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A construction of real quadratic fields of minimal type and primary symmetric parts of ELE type

By

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Abstract

This article is an announcement of our recent papers [6] and [5]. For a non-square positive integer d with $4 \nmid d$, put $\omega(d) := (1+\sqrt{d})/2$ if d is congruent to 1 modulo 4 and otherwise $\omega(d) := \sqrt{d}$. Let $a_1, a_2, \ldots, a_{\ell-1}$ be the symmetric part of the simple continued fraction expansion of $\omega(d)$. We say that the string $a_1, a_2, \ldots, a_{[\ell/2]}$ is the primary symmetric part of the simple continued fraction expansion of $\omega(d)$. The main purposes of this article are to introduce notions of "ELE type" and "pre-ELE type" for a finite string of positive integers, and to study properties for a non-square positive integer d such that the primary symmetric part of the simple continued fraction expansion of $\sqrt{d}$ with even period is of ELE type.

§ 1. Introduction

Let d be a non-square positive integer with $4 \nmid d$. Put $\omega(d) := (1+\sqrt{d})/2$ if $d \equiv 1 \pmod 4$ and otherwise $\omega(d) := \sqrt{d}$. Then it is well-known that the simple continued fraction expansion is of the form

$$\omega(d) = [a_0, \overline{a_1, a_2, \ldots, a_\ell}],$$

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where $\ell = \ell(d)$ is the minimal period. Moreover, the string of partial quotients $a_1, a_2, \ldots, a_{\ell-1}$ is symmetric, namely, partial quotients $a_1, \ldots, a_{\ell-1}$ are of the form

$$a_1, \ldots, a_{L-1}, a_L, a_{L-1}, \ldots, a_1, \text{ if } \ell = 2L \text{ is even,}$$
$$a_1, \ldots, a_{L-1}, a_L, a_L, a_{L-1}, \ldots, a_1, \text{ if } \ell = 2L+1 \text{ is odd.}$$

We call the string $a_1, \ldots, a_{L-1}, a_L$ the *primary symmetric part* of the simple continued fraction expansions of $\omega(d)$.

In [9], the first and fourth authors gave the following table: We arrange some values of d in ascending order of size in each period ℓ.

ℓ	d					
1	2	5	10	13	26	...
2	3	6	11	15	18	...
3	17	37	61	65	101	...
4	7	14	23	33	34	...
5	41	74	149	157	181	...
6	19	22	54	57	59	...
7	58	89	109	113	137	...
8	31	71	91	135	153	...
9	73	97	106	233	277	...
10	43	67	86	115	118	...
11	265	298	541	554	593	...
12	46	103	127	177	209	...
13	421	746	757	778	1021	...
14	134	179	190	201	251	...
15	193	281	481	1066	1417	...
16	94	191	217	249	302	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

For a positive integer ℓ, we denote d_ℓ the smallest non-square positive integer d with $4 \nmid d$ and $\ell = \ell(d)$:

$$d_1 = 2, \quad d_2 = 3, \quad d_3 = 17, \quad d_4 = 7, \quad d_5 = 41, \ldots.$$

Then, for $1 \leq \ell \leq 69868$, d_ℓ is square-free and the class number of $\mathbb{Q}(\sqrt{d_\ell})$ is equal to 1 except for the six cases: $\ell = 7,\ 11,\ 49,\ 225,\ 299,\ 1032$. To find many real quadratic fields of class number 1, we are interested in how to construct d_ℓ and in properties of d_ℓ.

In order to analyze in detail, we proceed with further experiments. Let d'_ℓ be the smallest integer d such that the minimal periods of the simple continued fraction expansions of $\omega(d)$ are equal to a fixed positive integer ℓ where d runs through non-square positive integers with $d \equiv 2, 3 \pmod 4$. Then the following hold for each even positive integer ℓ with $8 \leq \ell \leq 73478$; i) d'_ℓ is square-free, ii) the class number of $\mathbb{Q}(\sqrt{d'_\ell})$ is equal to 1, iii) $\mathbb{Q}(\sqrt{d'_\ell})$ is of minimal type, iv) the primary symmetric part of the simple continued fraction expansion of $\omega(d'_\ell)$ is of ELE type. We will define "minimal type" for a positive integer and for a real quadratic field in Section 4.

This paper is organized as follows. In Section 2, we introduce a notion of ELE type for a finite string of positive integers. In Section 3, in order to construct strings of ELE type, we define pre-ELE type for a finite string. In the final section, Section 4, we give an infinite family of real quadratic fields with period ℓ of minimal type for each even $\ell \geq 6$, as an application of our results.

§ 2. A string of ELE type

First we will see the following numerical results. For a positive integer ℓ, we define

$$\mathrm{CF}_\ell := \{d \in \mathbb{N} \mid d \text{ is not square}, \ 4 \nmid d, \ \ell(d) = \ell\}.$$

For the 100 smallest integers

$$d_\ell = d_\ell^{(0)} < d_\ell^{(1)} < \cdots < d_\ell^{(99)}$$

in CF_ℓ, we denote the simple continued fraction expansion of $\omega(d_\ell^{(i)})$ by

$$\omega(d_\ell^{(i)}) = [a_0^{(i)}, \overline{a_1^{(i)}, \ldots, a_\ell^{(i)}}].$$

Let us plot

$$(x, y, z) = (i, j, a_j^{(i)}), \quad 0 \leq i \leq 99, \quad 1 \leq j \leq L := [\ell/2]$$

in three dimensional space and connect them for each i. Here, $[x]$ denotes the largest integer $\leq x$ for a real number x. The figures (a)-(d) in the next page are the cases when $\ell = 100, 101, 102$ and 103, respectively. We can observe that the graphs of even cases are characteristic. Our motivation is to investigate why the ends of graphs are extremely large. To see this, we will define a string of ELE type as follows.

