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The local and global ε-conjectures for the rank two case

By

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Abstract

This is a research anouncement of the author's article "Local ε-isomorphisms for rank two p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a functional equation of Kato's Euler system" (preprint, arXiv:1502.04924). In §1, we first briefly recall the formulation of the local ε-conjecture, and state our theorem concerning this conjecture for the rank two case. In §2, we first briefly recall the generalized Iwasawa main conjecture and the global ε-conjecture, and state our theorem concerning the global ε-conjecture for the rank two case.

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§ 1. Introduction

In his celebrated article [Ka04] on the Iwasawa main conjecture for modular forms, Kato constructed an Euler system associated to any elliptic cuspidal Hecke eigen nor-

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malized new form $f = \sum_{n\geqq 1} a_n q^n$. An Euler system for f is a compatible system of elements of Galois cohomologies of the p-adic representations $T_f|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))}$, where T_f is the integral p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to f. His Euler system is crucial for the (generalized) Iwasawa main conjecture for f since it connects the Galois cohomologies (e.g. Selmer groups) with the (p-adic) L-function associated to $f^* = \sum_{n\geqq 1} \bar{a}_n q^n$, where $\bar{a}_n$ is the complex conjugate of a_n. By this property, his Euler system is considered to be an incarnation (called the zeta element) of the L-function in the world of Galois cohomology. As a vast generalization of this picture, he [Ka93a] formulated a conjecture called the generalized Iwasawa main conjecture, which predicts that such zeta elements interpolating the L-functions of motives exist for all the families of p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which are unramified outside a finite set of primes. Since the zeta elements are incarnations of the L-functions, many properties satisfied by the L-functions are expected to have the corresponding ones for the zeta elements. In [Ka93b], he formulated a conjecture called the global ε-conjecture concerning the functional equation satisfied by the zeta elements. To formulate this conjecture, he also formulated another conjecture called the local ε-conjecture, which predicts the existence of incarnations (called the local ε-isomorphisms) of the local ε-factors for all the families of p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ for any prime l.

In this research anouncement, we briefly recall the formulations of these conjectures and explain our results concerning these conjectures for the rank two case. In particular, we explain our theorem concerning the global ε-conjecture for the cyclotomic deformation of T_f for f as above, which we call a functional equation of Kato's Euler system in [Na2].

Notation

Here, we fix notations which we use throughout this article. For a field F, let G_F be the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ of F. For each $m \in \mathbb{Z}_{\geqq 0}$, let $\mu_m(F)$ be the group of the m-th roots of unity in F. For each prime p, we set $\Gamma(F, \mathbb{Z}_p(1)) := \varprojlim_{n\geqq 1} \mu_{p^n}(F)$. For $F = \mathbb{Q}_p$, let $W_{\mathbb{Q}_p} \subseteq G_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$, $I_p \subseteq W_{\mathbb{Q}_p}$ the inertia subgroup. Let $\mathrm{rec}_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \xrightarrow{\sim} W_{\mathbb{Q}_p}^{\mathrm{ab}}$ be the reciprocity map of the local class field theory normalized so that $\mathrm{rec}_{\mathbb{Q}_p}(p)$ is a lift of the geometric Frobenius $\mathrm{Fr}_p \in G_{\mathbb{F}_p}$. Set $\Gamma := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. Let $\chi \colon \Gamma \xrightarrow{\sim} \mathbb{Z}_p^\times$ be the p-adic cyclotomic character, which we also see as a character of $G_{\mathbb{Q}_p}$. Set $H_{\mathbb{Q}_p} := \mathrm{Ker}(\chi) \subseteq G_{\mathbb{Q}_p}$. For each $b \in \mathbb{Z}_p^\times$, define $\sigma_b \in \Gamma$ by $\chi(\sigma_b) = b$. For a perfect ring R of characteristic p, we denote by $W(R)$ the ring of Witt vectors, on which the lift φ of the p-th power Frobenius on R acts. Let $[-] \colon R \to W(R)$ be the Teichmüller lift. Let $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}}_p$ be the p-adic completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, $\mathcal{O}_{\mathbb{C}_p}$ the ring of integres of $\mathbb{C}_p$. We define $\widetilde{\mathbf{E}}^+ := \varprojlim_{n\geqq 0} \mathcal{O}_{\mathbb{C}_p}/p$ to be the projective limit with respect to p-th power map. We set $\widetilde{\mathbf{E}} := \mathrm{Frac}(\widetilde{\mathbf{E}}^+)$, $\widetilde{\mathbf{A}}^+ := W(\widetilde{\mathbf{E}}^+)$,

$\widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}})$, $\widetilde{\mathbf{B}}^+ := \widetilde{\mathbf{A}}^+[1/p]$, and $\widetilde{\mathbf{B}} := \widetilde{\mathbf{A}}[1/p]$. Let $\theta\colon \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ be the continuous $\mathbb{Z}_p$-algebra homomorphism defined by $\theta([(\bar{x}_n)_{n\geq 0}]) := \lim_{n\to\infty} x_n^{p^n}$ for any $(\bar{x}_n)_{n\geq 0} \in \widetilde{\mathbf{E}}^+$, where $x_n \in \mathcal{O}_{\mathbb{C}_p}$ is a lift of $\bar{x}_n \in \mathcal{O}_{\mathbb{C}_p}/p$. Set $\mathbf{B}^+_{\mathrm{dR}} := \varprojlim_{n\geq 1} \left(\widetilde{\mathbf{A}}^+/\mathrm{Ker}(\theta)^n[1/p]\right)$. For each $\mathbb{Z}_p$-basis $\zeta = (\zeta_{p^n})_{n\geq 0} \in \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_p(1))$, we set $t_\zeta := \log([(\bar{\zeta}_{p^n})_{n\geq 0}]) \in \mathbf{B}^+_{\mathrm{dR}}$, which is a uniformizer of $\mathbf{B}^+_{\mathrm{dR}}$. Set $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}^+_{\mathrm{dR}}[1/t_\zeta]$.

§ 2. The local ε-conjecture for the rank two case

§ 2.1. Review of the local ε-conjecture

In this subsection, we briefly recall the formulation of the local ε-conjecture. See the original articles [Ka93b], [FK06] (the latter one includes the non-commutative version) or [Na1] for more details.

