathrm{dR}}(F_m,\mathbb{Q})$. Here H_{dR} denotes the algebraic de Rham cohomology. By [Gr, Theorem in Appendix by Rohrlich], we have for all $\gamma \in H_1^{\mathrm{B}}(F_m(\mathbb{C}),\mathbb{Q})$

$$\frac{\int_\gamma \eta_{\frac{i}{m},\frac{j}{m}}}{B(\frac{i}{m},\frac{j}{m})} \in \mathbb{Q}(\zeta_m), \tag{2.1}$$

where $B(\alpha,\beta)$ is the beta function. Moreover, we see that we can take γ_0 so that

$$\int_{\gamma_0} \eta_{\frac{i}{m},\frac{j}{m}} = B(\tfrac{i}{m},\tfrac{j}{m}). \tag{2.2}$$

In fact, $\gamma_0 := m\gamma_m$ with γ_m in [Ot, Proposition 4.9] satisfies (2.2).

§ 2.2. The absolute Frobenius action

Let $W_p \subset \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ be the Weil group. In [Co], Coleman calculated its action on $H^1_{\mathrm{dR}}(F_m,\overline{\mathbb{Q}_p})$ in terms of a modified p-adic Γ-function. In this subsection, we rewrite it by using another p-adic Γ-function which is characterized as follows.

Definition 2.1 ([Ka, Lemma 2, Definition 1])**.**

1. When $z \in \mathbb{Z}_p$, we denote Morita's p-adic Γ-function by $\Gamma_p(z)$. Namely,

$$\Gamma_p(z) := \lim_{\mathbb{N}\ni n\to z}\left((-1)^n \prod_{k=1,\ p\nmid k}^{n-1} k\right) \quad (z \in \mathbb{Z}_p).$$

2. When $p > 2$, there exists a continuous function $\Gamma_p \colon \mathbb{Q}_p - \mathbb{Z}_p \to \overline{\mathbb{Q}_p}$ satisfying the following conditions.

$$\begin{aligned} |\Gamma_p(z)|_p &= 1, \\ \log_p(\Gamma_p(z+1)) &= \log_p(z) + \log_p(\Gamma_p(z)), \\ \log_p(\Gamma_p(2z)) &= (2z-\tfrac{1}{2})\log_p 2 + \log_p(\Gamma_p(z)) + \log_p(\Gamma_p(z+\tfrac{1}{2})). \end{aligned}$$

Here we denote Iwasawa's p-adic log function by $\log_p$.

We define the function $\Gamma_p \colon \mathbb{Q}_p \to \overline{\mathbb{Q}_p}$ by gathering the above Γ_p's.

The following is a p-adic analogue of (E). For the proof, see [GK, Lemma 2.3, in the case $z \in \mathbb{Z}_p$], [Ka, Proof of Lemma 4, in the case $z \in \mathbb{Q}_p$].

Proposition 2.2. *For $z \in \mathbb{Q}_p$, we have*

$$\Gamma_p(z)\Gamma_p(1-z) \in \mu_\infty.$$

If $z \in \mathbb{Z}_p$, then we have moreover,

$$\Gamma_p(z)\Gamma_p(1-z) = \pm 1.$$

Remark. Coleman [Co] also generalized Morita's p-adic Γ-function to a continuous function on $\mathbb{Q}_p$. We denote it by Γ_{col}. Then by [Ka, (21)] we see that

$$\Gamma_p(z) = \Gamma_{\mathrm{col}}(z)\Gamma_p(z_p) \qquad (z \in \mathbb{Q}_p).$$

Here we define the fractional part z_p for $z \in \mathbb{Q}_p$ by $z_p \in \mathbb{Z}[\frac{1}{p}] \cap [0,1)$, $z_p \equiv z \bmod \mathbb{Z}_p$.