For a finite string $a_1, \ldots, a_L$ ($L \geq 2$), we define nonnegative integers q_i, r_i by using the following recurrence equations:

$$\begin{cases} q_0 = 0, \quad q_1 = 1, \quad q_i = a_{i-1}q_{i-1} + q_{i-2} \ (2 \leq i \leq L+1), \\ r_0 = 1, \quad r_1 = 0, \quad r_i = a_{i-1}r_{i-1} + r_{i-2} \ (2 \leq i \leq L+1). \end{cases} \tag{2.1}$$

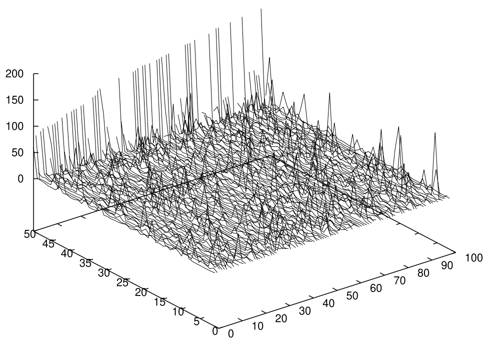
Moreover, define integers u_1, u_2, w, v_1, v_2, z, δ by

$$(r_L^2 - (-1)^L)(r_{L+1} + r_{L-1}) = q_L v_1 + u_1 \ (0 \leq u_1 < q_L), \tag{2.2}$$

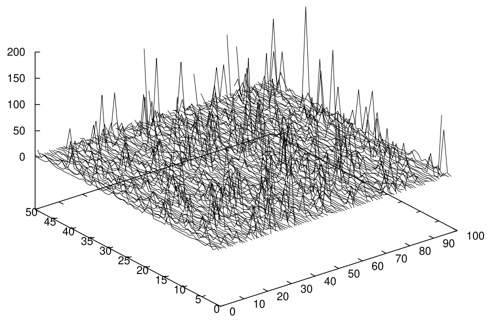
$$(-1)^L (r_L - q_{L-1}) r_L = q_L z + w \ (0 \leq w < q_L), \tag{2.3}$$

$$(-1)^L (q_L - r_{L+1}) + z = q_L v_2 + u_2 \ (0 \leq u_2 < q_L), \tag{2.4}$$

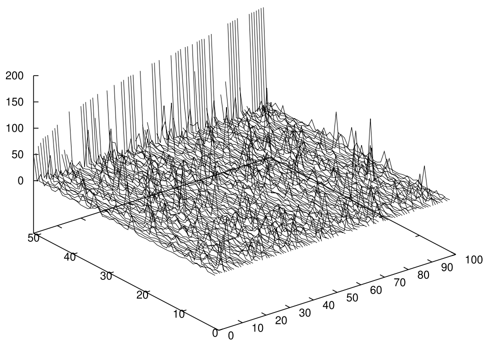
$$\delta = \begin{cases} 0 & \text{if } u_1 \leq u_2, \\ 1 & \text{if } u_1 > u_2. \end{cases}$$



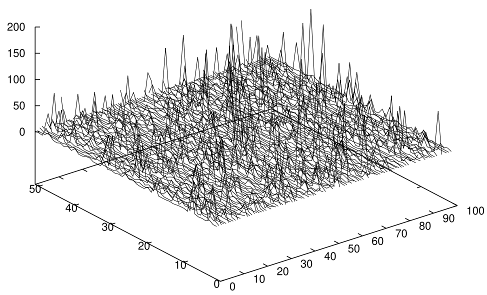
(a) $\ell = 100$, $n = 100$



(b) $\ell = 101$, $n = 100$



(c) $\ell = 102$, $n = 100$



(d) $\ell = 103$, $n = 100$

We put

$$\gamma := q_L(\delta q_L + u_2 - u_1) + w, \tag{2.5}$$

$$\mu := \frac{1}{q_L}\{\gamma(q_{L+1} + q_{L-1}) + 2(q_{L-1} - r_L)\}. \tag{2.6}$$

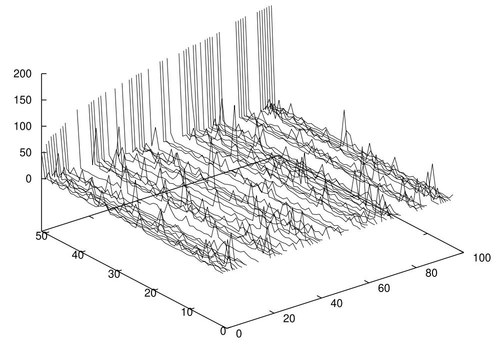
Definition 2.1. Let $L \geq 2$ and let $a_1, a_2, \ldots, a_L$ be a string of positive integers. If

"$a_L \geq 2$ and $\mu = a_L$" or "$a_L \geq 4$ and $\mu = a_L + 2$"

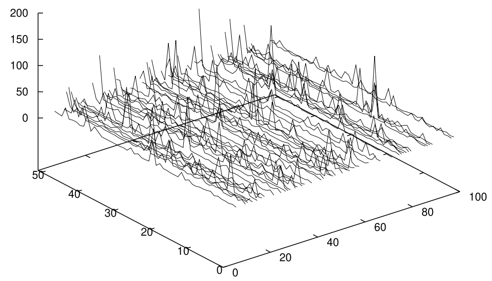
holds, we say that $a_1, a_2, \ldots, a_L$ is a *string with extremely large end* (for convenience, $a_1, a_2, \ldots, a_L$ is of *ELE type*). Specially $a_1, a_2, \ldots, a_L$ is said to be of *ELE*$_1$ *type* (resp. *ELE*$_2$ *type*) if the former conditions (resp. the latter conditions) hold.