The local ε-conjecture is formulated by using the theory of the determinant functor. In our work, we use the Knudsen-Mumford's one [KM76], which we briefly recall here (see also §3.1 of [Na1]).

Let R be a commutative ring. We denote by $\mathbf{D}^-(R)$ the derived category of bounded below complexes of R-modules, by $\mathbf{D}_{\mathrm{perf}}(R)$ the full subcategory of perfect complexes of R-modules, i.e. consisting of the complexes which are quasi-isomorphic to some complexes $P^\bullet$ such that $P^{\pm i} = 0$ for any sufficiently large $i > 0$ and P^i are finite projective for any $i \in \mathbb{Z}$. We denote by $\mathbf{P}_{\mathrm{fg}}(R)$ the category of finite projective R-modules. For any $P \in \mathbf{P}_{\mathrm{fg}}(R)$, we denote by r_P its R-rank, by $P^\vee := \mathrm{Hom}_R(P, R)$ its dual. We define a category $\mathcal{P}_R$ as follows. The objects are the pairs (L, r) where L is an invertible R-module and $r : \mathrm{Spec}(R) \to \mathbb{Z}$ is a locally constant function. The morphisms are defined by $\mathrm{Mor}_{\mathcal{P}_R}((L, r), (M, s)) := \mathrm{Isom}_R(L, M)$ if $r = s$, or empty otherwise. We call objects of this category graded invertible R-modules. For $(L, r), (M, s)$, define its product by $(L, r) \boxtimes (M, s) := (L \otimes_R M, r + s)$. The unit object for the product is $\mathbf{1}_R := (R, 0)$. For each (L, r), we define $(L, r)^{-1} := (L^\vee, -r)$ to be the inverse of (L, r) by the canonical isomorphism $i_{(L,r)}\colon (L, r) \boxtimes (L^\vee, -r) \xrightarrow{\sim} \mathbf{1}_R$. For a category $\mathcal{C}$, denote by $(\mathcal{C}, \mathrm{is})$ the category such that the objects are the same as $\mathcal{C}$ and the morphisms are all the isomorphisms in $\mathcal{C}$. Define a functor Det_R by

$$\mathrm{Det}_R\colon (\mathbf{P}_{\mathrm{fg}}(R), \mathrm{is}) \to \mathcal{P}_R : P \mapsto \mathrm{Det}_R(P) := (\det_R P, r_P),$$

where we set $\det_R P := \wedge_R^{r_P} P$. For a short exact sequence $0 \to P_1 \to P_2 \to P_3 \to 0$ in $\mathbf{P}_{\mathrm{fg}}(R)$, we always identify $\mathrm{Det}_R(P_1) \boxtimes \mathrm{Det}_R(P_3)$ with $\mathrm{Det}_R(P_2)$ by the following functorial isomorphism (put $r_i := r_{P_i}$)

$$\mathrm{Det}_R(P_1) \boxtimes \mathrm{Det}_R(P_3) \xrightarrow{\sim} \mathrm{Det}_R(P_2) \tag{2.1}$$

induced by

$$(x_1 \wedge \cdots \wedge x_{r_1}) \otimes (\overline{x_{r_1+1}} \wedge \cdots \wedge \overline{x_{r_2}}) \mapsto x_1 \wedge \cdots \wedge x_{r_1} \wedge x_{r_1+1} \wedge \cdots \wedge x_{r_2}$$

where $x_1, \cdots, x_{r_1}$ (resp. $\overline{x_{r_1+1}}, \cdots, \overline{x_{r_2}}$) are local sections of P_1 (resp. P_3) and $x_i \in P_2$ ($i = r_1+1, \cdots, r_2$) is a lift of $\overline{x_i} \in P_3$. For a bounded complex $P^\bullet$ in $\mathbf{P}_{\mathrm{fg}}(R)$, define $\mathrm{Det}_R(P^\bullet) \in \mathcal{P}_R$ by

$$\mathrm{Det}_R(P^\bullet) := \boxtimes_{i\in\mathbb{Z}} \mathrm{Det}_R(P^i)^{(-1)^i}.$$

By the result of [KM76], Det_R naturally extends to a functor

$$\mathrm{Det}_R \colon (\mathbf{D}_{\mathrm{perf}}(R), \mathrm{is}) \to \mathcal{P}_R$$

which we call the determinant functor.

Now, we start to recall the local ε-conjecture. Fix a prime p. In this article, we assume that $p \neq 2$ for simplicity (we can obtain similar results when $p = 2$, see [Na2]).

From now on until the end of the article, we use the notation R to represent a commutative topological $\mathbb{Z}_p$-algebra satisfying one of the following conditions (i) and (ii).

(i) R is a $\mathrm{Jac}(R)$-adically complete noetherian semi-local ring equipped with the $\mathrm{Jac}(R)$-adic topology such that $R/\mathrm{Jac}(R)$ is a finite ring, where $\mathrm{Jac}(R)$ is the Jacobson radical of R,

(ii) R is a finite extension of $\mathbb{Q}_p$, which is equipped with the p-adic topology.

We note that a ring R (satisfying (i) or (ii)) satisfies (i) if and only if $p \notin R^\times$. We use the notation L instead of R if we consider the case where the condition (ii) is satisfied.

In this article, we mainly treat representations (of $G_{\mathbb{Q}_p}$ or $\mathrm{GL}_2(\mathbb{Q}_p)$, etc.) defined over such a ring R. Let G be a topological group. We say that T is an R-representation of G if T is a finite projective R-module with a continuous R-linear G-action. For a continuous homomorphism $\delta \colon G \to R^\times$, we define $R(\delta) := R\mathbf{e}_\delta$ to be the R-representation of rank one with a fixed basis $\mathbf{e}_\delta$ on which G acts by $g(\mathbf{e}_\delta) := \delta(g)\mathbf{e}_\delta$. For an R-representation T of G, we set $T(\delta) := T \otimes_R R(\delta)$ for any $\delta \colon G \to R^\times$, and denote by $\mathrm{C}^\bullet_{\mathrm{cont}}(G, T)$ the complex of continuous cochains of G with its values in T, i.e. defined by $\mathrm{C}^i_{\mathrm{cont}}(G, T) := \{c \colon G^{\times i} \to T\colon \text{ continuous maps } \}$ for each $i \geqq 0$ with the usual boundary map. We also regard $\mathrm{C}^\bullet_{\mathrm{cont}}(G, T)$ as an object of $\mathbf{D}^-(R)$.