Let Φ_τ be the action on $H^1_{\mathrm{dR}}(F_m, \overline{\mathbb{Q}_p})$ associated with $\tau \in W_p$, which is denoted by $\rho_{\mathrm{cris}}(\tau)$ in [Co]. When $p \nmid m$, F_m has good reduction at p, so Φ_τ is defined via the canonical isomorphism

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(F_m, \overline{\mathbb{Q}_p}) & \cong & H^1_{\mathrm{cris}}(F_m \times \mathbb{F}_p, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}_p}, \\ \Phi_\tau & \leftrightarrow & \Phi_{\mathrm{cris}}^{\deg \tau} \otimes \tau, \end{array}$$

where Φ_{cris} denotes the absolute Frobenius action on the cristalline cohomology group $H^1_{\mathrm{cris}}(F_m \times \mathbb{F}_p, \mathbb{Z}_p)$. When $p \mid m$, we need a more delicate argument.

We need a few more notations in [Co]. We denote the fractional part in $(0,1]$ of $\alpha \in \mathbb{Q}$ by $\langle \alpha \rangle$ and put $\varepsilon(\alpha,\beta) := \langle \alpha \rangle + \langle \beta \rangle - \langle \alpha+\beta \rangle$ for $\alpha, \beta \in \mathbb{Q}$. The following Theorem is a corollary of [Co, Theorems 1.7, 3.13]. For simplicity, we denote by the same symbol $\eta_{\frac{i}{m},\frac{j}{m}}$ its class in $H^1_{\mathrm{dR}}(F_m, \mathbb{Q})$.

Theorem 2.3 ([Ka, Theorem 1]). *For $\tau \in W_p$ and for $i, j, m \in \mathbb{N}$ with $i, j < m$, $i + j \neq m$, there exists $\gamma(\tau, \frac{i}{m}, \frac{j}{m}) \in \mathbb{Q}_p$ satisfying*

$$\Phi_\tau(\eta_{\frac{i}{m},\frac{j}{m}}) = \gamma(\tau, \tfrac{i}{m}, \tfrac{j}{m})\eta_{\tau(\frac{i}{m}),\tau(\frac{j}{m})}.$$

More precisely, we have the following.

1. *When $(p,m) = 1$, $\deg \tau = 1$, we have*

$$\gamma(\tau, \tfrac{i}{m}, \tfrac{j}{m}) = \frac{p^{1-\varepsilon(\frac{i}{m},\frac{j}{m})}}{\langle \frac{i}{m} + \frac{j}{m} \rangle^{\varepsilon(\frac{i}{m},\frac{j}{m})}} \frac{(-1)^{\varepsilon(\tau(\frac{i}{m}),\tau(\frac{j}{m}))} \langle \tau(\frac{i}{m}) + \tau(\frac{j}{m}) \rangle^{\varepsilon(\tau(\frac{i}{m}),\tau(\frac{j}{m}))}}{B_p\langle \tau(\frac{i}{m}), \tau(\frac{j}{m}) \rangle}.$$

   Here we put $B_p\langle \alpha, \beta \rangle := \frac{\Gamma_p(\langle \alpha \rangle)\Gamma_p(\langle \beta \rangle)}{\Gamma_p(\langle \alpha+\beta \rangle)}$.

2. *When* $p > 2$, $p \mid m$, $(p, ij(i+j)) = 1$, *and* $\deg \tau = 1$, *we have*

$$\gamma(\tau, \tfrac{i}{m}, \tfrac{j}{m}) \equiv p^{\frac{1}{2}} \frac{B_p(\frac{i}{m}, \frac{j}{m})}{B_p(\tau(\frac{i}{m}), \tau(\frac{j}{m}))} \bmod \mu_\infty.$$

Here put $B_p(\alpha, \beta) := \frac{\Gamma_p(\alpha)\Gamma_p(\beta)}{\Gamma_p(\alpha+\beta)}$.

Remark. In order to derive Theorem 2.3 from [Co, Theorems 1.7, 3.13], we used the fact that $\Gamma_p(\frac{1}{2})$ is a root of unity, which follows from Proposition 2.2.

