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A new geometric construction of a family of Galois representations associated to modular forms: research announcement

By

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Abstract

This is a research announcement of the results in [Mih15]. For an odd prime p dividing an integer $N \geq 5$, we define an inverse system of sheaves of torsion $\mathbb{Z}_p$-modules on a modular curve of level $\Gamma_1(N)$. The representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to any cuspidal eigenform is obtained as a twist of a quotient of its cohomology. We construct a family of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to cuspidal eigenforms of finite bounded slope as a quotient of a twist of a scalar extension of its cohomology.

Contents

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References

§ 1. Introduction

This article is a research announcement of the results of the paper [Mih15]. Let p be an odd prime number, and $N \geq 5$ an integer divisible by p. We fix a $\mathbb{Q}$-algebra isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$. We give a new geometric construction of a p-adic family of Galois representations associated to a family of normalised cuspidal eigenforms of level $\Gamma_1(N)$ of finite bounded slope. We give a sketch of the construction.

Let S_N denote the set of normalised cuspidal eigenforms of weight ≥ 2 and level $\Gamma_1(N)$, $S_{k,N} \subset S_N$ the subset of eigenforms of weight $k \geq 2$, $Y_1(N)'$ the moduli space over $\mathbb{Q}$ of pairs (E, β) of an elliptic curve E and a surjective morphism $\beta \colon E[N] \twoheadrightarrow \underline{\mathbb{Z}/N\mathbb{Z}}$ of group schemes, $\pi'_N \colon E_1(N)' \to Y_1(N)'$ the universal elliptic curve, and F'_k the smooth p-adic sheaf $\mathrm{Sym}^{k-2}(\mathrm{R}^1(\pi'_N)_*(\underline{\mathbb{Z}_p})_{E_1(N)'})$ on $Y_1(N)'_{\text{ét}}$. As an analogous result by P. Deligne on the geometric construction of the 2-dimensional Galois representation V_f over $\mathbb{Q}_p(f) := \mathbb{Q}_p(a_n(f) \mid n \in \mathbb{N}) \subset \overline{\mathbb{Q}}_p$ associated to any $f \in S_{k,N}$ in [Del69], B. H. Gross gave another geometric construction of V_f as a twist of a quotient of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F'_k) = \mathrm{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}(\mathrm{R}^1(\pi'_N)_*(\underline{\mathbb{Q}_p})_{E_1(N)'}))$ in [Gro90] Proposition 11.4. We note that B. H. Gross assumed $k = 2$ in [Gro90] Proposition 11.4, but the assumption is easily removed (cf. [Mih15]).

In order to construct a family of Galois representations interpolating $(V_f)_{f \in S_N}$, we "interpolate" $(F'_k)_{k \geq 2}$ first. We put the quotation marks " " because the rank function $\mathbb{N} \cap [2, \infty) \to \mathbb{N}$, $k \mapsto \mathrm{rank}_{\mathbb{Z}_p}(F'_k) = k - 1$ is not constant on $k \geq 2$, and hence $(F'_k)_{k \geq 2}$ is not interpolated in the naive sense. For this purpose, we introduce a notion of a profinite smooth $\mathbb{Z}_p$-sheaf (Definition 2.4), which is a generalisation of that of a smooth p-adic sheaf. In particular, every smooth p-adic sheaf can be naturally regarded as a profinite smooth $\mathbb{Z}_p$-sheaf. We explain how the "interpolation" precisely goes using profinite smooth $\mathbb{Z}_p$-sheaves. There is a functorial correspondence from a finitely generated torsion $\mathbb{Z}_p$-module M endowed with a continuous action ρ of a subgroup $\mathrm{G}\hat{\Gamma}_\epsilon(N) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ (Example 2.5) to a locally constant étale sheaf $\underline{(M, \rho)}_{Y_1(N)'}$ of torsion $\mathbb{Z}_p$-modules on $Y_1(N)'$. We extend it to a functorial correspondence from a profinite $\mathbb{Z}_p[\mathrm{G}\hat{\Gamma}_\epsilon(N)]$-module (Definition 2.2) to a profinite smooth $\mathbb{Z}_p$-sheaf on $Y_1(N)'$. One of the most important fact is that the $\mathbb{Z}_p$-module $L'_k := \mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)$ endowed with the natural continuous action of $\mathrm{G}\hat{\Gamma}_\epsilon(N)$ corresponds to F'_k for any $k \geq 2$. Therefore it suffices to "interpolate" $(L'_k)_{k \geq 2}$ by a profinite $\mathbb{Z}_p[\mathrm{G}\hat{\Gamma}_\epsilon(N)]$-module. In order to equip the cohomology of the corresponding profinite smooth $\mathbb{Z}_p$-sheaf with an action of Hecke operators, we need to "interpolate" $(L'_k)_{k \geq 2}$ together with continuous actions of a certain multiplicative submonoid $\Pi_0(p) \subset \mathrm{M}_2(\mathbb{Z}_p)$ (Definition 3.1) containing the image of $\mathrm{G}\hat{\Gamma}_\epsilon(N)$. For each $k \geq 2$, we construct a continuous action ρ_{k-2} of $\Pi_0(p)$ on $\mathbb{Z}_p^{\mathbb{N}}$ for which $\mathbb{Z}_p^{\mathbb{N}}$ admits a continuous surjective $\mathbb{Z}_p$-linear $\Pi_0(p)$-equivariant homomorphism onto a lattice L_k of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L'_k \cong \mathrm{Sym}^{k-2}(\mathbb{Q}_p^2)$ and whose matrix presentation is given in an analytic way on k (Proposition 3.8). Interpolating $(\rho_{k-2})_{k\geq 2}$, we obtain a continuous action $\rho_{\bullet-2}$ of $\Pi_0(p)$ on the topological Λ_0-module $\Lambda_0^{\mathbb{N}}$ (Proposition 3.10), where Λ_0 denotes the topological $\mathbb{Z}_p$-algebra of rigid analytic functions on the weight space $W := \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$. We verify that V_f is obtained as a twist of a quotient of the cohomology of the profinite smooth $\mathbb{Z}_p$-sheaf F associated to $(\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})$ for any $f \in S_N$(Proposition 3.13). We denote by $\mathbb{H}(N)$ a natural torsionfree quotient of the cohomology of F, and regard it as a profinite module over the profinite Λ_0-algebra $\Lambda_0\mathbb{T}_N$ topologically generated by Hecke operators. We then show that a suitable quotient $\mathbb{H}(N)^{<s}$ of a scalar extension of $\mathbb{H}(N)$ for a fixed upper bound s of slopes (cf. [Col97] §0 and [CM98] §6.2) is a finitely generated module over a profinite Λ_0-algebra $\Lambda_0\mathbb{T}_N^{<s}$ topologically generated by Hecke operators restricted to the subspace of modular forms of slope $< s$, and V_f is obtained as a twist of a quotient of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{H}(N)^{<s}$ for any $f \in S_N$ of slope $< s$ (Theorem 3.15). Finally, for a compact Λ_0-algebra Λ_1 with suitable conditions called a Λ-*adic domain* (Definition 3.16), we introduce a set $\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$ of "characters on Λ_1 of weight ≥ 2", and define a continuous Galois representation $\mathbb{V}_f$ over Λ_1 as the quotient of a twist of $\mathbb{H}(N)^{<s}$ by a normalised cuspidal Λ_1-adic eigenform f (Definition 3.19). Then we state the following main theorem:

Theorem 3.21 ([Mih15]). *Suppose $p^s \mid N$. For any normalised Λ_1-adic cuspidal eigenform f of level $\Gamma_1(N)$ of slope $< s$, $\mathrm{Frac}(\Lambda_1) \otimes_{\Lambda_1} \mathbb{V}_f$ is a 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathrm{Frac}(\Lambda_1)$. Furthermore, there is a finite subset $\Sigma_s \subset \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$ satisfying the following for any normalised Λ_1-adic cuspidal eigenform f of level $\Gamma_1(N)$, character χ, and slope $< s$:*

(i) *For any $\varphi \in \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s$, the specialisation f_φ of f at φ is a normalised cuspidal eigenform of weight $\mathrm{wt}(\varphi)$, level $\Gamma_1(N)$, character $\varphi \circ \chi$, and slope $< s$.*

(ii) *For any $\varphi \in \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s$, the specialisation $\overline{\mathbb{Q}}_p(\varphi) \otimes_{\Lambda_1} \mathbb{V}_f$ of $\mathbb{V}_f$ at φ is isomorphic to $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p(f_\varphi)} V_{f_\varphi}$, where $\overline{\mathbb{Q}}_p(\varphi)$ denotes $\overline{\mathbb{Q}}_p$ regarded as a Λ_1-algebra through φ.*

(iii) *The $\mathrm{C}(\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s, \overline{\mathbb{Q}}_p)[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $\mathrm{C}\left(\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s, \overline{\mathbb{Q}}_p\right) \otimes_{\Lambda_1} \mathbb{V}_f$ is free of rank 2 as a $\mathrm{C}(\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s, \overline{\mathbb{Q}}_p)$-module.*

We note that for any $f \in S_N$, there are a Λ-adic domain Λ_1 and a normalised cuspidal Λ_1-adic eigenform which admits a specialisation to f (Remark 3.20). Therefore Theorem 3.21 yields p-adic families interpolating $(V_f)_{f \in S_N}$.