Now, we fix another prime l (we consider the both $l \neq p$ and $l = p$ cases). Let T be an R-representation of $G_{\mathbb{Q}_l}$. We set $\mathrm{H}^i(\mathbb{Q}_l, T) := \mathrm{H}^i(\mathrm{C}^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T))$. For each $r \in \mathbb{Z}$, we set $T(r) := T \otimes_{\mathbb{Z}_p} \Gamma(\overline{\mathbb{Q}}_l, \mathbb{Z}_p(1))^{\otimes r}$. We denote by $T^* := T^\vee(1)$ the Tate dual of T. By the classical theory of the Galois cohomology of local fields, it is known that one has $\mathrm{C}^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T) \in \mathbf{D}_{\mathrm{perf}}(R)$. Using the determinant functor, we define an invertible R-module $\Delta_{R,1}(T)$ by

$$\Delta_{R,1}(T) := \mathrm{Det}_R(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T)),$$

which is of degree $-r_T$ (resp. of degree 0) when $l = p$ (resp. when $l \neq p$) by the Euler-Poincaré formula. In particular, when $l = p$, $\Delta_{R,1}(T)$ is in general not of degree zero, which means that we can't take any canonical basis of $\Delta_{R,1}(T)$ (e.g. for any $a \in R^\times$, the isomorphism $T \xrightarrow{\sim} T : x \mapsto ax$ induces an isomorphism $\Delta_{R,1}(T) \xrightarrow{\sim} \Delta_{R,1}(T) : x \mapsto a^r x$).

To obtain an invertible module of degree zero, we need another kind of invetible R-module $\Delta_{R,2}(T)$ defined as follows. For $a \in R^\times$ ($a \in \mathcal{O}_L^\times$ if $R = L$), we set

$$R_a := \{x \in W(\overline{\mathbb{F}}_p)\hat{\otimes}_{\mathbb{Z}_p} R | (\varphi \otimes \mathrm{id}_R)(x) = (1 \otimes a) \cdot x\}.$$

Using the fact that $\mathrm{H}^1(G_{\mathbb{F}_p}, \overline{\mathbb{F}}_p^\times) = \mathrm{H}^1(G_{\mathbb{F}_p}, \overline{\mathbb{F}}_p) = 0$, we can easily show that R_a is an invertible R-module. For T as above, we freely regard $\det_R T$ as a continuous homomorphism $\det_R T \colon G^{\mathrm{ab}}_{\mathbb{Q}_l} \to R^\times$. We define a constant $a_l(T) \in R^\times$ by

$$a_l(T) := \det{}_R T(\mathrm{rec}_{\mathbb{Q}_l}(p)),$$

and define an invertible F-module $\Delta_{R,2}(T)$ by

$$\Delta_{R,2}(T) := \begin{cases} (R_{a_l(T)}, 0) & (l \neq p) \\ (\det_R T \otimes_R R_{a_p(T)}, r_T) & (l = p). \end{cases}$$

Finally, we set

$$\Delta_R(T) := \Delta_{R,1}(T) \boxtimes \Delta_{R,2}(T),$$

which we call the local fundamental line.

We remark that the local fundamental line is of degree zero, and is compatible with functorial operations as follows. For any $R \to R'$, one has a canonical isomorphism

$$\Delta_R(T) \otimes_R R' \xrightarrow{\sim} \Delta_{R'}(T \otimes_R R'). \tag{2.2}$$

For any exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of R-representations of $G_{\mathbb{Q}_l}$, one has a canonical isomorphism

$$\Delta_R(T_2) \xrightarrow{\sim} \Delta_R(T_1) \boxtimes \Delta_R(T_3). \tag{2.3}$$

One has the following canonical isomorphism

$$\Delta_R(T) \xrightarrow{\sim} \begin{cases} \Delta_R(T^*)^\vee & (l \neq p) \\ \Delta_R(T^*)^\vee \boxtimes (R(r_T), 0) & (l = p) \end{cases} \tag{2.4}$$

which is induced by the Tate duality $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T^*), R)[-2]$ (see §2.1 of [Na2] for the precise definition).

The local ε-conjecture is concerned with the existence of a compatible family of trivializations called the local ε-isomorphisms

$$\varepsilon_{R,\zeta}(T)\colon \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$$

for all the pairs (R, T) as above, which interpolates the given trivializations

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)\colon \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$$

called the de Rham ε-isomorphisms.

The de Rham ε-isomorphism $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)$ is defined for all the pairs $(L, V) = (R, T)$ such that V is de Rham (resp. arbitrary) if $l = p$ (resp. if $l \neq p$), whose definition we briefly recall now. Here, $\zeta := \{\zeta_{l^n}\}_{n \geqq 0}$ is a fixed system of primitive l^n-th root of unity $\zeta_{l^n} \in \overline{\mathbb{Q}}_p^\times$ such that $\zeta_{l^{n+1}}^l = \zeta_{l^n}$ for any $n \geqq 1$. As explained below, this additional data ζ plays a similar role as a fixed additive character in the classical theory of local constants.

To recall the definition of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)\colon \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$, we first recall the ε-constants defined for Weil-Deligne representations of the Weil group $W_{\mathbb{Q}_l}$ of $\mathbb{Q}_l$. Let K be a field of characteristic zero which contains all the l-power roots of unity. For a $\mathbb{Z}_l$-basis $\zeta = \{\zeta_{l^n}\}_{n \geqq 0} \in \Gamma(K, \mathbb{Z}_l(1)) := \varprojlim_{n \geqq 0} \mu_{l^n}(K)$, we define an additive character

$$\psi_\zeta\colon \mathbb{Q}_l \to K^\times$$

by

$$\psi_\zeta\left(\frac{1}{l^n}\right) := \zeta_{l^n}$$

for any $n \geqq 0$. By the theory of local constants [De73], one can attach a constant

$$\varepsilon(\rho, \psi, dx) \in K^\times$$

to any smooth K-representation $\rho = (M, \rho)$ of $W_{\mathbb{Q}_l}$ (i.e. M is a finite dimensional K-vector space with a K-linear smooth action ρ of $W_{\mathbb{Q}_l}$), which depends on the choices of an additive character $\psi\colon \mathbb{Q}_l \to K^\times$ and a Haar measure dx (with its values in K) on $\mathbb{Q}_l$. In this article, we consider this constant only for the pair (ψ_ζ, dx) such that $\int_{\mathbb{Z}_l} dx = 1$, which we denote by

$$\varepsilon(\rho, \zeta) := \varepsilon(\rho, \psi_\zeta, dx)$$

for simplicity. For a Weil-Deligne representation $M = (M, \rho, N)$ of $W_{\mathbb{Q}_l}$ over K (i.e. $\rho := (M, \rho)$ is a smooth K-representation of $W_{\mathbb{Q}_l}$ with a K-linear endomorphism N :