§ 3. The absolute Frobenius action on p-adic periods of Fermat curves

We define p-adic periods of Fermat curves in terms of the comparison isomorphisms of p-adic Hodge theory, which is developed by many mathematicians (for example, [Fo1], [Fo2], [Fa], [Ts]). For simplicity, we fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Let $B_{\mathrm{cris}}, B_{\mathrm{dR}}$ be Fontaine's p-adic period rings. Note that the composite ring $B_{\mathrm{cris}}\overline{\mathbb{Q}_p} \subset B_{\mathrm{dR}}$ is well-defined since there exist canonical embeddings $\overline{\mathbb{Q}_p}, B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$. We obtain a period-ring-valued pairing

$$\int_p \colon H_1^{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q}) \times H^1_{\mathrm{dR}}(F_m, \mathbb{Q}) \to B_{\mathrm{dR}}, \qquad (\gamma, \eta) \mapsto \int_{p,\gamma} \eta$$

by combining the comparison isomorphisms $H^1_{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H^1_{\text{ét}}(F_m \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$, $H^1_{\text{ét}}(F_m \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^1_{\mathrm{dR}}(F_m, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{\mathrm{dR}}$ and the pairing $H_1^{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q}) \times H^1_{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}$. This is a p-adic counterpart of the usual period: $H_1^{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q}) \times H^1_{\mathrm{dR}}(F_m, \mathbb{Q}) \to \mathbb{C}$. We see that the image under the pairing $\int_p$ is in $B_{\mathrm{cris}}\overline{\mathbb{Q}_p}$ since the Jacobian variety of F_m has CM and so has potentially good reduction. Therefore we can reduce the coefficient ring B_{dR} to $B_{\mathrm{cris}}\overline{\mathbb{Q}_p}$, and get the following isomorphism compatible with the action of $\tau \in W_p$:

$$\begin{array}{ccc} H^1_{\text{ét}}(F_m \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}\overline{\mathbb{Q}_p} & \cong & H^1_{\mathrm{dR}}(F_m, \overline{\mathbb{Q}_p}) \otimes_{\overline{\mathbb{Q}_p}} B_{\mathrm{cris}}\overline{\mathbb{Q}_p}, \\ 1 \otimes \Phi_\tau & \leftrightarrow & \Phi_\tau \otimes \Phi_\tau. \end{array} \tag{3.1}$$

Here the action Φ_τ on $B_{\mathrm{cris}}\overline{\mathbb{Q}_p} = B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} \overline{\mathbb{Q}_p}$ is defined as $\Phi_\tau := \Phi_{\mathrm{cris}}^{\deg \tau} \otimes \tau$ with Φ_{cris} the action of the absolute Frobenius on B_{cris}. We note that $\tau \in W_p$ acts on $H_1^{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q}), H^1_{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q})$ trivially since we embedded $H^1_{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q}) \hookrightarrow H^1_{\text{ét}}(F_m \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$. Therefore we can derive the following "reciprocity law on p-adic periods of Fermat curves" from Theorem 2.3.

Theorem 3.1 ([Ka, Theorem 2]). *Let* i, j, m *be as in Theorem* 2.3, *and* $\gamma \in H_1^{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q})$, $\tau \in W_p$.

1. *When* $(p,m)=1$ *and* $\deg\tau\geq 0$, *we have*

$$\Phi_\tau\left(\int_{p,\gamma}\eta_{\frac{i}{m},\frac{j}{m}}\right)=\left(\prod_{k=1}^{\deg\tau}\frac{(-1)^{\varepsilon(\langle\frac{p^ki}{m}\rangle,\langle\frac{p^kj}{m}\rangle)}p^{1-\varepsilon(\langle\frac{p^{k-1}i}{m}\rangle,\langle\frac{p^{k-1}j}{m}\rangle)}}{B_p\langle\frac{p^ki}{m},\frac{p^kj}{m}\rangle}\right)$$
$$\times\frac{\langle\tau(\frac{i}{m})+\tau(\frac{j}{m})\rangle^{\varepsilon(\tau(\frac{i}{m}),\tau(\frac{j}{m}))}}{\langle\frac{i}{m}+\frac{j}{m}\rangle^{\varepsilon(\frac{i}{m},\frac{j}{m})}}\int_{p,\gamma}\eta_{\tau(\frac{i}{m}),\tau(\frac{j}{m})}.$$

2. *When* $p>2$, $p\mid m$, $(p,ij(i+j))=1$, *we have*

$$\Phi_\tau\left(\frac{\int_{p,\gamma}\eta_{\frac{i}{m},\frac{j}{m}}}{B_p(\frac{i}{m},\frac{j}{m})}\right)\equiv p^{\frac{\deg\tau}{2}}\frac{\int_{p,\gamma}\eta_{\tau(\frac{i}{m}),\tau(\frac{j}{m})}}{B_p(\tau(\frac{i}{m}),\tau(\frac{j}{m}))}\bmod\mu_\infty.$$

§ 4. A reciprocity law on a period-ring-valued beta function

In this section, we define a period-ring-valued beta function and study its properties. In particular, we give a reciprocity law on its special values. Moreover, we explain that this reciprocity law (Theorem 4.2) is a refinement of (1.4).