Remark 2.1. There is no string of ELE type with length 2.

Here let us look at some graphs again. Dividing the graph in (c) into the case of ELE type and the case of not ELE type (see Figs. (e) and (f)), we expect that "ELE type" has caught the graphs whose ends are extremely large.



(e) $\ell = 102$, $n = 100$, ELE type



(f) $\ell = 102$, $n = 100$, not ELE type

We now state one of the main results of this article. Let d be a non-square positive integer and assume that the simple continued fraction expansion of $\sqrt{d}$ is

$$\sqrt{d} = [a_0, \overline{a_1, \dots, a_{L-1}, a_L, a_{L-1}, \dots, a_1, 2a_0}]$$

with minimal even period $2L$ (≥ 4). We determine quadratic irrationals ω_n $(0 \leq n \leq 2L)$ such that

$$\omega_0 := \sqrt{d}, \quad \omega_n = a_n + \frac{1}{\omega_{n+1}}, \quad a_n = [\omega_n],$$

where $a_n = a_{n-L}$ $(L+1 \leq n \leq 2L-1)$ and $a_{2L} = 2a_0$. Then we can write uniquely $\omega_n = (P_n + \sqrt{d})/Q_n$ with some positive integers P_n, Q_n for each $n \geq 1$ (cf. [7, Section 2]).

Theorem 1. *Under the above setting, assume that $L \geq 3$ and $d \neq 19$. Then the following four conditions are equivalent*:

(i) *d is a positive integer with period $2L$ of minimal type for $\sqrt{d}$ and the primary symmetric part $a_1, a_2, \dots, a_L$ of the simple continued fraction expansion of $\sqrt{d}$ is of ELE type*;

(ii) *d is a positive integer with period $2L$ of minimal type for $\sqrt{d}$, and either*

$$r_L = 2q_{L-1},\ a_L \equiv (-1)^{L-1} q_{L-1} r_{L-1} \pmod{q_L} \text{ and } a_L \geq 2$$

or

$$r_L = 2q_{L-1} - q_L,\ a_L \equiv (-1)^{L-1} q_{L-1}(q_{L-1} + r_{L-1}) \pmod{q_L} \text{ and } a_L \geq 4$$

holds;

(iii) $Q_L = 2$;

(iv) $a_L = a_0$, *or* $a_L = a_0 - 1$.

The proof of Theorem 1 is omitted. We give some remarks.

Remark 2.2. (1) It is known by a classical result (see Perron [11, Satz 3.14]) that one of the three conditions

$$a_L = a_0,\ a_L = a_0 - 1 \text{ or } a_L \leq \frac{2a_0}{3}$$

holds under the above setting.

(2) In the case $d = 19$, $\sqrt{19} = [4, \overline{2, 1, 3, 1, 2, 8}]$, all of conditions (i)-(iv) hold except for $a_L \geq 4$.

(3) Golubeva proved that (iii) yields the equation and the congruence in (ii), d being a prime number congruent to 3 modulo 4 ([2, Theorem 1]).

(4) The implication (iii) $\Rightarrow$ (iv) is shown in the proof of [11, Satz 3.14] or [2, p.1279].

The next theorem gives a way of constructing every positive integer d satisfying the condition (i) of Theorem 1

Theorem 2. *Assume that a string* $a_1, a_2, \ldots, a_L\,(L \geq 3)$ *is of ELE type. In addition, we assume*

$$2a_L > a_1, a_2, \ldots, a_{L-1} \tag{2.7}$$

$$(\textit{resp. } 2a_L + 2 > a_1, a_2, \ldots, a_{L-1}), \tag{2.8}$$

and put $\varepsilon := 0$ (*resp.* $\varepsilon := 1$) *if* $a_1, a_2, \ldots, a_L$ *is of* ELE_1 *type* (*resp.* ELE_2 *type*).

(1) *There does not exist a positive integer* d, $d \equiv 1 \pmod 4$, *with period* $2L$ *of minimal type for* $(1 + \sqrt{d})/2$ *whose simple continued fraction expansion has the symmetric part* $a_1, \ldots, a_{L-1}, a_L, a_{L-1}, \ldots, a_1$.

(2) *Put*

$$d := (a_L + \varepsilon)^2 + \frac{2r_{L+1} + \varepsilon r_L}{q_L}.$$

Then d *is a positive integer with*

$$d \equiv \begin{cases} 2 \pmod 4 & \textit{if } a_L \textit{ is even}, \\ 3 \pmod 4 & \textit{if } a_L \textit{ is odd}. \end{cases}$$

Furthermore, the simple continued fraction expansion of $\sqrt{d}$ *is*

$$\sqrt{d} = [a_L + \varepsilon, \overline{a_1, \ldots, a_{L-1}, a_L, a_{L-1}, \ldots, a_1, 2a_L + 2\varepsilon}]$$

and d *is a positive integer with period* $2L$ *of minimal type for* $\sqrt{d}$.