$M \to M$ such that $\widetilde{\mathrm{Fr}_l} \circ N = l^{-1} \cdot N \circ \widetilde{\mathrm{Fr}_l}$ for any lift $\widetilde{\mathrm{Fr}_l} \in W_{\mathbb{Q}_l}$ of the geometric Frobenius $\mathrm{Fr}_l \in G_{\mathbb{F}_l}$), its ε-constant is defined by

$$\varepsilon(M, \zeta) := \varepsilon(\rho, \zeta) \cdot \det_K(-\mathrm{Fr}_l | M^{I_l}/(M^{N=0})^{I_l}).$$

Now, we recall the definition of de Rham ε-isomorphism $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V) \colon \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$ for any (resp. de Rham) L-representation V of $G_{\mathbb{Q}_l}$ when $l \neq p$ (resp. $l = p$). Fix a $\mathbb{Z}_l$-basis $\zeta = \{\zeta_{l^n}\}_{n \geqq 0} \in \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1))$. By the Grothendieck's local monodromy theorem (resp. the p-adic local monodromy theorem and the Fontaine's functor $\mathbf{D}_{\mathrm{pst}}(-)$) when $l \neq p$ (resp. $l = p$), one can functorially define a Weil-Deligne representation

$$W(V) = (W(V), \rho, N)$$

of $W_{\mathbb{Q}_l}$ over L. We set $L_\infty := L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{l^\infty})$, and decompose it $L_\infty = \prod_\tau L_\tau$ into the product of fields L_τ. Then, we define a constant

$$\varepsilon_L(W(V), \zeta) \in L_\infty^\times$$

as the product of the ε-constants $\varepsilon(W(V)_\tau, \zeta_\tau) \in L_\tau^\times$ of $W(V)_\tau := W(V) \otimes_L L_\tau$ for all τ, where $\zeta_\tau \in \Gamma(L_\tau, \mathbb{Z}_l(1))$ is the image of ζ in L_τ. Set

$$\mathbf{D}_{\mathrm{st}}(V) := W(V)^{I_l}, \mathbf{D}_{\mathrm{cris}}(V) := \mathbf{D}_{\mathrm{st}}(V)^{N=0}$$

on which the Frobenius $\varphi_l := \mathrm{Fr}_l$ naturally acts. Note that one has $\mathbf{D}_{\mathrm{cris}}(V) = V^{I_l}$ if $l \neq p$. We set

$$\mathbf{D}_{\mathrm{dR}}(V) := (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}, \mathbf{D}^i_{\mathrm{dR}}(V) := (t^i \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}} \text{ and } t_V := \mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}^0_{\mathrm{dR}}(V)$$

(resp. $\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}^i_{\mathrm{dR}}(V) = t_V := 0$) when $l = p$ (resp. $l \neq p$).

Using these preliminaries, we first define an isomorphism

$$\theta_L(V) \colon \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(V) \boxtimes \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(V))$$

as that naturally induced by the following exact sequence of L-vector spaces

$$\begin{aligned}(2.5) \quad 0 \to \mathrm{H}^0(\mathbb{Q}_l, V) \to \mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{(a)} \mathbf{D}_{\mathrm{cris}}(V) \oplus t_V \xrightarrow{(b)} \mathrm{H}^1(\mathbb{Q}_l, V) \\ \xrightarrow{(c)} \mathbf{D}_{\mathrm{cris}}(V^*)^\vee \oplus \mathbf{D}^0_{\mathrm{dR}}(V) \xrightarrow{(d)} \mathbf{D}_{\mathrm{cris}}(V^*)^\vee \to \mathrm{H}^2(\mathbb{Q}_l, V) \to 0,\end{aligned}$$

where the map (a) is the sum of $1 - \varphi_l \colon \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{cris}}(V)$ and the canonical map $\mathbf{D}_{\mathrm{cris}}(V) \to t_V$, and the map (d) is the dual of (a) for V^*. When $l = p$, the maps (b) and (c) are respectively defined by using Bloch-Kato's exponentials

$$\exp \colon t_V \to \mathrm{H}^1(\mathbb{Q}_p.V) \quad \text{and} \quad \exp_f \colon \mathbf{D}_{\mathrm{cris}}(V) \to \mathrm{H}^1(\mathbb{Q}_p.V)$$

and its duals for V^*

$$\exp^* := (\exp)^\vee\colon \mathrm{H}^1(\mathbb{Q}_p, V) \to \mathbf{D}^0_{\mathrm{dR}}(V) \quad \text{and} \quad (\exp_f)^\vee\colon \mathrm{H}^1(\mathbb{Q}_p, V) \to \mathbf{D}_{\mathrm{cris}}(V^*)^\vee$$

(see [Ka93b], [FK06] and [Na1] for the precise definition). We remark that the determinant of $1-\varphi_l\colon \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{cris}}(V)$ is nothing else than the L-constant associated to V, hence the isomorphism $\theta_L(V)$ contains information on the L-constants associated to V and V^*, and the determinants of Bloch-Kato's exponentials.