We recall preceding studies on a family of Galois representations associated to modular forms. H. Hida constructed and studied a p-adic family of ordinary modular forms called a *Hida family* in [Hid86-1] and [Hid86-2]. Two constructions of a family of Galois representations associated to a Hida family are known in the case where the modular forms are ordinary. One was given in a geometric way by H. Hida as the inverse limit of the Tate modules of the Jacobian varieties of the compactifications $X_1(Np^r)$ of the modular curves $Y_1(Np^r)$ with $r \in \mathbb{N}$ in [Hid86-2] Theorem 2.1. The works of H. Hida yielded many fundamental frameworks in the study of a family of Galois representations such as Hida theory. The other one was given by A. Wiles by gluing pseudo-representations associated to $(V_f)_{f \in S_N}$ in [Wil88] Theorem 2.2.1. This work contributed to his subsequent studies on the Galois representation associated to a Hilbert cusp form, which formed the kernel of the modular theoretic approach to Iwasawa main conjecture for a totally real field in [Wil90]. R. Coleman constructed in [Col97] Corollary B5.7.1 a p-adic family, which is called a *Coleman family*, of modular forms of weights ≥ 2 and level $\Gamma_1(N_0 p)$ for an integer $N_0 \geq 4$ coprime to p. It interpolates modular forms of finite slopes, while a Hida family interpolates modular forms of slope 0. R. Coleman and B. Mazur defined a rigid analytic curve C_p called *the eigencurve of tame level* 1 in [CM98] §6.1 Definition 1 by using the universal deformation ring parametrising pseudo-representations, and partially generalised the construction of a Coleman family to modular forms of tame level 1, i.e. levels in $\{\Gamma_1(p^r) \mid r \in \mathbb{N}\}$, in [CM98] Theorem 6.2.1 so that the eigencurve parametrises Coleman families. As is mentioned in [CM98] pp. 4-5, a family of Galois representations over the complement of a discrete subset of C_p can be constructed by gluing pseudo-representations.

We remark the relation between our construction and the eigencurve. As we mentioned above, the eigencurve in [CM98] parametrises a family of modular forms of weights ≥ 2 and levels in $\{\Gamma_1(p^r) \mid r \in \mathbb{N}\}$, while we can deal only with a family of modular forms of weights ≥ 2 and level $\Gamma_1(N)$ in our construction. Therefore the eigencurve is not applicable to our construction. On the other hand, M. Emerton referred in [Eme] Theorem 2.23 to a construction of another eigencurve parametrising a family of modular forms of level $\Gamma_1(N)$, which is applicable to our construction. Namely, it yields an example of a normalised cuspidal Λ_1-adic eigenform for the compact Λ-algebra Λ_1 associated to an open disc on the eigencurve (Example 3.20). In addition, our construction is partially related to one of the open questions in [CM98], which asked whether the family of Galois representations associated to a Coleman family is obtained as the Pontryagin dual of the direct limit of étale cohomologies of a tower of modular curves of levels $\Gamma_1(Np^r)$ with $r \in \mathbb{N}$. Although we do not answer this question literally, our result gives a realisation of the family of Galois representations as the étale cohomology of a modular curve.

We emphasise that even if we restrict it to the case where modular forms are ordinary, our construction differs from the constructions by H. Hida and A. Wiles recalled above. To begin with, we compare our construction with that by H. Hida. For this sake, we interpret his construction in terms of the cohomology of a profinite smooth $\mathbb{Z}_p$-sheaf T_∞ on $Y_1(N)$. For each $r \in \mathbb{N}$, the Tate module of the Jacobian variety of $X_1(Np^r)$ is naturally isomorphic to the dual of $\mathrm{H}^1_{\text{ét}}(X_1(Np^r)_{\overline{\mathbb{Q}}}, (\mathbb{Z}_p)_{X_1(Np^r)})$. There is a natural injective homomorphism from the inverse limit of the Tate modules of the Jacobian varieties of $X_1(Np^r)$ to the inverse limit of $(\mathrm{H}^1_{\text{ét}}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)))_{r \in \mathbb{N}}$ by the trace maps, which can be interpreted as the cohomology of a profinite smooth $\mathbb{Z}_p$-sheaf T_∞ on $Y_1(N)$ by Shapiro's lemma. The construction of T_∞ is valid for $Y_1(N)'$, and we denote by T'_∞ the resulting profinite smooth $\mathbb{Z}_p$-sheaf. The cohomology of T_∞ is naturally isomorphic to a twist of the cohomology of T'_∞. It is an interesting question whether T'_∞ is related to our profinite smooth $\mathbb{Z}_p$-sheaf F, e.g. whether there is a natural epimorphism $F \twoheadrightarrow T'_\infty$, but we have no idea for it. Next, we compare our construction with that by A. Wiles. We realised a family of Galois representations as a quotient of a twist of a cohomology, while A. Wiles did not in the construction given by gluing pseudo-representations. As is mentioned above, it is still an open question whether there is an appropriate way to interpret the construction of a p-adic family of Galois representations associated to modular forms using pseudo-representations in terms of cohomologies. Therefore our construction is quite different from that by A. Wiles.

We explain the contents of this article. First, §2 consists of two subsections. We introduce notions of a profinite module and of a profinite smooth $\mathbb{Z}_p$-sheaf in §2.1 and §2.2 respectively. As a profinite extension of the correspondence between representations and local systems, we give a method of constructing a profinite smooth $\mathbb{Z}_p$-sheaf from a profinite module. Next, §3 consists of four subsections. We construct a continuous action $\rho_{\kappa-2}$ of $\Pi_0(p)$ on $\mathbb{Z}_p^{\mathbb{N}}$ analytically parametrised by $\kappa \in W$ for which $\mathbb{Z}_p^{\mathbb{N}}$ admits a continuous surjective $\mathbb{Z}_p$-linear $\Pi_0(p)$-equivariant homomorphism onto a lattice of $\mathrm{Sym}^{k-2}(\mathbb{Q}_p^2)$ in the case $\kappa = k$ for an integer $k \geq 2$ in §3.1. We construct a huge profinite $\mathbb{Z}_p[\Pi_0(p)]$-module $(\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})$ interpolating $(\mathbb{Z}_p^{\mathbb{N}}, \rho_{\kappa-2})_{\kappa \in W}$ in §3.2. As a natural torsionfree quotient of the cohomology of the profinite smooth $\mathbb{Z}_p$-sheaf F associated to $(\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})$, we define a profinite $\Lambda_0 \mathbb{T}_N[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $\mathbb{H}(N)$ in §3.3. In §3.4, for a fixed upper bound $s \in \mathbb{N} \backslash \{0\}$ of slopes, we construct a profinite $\Lambda_0 \mathbb{T}_N^{<s}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $\mathbb{H}(N)^{<s}$ by using $\mathbb{H}(N)$. We state its finiteness as a $\Lambda_0 \mathbb{T}_N^{<s}$-module in Theorem 3.15. We introduce the notions of a Λ-adic domain and a modular form over a Λ-adic domain in Definition 3.16 and Definition 3.19 respectively. As a main result, we construct a p-adic family of Galois representations associated to modular forms of slope $< s$ over a Λ-adic domain in Theorem 3.21.

§ 2. Representations and Local Systems

In this section, let R denote a topological ring, and G a topological monoid. We mainly consider the case where R is $\mathbb{Z}_p$ or the Iwasawa algebras in §3. We introduce notions of a profinite $R[G]$-module and of a profinite smooth R-sheaf on a Noetherian scheme. We give a method of constructing a profinite smooth R-sheaf on $Y_1(N)'$ from a profinite $R[G]$-module in the case where G is a certain subgroup $\mathrm{G}\hat{\Gamma}_\epsilon(N) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$.

§ 2.1. Profinite Modules

A *topological R-module* is a left R-module M endowed with a topology for which the addition $M \times M \to M$ and the scalar multiplication $R \times M \to M$ are continuous.

Example 2.1. Let M be a left R-module. Then M admits a natural topology called *the canonical topology*, which is the strongest topology for which M forms a topological R-module. If M is a finitely generated free R-module, then the natural R-linear isomorphism $R^S \to M$ is a homeomorphism with respect to the direct product topology on R^S and the canonical topology on M for any R-linear basis $S \subset M$.

A topological R-module is said to be a *discrete R-module* if its underlying topology is the discrete topology, is said to be a *finite R-module* if it is a discrete R-module whose underlying set is a finite set, and is said to be a *profinite R-module* if it is homeomorphically isomorphic to the inverse limit of finite R-modules. For a topological R-module M, we denote by O_M the set of open R-submodules of M. For any profinite R-module M, O_M forms a fundamental system of neighbourhoods of $0 \in M$. Suppose that R is commutative. For profinite R-modules M_0 and M_1, we set $M_0 \hat{\otimes}_R M_1 := \varprojlim_{(L_0,L_1) \in O_{M_0} \times O_{M_1}} (M_0/L_0) \otimes_R (M_1/L_1)$, and endow it with the inverse limit topology of the discrete topologies. Then $M_0 \hat{\otimes}_R M_1$ forms a profinite R-module. We call $M_0 \hat{\otimes}_R M_1$ the *completed tensor product* of M_0 and M_1. We remark that the completed tensor product preserves the surjectivity.

In continuation, suppose that R is commutative. A *topological R-algebra* is a topological ring A endowed with a continuous ring homomorphism $R \to A$ whose image lies in the centre. Every topological R-algebra is a topological R-module with respect to the natural scalar multiplication. A topological R-algebra is said to be a *discrete R-algebra* if its underlying topology is the discrete topology, is said to be a *finite R-algebra* if it is a discrete R-algebra whose underlying set is a finite set, and is said to be a *profinite R-algebra* if it is homeomorphically isomorphic to the inverse limit of finite R-algebras. The underlying topological R-module of a finite R-algebra is a finite R-module, and hence the underlying topological R-module of a profinite R-algebra is a profinite R-module. The completed tensor product of the underlying profinite R-modules of profinite R-algebras naturally forms a profinite R-algebra.

Definition 2.2. A *topological* $R[G]$*-module* is a pair (M, ρ) of a topological R-module M and a continuous R-linear action $\rho\colon G \times M \to M$. A topological $R[G]$-module is said to be *a discrete* $R[G]$*-module* if its underlying topology is the discrete topology, is said to be *a finite* $R[G]$*-module* if it is a discrete $R[G]$-module whose underlying set is a finite set, and is said to be a *profinite* $R[G]$*-module* if it is homeomorphically isomorphic to the inverse limit of finite $R[G]$-modules.

The underlying topological R-module of a finite $R[G]$-module is a finite R-module, and hence the underlying topological R-module of a profinite $R[G]$-module is a profinite R-module.