(3) *Let* d *be as in* (2). *Then we have*

$$(-1)^n Q_n = -\frac{2r_{L+1} + \varepsilon r_L}{q_L} q_n^2 + 2(a_L + \varepsilon) q_n r_n + r_n^2 \ (1 \leq n \leq 2L-1). \tag{2.9}$$

Moreover, let m_d *be the Yokoi invariant of* d *defined below. Then we have* $m_d = 2q_L^2$ *if* L *is even, and* $m_d = 2q_L^2 - 1$ *if* L *is odd.*

The proof of Theorem 2 is also omitted. We give some remarks.

Remark 2.3. (1) In the case $n = L$ of (2.9), we have the condition (iii), $Q_L = 2$, of Theorem 1. The values of Q_n are related to the class number one problem (cf. Louboutin [10]). They will be studied on another occasion.

(2) Let d be a non-square positive integer with $d \equiv 2, 3 \pmod 4$. We let $d = d_1 d_2^2$ be a factorization of d into positive integers with d_1 square-free, and consider a real quadratic field $K = \mathbb{Q}(\sqrt{d_1})$. Let $\mathcal{O}_{d_2}$ be the order of conductor d_2 in K, that is, the subring of the ring $\mathcal{O}_K$ of integers in K, containing 1, with finite index $(\mathcal{O}_K : \mathcal{O}_{d_2}) = d_2$. By [9, Lemma 2.3], the discriminant of $\mathcal{O}_{d_2}$ is $4d$. Thus we consider the real quadratic order of discriminant $4d$ (cf. [9, Remark 2.4]). We denote by $E_d > 1$ the fundamental unit of $\mathcal{O}_{d_2}$. Then we can write uniquely $E_d = (T + U\sqrt{d})/2$ with positive integers T, U. We define an integer $m_d \, (\geq 0)$ by $m_d = [U^2/T]$ and call it the *Yokoi invariant of* d ([9, Definition 2.1]). By a theorem of Yokoi ([9, Theorem 2.1 [B]]) for a non-square positive integer, it holds that $m_d d < E_d < (m_d + 1)d$ if $d > 13$. Thus the quantity m_d gives a size of the fundamental unit E_d for d. The value of m_d gives a rough size of E_d instead of the regulator $\log E_d$.

§ 3. A string of pre-ELE type

In this section, we examine a construction of primary symmetric parts of ELE type. Theorem 1 implies that the primary symmetric part $a_1, a_2, \ldots, a_L$ is of ELE_1 type (resp. ELE_2 type) only if the string $\langle a_1, a_2, \ldots, a_{L-1} \rangle$ satisfies

$$r_L = 2q_{L-1} \ (\text{resp. } r_L = 2q_{L-1} - q_L).$$

Definition 3.1. For a string of $m \, (\geq 1)$ positive integers $\langle a_1, a_2, \ldots, a_m \rangle$, we define q_n and r_n $(0 \leq n \leq m+1)$ by using (2.1) inductively. If either $r_{m+1} = 2q_m$ or $r_{m+1} = 2q_m - q_{m+1}$ holds, we say that $\langle a_1, a_2, \ldots, a_m \rangle$ is of *pre-ELE type with length* m. Specially $\langle a_1, a_2, \ldots, a_m \rangle$ is said to be of *pre-*ELE_1 *type* (resp. *pre-*ELE_2 *type*) *with length* m if $r_{m+1} = 2q_m$ (resp. $r_{m+1} = 2q_m - q_{m+1}$) holds.

We can show basic properties for finite strings of pre-ELE type.

Proposition 3.1. *Let a, b be positive integers and let $A = \langle a_1, \ldots, a_m \rangle$ be a string of $m\,(\geq 1)$ positive integers. We denote the reversed string of A by $\overleftarrow{A} := \langle a_m, \ldots, a_2, a_1 \rangle$. Then the following properties hold.*

(1) *There does not exist a string of pre-ELE$_1$ type with length 1. A is of pre-ELE$_2$ type with length 1 if and only if $A = \langle 1 \rangle$.*

(2) *A is of pre-ELE$_1$ type with length 2 if and only if $A = \langle a_1, 2a_1 \rangle$. Moreover, A is of pre-ELE$_2$ type with length 2 if and only if $A = \langle 2, 1 \rangle$.*

(3) *Assume $m \geq 2$. If A is of pre-ELE$_1$ type then either $a_m = 2a_1$ or $a_m = 2a_1 + 1$ holds.*

(4) *If A is of pre-ELE$_2$ type then $a_m = 1$ holds.*

(5) *A is of pre-ELE$_1$ type with length 3 if and only if $A = \langle a_1, 1, 2a_1 + 1 \rangle$. Moreover, A is of pre-ELE$_2$ type with length 3 if and only if $A = \langle 2, 2, 1 \rangle$.*

(6) *If $\langle a, A, b, 1 \rangle$ is of pre-ELE$_2$ type, $a \geq 2$ and $b \geq 2$, then $a = b = 2$.*

(7) *If $b = 1$ or 2 then $\langle 1, A, b, 1 \rangle$ is not of pre-ELE$_2$ type.*

(8) *$\langle \overleftarrow{A}, 1 \rangle$: pre-ELE$_2$ type $\iff$ $\langle A, 1 \rangle$: pre-ELE$_2$ type.*

(9) *$\langle 2, A, 1, 1 \rangle$ is not of pre-ELE$_2$ type.*

(10) *$\langle a, \overleftarrow{A}, 2a \rangle$: pre-ELE$_1$ type $\iff$ A : pre-ELE$_1$ type.*