Define a constant $\Gamma_L(V) \in \mathbb{Q}^\times$ by

$$\Gamma_L(V) := \begin{cases} 1 & (l \neq p) \\ \prod_{r\in\mathbb{Z}} \Gamma^*(r)^{-\dim_L \mathrm{gr}^{-r}\mathbf{D}_{\mathrm{dR}}(V)} & (l = p), \end{cases}$$

where we set

$$\Gamma^*(r) := \begin{cases} (r-1)! & (r \geqq 1) \\ \frac{(-1)^r}{(-r)!} & (r \leqq 0). \end{cases}$$

We next define an isomorphism

$$\theta_{\mathrm{dR},L}(V,\zeta) : \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(V)) \xrightarrow{\sim} \Delta_{L,2}(V)$$

as that naturally induced by the isomorphism

$$\det{}_L \mathbf{D}_{\mathrm{dR}}(V) = L \xrightarrow{\sim} L_{a_l(V)} : x \mapsto \varepsilon_L(W(V),\zeta)\cdot x$$

when $l \neq p$ (we remark that one has $\varepsilon_L(W(V),\zeta) \in L_{a_l(V)}$ when $l \neq p$), by the inverse of the isomorphism

$$L_{a_p(T)} \otimes_L \det{}_L V \xrightarrow{\sim} \det{}_L \mathbf{D}_{\mathrm{dR}}(V) (\subseteq \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \det{}_L V)\colon x \mapsto \frac{1}{\varepsilon_L(W(V),\zeta)} \cdot \frac{1}{t_\zeta^{h_V}} \cdot x$$

when $l = p$, where we set $h_V := \sum_{r\in\mathbb{Z}} r \cdot \dim_L \mathrm{gr}^{-r}\mathbf{D}_{\mathrm{dR}}(V)$. Finally, we define the de Rham ε-isomorphism

$$\varepsilon^{\mathrm{dR}}_{L,\zeta}(V)\colon \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$$

as the following composites

$$\begin{aligned} \varepsilon^{\mathrm{dR}}_{L,\zeta}(V)\colon \mathbf{1}_L &\xrightarrow{\Gamma_L(V)\cdot\theta_L(V)} \Delta_{L,1}(V) \boxtimes \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(V)) \\ &\xrightarrow{\mathrm{id}\boxtimes\theta_{\mathrm{dR},L}(V,\zeta)} \Delta_{L,1}(V) \boxtimes \Delta_{L,2}(V) = \Delta_L(V). \end{aligned}$$

The local ε-conjecture (Conjecture 1.8 [Ka93b], Conjecture 3.4.3 [FK06], and Conjecture 3.8 [Na1]) is the following, which says that the de Rham ε-isomorphisms can be p-adically interpolated to all the families of p-adic representations, which is now a theorem when $l \neq p$ by [Ya09].

Conjecture 2.1. Fix a prime l. Then, there exists a unique compatible family of isomorphisms

$$\varepsilon_{R,\zeta}(T)\colon \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$$

for all the triples (R,T,ζ) such that T is an R-representation of $G_{\mathbb{Q}_l}$ and ζ is a $\mathbb{Z}_l$-basis of $\Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1))$, which satisfies the following properties.

(1) For any continuous $\mathbb{Z}_p$-algebra homomorphism $R \to R'$, one has

$$\varepsilon_{R,\zeta}(T) \otimes \mathrm{id}_{R'} = \varepsilon_{R',\zeta}(T \otimes_R R')$$

under the canonical isomorphism (2.2).

(2) For any exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of R-representations of $G_{\mathbb{Q}_l}$, one has

$$\varepsilon_{R,\zeta}(T_2) = \varepsilon_{R,\zeta}(T_1) \boxtimes \varepsilon_{R,\zeta}(T_3)$$

under the canonical isomorphism (2.3).

(3) For each $a \in \mathbb{Z}_l^\times$, one has

$$\varepsilon_{R,\zeta^a}(T) = \det{}_R T(\mathrm{rec}_{\mathbb{Q}_l}(a)) \cdot \varepsilon_{R,\zeta}(T).$$

(4) One has the following commutative diagrams

$$\begin{tikzcd}
\Delta_R(T) \arrow[r, "(2.4)"] & \Delta_R(T^*)^\vee \arrow[d, "\varepsilon_{R,\zeta}(T^*)^\vee"] \\
\mathbf{1}_R \arrow[u, "\varepsilon_{R,\zeta^{-1}}(T)"] \arrow[r, "\mathrm{id}"] & \mathbf{1}_R
\end{tikzcd}$$

when $l \neq p$, and

$$\begin{tikzcd}
\Delta_R(T) \arrow[r, "(2.4)"] & \Delta_R(T^*)^\vee \boxtimes (R(r_T), 0) \arrow[d, "\varepsilon_{R,\zeta}(T^*)^\vee \boxtimes [\mathbf{e}_{r_T} \mapsto 1]"] \\
\mathbf{1}_R \arrow[u, "\varepsilon_{R,\zeta^{-1}}(T)"] \arrow[r, "\mathrm{can}"] & \mathbf{1}_R \boxtimes \mathbf{1}_R
\end{tikzcd}$$

when $l = p$.

(5) For any triple (L, V, ζ) such that V is arbitrary (resp. de Rham) if $l \neq p$ (resp. if $l = p$), one has

$$\varepsilon_{L,\zeta}(V) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(V).$$

Remark 2.2.

(1) There is a non-commutative coefficient version of this conjecture, but we only consider the commutative case in this article. See [FK06] for the non-commutative version.

(2) When $l \neq p$, this conjecture has been already proved by Yasuda [Ya09]. As we recalled above, when $l \neq p$, one can define the ε-constant for any L-representation of $G_{\mathbb{Q}_l}$. Hence, the uniqueness of local ε-isomorphims seems to be trivial, and the conjecture is essentially reduced to the problem on the continuity of ε-constants when p-adic representations vary in p-adic families. In fact, Yasuda proved that the correspondence

$$(L, V, \zeta) \mapsto \varepsilon_{0,L}(V, \zeta) := \det_L(-\varphi_l | V^{I_l}) \cdot \varepsilon_L(W(V), \zeta) \in L_{a_l(V)}$$

defined for all the triples (L, V, ζ) as in the condition (5) (for $l \neq p$) in Conjecture 2.1 uniquely extends to a correspondence

$$(R, T, \zeta) \mapsto \varepsilon_{0,R}(T, \zeta) \in R_{a_l(T)}$$

for all the triples (R, T, ζ) as in the conjecture.