Definition 4.1. Let i,j,m be natural numbers satisfying $i,j<m$, $i+j\neq m$. We take $\gamma\in H_1^{\mathrm{B}}(F_m(\mathbb{C}),\mathbb{Q})$ so that $\int_\gamma\eta_{\frac{i}{m},\frac{j}{m}}\neq 0$. We define $\mathfrak{B}(\frac{i}{m},\frac{j}{m})\in B_{\mathrm{cris}}\overline{\mathbb{Q}_p}$ as follows.

1. When $(p,m)=1$, we put $\mathfrak{B}(\frac{i}{m},\frac{j}{m}):=\dfrac{B(\frac{i}{m},\frac{j}{m})}{\int_\gamma\eta_{\frac{i}{m},\frac{j}{m}}}\langle\frac{i}{m}+\frac{j}{m}\rangle^{\varepsilon(\frac{i}{m},\frac{j}{m})}\int_{p,\gamma}\eta_{\frac{i}{m},\frac{j}{m}}$.

2. When $p>2$, $p\mid m$, $(p,ij(i+j))=1$, we put $\mathfrak{B}(\frac{i}{m},\frac{j}{m}):=\dfrac{B(\frac{i}{m},\frac{j}{m})}{\int_\gamma\eta_{\frac{i}{m},\frac{j}{m}}}\dfrac{\int_{p,\gamma}\eta_{\frac{i}{m},\frac{j}{m}}}{B_p(\frac{i}{m},\frac{j}{m})}$.

Rohrlich's formula (2.1) states that $\frac{B(\frac{i}{m},\frac{j}{m})}{\int_\gamma\eta_{\frac{i}{m},\frac{j}{m}}}\in\overline{\mathbb{Q}}$, so $\mathfrak{B}(\frac{i}{m},\frac{j}{m})\in B_{\mathrm{cris}}\overline{\mathbb{Q}_p}$. By a basic argument in the CM-theory, we see that the ratio "$\int_\gamma\eta_{\frac{i}{m},\frac{j}{m}}:\int_{p,\gamma}\eta_{\frac{i}{m},\frac{j}{m}}$" does not depend on the choice of m,γ ([Ka, Lemma 3]). Hence the definition of $\mathfrak{B}(\frac{i}{m},\frac{j}{m})$ also does not depend on m,γ. We added the remaining term ($\langle\frac{i}{m}+\frac{j}{m}\rangle^{\varepsilon(\frac{i}{m},\frac{j}{m})}$ or $B_p(\frac{i}{m},\frac{j}{m})$) in order to simplify formulas in Theorem 4.2 and Lemma 4.3-(3). The following Theorem follows from Theorem 3.1 and (2.2).

Theorem 4.2 ([Ka, Theorem 3]). *Let* p *be a prime,* $i,j,m\in\mathbb{N}$ *with* $i,j<m$, $i+j\neq m$, *and* $\tau\in W_p$.

1. *When* $(p,m)=1$ *and* $\deg\tau\geq 0$, *we have*

$$\Phi_\tau(\mathfrak{B}(\tfrac{i}{m},\tfrac{j}{m}))=\left(\prod_{k=1}^{\deg\tau}\frac{(-1)^{\varepsilon(\langle\frac{p^ki}{m}\rangle,\langle\frac{p^kj}{m}\rangle)}p^{1-\varepsilon(\langle\frac{p^{k-1}i}{m}\rangle,\langle\frac{p^{k-1}j}{m}\rangle)}}{B_p\langle\frac{p^ki}{m},\frac{p^kj}{m}\rangle}\right)\mathfrak{B}(\tau(\tfrac{i}{m}),\tau(\tfrac{j}{m})).$$

2. *When $p > 2$, $p \mid m$, $(p, ij(i+j)) = 1$, we have*

$$\Phi_\tau(\mathfrak{B}(\tfrac{i}{m}, \tfrac{j}{m})) \equiv p^{\frac{\deg \tau}{2}} \mathfrak{B}(\tau(\tfrac{i}{m}), \tau(\tfrac{j}{m})) \bmod \mu_\infty.$$

Finally in this section, we shall derive (1.4) (= Corollary 4.4) from Theorem 4.2.