(11) *$\langle a, \overleftarrow{A}, 2a + 1 \rangle$: pre-ELE$_1$ type $\iff$ A : pre-ELE$_2$ type.*

(12) *Assume $a \geq 2$. Then,*

$$\langle 1, A, a + 1, 1 \rangle : \text{pre-ELE}_2 \text{ type} \iff \langle A, a \rangle : \text{pre-ELE}_1 \text{ type}.$$

(13) *Assume $a \geq 2$. Then,*

$$\langle a + 1, \overleftarrow{A}, 1, 1 \rangle : \text{pre-ELE}_2 \text{ type} \iff \langle A, a \rangle : \text{pre-ELE}_1 \text{ type}.$$

(14) *$\langle 2, A, 2, 1 \rangle$: pre-ELE$_2$ type $\iff$ $\langle A, 1 \rangle$: pre-ELE$_2$ type.*

Proof. We shall prove only (2), (3), (8) and (10).

We calculate nonnegative integers q_n and r_n ($1 \leq n \leq m + 1$) by using (2.1) from the string A. Then it is known that

$$\begin{pmatrix} q_{n+1} & q_n \\ r_{n+1} & r_n \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad (1 \leq n \leq m). \tag{3.1}$$

(See, for example, Halter-Koch [4, Chapter 2] or [8, (2.5)].)

(2) Let $m = 2$. Then we have the following table:

n	0	1	2	3
q_n	0	1	a_1	$a_1a_2 + 1$
r_n	1	0	1	a_2

Hence, A is of pre-ELE$_1$ type if and only if $a_2 = 2a_1$, and then we have $A = \langle a_1, 2a_1\rangle$. Moreover, A is of pre-ELE$_2$ type if and only if "$a_2 = 1$ and $a_1 = 2$" because we have

$$r_3 = 2q_2 - q_3 \iff a_2 = 2a_1 - a_1a_2 - 1 \iff a_2 + 1 = a_1(2 - a_2),$$

and then we obtain $A = \langle 2, 1\rangle$.

(3) If $m = 2$, then our assertion follows from (2). So we assume $m \geq 3$. Let $\tilde{q}_n$ and $\tilde{r}_n$ $(1 \leq n \leq m+1)$ be nonnegative integers calculated by using (2.1) from the reversed string $\overleftarrow{A}$ of A. Then by (3.1), we have

$$\begin{pmatrix} \tilde{q}_{m+1} & \tilde{q}_m \\ \tilde{r}_{m+1} & \tilde{r}_m \end{pmatrix} = \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

By taking the transpose of matrices in both sides, we get

$$\begin{pmatrix} \tilde{q}_{m+1} & \tilde{r}_{m+1} \\ \tilde{q}_m & \tilde{r}_m \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_{m+1} & q_m \\ r_{m+1} & r_m \end{pmatrix}$$

so that $\tilde{r}_{m+1} = q_m$, $\tilde{r}_m = r_m$. Hence,

$$q_m = \tilde{r}_{m+1} = a_1\tilde{r}_m + \tilde{r}_{m-1} = a_1r_m + \tilde{r}_{m-1}$$

and then we obtain

$$a_mr_m + r_{m-1} = r_{m+1} = 2q_m = 2a_1r_m + 2\tilde{r}_{m-1}. \tag{3.2}$$

Here we remark that $r_{m-1} > 0$ and $\tilde{r}_{m-1} > 0$ by $m \geq 3$. Therefore, on the one hand, the first inequality and (3.2) yield that

$$a_mr_m < 2a_1r_m + 2\tilde{r}_{m-1} \leq 2a_1r_m + 2\tilde{r}_m = (2a_1 + 2)r_m.$$

Hence, $a_m < 2a_1 + 2$ holds. On the other hand, $r_m \geq r_{m-1}$, (3.2) and $\tilde{r}_{m-1} > 0$ yield that

$$a_mr_m + r_m \geq 2a_1r_m + 2\tilde{r}_{m-1} > 2a_1r_m.$$

Hence, $a_m > 2a_1 - 1$ holds. Thus either $a_m = 2a_1$ or $a_m = 2a_1 + 1$ holds.

(8) For brevity, we put

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} := \begin{pmatrix} q_{m+1} & q_m \\ r_{m+1} & r_m \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix}.$$

Then it holds that

$$\begin{pmatrix} x & z \\ y & w \end{pmatrix} = \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x+z & x \\ y+w & y \end{pmatrix}, \quad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x+y & x \\ z+w & z \end{pmatrix}.$$

Hence we have

$$\begin{aligned}\langle \overleftarrow{A}, 1\rangle : \text{pre-ELE}_2 \text{ type} &\iff (y+w)-2x+(x+z)=0\\ &\iff (z+w)-2x+(x+y)=0\\ &\iff \langle A, 1\rangle : \text{pre-ELE}_2 \text{ type.}\end{aligned}$$