(3) When $l = p$, this conjecture is much more difficult and deeper than that for $l \neq p$ by the following two reasons. The first reason is that there are many L-representations of $G_{\mathbb{Q}_p}$ which are not de Rham, for which one cannot define the L-and ε-constants. Therefore, even the uniqueness of the local ε-isomorphisms never be trivial, but it seems to be reasonable to expect this uniqueness since it is expected that the set of points corresponding to de Rham representations is Zariski dense in any universal deformation space (see [Na14a]). The second reason is that the de Rham ε-isomorphisms for de Rham representations are much more complicated than that for $l \neq p$ since Bloch-Kato exponentials appear in the definition. Bloch-Kato exponentials are in general difficult to explicitly describe, but is very important in number theory since (the duals of) these relate the zeta element to the special value of the associated L-function. Up to now, the conjecture for $l = p$ has been proved only in some special cases before the article [Na2]. For the rank one case, it was proved by Kato [Ka93b] (see also [Ve13]) using Coleman homomorphism. For the cyclotomic deformation (or a more general twists of) crystalline representations, it was proved by Benois-Berger [BB08] and Loeffler-Venjakob-Zerbes [LVZ13] using Perrin-Riou exponential map and the theory of Wach modules. For the trianguline case, it was proved by the author in the previous article [Na1], where we generalized the conjecture for rigid analytic families of (φ, Γ)-modules over the Robba ring, and proved the generalized version of the conjecture for trianguline case which essentially includes all the known cases [Ka93b], [BB08], [LVZ13], [Ve13].

Before proceeding to the next subsection, we recall the notion of cyclotomic deformation (see §2.1 [Na2]), which we need to explain the idea of the proof of the theorem in the next subsection, and (for the convenience for the readers) then roughly explain the construction of local ε-isomorphism for a rank one case. For R such that $p \notin R^{\times}$, we define $\Lambda_R(\Gamma) := \varprojlim_{\Gamma' \subseteq \Gamma} R[\Gamma/\Gamma']$ to be the projective limit with respect to open subgroups Γ' of Γ. For $R = L$, we set $\Lambda_L(\Gamma) := \Lambda_{\mathcal{O}_L}(\Gamma)[1/p]$. Denote by $[-] : \Gamma \to \Lambda_R(\Gamma)^{\times}$ the natural homomorphism defined as the limit of $\Gamma \to R[\Gamma/\Gamma']^{\times} : \gamma \mapsto [\bar{\gamma}]$. For an R-representation T of $G_{\mathbb{Q}_l}$, we define a $\Lambda_R(\Gamma)$-representation $\mathbf{Dfm}(T)$ called the cyclotomic deformation of T by

$$\mathbf{Dfm}(T) := T \otimes_R \Lambda_R(\Gamma),$$

on which $G_{\mathbb{Q}_l}$ acts by

$$g(x \otimes \lambda) := g(x) \otimes [\bar{g}]^{-1} \cdot \lambda,$$

where $\bar{g} \in \Gamma$ is the image of $g \in G_{\mathbb{Q}_l}$. For an R-representation T of $G_{\mathbb{Q}_l}$, we set

$$\Delta_R^{\mathrm{Iw}}(T) := \Delta_{\Lambda_R(\Gamma)}(\mathbf{Dfm}(T)),\ \mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Q}_l, T) := \mathrm{H}^i(\mathbb{Q}_l, \mathbf{Dfm}(T)),$$

and set (conjecturally)

$$\varepsilon^{\mathrm{Iw}}_{R,\zeta}(T) := \varepsilon_{\Lambda_R(\Gamma),\zeta}(\mathbf{Dfm}(T))$$

(precisely, when $R = L$, the ring $\Lambda_L(\Gamma)$ satisfies neither the condition (i) nor the condition (ii) in page 4, in this case, we define $\Delta_L^{\mathrm{Iw}}(T)$, etc. by $\Delta_L^{\mathrm{Iw}}(T) := \Delta^{\mathrm{Iw}}_{\mathcal{O}_L}(T_0) \otimes_{\mathcal{O}_L} L$, etc. , where T_0 is a $G_{\mathbb{Q}_l}$ stable $\mathcal{O}_L$-lattice in T). Define an involution $\eta \colon \Lambda_R(\Gamma) \xrightarrow{\sim} \Lambda_R(\Gamma)$ by $[\gamma] \mapsto [\gamma]^{-1}$. For a $\Lambda_R(\Gamma)$-module M, we set $M^{\eta} := M \otimes_{\Lambda_R(\Gamma),\eta} \Lambda_R(\Gamma)$. Then, one has a canonical isomorphism

$$\Delta_R^{\mathrm{Iw}}(T^*)^{\eta} \xrightarrow{\sim} \Delta_{\Lambda_R(\Gamma)}((\mathbf{Dfm}(T))^*).$$