Lemma 4.3 ([Ka, Lemma 1, Lemma 5, (32)])**.**

(1) *For any $\alpha \in \mathbb{Q} \cap (0,1)$, we can write $\Gamma(\alpha)\Gamma(1-\alpha)$ in the form of a product of rational powers of $(1-2\beta)B(\beta,\beta)B(1-\beta,1-\beta)$ with $\beta \in \{2^k\alpha \mid k = 0,1,2,\dots\}$. More precisely, we obtain the following expressions.*

(a) *Write $\alpha = \frac{a}{m}$ so that $(a,m) = 1$, $0 < a < m$ and $m = 2^t m_0$ so that $(2, m_0) = 1$. First assume that $\frac{a}{m_0} \neq 1$. Then taking the smallest $f_{m_0} \in \mathbb{N}$ satisfying $2^{f_{m_0}} \equiv 1 \bmod m_0$, we can write*

$$\begin{aligned}&(\Gamma(\tfrac{a}{m})\Gamma(\tfrac{m-a}{m}))^{2^t(2^{f_{m_0}}-1)}\\ &= \pm \prod_{k=1}^{t} \left(\tfrac{2^{k-1}m_0-a}{2^{k-1}m_0} B(\tfrac{a}{2^k m_0}, \tfrac{a}{2^k m_0}) B(\tfrac{2^k m_0 - a}{2^k m_0}, \tfrac{2^k m_0 - a}{2^k m_0})\right)^{2^{k-1}(2^{f_{m_0}}-1)}\\ &\times \prod_{l=0}^{f_{m_0}-1} \left(\tfrac{m_0 - 2^{l+1}a}{m_0} B(\tfrac{2^l a}{m_0}, \tfrac{2^l a}{m_0}) B(\tfrac{m_0 - 2^l a}{m_0}, \tfrac{m_0 - 2^l a}{m_0})\right)^{2^{f_{m_0}-1-l}}.\end{aligned}$$

(b) *When $\alpha = \frac{1}{2^t}$, we have*

$$(\Gamma(\tfrac{1}{2^t})\Gamma(\tfrac{2^t-1}{2^t}))^{2^{t-1}} = B(\tfrac{1}{2}, \tfrac{1}{2}) \prod_{k=2}^{t} \left(\tfrac{2^{k-1}-1}{2^{k-1}} B(\tfrac{1}{2^k}, \tfrac{1}{2^k}) B(\tfrac{2^k-1}{2^k}, \tfrac{2^k-1}{2^k})\right)^{2^{k-2}}.$$

(2) *We define the p-adic counterpart $(2\pi i)_p \in B_{\mathrm{dR}}^\times$ of $2\pi i \in \mathbb{C}$ as the p-adic period of the second cohomology of the projective line $\mathbb{P}^1$. Namely, we put*

$$(2\pi i)_p := \frac{(2\pi i)\int_{p,c_b} c_{dr}}{\int_{c_b} c_{dr}},$$

where c_b, c_{dr} are bases of $H_2^{\mathrm{B}}(\mathbb{P}^1(\mathbb{C}), \mathbb{Q}), H_{\mathrm{dR}}^2(\mathbb{P}^1, \mathbb{Q})$, respectively. Then we have $(2\pi i)_p \in B_{\mathrm{cris}}$, $\Phi_{\mathrm{cris}}((2\pi i)_p) = p(2\pi i)_p$. Moreover we have

$$\frac{\int_{\gamma_1} \eta_{\frac{i}{m},\frac{j}{m}} \int_{\gamma_2} \eta_{\frac{m-i}{m},\frac{m-j}{m}}}{2\pi i} = \frac{\int_{p,\gamma_1} \eta_{\frac{i}{m},\frac{j}{m}} \int_{p,\gamma_2} \eta_{\frac{m-i}{m},\frac{m-j}{m}}}{(2\pi i)_p} \in \mathbb{Q}(\zeta_m)$$

for all $i, j, m \in \mathbb{N}$ with $i, j < m$, $i + j \neq m$ and for all $\gamma_1, \gamma_2 \in H_1^{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q})$.