Example 2.3. We endow $\mathrm{M}_2(R)$ with the canonical topology (Example 2.1) as an R-module. Then $\mathrm{M}_2(R)$ forms a topological monoid with respect to the multiplication, and R^2 is a topological $R[\mathrm{M}_2(R)]$-module with respect to the canonical topology as an R-module and the natural action $\rho_{R^2} : \mathrm{M}_2(R) \times R^2 \to R^2$. For each $n \in \mathbb{N}$, we denote by $\mathrm{Sym}^n(R^2, \rho_{R^2}) = (\mathrm{Sym}^n(R^2), \mathrm{Sym}^n(\rho_{R^2}))$ the topological $R[\mathrm{M}_2(R)]$-module obtained as the n-th symmetric tensor product of (R^2, ρ_{R^2}) over R endowed with the canonical topology as an R-module. We identify $\mathrm{Sym}^n(R^2)$ with the R-module $\bigoplus_{i=0}^n RT_1^i T_2^{n-i} \subset R[T_1, T_2]$ of homogeneous polynomials of degree n, by putting

$$T_1^i T_2^{n-i} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}^i \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{n-i} \in \mathrm{Sym}^n(R^2)$$

for each $(n, i) \in \mathbb{N} \times \mathbb{N}$ with $i \leq n$.

For a topological $R[G]$-module (M, ρ), we denote by $O_{(M,\rho)}$ the set of open $R[G]$-submodules of (M, ρ). For any profinite $R[G]$-module (M, ρ), $O_{(M,\rho)}$ forms a fundamental system of neighbourhoods of $0 \in M$, and hence is cofinal in O_M. Let (M_0, ρ_0) and (M_1, ρ_1) be profinite $R[G]$-modules. Then the continuous actions ρ_0 and ρ_1 induce a continuous action $\rho_0 \hat{\otimes} \rho_1 \colon G \times (M_0 \hat{\otimes}_R M_1) \to M_0 \hat{\otimes}_R M_1$, for which $(M_0, \rho_0) \hat{\otimes}_R (M_1, \rho_1) := (M_0 \hat{\otimes}_R M_1, \rho_0 \hat{\otimes} \rho_1)$ forms a profinite $R[G]$-module. When (M_0, ρ_0) is the underlying topological $R[G]$-module of a commutative profinite R-algebra A endowed with the trivial action of G, then we regard $A \hat{\otimes}_R (M_1, \rho_1)$ as a profinite $A[G]$-module with respect to the natural continuous action of A.

§ 2.2. Profinite Smooth Sheaves

Continuing from §2.1, let R denote a commutative topological ring. Suppose that R itself is a profinite R-algebra so that R/I forms a finite R-algebra for any $I \in O_R$.

Definition 2.4. Let S be a Noetherian scheme. A *finite smooth* R*-sheaf on* S is a sheaf on $S_{\text{ét}}$ of R-modules which is representable by a finite étale S-scheme and is annihilated by an $I \in O_R$. A *profinite smooth R-sheaf on S* is an inverse system $F = (F_H)_{H \in \mathbb{H}}$ of finite smooth R-sheaves on S indexed by a directed set $\mathbb{H}$. For a profinite smooth R-sheaf $F = (F_H)_{H \in \mathbb{H}}$ on S, we set

$$\mathbb{H}^*_{\text{ét}}(S, F) := \varprojlim_{H \in \mathbb{H}} \mathrm{H}^*_{\text{ét}}(S, F_H),$$

and endow it with the inverse limit topology of the discrete topologies. We call it *the cohomology of F*.

We note that the notion of a profinite sheaf in [Woj12] 14.6 Definition 18 coincides with that of a profinite smooth $\mathbb{Z}_\ell$-sheaf. We explain a method of constructing a profinite smooth R-sheaf. A similar construction of a profinite smooth $\mathbb{Z}_p$-sheaf (resp. a smooth sheaf) is given in [Oht93] 2.3 (resp. [HT01] III 2). To begin with, suppose that G is a discrete finite group. Let Y_1 be a Noetherian scheme with a G-torsor $Y \to Y_1$, where G acts on Y from the right. Let (M, ρ) be a finite $R[G]$-module. We construct a finite smooth R-sheaf $\underline{(M, \rho)}_{Y_1}$ on Y_1 associated to (M, ρ). For a scheme X and a set I, we denote by $X \times I$ the disjoint union of copies of X indexed by I. We consider the right action of G on M given by setting $mg := \rho(g^{-1}, m)$ for each $(m, g) \in M \times G$. We endow $Y \times M$ with the right diagonal action of G over Y_1. We define $\underline{(M, \rho)}_{Y_1}$ as the sheaf on $(Y_1)_{\text{ét}}$ represented by $(Y \times M)/G$, which is a finite smooth R-sheaf with respect to the natural structure given in the following way: Since $Y \to Y_1$ is a G-torsor, we have a natural isomorphism $Y \times G \to Y \times_{Y_1} Y,\ (y, g) \mapsto (y, yg)$ over Y_1, and it induces an isomorphism $(Y \times M) \times_{Y_1} (Y \times M) \cong Y \times (G \times M \times M)$ over Y_1, which is $(G \times G)$-equivariant with respect to the right action of $G \times G$ on the source given in a natural way and on the target defined by setting $(y, (g, m_1, m_2))(g_1, g_2) := (yg_1, (g_1^{-1} g g_2, m_1 g_1, m_2 g_2))$ for $((y, (g, m_1, m_2)), (g_1, g_2)) \in (Y \times (G \times M \times M)) \times (G \times G)$. We obtain

$$\begin{aligned}\underline{(M, \rho)}_{Y_1} \times_{Y_1} \underline{(M, \rho)}_{Y_1} &\cong ((Y \times M) \times_{Y_1} (Y \times M))/(G \times G) \\ &\cong (Y \times (G \times M \times M))/(G \times G).\end{aligned}$$

The map $G \times M \times M \to M,\ (g, m_1, m_2) \mapsto m_1 + \rho(g, m_2)$ defines a morphism $(Y \times M) \times_{Y_1} (Y \times M) \cong Y \times (G \times M \times M) \to Y \times M$, which is $(G \times G)$-equivariant with respect to the right action of $G \times G$ on the target given by the first projection $G \times G \to G$. It induces an addition $\underline{(M, \rho)}_{Y_1} \times_{Y_1} \underline{(M, \rho)}_{Y_1} \to \underline{(M, \rho)}_{Y_1}$. The map $M \times R \to M,\ (m, r) \mapsto rm$ defines a G-equivariant morphism $Y \times (M \times R) \to Y \times M$, and induces a scalar multiplication $\underline{(M, \rho)}_{Y_1} \times R \to \underline{(M, \rho)}_{Y_1}$ compatible with the addition.

Now suppose that G is a profinite group. Let $\mathbb{H}$ be a fundamental system of neighbourhoods of $1 \in G$ consisting of open normal subgroups of G. Let $(Y_H)_{H \in \mathbb{H}}$

be an inverse system of Noetherian schemes endowed with right actions of G such that H acts trivially on Y_H and Y_H forms a (G/H)-torsor over $Y_1 := Y_G$ for any $H \in \mathbb{H}$. Let (M, ρ) be a profinite $R[G]$-module. For each $L \in O_{(M,\rho)}$, we denote by $H_L \subset G$ the open normal subgroup $\{g \in G \mid \rho(g, m) - m \in L, {}^\forall m \in M\}$. For each $L \in O_{(M,\rho)}$ and $H \in \mathbb{H}$ with $H \subset H_L$, we have a finite smooth R-sheaf $(\underline{(M,\rho)/L})_{Y_1}$ on Y_1 given by the (G/H)-torsor $Y_H \to Y_1$, which is independent of the choice of H up to natural isomorphism because $Y_{H'} \to Y_H$ is an (H/H')-torsor for any $H' \in \mathbb{H}$ with $H' \subset H$. For each $(L_0, L_1) \in O_{(M,\rho)} \times O_{(M,\rho)}$ with $L_0 \subset L_1$, the canonical projection $(M,\rho)/L_0 \twoheadrightarrow (M,\rho)/L_1$ induces a morphism $(\underline{(M,\rho)/L_0})_{Y_1} \to (\underline{(M,\rho)/L_1})_{Y_1}$ of sheaves on Y_1, for which $\underline{(M,\rho)}_{Y_1} := ((\underline{(M,\rho)/L})_{Y_1})_{L \in O_{(M,\rho)}}$ forms a profinite smooth R-sheaf on Y_1.

Example 2.5. We give an explicit example of data $(G, \mathbb{H}, (Y_H)_{H \in \mathbb{H}})$. Let $N \geq 5$ be an integer. As in §1, we denote by $Y_1(N)'$ the moduli space over $\mathbb{Q}$ of a pair (E, β) of an elliptic curve E and a surjective morphism $E[N] \twoheadrightarrow \underline{\mathbb{Z}/N\mathbb{Z}}$ of finite group schemes. We consider the case

$$G = \mathrm{G}\hat{\Gamma}_\epsilon(N) := \begin{pmatrix} \hat{\mathbb{Z}} & \hat{\mathbb{Z}} \\ N\hat{\mathbb{Z}} & 1 + N\hat{\mathbb{Z}} \end{pmatrix} \cap \mathrm{GL}_2(\hat{\mathbb{Z}}).$$

Let $M \geq 1$ be an integer with $N \mid M$. We put

$$H_M = \mathrm{G}\hat{\Gamma}(M) := \begin{pmatrix} 1 + M\hat{\mathbb{Z}} & M\hat{\mathbb{Z}} \\ M\hat{\mathbb{Z}} & 1 + M\hat{\mathbb{Z}} \end{pmatrix} \cap \mathrm{GL}_2(\hat{\mathbb{Z}}).$$