Now, for the convenience of readers, we very roughly explain the construction of the local ε-isomorhism for $\mathbf{Dfm}(\mathbb{Z}_p(1))$. See [Ka93b] or [Na1] for more details. In this case, the local ε-isomorhism is essentially the classical Coleman homomorphism, and is intimately related with the construction of the p-adic L-function from the norm compatible system of cyclotomic units, which we briefly recall now. We have a canonical $\Lambda_{\mathbb{Z}_p}(\Gamma)$-equivariant injection $\varprojlim_n (\mathbb{Z}_p[\zeta_{p^n}]^{\times})^{\wedge} \hookrightarrow \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, \mathbb{Z}_p(1))$ by Kummer theory ($(-)^{\wedge}$ is the p-adic completion of $(-)$), it is known that its cokernel and $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q}_p, \mathbb{Z}_p(1))$ are very small (both are isomorpchic to $\mathbb{Z}_p$ as $\Lambda_{\mathbb{Z}_p}(\Gamma)$-module), and $\mathrm{H}^0_{\mathrm{Iw}}(\mathbb{Q}_p, \mathbb{Z}_p(1))$ is zero (we remark that $\mathrm{H}^0_{\mathrm{Iw}}(\mathbb{Q}_p, T)$ is always zero for any T). Hence, the problem is reduced to construct a suitable $\Lambda_{\mathbb{Z}_p}(\Gamma)$-equivariant morphism $\varprojlim_n (\mathbb{Z}_p[\zeta_{p^n}]^{\times})^{\wedge} \to \Lambda_{\mathbb{Z}_p}(\Gamma)$. For any $u = (u_n)_{n \geqq 1} \in \varprojlim_n (\mathbb{Z}_p[\zeta_{p^n}]^{\times})^{\wedge}$, Coleman defined an element $g_u \in \mathbb{Z}_p[[T]]^{\times}$ characterized by $N(g_u) = g_u$ and $g_u(\zeta_{p^n} - 1) = u_n$ for any $n \geqq 1$, where $N : \mathbb{Z}_p[[T]]^{\times} \to \mathbb{Z}_p[[T]]^{\times}$ is the norm with respect to the finite flat morphism $\mathbb{Z}_p[[T]] \to \mathbb{Z}_p[[T]] : f(T) \mapsto f((1+T)^p - 1)$. Here, we give a technical remark that the choice of the parameter T depends on the choice of $\zeta = (\zeta_{p^n})_{h \geqq 1} \in \mathbb{Z}_p(1)$, which also results in the dependence on ζ for the local ε-isomorphism of $\mathbf{Dfm}(\mathbb{Z}_p(1))$. If we put $G_u(T) := (1+T)\frac{g'_u(T)}{g_u(T)}$, then there exists a unique $\lambda_u \in \Lambda_{\mathbb{Z}_p}(\Gamma)$ such that $\lambda_u \cdot (1+T) = G_u(T) - G_u((1+T)^p - 1)$, where $\mathbb{Z}_p[[T]]$ is equipped with an action of $\Lambda_{\mathbb{Z}_p}(\Gamma)$ defined by $[\gamma]\cdot f(T) := f((1+T)^{\chi(\gamma)}-1)$ for any $\gamma \in \Gamma$. Then, the local ε-isomorphism for $\mathbf{Dfm}(\mathbb{Z}_p(1))$ is induce by the morphism $\varprojlim_n (\mathbb{Z}_p[\zeta_{p^n}]^\times)^\wedge \to \Lambda_{\mathbb{Z}_p}(\Gamma) : u \mapsto \lambda_u$, which is essentially the classical Coleman homomorphism. If we see its kernel and cokernel in details, we can show that this map induces an isomorphism $\Lambda_{\mathbb{Z}_p}(\Gamma) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathbb{Z}_p}(\mathbb{Z}_p(1))$. Finally, for any non-torsion $a \in \mathbb{Z}_p^\times$, we have $u_a := \left(\frac{\zeta^a_{p^n}-1}{\zeta_{p^n}-1}\right)_{n\geqq 1} \in \varprojlim_n (\mathbb{Z}_p[\zeta_{p^n}]^\times)$, and remark that the pseudo measure $\mu_{KL} := \frac{\lambda_{u_a}}{1-[\gamma_a]} \in \Lambda_{\mathbb{Z}_p}(\Gamma)[\frac{1}{1-[\gamma_a]}]$ is independent of the choice of $a \in \mathbb{Z}_p^\times$ and is nothing else the pseudo measure corresponding to the Kubota-Leopoldt p-adic L-fucntion.

§ 2.2. Result on the local ε-conjecture for the rank two case

Our result on the local ε-conjecture is the following (Theorem 1.1 [Na2]), which proves many parts of the conjecture for the rank two case.

Theorem 2.3. *Assume that $l = p$. For all the pairs (R, T, ζ) as in Conjecture* 2.1 *such that T are of rank one or two, one can uniquely define R-linear isomorphisms*

$$\varepsilon_{R,\zeta}(T)\colon \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$$

satisfying (1), (2), (3), (4) *in Conjecture* 2.1 *and the following: for any pair (L, V) such that V is de Rham of rank one or two,*

(i) *if V is trianguline, then we have*

$$\varepsilon_{L,\zeta}(V) = \varepsilon^{\mathrm{dR}}_{L,\zeta}(V),$$

(ii) *if V is non-trianguline with distinct Hodge-Tate weights $\{0, k\}$ for $k \geqq 1$, then we have*

$$\varepsilon_{\overline{\mathbb{Q}}_p,\zeta}(V(-r)(\delta)) = \varepsilon^{\mathrm{dR}}_{\overline{\mathbb{Q}}_p,\zeta}(V(-r)(\delta))$$

for any pair (r, δ) such that $0 \leqq r \leqq k-1$ and $\delta\colon \Gamma \to \overline{\mathbb{Q}}_p^\times$ is a homomorphism with finite image.

In this article, we don't recall the definition of trianguline representations. When V is a two dimensional de Rham L-representation of $G_{\mathbb{Q}_p}$, V is trianguline if and only if the associated Weil-Deligne representation $W(V)$ is absolutely reducible. In the trianguline case, the ε-constant of $W(V)$ can be easily written by using some Gauss sums, and the theory of Perrin-Riou exponential (a map interpolating Bloch-Kato exponentials) is fully developed. By these facts, the conjecture for the trianguline case, i.e. (i) in the theorem, is relatively easy, and is known even for higher rank case by our previous work [Na1]. In the non-trianguline case, i.e. the condition (ii) in the theorem, the calculation of ε-constant is much more difficult than that in the trianguline case, and the theory of Perrin-Riou exponential is not known enough in the non-trianguline case (see [Na14b] for the definition of Perrin-Riou exponential). By these difficulties, any results on the local ε-conjecture were not known (at least to the author) in the non-trianguline case before the article [Na2].

In [Na2], we overcome these difficulties by relating the rank two case of the local ε-conjecture to the theory of p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([Co10], [Pa13], [CDP14]) which gives us a canonical bijection between the isomorphism class of absolutely irreducible two dimensional L-representations of $G_{\mathbb{Q}_p}$ and that of unitary L-Banach admissible non-ordinary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Here, we explain the idea of the proof of the theorem (see §3 [Na2] for details). For simplicity, we assume that V is a two dimensional absolutely irreducible L-representation.