(3) *Let i, j, m be natural numbers satisfying $i, j < m$, $i+j \neq m$. Assume that $(p, m) = 1$ or $p > 2$, $p \mid m$, $(p, ij(i+j)) = 1$. Then we have*

$$\frac{\mathfrak{B}(\frac{i}{m}, \frac{j}{m})\mathfrak{B}(\frac{m-i}{m}, \frac{m-j}{m})}{(2\pi i)_p} \equiv \left(1 - \tfrac{i}{m} - \tfrac{j}{m}\right)^{\flat} \frac{B(\frac{i}{m}, \frac{j}{m})B(\frac{m-i}{m}, \frac{m-j}{m})}{2\pi i} \bmod \mu_\infty.$$

Here we define $z^\flat \in \mathbb{Z}_p^\times$ for $z \in \mathbb{Q}_p^\times$ as $z^\flat := zp^{-\mathrm{ord}_p z}$.

(4) *We can write $u_{\mathbb{Q}}(\sigma_{\pm\frac{a}{m}})$, which is equal to $(\frac{2\pi}{\Gamma(\frac{a}{m})\Gamma(\frac{m-a}{m})})^2$, as a product of rational powers of $\frac{\mathfrak{B}(\alpha,\alpha)\mathfrak{B}(1-\alpha,1-\alpha)}{(2\pi i)_p}$ with $\alpha \in \mathbb{Q} \cap (0,1)$ up to multiplication by a root of unity.*

Remark. The referee of this paper suggested that there is a simpler formula than Lemma 4.3-(1) as follows:

$$\Gamma(\tfrac{a}{m})^m = \Gamma(a) \prod_{k=1}^{m-1} B(\tfrac{a}{m}, \tfrac{ka}{m}).$$

The author thanks the referee for his helpful suggestions.

Sketch of Proof. (1) follows from functional equations $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \Gamma(z+1) = z\Gamma(z)$, (3) from (2), Proposition 2.2, and (4) from (1), (3), respectively. We provide a sketch of the proof of (2). We put $\eta_1 := \eta_{\frac{i}{m}, \frac{j}{m}}$, $\eta_2 := \eta_{\frac{m-i}{m}, \frac{m-j}{m}}$ and take $\gamma_i \in H_1^{\mathrm{B}}(F_m(\mathbb{C}), \mathbb{Q})$ so that $\int_{\gamma_i} \eta_i \neq 0$. Then there exists $\gamma_i^* \in H_{\mathrm{B}}^1(F_m(\mathbb{C}), \mathbb{Q}(\zeta_m))$ satisfying

$$\begin{array}{ccccc} H_{\mathrm{B}}^1(F_m(\mathbb{C}), \mathbb{Q}(\zeta_m)) \otimes_{\mathbb{Q}(\zeta_m)} \mathbb{C} & \cong & H_{\mathrm{dR}}^1(F_m, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} & & \\ \gamma_i^* \otimes \int_{\gamma_i} \eta_i & \leftrightarrow & \eta_i \otimes 1 & & (i = 1, 2) \end{array}$$

since η_i are simultaneous eigenvectors for CM. Besides, the de Rham isomorphism is a ring isomorphism under the cup products, hence we have

$$\begin{array}{ccc} H_{\mathrm{B}}^2(F_m(\mathbb{C}), \mathbb{Q}(\zeta_m)) \otimes_{\mathbb{Q}(\zeta_m)} \mathbb{C} & \cong & H_{\mathrm{dR}}^2(F_m, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \\ (\gamma_1^* \cup \gamma_2^*) \otimes (\int_{\gamma_1} \eta_1 \int_{\gamma_2} \eta_2) & \leftrightarrow & (\eta_1 \cup \eta_2) \otimes 1. \end{array}$$