(10) We have

$$\begin{aligned}\begin{pmatrix} a & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} x & z\\ y & w\end{pmatrix}\begin{pmatrix} 2a & 1\\ 1 & 0\end{pmatrix} &= \begin{pmatrix} ax+y & az+w\\ x & z\end{pmatrix}\begin{pmatrix} 2a & 1\\ 1 & 0\end{pmatrix}\\ &= \begin{pmatrix} 2a^2x+2ay+az+w & ax+y\\ 2ax+z & x\end{pmatrix},\end{aligned}$$

and hence

$$\begin{aligned}\langle a, \overleftarrow{A}, 2a\rangle : \text{pre-ELE}_1 \text{ type} &\iff (2ax+z)-(2ax+2y)=0\\ &\iff z-2y=0\\ &\iff A : \text{pre-ELE}_1 \text{ type,}\end{aligned}$$

as desired. □

Definition 3.2. Let a, c be positive integers and let A, B be strings of positive integers with length ≥ 1. By using Proposition 3.1 (10)-(14), we define 5 growth transformations for a finite string of pre-ELE type:

$$\begin{aligned}
e_a(A) &:= \langle a, \overleftarrow{A}, 2a\rangle, && (\text{pre-ELE}_1 \text{ type} \longrightarrow \text{pre-ELE}_1 \text{ type})\\
o_a(A) &:= \langle a, \overleftarrow{A}, 2a+1\rangle, && (\text{pre-ELE}_2 \text{ type} \longrightarrow \text{pre-ELE}_1 \text{ type})\\
\text{For } c\geq 2,\ F(\langle B, c\rangle) &:= \langle 1, B, c+1, 1\rangle, && (\text{pre-ELE}_1 \text{ type} \longrightarrow \text{pre-ELE}_2 \text{ type})\\
\text{For } c\geq 2,\ G(\langle B, c\rangle) &:= \langle c+1, \overleftarrow{B}, 1, 1\rangle, && (\text{pre-ELE}_1 \text{ type} \longrightarrow \text{pre-ELE}_2 \text{ type})\\
H(\langle B, 1\rangle) &:= \langle 2, B, 2, 1\rangle. && (\text{pre-ELE}_2 \text{ type} \longrightarrow \text{pre-ELE}_2 \text{ type})
\end{aligned}$$

Theorem 3. *Every finite string A of pre-ELE type can be obtained by the finite compositions of possible* 5 *growth transformations e_a, o_a, F, G and H starting from one of the three "kernel" $\langle\ \rangle$, $\langle 1\rangle$ and $\langle 2, 1\rangle$. Furthermore this "growth decomposition" of A is unique.*

Outline of proof. Let A be any string of pre-ELE type with length $m\,(\geq 1)$. By using Proposition 3.1 (3), (4), (6), (7) and (9)-(14), we can show that there exists a string B of pre-ELE type with length ≤ 3 such that

$$A = (f_n \circ f_{n-1} \circ \cdots \circ f_1)(B),$$

where f_i are growth transformations. Assume that B is of pre-ELE_1 type. Then by Proposition 3.1 (1), the length of B is 2 or 3. If the length of B is 2 (resp. 3) then Proposition 3.1 (2) (resp. (5)) implies that $B = \langle a, 2a\rangle$ (resp. $B = \langle a, 1, 2a+1\rangle$) with some positive integer a and we have $B = e_a(\langle\ \rangle)$ (resp. $B = o_a(\langle 1\rangle)$). Assume that B is of pre-ELE_2 type. If the length of B is 1 (resp. 2, 3) then Proposition 3.1 (1) (resp. (2), (5)) implies that we have $B = \langle 1\rangle$ (resp. $B = \langle 2, 1\rangle$, $B = \langle 2, 2, 1\rangle = H(\langle 1\rangle)$). Therefore, we see that one of the three strings $\langle\ \rangle$, $\langle 1\rangle$ and $\langle 2, 1\rangle$ appears as a starting point. (We call this decomposition a *growth decomposition* and call the above strings 3 *kernels of growth decomposition.*)

As for the uniqueness of a growth decomposition, we can argue as follows. Note that for a growth transformation f and for a finite string A of pre-ELE type, the length of $f(A)$ is increasing by two. By looking at the last two numbers of $f(A)$, we see that each growth transformation is distinguishable. Hence, for growth transformations f, g and for finite strings A, B of pre-ELE type, it holds that

$$f(A) = g(B) \implies f = g,\ A = B. \tag{3.3}$$

Let A be a string of pre-ELE type, K, K' two kernels and assume that a growth decomposition of A is

$$A = (f_n \circ f_{n-1} \circ \cdots \circ f_1)(K) = (g_{n'} \circ g_{n'-1} \circ \cdots \circ g_1)(K').$$

First, assume that the length of A is odd. Then by the definition of growth decomposition, we have $K = K' = \langle 1\rangle$. Since the length of A is equal to $2n+1 = 2n'+1$, we get $n = n'$. Hence by (3.3), we obtain $f_i = g_i$ for all $i\,(1 \le i \le n)$. Next, assume that the length of A is even. Then the possible kernel becomes $\langle\ \rangle$ or $\langle 2, 1\rangle$. If we assume $K \ne K'$ then we may have $K = \langle\ \rangle$, $K' = \langle 2, 1\rangle$. Since the length of A is equal to $2n = 2n'+2$, we get $n' = n-1$. Hence by (3.3), we obtain $f_i = g_{i-1}$ for all $i\,(2 \le i \le n)$ and then

$$f_1(\langle\ \rangle) = \langle 2, 1\rangle.$$

Since the lengths of $\langle\ \rangle$ and $\langle 2, 1\rangle$ are 0 and 2 respectively, it follows from the definition of growth transformation that $f_1 = e_a$ for some positive integer a. Therefore, $\langle a, 2a\rangle = \langle 2, 1\rangle$ and this is impossible. Hence, $K = K'$ so that $n = n'$. Then (3.3) yields that $f_i = g_i$ for all $i\,(1 \le i \le n)$. □