We denote by $P_M(X_M) \in \mathbb{Z}[X_M]$ the M-th cyclotomic polynomial, and by $Y(M)$ the moduli space over $\mathbb{Q}[X_M]/(P_M(X_M))$ of pairs $(E, (\alpha_1, \alpha_2))$ of an elliptic curve E and a $(\underline{\mathbb{Z}/M\mathbb{Z}})$-linear basis (α_1, α_2) of the finite group scheme $E[M]$ with $\langle \alpha_1, \alpha_2 \rangle_E = X_M$, where $\langle \cdot, \cdot \rangle_E \colon E[M] \times E[M] \to \mathbb{G}_{\mathrm{m}}[M]$ denotes the Weil pairing. For a scheme S over $\mathbb{Q}$, an elliptic curve E over S, and a $(\underline{\mathbb{Z}/N\mathbb{Z}})_S$-linear basis (α_1, α_2) of $E[N]$, we consider the surjective morphism $\beta_{\alpha_1,\alpha_2} \colon E[N] \twoheadrightarrow (\underline{\mathbb{Z}/N\mathbb{Z}})_S,\ c_1\alpha_1 + c_2\alpha_2 \mapsto c_2$ of group schemes over S. Put $M = mN$. The correspondence $(E, (\alpha_1, \alpha_2)) \rightsquigarrow (E, \beta_{m\alpha_1, m\alpha_2})$ gives a morphism $Y(M) \to Y_1(N)'$ over $\mathbb{Q}$, which forms a (G/H_M)-torsor with respect to the natural right action of $\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ on $Y(M)$. We obtain an inverse system $(Y(mN))_{m \geq 1}$ of schemes over $Y_1(N)'$. Put $\mathbb{H} := \{H_{mN} \mid m \geq 1\}$. For each $H \in \mathbb{H}$, taking a unique $m \geq 1$ with $H_{mN} = H$, we put $Y_H := Y(mN)$. The inverse system $(Y_H)_{H \in \mathbb{H}}$ satisfies the required conditions.

For a prime number p dividing N, the profinite smooth $\mathbb{Z}_p$-sheaf on $Y_1(N)'$ associated to the profinite $\mathbb{Z}_p[\mathrm{G}\hat{\Gamma}_\epsilon(N)]$-module $L'_k := \mathrm{Sym}^{k-2}(\mathbb{Z}_p^2, \rho_{\mathbb{Z}_p^2})$ represents the smooth $\mathbb{Z}_p$-sheaf $F'_k := \mathrm{Sym}^{k-2}(\mathrm{R}^1(\pi'_N)_*(\underline{\mathbb{Z}_p})_{E_1(N)'})$ appeared in §1 for any integer $k \geq 2$.

§ 3. Interpolation of Sheaves and Cohomologies

In the following, let p denote an odd prime number dividing an integer $N \geq 5$. We interpolate the smooth $\mathbb{Z}_p$-sheaves $(F'_k)_{k \geq 2}$ "analytically" parametrised by weights $k \geq 2$. For this sake, we construct a profinite $\mathbb{Z}_p[\mathrm{G}\hat{\Gamma}_\epsilon(N)]$-module interpolating $(L'_k)_{k \geq 2}$, which corresponds to a profinite smooth $\mathbb{Z}_p$-sheaf whose cohomology admits certain specialisation maps to $(\mathrm{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F'_k))_{k \geq 2}$.

§ 3.1. Infinite Dimensional Extension of Sym^{k-2}

Definition 3.1. Let $\Pi_0(p)$ denote the multiplicative submonoid

$$\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \subset \mathrm{M}_2(\mathbb{Z}_p)$$

equipped with the continuous monoid homomorphism $\mathrm{G}\hat{\Gamma}_\epsilon(N) \to \Pi_0(p)$ induced by the canonical projection $\mathrm{M}_2(\hat{\mathbb{Z}}) \twoheadrightarrow \mathrm{M}_2(\mathbb{Z}_p)$.

We use not only $\mathrm{G}\hat{\Gamma}_\epsilon(N)$ but also $\Pi_0(p)$ because actions of matrices such as $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \Pi_0(p)$ are necessary to define actions of Hecke operators on the cohomologies of the profinite smooth $\mathbb{Z}_p$-sheaves associated to profinite $\mathbb{Z}_p[\mathrm{G}\hat{\Gamma}_\epsilon(N)]$-modules.

Let $k \geq 2$. Since L'_k does not satisfy good congruence relations, we take another lattice L_k of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L'_k \cong \mathrm{Sym}^{k-2}(\mathbb{Q}_p^2)$.

Lemma 3.2. *The* $\mathbb{Z}_p$*-submodule* $\mathrm{Sym}_0^{k-2}(\mathbb{Z}_p^2) \subset \mathrm{Sym}^{k-2}(\mathbb{Z}_p^2)$ *generated by*

$$e_{k,i} := \binom{k-2}{i} T_1^i T_2^{k-2-i} \in \bigoplus_{i=0}^{k-2} \mathbb{Z}_p T_1^i T_2^{k-2-i}$$

(*Example* 2.3) *with* $i \in \mathbb{N} \cap [0, k-2]$ *is stable under the action* ρ_{k-2} *of* $\mathrm{M}_2(\mathbb{Z}_p)$.

We denote by L_k the profinite $\mathbb{Z}_p[\Pi_0(p)]$-module $\mathrm{Sym}_0^{k-2}(\mathbb{Z}_p^2)$ on which $\Pi_0(p)$ acts by the restriction of ρ_{k-2}. The following lemma ensures that the new lattice above works well when one considers an interpolation along weights:

Lemma 3.3. *For any* $r \geq 1$ *and* $k_1 \geq k_0 \geq 2$ *with* $k_1 - k_0 \in p^{r-1}(p-1)\mathbb{Z}$*, the natural projection*

$$L_{k_1}/p^r \to L_{k_0}/p^r, \ \sum_{i=0}^{k_1-2} \overline{\alpha}_i e_{k_1,i} \mapsto \sum_{i=0}^{k_0-2} \overline{\alpha}_i e_{k_0,i}$$

is a surjective $(\mathbb{Z}/p^r\mathbb{Z})$*-linear* $\Pi_0(p)$*-equivariant homomorphism.*

We remark that the natural projection in Lemma 3.3 is an analogue of the $\mathrm{M}_2(\mathbb{Z}_p)$-equivariant projections $\mathrm{Sym}^{k_1-2}(\mathbb{Z}_p^2)/p^r \twoheadrightarrow (\mathrm{Sym}^0(\mathbb{Z}_p^2)/p^r)(k_1-2) \cong (\mathbb{Z}/p^r\mathbb{Z})(k_1-2)$ introduced right after [Hid86-2] Lemma 4.3 and $\mathrm{Sym}^{k_1-2}(\mathbb{Z}_p^2)/p \twoheadrightarrow \mathrm{Sym}^0(\mathbb{Z}_p^2)/p \cong \mathbb{Z}/p\mathbb{Z}$ for $k_1 \geq 2$ introduced in the proof of [Hid93] 7.2 Theorem 2.

We denote by W the Abelian group of weights, i.e. continuous group endomorphisms of $\mathbb{Z}_p^\times$. For each $(\kappa, u) \in W \times \mathbb{Z}_p^\times$, we put $u^\kappa := \kappa(u)$. We regard $\mathbb{Z}$ as a subgroup of W through the pairing $\mathbb{Z} \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$, $(k,u) \mapsto u^k$. We call W *the weight space*, and will use it as the parametre space of the interpolation. The reduction $\mathbb{Z}_p^\times \twoheadrightarrow \mathbb{F}_p^\times$ induces a natural identification $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1+p\mathbb{Z}_p)$ through a unique Teichmüller lifting $[\cdot]\colon \mathbb{F}_p^\times \hookrightarrow \mathbb{Z}_p^\times$. For each $\kappa \in W$, we identify $\kappa|_{\mathbb{F}_p^\times}$ with an element of $\mathbb{Z}/(p-1)\mathbb{Z}$ through the pairing $(\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$, $(k, \overline{u}) \mapsto [\overline{u}^k]$, and $\kappa|_{1+p\mathbb{Z}_p}$ with an element of $\mathbb{Z}_p$ through the pairing $\mathbb{Z}_p \times (1+p\mathbb{Z}_p) \to \mathbb{Z}_p^\times$, $(k,u) \mapsto u^k$. We obtain a natural identification

$$\begin{aligned} W &\cong (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p \cong (\mathbb{Z}/(p-1)\mathbb{Z}) \times \bigsqcup_{\zeta=0}^{p-1} (\zeta + p\mathbb{Z}_p) \\ &\cong ((\mathbb{Z}/(p-1)\mathbb{Z}) \times (\mathbb{N} \cap [0,p-1])) \times p\mathbb{Z}_p \cong (p\mathbb{Z}_p)^{\sqcup p(p-1)} \end{aligned}$$

of sets, through which we endow W with the disjoint union topology of copies of $p\mathbb{Z}_p$.

We construct a system $(\rho_\kappa)_{\kappa \in W}$ of continuous actions of $\Pi_0(p)$ on $\mathbb{Z}_p^{\mathbb{N}}$ such that $(\mathbb{Z}_p^{\mathbb{N}}, \rho_\kappa)$ is a profinite $\mathbb{Z}_p[\Pi_0(p)]$-module for any $\kappa \in W$, the matrix coefficients of ρ_κ form "rigid analytic functions" on $\kappa \in W \cong (p\mathbb{Z}_p)^{\sqcup p(p-1)}$, and L_k is obtained as a quotient of $(\mathbb{Z}_p^{\mathbb{N}}, \rho_k)$ for any $k \geq 2$. Here a "rigid analytic function" on W means a function lying in the ring Λ_0 which we will introduce later in Definition 3.9.