Since we may identify the second cohomology group of F_m with that of $\mathbb{P}^1$, we have $\int_{\gamma_1} \eta_1 \int_{\gamma_2} \eta_2 \in 2\pi i \cdot \mathbb{Q}(\zeta_m)$. The same holds true even if we replace the above isomorphism by its p-adic analogue: $H_{\mathrm{B}}^2(F_m(\mathbb{C}), \mathbb{Q}(\zeta_m)) \otimes_{\mathbb{Q}(\zeta_m)} B_{\mathrm{cris}\overline{\mathbb{Q}_p}} \cong H_{\mathrm{dR}}^2(F_m, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{\mathrm{cris}\overline{\mathbb{Q}_p}}$. Hence it suffices to show that $\eta_1 \cup \eta_2 \neq 0$. In [Ka, Lemma 5], we proved this by a direct computation, although there is an alternative proof using the non-degeneracy of the cup product and the complex multiplication. $\square$

Corollary 4.4 ([Ka, Corollaries 1, 2]). *We have*

$$u_{\mathbb{Q}}(\sigma) \in \overline{\mathbb{Q}} \quad (\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})),$$
$$\tau(u_{\mathbb{Q}}(\sigma)) \equiv u_{\mathbb{Q}}(\tau\sigma) \bmod \mu_\infty \quad (\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q}),\ \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})).$$

Proof. The algebraicity of $u_{\mathbb{Q}}(\sigma)$ follows from (2.1), Lemma 4.3-(1), (2) immediately. In order to prove the reciprocity law for $u_{\mathbb{Q}}(\sigma)$, it suffices to prove the reciprocity law for $\frac{\mathfrak{B}(\alpha,\alpha)\mathfrak{B}(1-\alpha,1-\alpha)}{(2\pi i)_p}$ by Lemma 4.3-(4). When $\tau \in W_p$, this reciprocity law follows from Theorem 4.2 since Φ_τ is τ-semilinear. We note that we regard W_p as a subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, and that we may change p and the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. Therefore, by the Chebotarev density theorem, we obtain the reciprocity law for all τ. $\square$

Remark. It is not necessary to define the period-ring-valued beta function $\mathfrak{B}$ in order just to derive the results in Corollary 4.4. The algebraicity follows from Lemma 4.3-(2) in the classical setting (without its p-adic analogue), and the reciprocity law from Coleman's results directly. The motivation of stating the above Corollary is to see that the reciprocity law of $\mathfrak{B}$ is a refinement of (1.3), (1.4).

§ 5. Toward generalizations for totally real fields

In this section, we give a short survey on some results in [Shin2],[Yo],[KY1],[KY2]. Let F be a totally real algebraic number field of degree n, K a finite abelian extension of F with conductor $\mathfrak{f}$, $G := \mathrm{Gal}(K/F)$. We denote the Artin map by $\mathrm{Art}\colon C_{\mathfrak{f}} \to G$, where $C_{\mathfrak{f}}$ is the narrow ray class group modulo $\mathfrak{f}$. Then the partial zeta function associated with $\sigma \in G$ is given by $\zeta(s,\sigma) = \sum_{c\in \mathrm{Art}^{-1}(\sigma)} \zeta(s,c)$ with

$$\zeta(s,c) := \sum_{\mathfrak{a}\subset \mathcal{O}_F, \mathfrak{a}\in c} N\mathfrak{a}^{-s}.$$

Strictly speaking, $\zeta(s,c)$ is defined as the analytic continuation of this function and the same applies hereinafter. Shintani provided an explicit formula of the following form (for detail, see [Shin2] or [Yo, Theorem 3.3, Chapter II]):

$$\zeta'(0,c) = \sum_{i=1}^{n} \left[\frac{d}{ds} \sum_{z\in R} \iota_i(z)^{-s} \right]_{s=0} + \text{“some correction terms”},$$

where $\iota_1,\dots,\iota_n$ are all embeddings of F into $\mathbb{R}$ and R is a certain set of totally positive elements in F depending on c. (By using the notation in [Yo, Lemma 3.2, Chapter II], we can write $R = \coprod_{j\in J}\{z \in (\mathfrak{a}_\mu\mathfrak{f})^{-1} \cap C_j,\ (z)\mathfrak{a}_\mu\mathfrak{f} = c \text{ in } C_{\mathfrak{f}}\}$). We note that $\sum_{z\in R} \iota_i(z)^{-s}$ can be written as a finite sum of Barnes' multiple zeta functions, so its derivative value at $s = 0$ can be written in terms of log of Barnes' multiple gamma functions.