We give some growth decompositions. For a positive integer L with $4 \le L \le 26$, let d'_{2L} denote the smallest positive integers d such that $d \equiv 2,\ 3 \pmod 4$ and the minimal period of the simple continued fraction expansion of $\sqrt{d}$ is equal to $2L$. Let

$$\sqrt{d'_{2L}} = [a_0, \overline{a_1, \ldots, a_{L-1}, a_L, a_{L-1}, \ldots, a_1, 2a_0}]$$

L	d'_{2L}	the growth decomposition of A
4	31	$o_1(\langle 1\rangle)$
5	43	$F \circ e_1(\langle\ \rangle)$
6	46	$(e_1 \circ o_1)(\langle 1\rangle)$
7	134	$(F \circ e_1 \circ e_1)(\langle\ \rangle)$
8	94	$(F \circ e_2 \circ o_1)(\langle 1\rangle)$
9	139	$(e_1 \circ o_1 \circ F \circ e_3)(\langle\ \rangle)$
10	151	$(G \circ e_1 \circ e_2 \circ o_3)(\langle 1\rangle)$
11	166	$(e_1 \circ o_3 \circ F \circ o_1)(\langle 2, 1\rangle)$
12	271	$(H \circ G \circ o_2 \circ G \circ o_4)(\langle 1\rangle)$
13	211	$(F \circ e_1 \circ o_4 \circ G \circ e_2 \circ e_1)(\langle\ \rangle)$
14	334	$(o_3 \circ G \circ e_1 \circ e_1 \circ o_5 \circ H)(\langle 1\rangle)$
15	379	$(e_2 \circ o_3 \circ G \circ e_1 \circ e_2 \circ e_1 \circ e_6)(\langle\ \rangle)$
16	463	$(o_1 \circ F \circ e_6 \circ e_1 \circ o_2 \circ G \circ o_1)(\langle 1\rangle)$
17	331	$(G \circ e_2 \circ o_5 \circ F \circ o_2 \circ H \circ G \circ e_1)(\langle\ \rangle)$
18	478	$(F \circ e_6 \circ o_1 \circ G \circ o_1 \circ F \circ e_1 \circ o_2)(\langle 1\rangle)$
19	619	$(F \circ e_7 \circ o_1 \circ G \circ e_2 \circ o_4 \circ F \circ e_1 \circ e_1)(\langle\ \rangle)$
20	526	$(e_1 \circ e_7 \circ e_3 \circ e_2 \circ e_1 \circ o_1 \circ H \circ F \circ o_3)(\langle 1\rangle)$
21	571	$(e_1 \circ e_4 \circ o_1 \circ H \circ G \circ o_1 \circ F \circ e_7 \circ e_3 \circ e_1)(\langle\ \rangle)$
22	766	$(F \circ e_2 \circ e_5 \circ e_1 \circ e_1 \circ e_1 \circ o_1 \circ F \circ e_1 \circ o_8)(\langle 1\rangle)$
23	694	$(o_2 \circ F \circ e_4 \circ e_1 \circ e_3 \circ o_1 \circ G \circ o_1 \circ H \circ G \circ e_8)(\langle\ \rangle)$
24	631	$(e_8 \circ e_1 \circ e_1 \circ e_2 \circ e_1 \circ o_4 \circ G \circ o_1 \circ H \circ G \circ o_2)(\langle 1\rangle)$
25	1051	$(H^2 \circ F \circ e_1 \circ e_1 \circ e_4 \circ e_2 \circ o_{10} \circ G \circ e_3 \circ e_6 \circ e_1)(\langle\ \rangle)$
26	751	$(e_2 \circ e_1 \circ o_8 \circ H \circ G \circ e_3 \circ o_2 \circ F \circ o_4 \circ G \circ e_1 \circ o_1)(\langle 1\rangle)$

Table 3.1. Some growth decompositions

be the simple continued fraction expansion of $\sqrt{d'_{2L}}$ and put $A := \langle a_1, \ldots, a_{L-1}\rangle$. Then the string A, a_L becomes of ELE type, as we have stated in Introduction. In Table 3.1, for each L with $4 \leq L \leq 26$ we list the value of d'_{2L} and the growth decomposition of A, which is of pre-ELE type.

In the last of this section, we show that there exist strings each of pre-ELE$_1$ type or of pre-ELE$_2$ type with any length (≥ 2).

Proposition 3.2. *Let k be a nonnegative integer.*

(1) *A string $\langle \underbrace{2, \ldots, 2, 2}_{k}, 1\rangle$ is of pre-ELE$_2$ type with length $k+1$.*

(2) *For any positive integer a, strings $\langle a, 2a\rangle$ and $\langle a, 1, \underbrace{2, 2, \ldots, 2}_{k}, 2a+1\rangle$ are of pre-ELE$_1$ type with length 2 and $k+3$, respectively.*

Proof. For brevity, we put $A := \langle 2, \ldots, 2, 2, 1\rangle$.

(1) By Proposition 3.1 (1), (2), both $\langle 1\rangle$ and $\langle 2, 1\rangle$ are of pre-ELE$_2$ type. When k is even (resp. odd), it follows from the definition of H that $A = H^{\frac{k}{2}}(\langle 1\rangle)$ (resp.

$A = H^{\frac{k-1}{2}}(\langle 2,1\rangle))$. Hence we see from Proposition 3.1 (14) that A is of pre-ELE$_2$ type.