Definition 3.4. Let $\kappa \in W$. For each $r \in \mathbb{N}$, let $\kappa^{(r)}$ denote the smallest integer with $\kappa^{(r)} \geq 0$ and $\kappa - \kappa^{(r)} \in p^r(p-1)W$. We define a continuous action ρ_κ on $\mathbb{Z}_p^{\mathbb{N}}$ through the homeomorphic $\mathbb{Z}_p$-linear isomorphism

$$\begin{aligned} \mathbb{Z}_p^{\mathbb{N}} &\to \varprojlim_{r \in \mathbb{N}} \varprojlim_{m \in \mathbb{N}} \left(L_{\kappa^{(r)}+2+p^r(p-1)m} / p^{r+1} \right) \\ (\alpha_i)_{i \in \mathbb{N}} &\mapsto \left(\left(\sum_{i=0}^{\kappa^{(r)}+2+p^r(p-1)m-2} \overline{\alpha}_i e_{\kappa^{(r)}+2+p^r(p-1)m, i} \right)_{m \in \mathbb{N}} \right)_{r \in \mathbb{N}} \end{aligned}$$

of topological $\mathbb{Z}_p$-modules so that $(\mathbb{Z}_p^{\mathbb{N}}, \rho_\kappa)$ is a profinite $\mathbb{Z}_p[\Pi_0(p)]$-module, where the transition maps of the double inverse system $((L_{\kappa^{(r)}+2+p^r(p-1)m}/p^{r+1})_{m \in \mathbb{N}})_{r \in \mathbb{N}}$ are given by the natural projections in Lemma 3.3 and the canonical projections.

The action ρ_κ of $\Pi_0(p)$ on $\mathbb{Z}_p^{\mathbb{N}}$ defined above is given by the matrix presentation whose entries are explicitly described as "rigid analytic functions" on $\kappa \in W$ as follows:

Theorem 3.5. *Let $\kappa \in W$. For any $(A, \alpha, i) \in \Pi_0(p) \times \mathbb{Z}_p^{\mathbb{N}} \times \mathbb{N}$ with $A = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix}$ and $\alpha = (\alpha_j)_{j=0}^{\infty}$, the infinite sum*

$$\rho_\kappa(A, \alpha)_i := \sum_{j=0}^{\infty} \alpha_j \sum_{h=0}^{\min\{i,j\}} \begin{pmatrix} i \\ h \end{pmatrix} \left(\prod_{m=i}^{i+j-h-1} (\kappa|_{1+p\mathbb{Z}_p} - m) \right) a^h b^{i-h} \frac{c^{j-h}}{(j-h)!} d^{\kappa-i-j+h}$$

converges in $\mathbb{Z}_p$, and the equality $\rho_\kappa(A, \alpha) = (\rho_\kappa(A, \alpha)_i)_{i \in \mathbb{N}}$ holds.

Remark 3.6. Let $k \geq 2$ be an integer. R. Pollack and G. Stevens defined a continuous right action of the topological monoid

$$\Sigma_0(p) := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \cap \mathrm{GL}_2(\mathbb{Q}_p)$$

corresponding to the weight k on the topological $\mathbb{Z}_p$-algebra $\mathbb{D}(\mathbb{Z}_p)$ of $\mathbb{Q}_p$-valued distributions on $\mathbb{Z}_p$ in [PS11] §3.3, and proved that the closed $\mathbb{Z}_p$-subalgebra $\mathbb{D}^0(\mathbb{Z}_p) \subset \mathbb{D}(\mathbb{Z}_p)$ of distributions with integral moments is stable under the right action of $\Sigma_0(p)$ in [PS11] Proposition 7.1. Moreover, it is easy to show that the closed $\mathbb{Z}_p$-submodule $\mathbb{D}_k^\dagger(\mathbb{Z}_p) \subset \mathbb{D}^0(\mathbb{Z}_p)$, which is canonically isomorphic to $\mathbb{Z}_p^{\mathbb{N}}$, of distributions m on $\mathbb{Z}_p$ with $m(z^i) \in \begin{pmatrix} k-2 \\ i \end{pmatrix} \mathbb{Z}_p$ for any $i \in \mathbb{N}$ is stable under the right action of $\Sigma_0(p)$. The induced right action of $\Sigma_0(p)$ on $\mathbb{Z}_p^{\mathbb{N}}$ is related to the action ρ_{k-2} of $\Pi_0(p)$ on $\mathbb{Z}_p^{\mathbb{N}}$ as follows. The map $\Pi_0(p) \cap \mathrm{GL}_2(\mathbb{Q}_p) \to \Sigma_0(p)^{\mathrm{op}}$, $A \mapsto \det(A)A^{-1}$ is a homeomorphic monoid isomorphism. Therefore we obtain a continuous left action ρ'_{k-2} of $\Pi_0(p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$ of weight $k \geq 2$ on $\mathbb{Z}_p^{\mathbb{N}}$. For any $k \geq 2$, the restriction of ρ_{k-2} to the submonoid $\Pi_0(p) \cap \mathrm{GL}_2(\mathbb{Q}_p) \subset \Pi_0(p)$ coincides with ρ'_{k-2}. Thus the construction of $(\mathbb{Z}_p^{\mathbb{N}}, \rho_{\kappa-2})_{\kappa \in W}$ is a generalisation of that of $(\mathbb{Z}_p^{\mathbb{N}}, \rho'_{k-2})_{k \geq 2}$ in the sense that the former one deals with possibly non-integer weights and $\Pi_0(p)$ while the latter one deals with integer weights ≥ 2 and $\Pi_0(p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$.

Remark 3.7. Another similar way to construct an infinite dimensional representation using the symmetric product is studied in [Yam07] §1.2 for Hilbert modular forms, but the direction of the extension is different from ours.

The following specialisation property immediately follows from Lemma 3.3 and Definition 3.4:

Proposition 3.8. *For any $k \geq 2$, the natural projection*

$$\varpi_k \colon (\mathbb{Z}_p^{\mathbb{N}}, \rho_{k-2}) \to L_k,\ (\alpha_i)_{i=0}^{\infty} \mapsto \sum_{i=0}^{k-2} \alpha_i e_{k,i}$$

is a continuous surjective $\mathbb{Z}_p$-linear $\Pi_0(p)$-equivariant homomorphism.

Let $k \geq 2$. Taking the continuous dual in an analogous way to the Schneider–Teitelbaum theory on representations of a profinite group ([ST02] Theorem 2.3), we obtain an exact sequence

$$0 \to \mathrm{Sym}^{k-2}(\mathbb{Q}_p^2, \rho_{\mathbb{Q}_p^2}) \to (\mathbb{Z}_p^{\mathbb{N}}, \rho_{k-2})^{\vee} \to (\ker(\varpi_{k-2}), \rho_{k-2}|_{\ker(\varpi_{k-2})})^{\vee} \to 0$$

of unitary Banach $\mathbb{Q}_p$-linear representations, and $(\ker(\varpi_{k-2}), \rho_{k-2}|_{\ker(\varpi_{k-2})})^{\vee}$ is an infinite dimensional irreducible unitary Banach $\mathbb{Q}_p$-linear representation. Thus $(\mathbb{Z}_p^{\mathbb{N}}, \rho_{k-2})$ is the continuous dual of an extension of an infinite dimensional irreducible unitary Banach $\mathbb{Q}_p$-linear representation by $\mathrm{Sym}^{k-2}(\mathbb{Q}_p^2, \rho_{\mathbb{Q}_p^2})$.

§ 3.2. Analytic Continuation over the Weight Space

Gluing $(\mathbb{Z}_p, \rho_{\kappa-2})_{\kappa \in W}$ along weights κ, we construct a profinite $\mathbb{Z}_p[\Pi_0(p)]$-module interpolating $(L_k)_{k \geq 2}$. For topological spaces U and V, we denote by $\mathrm{C}(U, V)$ the set of continuous maps $U \to V$.

Definition 3.9. We denote by $\Lambda_0 \subset \mathrm{C}(W, \mathbb{Z}_p)$ the $\mathbb{Z}_p$-subalgebra of $\mathbb{Z}_p$-valued functions on $W \cong (p\mathbb{Z}_p)^{\sqcup p(p-1)}$ whose restrictions to the copies of $p\mathbb{Z}_p$ admit Taylor expansions with coefficients in $\mathbb{Z}_p$. We identify Λ_0 with $\mathbb{Z}_p[[X]]^{p(p-1)}$, and endow it with the direct product topology of copies of $\mathbb{Z}_p[[X]]$. Here $\mathbb{Z}_p[[X]] \cong \varprojlim_{r \in \mathbb{N}} \mathbb{Z}_p[X]/(p, X)^r$ is regarded as a profinite $\mathbb{Z}_p$-algebra with respect to the inverse limit topology of the discrete topologies.

The continuous function $z \colon W \twoheadrightarrow \mathbb{Z}_p,\ \kappa \mapsto \kappa|_{1+p\mathbb{Z}_p}$ is contained in Λ_0. Since $\mathbb{N} \cap [2, \infty)$ is dense in W, the evaluation map $\Lambda_0 \hookrightarrow \prod_{k=2}^{\infty} \mathbb{Z}_p,\ f \mapsto (f(k))_{k=2}^{\infty}$ is a homeomorphic $\mathbb{Z}_p$-algebra isomorphism onto the closed image. For each $u \in \mathbb{Z}_p^{\times}$, we denote by $u^z \colon W \to \mathbb{Z}_p$ the continuous function defined by $u^z(\kappa) := u^{\kappa}$, which lies in $\Lambda_0^{\times}$. By the universality of the Iwasawa algebra, the continuous group homomorphism $1 + N\mathbb{Z}_p \hookrightarrow \Lambda_0^{\times},\ \gamma \mapsto \gamma^z$ induces a homeomorphic $\mathbb{Z}_p$-algebra isomorphism $\mathbb{Z}_p[[1 + N\mathbb{Z}_p]] \hookrightarrow \Lambda_0$ onto the closed image.