Yoshida provided a good expression of the correction terms in Shintani's formula. Namely, there exist $m \in \mathbb{N}$, elements a_k in F, and totally positive elements b_k in F $(k = 1, 2, \ldots, m)$ satisfying

$$\zeta'(0, c) = \sum_{i=1}^{n} \left[\frac{d}{ds} \sum_{z \in R} \iota_i(z)^{-s} \right]_{s=0} + \sum_{i=1}^{n} \sum_{k=1}^{m} \iota_i(a_k) \log \iota_i(b_k). \tag{5.1}$$

Yoshida defined the symbol X ([Yo, (3.9), Chapter III]) as

$$X(\iota_i(c)) := \left[\frac{d}{ds} \sum_{z \in R} \iota_i(z)^{-s} \right]_{s=0} + \sum_{k=1}^{m} \iota_i(a_k) \log \iota_i(b_k).$$

Therefore we get a canonical decomposition:

$$\exp(\zeta'(0, c)) = \prod_{i=1}^{n} \exp(X(\iota_i(c))), \quad \exp(\zeta'(0, \sigma)) = \prod_{i=1}^{n} \prod_{c \in \mathrm{Art}^{-1}(\sigma)} \exp(X(\iota_i(c))).$$

When $\iota_i = \mathrm{id}$, we write $X(c) := X(\iota_i(c))$. Note that we regard a number field as a subfield of $\mathbb{C}$.

Yoshida formulated the following Conjecture on the relation between his X-invariant and Shimura's period symbol p_K. By [Yo, Theorem 2.6, Chapter III], we see that this Conjecture is a generalization of Rohrlich's formula (2.1) and the Chowla-Selberg formula.

Conjecture 5.1 ([Yo, Conjecture 3.9, Chapter III]). *Let K be a CM-field and $p_K(\sigma, \tau)$ Shimura's period symbol for embeddings σ, τ of K into $\mathbb{C}$. Assume that $\sigma = \mathrm{id}$ and there exists a totally real subfield $F \subset K$ such that K/F is abelian and $\tau \in G := \mathrm{Gal}(K/F)$. Then we have*

$$p_K(\mathrm{id}, \tau) \equiv \pi^{-\mu(\tau)/2} \exp\left(\frac{1}{|G|} \sum_{\mathfrak{f} | \tilde{\mathfrak{f}}} \sum_{\omega \in (\hat{G}_-)_{\mathfrak{f}}} \frac{\omega(\tau)}{L(0, \omega)} \sum_{c \in C_{\mathfrak{f}}} \omega(c) X(c) \right) \bmod \overline{\mathbb{Q}}^{\times}.$$

Here we denote by $\tilde{\mathfrak{f}}$ the conductor of K/F, by $(\hat{G}_-)_{\mathfrak{f}}$ the set of primitive odd characters of G whose conductor is equal to $\mathfrak{f}$. We put $\mu(\tau) := 1, -1, 0$ if $\tau = \mathrm{id}$, the complex conjugation, otherwise, respectively.

Yoshida and the author ([KY1], [KY2]) also formulated p-adic analogues of Conjecture 5.1: In [KY1], we defined an invariant $X_p(\iota_i(c)) \in \overline{\mathbb{Q}_p}$ which is a p-adic analogue of Yoshida's X-invariant and conjectured a generalization of the Gross-Koblitz formula

[GK]. In [KY2], we studied motives attached to algebraic Hecke characters and conjectured a relation [KY2, Conjecture Q] between the Frobenius map on such motives and our X_p-invariant. As we remarked in [KY2, Remark 1], we can show that [KY2, Conjecture Q] holds true when $F = \mathbb{Q}$, by using some results in [Og]. When $F = \mathbb{Q}$, the results in [Og] are consistent with those in [Co] (for details, see [Og, Proof of Proposition 2.4], [Co, the third paragraph of §III]). In this sense, [KY2, Conjecture Q] is a generalization of Theorems 2.3, 3.1 with $p \nmid m$.

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