(2) By Proposition 3.1 (2), $\langle a, 2a\rangle$ is of pre-ELE$_1$ type. Moreover since $\langle a, 1, 2, 2, \ldots, 2, 2a+1\rangle = o_a(A)$, this is of pre-ELE$_1$ type by (1). □

§ 4. Application

The goal of this section is to give an application of our theorems.

First we will define "minimal type" for a positive integer and for a real quadratic field. Let d be a non-square positive integer and put $\omega_d = \sqrt{d}$ or $\omega_d = (1+\sqrt{d})/2$. Here we assume $d \equiv 1 \pmod 4$ if $\omega_d = (1+\sqrt{d})/2$. Then it is known that the simple continued fraction expansion is of the form

$$\omega_d = [a_0, \overline{a_1, a_2, \ldots, a_\ell}].$$

From the string of partial quotients $a_1, a_2, \ldots, a_{\ell-1}$, we define nonnegative integers q_n and r_n $(0 \le n \le \ell)$ by using (2.1) inductively. For brevity, we put

$$A := q_\ell,\ B := q_{\ell-1},\ C := r_{\ell-1},$$

and define linear polynomials $g(x), h(x)$ and a quadratic polynomial $f(x)$ by

$$g(x) = Ax - (-1)^\ell BC,\ h(x) = Bx - (-1)^\ell C^2,\ f(x) = g(x)^2 + 4h(x).$$

Furthermore, let s_0 be the least integer x for which $g(x) > 0$. Then we see from [7, Theorem 3.1], which is an improvement of results of Friesen [1, Theorem] and of Halter-Koch [3, Theorem 1A, Corollary 1A], that d can be written uniquely as

$$d = f(s)/4 \text{ (resp. } d = f(s)) \quad \text{if } \omega_d = \sqrt{d} \text{ (resp. } \omega_d = (1+\sqrt{d})/2),$$

with some integer $s \ge s_0$.

Definition 4.1 ([7, Definition 3.1]). Under the above setting, if $s = s_0$, that is, $d = f(s_0)/4$ (resp. $d = f(s_0)$) holds, then we say that d is a *positive integer with period ℓ of minimal type for* (*the simple continued fraction expansion of*) $\sqrt{d}$ (resp. $(1+\sqrt{d})/2$).

Furthermore, for a square-free positive integer $d > 1$, we say that $\mathbb{Q}(\sqrt{d})$ is a *real quadratic field with period ℓ of minimal type*, if d is a positive integer with period ℓ of minimal type for $\sqrt{d}$ when $d \equiv 2, 3 \pmod 4$, and if d is a positive integer with period ℓ of minimal type for $(1+\sqrt{d})/2$ when $d \equiv 1 \pmod 4$.

Remark 4.1. There exist exactly 51 real quadratic fields of class number 1 that are not of minimal type, with one more possible exception ([7]).

As for the existence of real quadratic fields of minimal type, the following have been known; i) only $\mathbb{Q}(\sqrt{5})$ is a real quadratic field with period 1 of minimal type, ii) there does not exist a real quadratic field with period $2, 3$ of minimal type, iii) there exist infinitely many real quadratic fields with period ℓ of minimal type for any even $\ell \geq 4$ with $8 \nmid \ell$. By using our theorems, we can remove the condition $8 \nmid \ell$ in the above iii). Namely, we can prove that there exist infinitely many real quadratic fields with period ℓ of minimal type for any even $\ell \geq 4$. Now we will state it more precisely. Let $L \geq 2$ and define positive integers q_{L-1} and q_L by using (2.1) from a string $\langle 2, \ldots, 2, 2, 1 \rangle$ with length $L - 1$.

Theorem 4. *Let $L \geq 3$ and $e_0 = 2, 3$. Then, for any positive integer h, there exist infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$, $d \equiv e_0 \pmod 4$ with period $2L$ of minimal type such that $h_d > h$,*

$$m_d = \begin{cases} 2q_L^2 & \text{if } L \text{ is even,} \\ 2q_L^2 - 1 & \text{if } L \text{ is odd} \end{cases} \tag{4.1}$$

and the primary symmetric part of the simple continued fraction expansion of $\sqrt{d}$ is of ELE_2 type. Here we denote the class number of $\mathbb{Q}(\sqrt{d})$ by h_d.

Outline of proof. For any positive integer t, we define

$$d(t) := q_L^2 t^2 + 4q_{L-1}t + 2.$$

From straightforward calculations, we can verify that for each t, $d(t)$ is a positive integer with period $2L$ of minimal type for $\sqrt{d(t)}$ satisfying (4.1) and the primary symmetric part of the simple continued fraction expansion of $\sqrt{d(t)}$ is

$$\underbrace{2, \ldots, 2, 2}_{L-2}, 1, q_L t,$$

which is of ELE_2 type (cf. Proposition 3.2). Moreover, q_L must be odd in this case. Then we have $d(t) \equiv 2, 3 \pmod 4$. If $d(t)$ is square-free, therefore, then $\mathbb{Q}(\sqrt{d(t)})$ is a real quadratic field with period $2L$ of minimal type. By using Nagell's result, we can show that there exist infinitely many positive integer t such that $d(t)$ is square-free and the class number $h_{d(t)}$ of $\mathbb{Q}(\sqrt{d(t)})$ is greater than given integer h (cf. [7, Proposition 6.1, Lemma 4.5]). $\square$

Remark 4.2. In [5], we give infinitely many real quadratic fields with even period $2L\,(\geq 8)$ of minimal type such that the primary symmetric part of the simple continued fraction expansion of $\sqrt{d}$ is of ELE_1 type.

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