We can define a continuous action $\rho_{\bullet-2}$ of $\Pi_0(p)$ on $\Lambda_0^{\mathbb{N}}$ replacing $\kappa|_{1+p\mathbb{Z}_p}$ and $d^{\kappa-i-j+h}$ in the explicit formula of ρ_{κ} in Theorem 3.5 by $z-2$ and $d^{z-2-i-j+h} := d^{-2-i-j+h} d^z$ respectively as follows:

Proposition 3.10. *For any $(A, f, i) \in \Pi_0(p) \times \Lambda_0^{\mathbb{N}} \times \mathbb{N}$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $f = (f_j)_{j=0}^{\infty}$, the infinite sum*

$$\rho_{\bullet-2}(A, f)_i := \sum_{j=0}^{\infty} f_j \sum_{h=0}^{\min\{i,j\}} \binom{i}{h} \left(\prod_{m=i}^{i+j-h-1} (z-2-m) \right) a^h b^{i-h} \frac{c^{j-h}}{(j-h)!} d^{z-2-i-j+h}$$

converges in Λ_0, and the map

$$\rho_{\bullet-2}\colon \Pi_0(p) \times \Lambda_0^{\mathbb{N}} \to \Lambda_0^{\mathbb{N}},\ (A, f) \mapsto (\rho_{\bullet-2}(A, f)_i)_{i\in\mathbb{N}}$$

makes $\Lambda_0^{\mathbb{N}}$ a profinite $\Lambda_0[\Pi_0(p)]$-module.

The profinite $\Lambda_0[\Pi_0(p)]$-module $(\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})$ interpolates not only $(\mathbb{Z}_p^{\mathbb{N}}, \rho_{k-2})_{k\geq 2}$ but also $(L_k)_{k\geq 2}$, i.e. the following holds:

Theorem 3.11. *The specialisation map*

$$e_k\colon \left(\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2}\right) \to L_k,\ (f_i)_{i=0}^{\infty} \mapsto \sum_{i=0}^{k-2} f_i(k) e_{k,i},$$

is a quotient map for any $k \geq 2$, and the evaluation map

$$\left(\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2}\right) \to \prod_{k=2}^{\infty} L_k,\ f \mapsto (e_k(f))_{k=2}^{\infty},$$

is a homeomorphic Λ_0-linear $\Pi_0(p)$-equivariant isomorphism onto the closed image.

Here a map $f\colon X \to Y$ between topological spaces is said to be *a quotient map* if f induces a homeomorphism between Y and the quotient space of X by the equivalence relation given by the subset $\{(x, x') \in X \times X \mid f(x) = f(x')\}$.

§ 3.3. Galois Representation over the Universal Hecke Algebra

For a finitely generated $\mathbb{Z}_p$-module M, let M_{free} denote the quotient of M by the $\mathbb{Z}_p$-submodule consisting of torsion elements. As in §1, we denote by S_N the set of normalised cuspidal eigenforms of weight ≥ 2 and level $\Gamma_1(N)$, and by V_f the Galois representation associated to $f \in S_N$ (cf. [Del69]). We define a profinite $\Lambda_0[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $\mathbb{H}(N)$ such that V_f is obtained from $\mathbb{H}(N)$ for any $f \in S_N$.

Definition 3.12. We put $F := \underline{(\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})}_{Y_1(N)'}$ and $F_k := \underline{(L_k, \rho_{\mathbb{Z}_p^2})}_{Y_1(N)'}$ for each $k \geq 2$. We endow $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)_{\mathrm{free}}$ with the p-adic topology, which coincides with the canonical topology and the quotient topology of $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)$ for each $k \geq 2$. We denote by $\mathbb{H}(N)$ the image of the continuous Λ_0-linear $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant homomorphism $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F) \to \prod_{k=2}^{\infty} \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)_{\mathrm{free}}$ induced by the specialisation maps e_k in Theorem 3.11. We endow $\mathbb{H}(N)$ with the relative topology of the target, and regard it as a profinite $\Lambda_0[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module.

We note that the topology of $\mathbb{H}(N)$ coincides with the quotient topology of the source. The definition above is inspired by the theory on direct integration of unitary representations over $\mathbb{C}$. One of my main ideas in interpolation is to find good submodules in the direct product of $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)_{\text{free}} \subset \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k) \cong \mathrm{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}\mathrm{R}^1(\pi'_N)_*(\underline{\mathbb{Q}_p})_{E_1(N)'})$ with $k \geq 2$. The following implies that V_f is obtained as a twist of a quotient of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{H}(N)$ for any $f \in S_N$, because V_f is isomorphic to a twist of the quotient of $\mathrm{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, \mathrm{Sym}^{k-2}\mathrm{R}^1(\pi'_N)_*(\underline{\mathbb{Q}_p})_{E_1(N)'})$ (cf. [Gro90] Proposition 11.4):

Proposition 3.13 ([Mih15]). *The specialisation map*

$$\mathrm{sp}_k \colon \mathbb{H}(N) \to \mathbb{H}^1_{\text{ét}}\left(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k\right)_{\text{free}}$$

given by the canonical projection is surjective for any $k \geq 2$.

Outline of the proof. By the definition of sp_k, it suffices to show the surjectivity of the map $\mathbb{H}^1_{\text{ét}}(e_k) \colon \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F) \to \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)$ induced by e_k. We have a natural equivalence $\mathrm{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, \underline{(M,\rho)}_{Y_1(N)'}) \cong \mathrm{H}^1(\Gamma_1(N), (M,\rho))$ between functors from the category of finite $\mathbb{Z}_p[\mathrm{G}\hat{\Gamma}_\epsilon(N)]$-modules (M,ρ). It is given through the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and the interpretation of $\mathrm{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, \underline{(M,\rho)}_{Y_1(N)'})$ (resp. $\mathrm{H}^1(\Gamma_1(N), (M,\rho))$) as the set of isomorphism classes of $\underline{(M,\rho)}_{Y_1(N)'}$-torsors (resp. (M,ρ)-torsors) in the category of étale sheaves over $Y_1(N)'_{\overline{\mathbb{Q}}}$ (resp. in the category of $\Gamma_1(N)$-sets). We denote by $\mathbb{H}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2}))$ and $\mathbb{H}^1(\Gamma_1(N), L_k)$ the inverse limits of the group cohomology corresponding to $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F)$ and $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)$ respectively. Owing to the first countability of $\Lambda_0^{\mathbb{N}}$ and L_k, we have natural isomorphisms $\mathrm{H}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})) \to \mathbb{H}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2}))$ and $\mathrm{H}^1(\Gamma_1(N), L_k) \to \mathbb{H}^1(\Gamma_1(N), L_k)$. Moreover, $\mathbb{H}^1_{\text{ét}}(e_k)$ corresponds to $\mathrm{H}^1(e_k) \colon \mathrm{H}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})) \to \mathrm{H}^1(\Gamma_1(N), L_k)$, which is surjective by Proposition 3.8. We note that the cohomological dimension of $\Gamma_1(N)$ with respect to the coefficient ring Λ_0 is 1, because $\Gamma_1(N)$ is isomorphic to the fundamental group of $H/\Gamma_1(N)$. Indeed, the fundamental group of a connected non-compact surface is a free group of finite rank, and the cohomological dimension of a free group is 1. □

Henceforth, we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and identify it with $\mathbb{C}$. For each $k \geq 2$, we denote by $\mathrm{M}_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ the $\overline{\mathbb{Q}}_p$-vector space of modular forms over $\overline{\mathbb{Q}}_p$ of weight k and level $\Gamma_1(N)$, by $\mathrm{T}_{k,N} \subset \mathrm{End}_{\overline{\mathbb{Q}}_p}(\mathrm{M}_k(\Gamma_1(N), \overline{\mathbb{Q}}_p))$ the $\mathbb{Z}_p$-algebra generated by Hecke operators, and by $\mathrm{T}_{\leq k,N} \subset \mathrm{End}_{\overline{\mathbb{Q}}_p}(\bigoplus_{k_0=2}^{k} \mathrm{M}_{k_0}(\Gamma_1(N), \overline{\mathbb{Q}}_p))$ the $\mathbb{Z}_p$-algebra generated by the endomorphisms given by the diagonal action of Hecke operators. Here a *Hecke operator* means one of the endomorphism T_ℓ for a prime number ℓ and S_n for an $n \in \mathbb{N}$ with $(n, N) = 1$. We define *the universal Hecke algebra* $\mathbb{T}_N$ as $\varprojlim_{k \geq 2} \mathrm{T}_{\leq k,N}$ and endow it with the inverse limit topology of the p-adic topologies. Then $\mathbb{T}_N$ is a profinite $\mathbb{Z}_p[[1 + N\mathbb{Z}_p]]$-algebra with respect to the structure morphism $\mathbb{Z}_p[[1 + N\mathbb{Z}_p]] \to \mathbb{T}_N$ associated to the continuous group homomorphism $1 + N\mathbb{Z}_p \to \mathbb{T}_N^\times$ sending $1 + N$ to S_{1+N} by the universality of the Iwasawa algebra. We put $\Lambda_0\mathbb{T}_N := \Lambda_0 \hat{\otimes}_{\mathbb{Z}_p[[1+N\mathbb{Z}_p]]} \mathbb{T}_N$.

Theorem 3.14 ([Mih15]). *The action of Hecke operators on* $\mathbb{H}(N)$ *induces a continuous* Λ_0*-linear* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-equivariant faithful action* $\mathbb{T}_N \times \mathbb{H}(N) \to \mathbb{H}(N)$*, for which* $\mathbb{H}(N)$ *forms a profinite* $\Lambda_0\mathbb{T}_N[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$*-module.*

Outline of the proof. The continuous action of $\Pi_0(p)$ on $\Lambda_0^{\mathbb{N}}$ induces an action of Hecke operators on $\mathbb{H}(N)$ compatible with the specialisation maps. It gives an action of the dense Λ_0-subalgebra of $\Lambda_0\mathbb{T}_N$ generated by Hecke operators, which is continuous with respect to the relative topology. The compactness of $\mathbb{H}(N)$ implies that the action extends to $\Lambda_0\mathbb{T}_N$. $\square$

§ 3.4. p-adic Family of Finite Slope

The profinite $\Lambda_0\mathbb{T}_N[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module $\mathbb{H}(N)$ does not seem to be finitely generated as a $\Lambda_0\mathbb{T}_N$-module, and we do not know whether the specialisation maps $\mathbb{H}(N) \twoheadrightarrow \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)_{\text{free}}$ with $k \geq 2$ can be obtained as the quotients by the corresponding ideals of $\Lambda_0\mathbb{T}_N$ or not. In order to deal with the specialisation in such a module theoretic way, we cut $\mathbb{H}(N)$ by a fixed bound s of slopes. Let $s \in \mathbb{N}\backslash\{0\}$ and $k \geq 2$. An $F \in \mathrm{M}_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ is said to be *of slope* $< s$ if every eigenvalue of T_p acting on $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{T}_{k,N}F$ is of norm $> |p|^s$. We denote by $\mathrm{M}_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)^{<s} \subset \mathrm{M}_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ the $\mathrm{T}_{k,N}$-stable $\overline{\mathbb{Q}}_p$-vector subspace consisting of modular forms of slope $< s$, by $T_p^{-1} \in \mathrm{End}_{\overline{\mathbb{Q}}_p}(\mathrm{M}_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)^{<s})$ the inverse of the restriction of T_p, by $\mathrm{T}^{<s}_{k,N} \subset \mathrm{End}_{\overline{\mathbb{Q}}_p}(\mathrm{M}_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)^{<s})$ the $\mathbb{Z}_p$-subalgebra generated by Heck operators and $p^sT_p^{-1}$, and by $\mathrm{T}^{<s}_{\leq k,N} \subset \mathrm{End}_{\overline{\mathbb{Q}}_p}(\bigoplus_{k_0=2}^{k} \mathrm{M}_{k_0}(\Gamma_1(N), \overline{\mathbb{Q}}_p)^{<s})$ the $\mathbb{Z}_p$-subalgebra generated by endomorphisms given by the diagonal action of Hecke operators and $p^sT_p^{-1}$. We set $\mathbb{T}^{<s}_N := \varprojlim_{k \geq 2} \mathrm{T}^{<s}_{\leq k,N}$. Then $\mathbb{T}^{<s}_N$ is a profinite $\mathbb{Z}_p[[1 + N\mathbb{Z}_p]]$-algebra. We put $\Lambda_0\mathbb{T}^{<s}_N := \Lambda_0 \hat{\otimes}_{\mathbb{Z}_p[[1+N\mathbb{Z}_p]]} \mathbb{T}^{<s}_N$, and regard it as a profinite $\Lambda_0\mathbb{T}_N$-algebra. We set $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)^{<s} := (\mathrm{T}^{<s}_{k,N} \otimes_{\mathrm{T}_{k,N}} \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)_{\text{free}})_{\text{free}}$. We denote by $\mathbb{H}(N)^{<s}$ the image of the natural continuous $\Lambda_0\mathbb{T}^{<s}_N$-linear $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant homomorphism $\Lambda_0\mathbb{T}^{<s}_N \hat{\otimes}_{\Lambda_0\mathbb{T}_N} \mathbb{H}(N) \to \prod_{k=2}^{\infty} \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)^{<s}$, and regard it as a profinite $\Lambda_0\mathbb{T}^{<s}_N[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module. The following theorem ensures that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{H}(N)^{<s}$ interpolates V_f for any $f \in S_N$ of slope $< s$:

Theorem 3.15 ([Mih15]). *If* $p^s \mid N$*, then* $\mathbb{H}(N)^{<s}$ *is finitely generated as a* $\Lambda_0\mathbb{T}^{<s}_N$*-module, and the canonical projections*

$$\mathrm{sp}^{<s}_k \colon \mathbb{H}(N)^{<s} \twoheadrightarrow \mathbb{H}^1_{\text{ét}}\left(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k\right)^{<s}, \ \widetilde{e}_k \colon \Lambda_0\mathbb{T}^{<s}_N \twoheadrightarrow \mathbb{T}^{<s}_{k,N}$$

induce a homeomorphic $\mathrm{T}^{<s}_{k,N}$*-linear* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-equivariant isomorphism*

$$\mathbb{H}(N)^{<s}/\ker(\widetilde{e}_k) \to \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k)^{<s}$$

with respect to the quotient topology of the source for any $k \geq 2$.

Outline of the proof. Since $\Gamma_1(N)$ is a finitely generated free group, there is a Λ_0-linear isomorphism $\mathrm{Z}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})) \cong (\Lambda_0^{\mathbb{N}})^d$ depending on a choice of a group isomorphism between $\Gamma_1(N)$ and the free group of rank $d \in \mathbb{N}$. Then the image of the composite of the zero-extension $(\Lambda_0^s)^d \hookrightarrow (\Lambda_0^{\mathbb{N}})^d$, the canonical projection $(\Lambda_0^{\mathbb{N}})^d \cong \mathrm{Z}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})) \twoheadrightarrow \mathrm{H}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2}))$, the natural isomorphism $\mathrm{H}^1(\Gamma_1(N), (\Lambda_0^{\mathbb{N}}, \rho_{\bullet-2})) \cong \mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F)$, and the natural map $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F) \twoheadrightarrow \mathbb{H}(N) \to \mathbb{H}(N)^{<s}$ generates $\mathbb{H}(N)^{<s}$ as a $\Lambda_0\mathbb{T}_N^{<s}$-module by a careful calculation of the p-adic valuations of the matrix coefficient of T_p on $\mathbb{H}^1_{\text{ét}}(Y_1(N)'_{\overline{\mathbb{Q}}}, F_k) \cong \mathrm{H}^1(\Gamma_1(N), L_k)$. The second assertion follows from a similar explicit computation. □

In order to cut $\mathbb{H}(N)^{<s}$ by "a family of eigenforms of slope $< s$" in a module theoretic way, we formulate a family of modular forms as a formal power series over a topological Λ_0-algebra Λ_1 called a Λ-adic domain.

Let Λ_1 be a topological Λ_0-algebra. A continuous $\mathbb{Z}_p$-algebra homomorphism $\Lambda_1 \to \overline{\mathbb{Z}}_p$ is said to be *a character of weight* $k \in \mathbb{Z}_p$ if its restriction on $1 + N\mathbb{Z}_p \cong \{[\gamma] \mid \gamma \in 1+N\mathbb{Z}_p\} \subset \mathbb{Z}_p[[1+N\mathbb{Z}_p]]$ coincides with the group homomorphism $1+N\mathbb{Z}_p \to \overline{\mathbb{Z}}_p^\times$, $\gamma \mapsto \gamma^k$. We denote by $\Omega(\Lambda_1)$ the Hausdorff space of continuous $\mathbb{Z}_p$-algebra homomorphisms $\Lambda_1 \to \overline{\mathbb{Z}}_p$, and by $\Omega(\Lambda_1)_S \subset \Omega(\Lambda_1)$ the subspace consisting of characters of weights in S for each $S \subset \mathbb{Z}_p$. For each $\varphi \in \Omega(\Lambda_1)_{\mathbb{Z}_p}$, we denote by $\mathrm{wt}(\varphi) \in \mathbb{Z}_p$ the weight of φ.

Definition 3.16 ([Mih15]). A compact topological Λ_0-algebra Λ_1 is said to be a Λ-*adic domain* if it satisfies the following conditions:

(i) The set $\{k \in \mathbb{N} \cap [2, \infty) \mid \Omega(\Lambda_1)_{\{k\}} \neq \emptyset\}$ is an infinite set.

(ii) For any infinite subset $\Sigma \subset \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$, the equality $\bigcap_{\varphi \in \Sigma} \ker(\varphi) = \{0\}$ holds.

The following justifies the terminology "a Λ-adic domain":

Proposition 3.17 ([Mih15]). *Every Λ-adic domain is a commutative profinite Λ_0-algebra whose underlying ring is an integral domain.*

Remark 3.18. Since Λ_0 is topologically of finite type as a topological $\mathbb{Z}_p$-algebra, it yields a $\mathbb{Q}_p$-analytic space W^{an} in the sense of Berkovich, whose $\mathbb{Q}_p$-rational points are naturally identified with W. Every Λ-adic domain finitely generated as a module over the Λ_0-algebra associated to an open ball in W^{an} is topologically of finite type as a topological $\mathbb{Z}_p$-algebra, and hence yields a $\mathbb{Q}_p$-analytic space over W^{an}. Every closed good $\mathbb{Q}_p$-analytic space étale over W^{an} admits an open covering by $\mathbb{Q}_p$-analytic spaces associated to such Λ-adic domains ([Mih15]). For more details about terminology on rigid analytic spaces such as a closed $\mathbb{Q}_p$-analytic space, a good $\mathbb{Q}_p$-analytic space, and an étale morphism between $\mathbb{Q}_p$-analytic spaces, see [Ber90] 3.1.2, [Ber93] 1.2.15, and [Ber93] Definition 3.3.4 respectively.

Let Λ_1 be a Λ-adic domain, and $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \Lambda_1^\times$ a group homomorphism. We denote by $\Lambda_1(\chi)$ the profinite $\Lambda_1[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module Λ_1 on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts through the character obtained as the composite of the restriction map $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$, the natural isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$, and χ.

Definition 3.19. A Λ_1*-adic form of level* $\Gamma_1(N)$*, character* χ*, and slope* $[< s]$ is an $f \in \Lambda_1[[q]]$ such that $f_\varphi := \sum_{h=0}^{\infty} \varphi(a_h(f))q^h \in \overline{\mathbb{Q}}_p[[q]]$ is a modular form of weight $\mathrm{wt}(\varphi)$, level $\Gamma_1(N)$, character $\varphi \circ \chi$, and slope $< s$ for all but finitely many $\varphi \in \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$. We denote by $\mathbb{M}(\Gamma_1(N), \chi, \Lambda_1)^{[<s]}$ the Λ_1-module of Λ_1-adic forms of level $\Gamma_1(N)$, character χ, and slope $[< s]$. A *normalised* Λ_1*-adic cuspidal eigenform of level* $\Gamma_1(N)$*, character* χ*, and slope* $< s$ is an $f \in \mathbb{M}(\Gamma_1(N), \chi, \Lambda_1)^{[<s]}$ with $a_p(f) \neq 0$ and $p^s a_p(f)^{-1} \in \Lambda_1$ as an element of $\mathrm{Frac}(\Lambda_1)$ such that f_φ is a normalised cuspidal eigenform for all but finitely many $\varphi \in \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$.

We show in [Mih15] that $\mathbb{M}(\Gamma_1(N), \chi, \Lambda_1)^{[<s]}$ is generically finitely generated in a similar way to the proof of [Hid93] §7.3 Theorem 1 by [Col97] Theorem B 3.5 and [Wan98] Theorem 2.5.

Example 3.20. The eigencurve given by M. Emerton in [Eme] Theorem 2.23 is equipped with a family of eigenforms of level $\Gamma_1(N)$ and finite slopes. Let C denote the cuspidal locus of the eigencurve. The normalisation of the canonical reduced closed analytic subvariety of C is a closed good $\mathbb{Q}_p$-analytic space, and hence admits an open covering by $\mathbb{Q}_p$-analytic spaces associated to Λ-adic domains by Remark 3.18. Let $f \in S_N$ be an eigenform of character $\chi_f\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ and slope $< s$, and $x_f \in C(\overline{\mathbb{Q}}_p)$ the point corresponding to f. Pulling back the family of eigenforms to a sufficiently small open neighbourhood of x_f given as the $\mathbb{Q}_p$-analytic space associated to a Λ-adic domain Λ_1, we obtain a group homomorphism $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \Lambda_1^\times$ whose specialisation at x_f coincides with χ_f and a normalised cuspidal Λ_1-adic eigenform of level $\Gamma_1(N)$, character χ, and slope $< s$ whose specialisation at x_f coincides with f, because the slope of eigenforms is locally constant on the eigencurve.

We denote by $\Lambda_1\mathbb{T}_{N,\chi}^{<s}$ "the Λ_1-algebra topologically generated by Hecke operators and $p^s T_p^{-1}$ restricted to $\mathbb{M}(\Gamma_1(N), \chi, \Lambda_1)^{[<s]}$". For each normalised Λ_1-adic cuspidal eigenform f of level $\Gamma_1(N)$, character χ, and slope $< s$, we denote by $\mathbb{V}_f$ the quotient of $(\Lambda_1\mathbb{T}_{N,\chi}^{<s} \hat{\otimes}_{\Lambda_0\mathbb{T}_N^{<s}} \mathbb{H}(N)^{<s})\hat{\otimes}_{\Lambda_1}\Lambda_1(\chi)$ by the continuous Λ_1-algebra homomorphism $\Lambda_1\mathbb{T}_{N,\chi}^{<s} \twoheadrightarrow \Lambda_1$ associated to f. Then $\mathbb{V}_f$ naturally forms a profinite $\Lambda_1[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module, and hence the action $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \times \mathbb{V}_f \to \mathbb{V}_f$ is continuous.

Theorem 3.21 ([Mih15]). *Suppose* $p^s \mid N$*. For any normalised* Λ_1*-adic cuspidal eigenform* f *of level* $\Gamma_1(N)$ *and slope* $< s$*,* $\mathrm{Frac}(\Lambda_1) \otimes_{\Lambda_1} \mathbb{V}_f$ *is a 2-dimensional representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *over* $\mathrm{Frac}(\Lambda_1)$*. Furthermore, there is a finite subset* $\Sigma_s \subset$

$\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$ *satisfying the following for any normalised* Λ_1*-adic cuspidal eigenform* f *of level* $\Gamma_1(N)$*, character* χ*, and slope* $< s$*:*

(i) *For any* $\varphi \in \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s$*,* f_φ *is a normalised cuspidal eigenform of weight* $\mathrm{wt}(\varphi)$*, level* $\Gamma_1(N)$*, character* $\varphi \circ \chi$*, and slope* $< s$*.*

(ii) *For any* $\varphi \in \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s$*,* $\overline{\mathbb{Q}}_p(\varphi) \otimes_{\Lambda_1} \mathbb{V}_f$ *is isomorphic to* $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p(f_\varphi)} V_{f_\varphi}$*, where* $\overline{\mathbb{Q}}_p(\varphi)$ *denotes* $\overline{\mathbb{Q}}_p$ *regarded as a* Λ_1*-algebra through* φ*.*

(iii) *The* $\mathrm{C}(\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s, \overline{\mathbb{Q}}_p)[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$*-module* $\mathrm{C}\left(\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s, \overline{\mathbb{Q}}_p\right) \otimes_{\Lambda_1} \mathbb{V}_f$ *is free of rank* 2 *as a* $\mathrm{C}(\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)} \backslash \Sigma_s, \overline{\mathbb{Q}}_p)$*-module.*

Outline of the proof. The generic finiteness of $\mathbb{M}(\Gamma_1(N), \chi, \Lambda_1)^{<s}$ ensures the finiteness of the set of normalised Λ_1-adic cuspidal eigenforms of level $\Gamma_1(N)$, character χ, and slope $< s$. Therefore it suffices to verify the existence of Σ_s satisfying (x) for any single $x \in \{\mathrm{i}, \mathrm{ii}, \mathrm{iii}\}$ and any single normalised Λ_1-adic cuspidal eigenform f of level $\Gamma_1(N)$, character χ, and slope $< s$. The assertion (i) follows from the definition of a Λ_1-adic form. The assertion (ii) follows from the latter assertion of Theorem 3.15. The specialisations of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{V}_f$ at φ are of 2-dimension for all but finitely many $\varphi \in \Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$ by the assertion (ii), and it ensures that there is a $D \in \Lambda_1 \setminus \{0\}$ such that $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_1)[D^{-1}] \otimes_{\Lambda_1} \mathbb{V}_f$ is a free $(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_1)[D^{-1}]$-module of rank 2. Therefore $\mathrm{Frac}(\Lambda_1) \otimes_{\Lambda_1} \mathbb{V}_f$ is of dimension 2, and the assertion (iii) follows from the fact that every $D \in \Lambda_1 \setminus \{0\}$ has at most finitely many zeros on $\Omega(\Lambda_1)_{\mathbb{N}\cap[2,\infty)}$ by the definition of a Λ-adic domain. □

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References

[Ber90] V. G. Berkovich, *Spectral Theory and Analytic Geometry over non-Archimedean Fields*, Mathematical Surveys and Monographs, Number 33, the American Mathematical Society, 1990.

[Ber93] V. G. Berkovich, *Étale Cohomology for non-Archimedean Analytic Spaces*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, Number 1, pp. 5–161, 1993.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis A Systematic Approach to Rigid Analytic Geometry*, Grundlehren der Mathematischen Wissenschaften 261, A Series of Comprehensive Studies in Mathematics, Springer, 1984.

[CM98] R. F. Coleman and B. Mazur, *The Eigencurve*, Galois Representations in Arithmetic Algebraic Geometry, London Mathematical Society Lecture Note Series, Number 254, pp. 1–113, Cambridge University Press, 1998.

[Col97] R. F. Coleman, *p-adic Banach Spaces and Families of Modular Forms*, Inventiones Mathematicae, Volume 127, Issue 3, pp. 417–479, 1997.

[Del69] P. Deligne, *Formes Modulaires et Représentations ℓ-adiques*, Séminaire Bourbaki, Volume 11, Number 355, pp. 139–172, 1968–69.

[Eme] M. Emerton, *Completed cohomology and the p-adic Langlands Program*, preprint.

[Gro90] B. H. Gross, *A Tameness Criterion for Galois Representations Associated to Modular Forms (mod p)*, Duke Mathematical Journal, Volume 61, Number 2, pp. 445–517, 1990.

[GS93] R. Greenberg and G. Stevens, *p-adic L-functions and p-adic Periods of Modular Forms*, Inventiones Mathematicae, Volume 111, Issue 2, pp. 407–447, 1993.

[Hid86-1] H. Hida, *Iwasawa Modules Attached to Congruences of Cusp Forms*, Annales Scientifiques de l'École Normale Supérieure, Volume 19, Issue 2, pp. 231–273, 1986.

[Hid86-2] H. Hida, *Galois Representations into* $\mathrm{GL}_2(Z_p[[X]])$ *Attached to Ordinary Cusp Forms*, Inventiones Mathematicae, Volume 85, pp. 545–613, 1986.

[Hid93] H. Hida, *Elementary Theory of L-functions and Eisenstein Series*, Student Texts 26, London Mathematical Society, 1993.

[HT01] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Annals of Mathematics Studies, Volume 151, Princeton University Press, 2001.

[Mih15] T. Mihara, *A New Geometric Construction of a Family of Galois Representations Associated to Modular Forms*, preprint, 2015.

[Oht93] M. Ohta, *On Cohomology Groups Attached to Towers of Algebraic Curves*, Journal of the Mathematical Society of Japan, Volume 45, Number 1, 1993.

[PS11] R. Pollack and G. Stevens, *Overconvergent Modular Symbols and p-adic L-functions*, Annales Scientifiques de l'École Normale Supérieure, Volume 44, fascicule 1, pp. 1–42, 2011.

[ST02] P. Schneider and J. Teitelbaum, *Banach Space Representations and Iwasawa Theory*, Israel Journal of Mathematics, Volume 127, Issue 1, pp. 359–380, 2002.

[Wan98] D. Wan, *Dimension Variation of Classical and p-adic Modular Forms*, Inventiones Mathematicae, Volume 133, pp. 449–463, 1998.

[Wil88] A. Wiles, *On Ordinary λ-adic Representations Associated to Modular Forms*, Inventiones Mathematicae, Volume 94, pp. 529–573, 1988.

[Wil90] A. Wiles, *The Iwasawa Conjecture for Totally Real Fields*, Annals of Mathematics, Volume 131, Number 3, pp. 493–540, 1990.

[Woj12] Z. Wojtkowiak, *On ℓ-adic Iterated Integrals V: Linear Independence, Properties of ℓ-adic Polylogarithms, ℓ-adic Sheaves*, The Arithmetic of Fundamental Groups, Contributions in Mathematical and Computational Sciences, Volume 2, pp. 339–374, 2012.

[Yam07] A. Yamagami, *On p-adic families of Hilbert cusp forms of finite slope*, Journal of Number Theory, Volume 123, Issue 2, pp. 363–387, 2007.