We first explain the idea of the definition of $\varepsilon^{\mathrm{Iw}}_{L,\zeta}(V) = \varepsilon_{\Lambda_L(\Gamma),\zeta}(\mathbf{Dfm}(V))$. In this case, one has $\mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Q}_p, V) = 0$ for $i = 0, 2$, and $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$ is free of rank two over $\Lambda_L(\Gamma)$. Hence, the essential part of $\Delta^{\mathrm{Iw}}_L(V)$ is $\wedge^2(\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V))$ (we forget the contribution of $\Delta^{\mathrm{Iw}}_{L,2}(V)$ for simplicity). Hence, it suffices to define a trivialization $\wedge^2(\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)) \xrightarrow{\sim} \Lambda_L(\Gamma)$. In this case, one has an L-Banach representation $\Pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to V via p-adic local Langlands correspondence. We set $g_p := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$. We define $\delta_V\colon \mathbb{Q}_p^\times \to L^\times$ as the character corresponding to $\det_L V$ via local class field theory, and $\Pi(V)^* := \mathrm{Hom}_{\mathrm{cont}}(\Pi(V), L)$ as the L-vector space of continuous L-linear homomorphisms. By the theory of Colmez (in particular, Théorème II.3.1 of [Co10]), one has canonical isomorphisms $\mathrm{H}^1_{\mathrm{Iw}}(V) \xrightarrow{\sim} (\Pi(V)^*)^{g_p=1}$ and $\mathrm{H}^1_{\mathrm{Iw}}(V^*) \xrightarrow{\sim} (\Pi(V)^*)^{g_p=\delta_V(p)}$, by which we identify the both sides. Here, we give a technical remark that these isomorphisms depends on the choice of $\zeta \in \mathbb{Z}_p(1)$, which also results in the dependence on ζ for our local ε-isomorpshism of $\mathbf{Dfm}(V)$. We note that we have an equality $w((\Pi(V)^*)^{g_p=1}) = (\Pi(V)^*)^{g_p=\delta_V(p)}$ for $w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$. Using the Tate duality for $\mathbf{Dfm}(V)$ and the canonical isomorphism $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V^*)^\eta \xrightarrow{\sim} \mathrm{H}^1(\mathbb{Q}_p, \mathbf{Dfm}(V)^*)$, we obtain a pairing $\{-,-\}_{\mathrm{Iw}}\colon \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V^*)^\eta \times \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \to \Lambda_L(\Gamma)$. Using these, we finally obtain the following pairing

$$\mathrm{H}^1_{\mathrm{Iw}}(V) \times \mathrm{H}^1_{\mathrm{Iw}}(V) \to \Lambda_L(\Gamma)\colon (x, y) \mapsto \{w(x), y\}_{\mathrm{Iw}}$$

This pairing is known to be anti-symmetric, and induces a $\Lambda_L(\Gamma)$-linear isomorphism $\wedge^2(\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)) \xrightarrow{\sim} \Lambda_L(\Gamma)$, which is the definition of our ε-isomorphism.

We next explain the idea of the proof of (ii) of the theorem. We assume that

V is de Rham as in (ii). For any $0 \leqq r \leqq k-1$ and δ as in (ii), the ε-isomorphism $\varepsilon_{\overline{\mathbb{Q}}_p,\zeta}(V(-r)(\delta))$ for $V(-r)(\delta)$ is obtained as the base change of $\varepsilon^{\mathrm{Iw}}_{L,\zeta}(V)$ by the condition (1) in the theorem. By the same base change, one has a canonical map $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V) \to \mathrm{H}^1(\mathbb{Q}_p, V(-r)(\delta)) : x \mapsto x_{-r,\delta}$, and similarly has $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V^*) \to \mathrm{H}^1(\mathbb{Q}_p, V^*(r)(\delta^{-1}))\colon y \mapsto y_{r,\delta^{-1}}$. By definition of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(-)$, the essential part of $\varepsilon^{\mathrm{dR}}_{L,\zeta}(V(-r)(\delta))$ can be described by using the dual exponential map $\mathrm{H}^1(\mathbb{Q}_p, V(-r)(\delta)) \xrightarrow{\exp^*} \mathbf{D}^0_{\mathrm{dR}}(V(-r)(\delta))$ and the ε-constant $\varepsilon_{\overline{\mathbb{Q}}_p}(W(V(-r)(\delta)), \zeta)$. On the other hand, by definition of $\varepsilon^{\mathrm{Iw}}_{L,\zeta}(V)$, the essential part of $\varepsilon_{L,\zeta}(V(-r)(\delta))$ can be described by using the action of w and the dual exponential map $\mathrm{H}^1(\mathbb{Q}_p, V^*(r)(\delta^{-1})) \xrightarrow{\exp^*} \mathbf{D}^0_{\mathrm{dR}}(V^*(r)(\delta^{-1}))$. Hence, to show (ii), it suffices to compare $\exp^*(x_{-r,\delta})$ with $\exp^*(w(x)_{r,\delta^{-1}})$ by using $\varepsilon_{\overline{\mathbb{Q}}_p}(W(V(-r)(\delta)), \zeta)$ for any $x \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V)$. We solve this problem by using Emerton's theorem ([Em]) on the compatibility of local and p-adic local Langlands correspondences. Let $\Pi(V)^{\mathrm{alg}} \subseteq \Pi(V)$ be the sub $L[\mathrm{GL}_2(\mathbb{Q}_p)]$-module consisting of locally algebraic vectors, which can be described by Colmez's locally algebraic Kirillov model. Using this model, we prove formulae (Proposition 3.17, Proposition 3.19 [Na2]) describing $\exp^*(x_{-r,\delta})$ (resp. $\exp^*(w(x)_{r,\delta^{-1}})$) by the pairing $[x, f_{r,\delta}] \in L$ (resp. $[w(x), g_{r,\delta}] \in L$) for some explicitly defined elements $f_{r,\delta}$ (resp. $g_{r,\delta}) \in \Pi(V)^{\mathrm{alg}}$, where $[-,-] : \Pi(V)^* \times \Pi(V) \to L\colon (f, x) \mapsto f(x)$ is the canonical $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant pairing. Since one has $[w(x), g_{r,\delta}] = [x, w(g_{r,\delta})]$, it suffices to compare $f_{r,\delta}$ with $w(g_{r,\delta})$ by using $\varepsilon_{\overline{\mathbb{Q}}_p}(W(V(-r)(\delta)), \zeta)$, where the theorem above of Emerton enters in. His theorem states that one has an isomorphism $\Pi(V)^{\mathrm{alg}} \xrightarrow{\sim} \pi(W(V)) \otimes_L \mathrm{Sym}^{k-1}(L^{\oplus 2})$, where $\pi(W(V))$ is the (model over L of the) irreducible smooth admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $W(V)$ via the classical local Langlands correspondence. We remark that $\pi(W(V))$ is supercuspidal under our assumption that V is non-trianguline. Then, it is well-known that the action of w on the classical Kirillov model of $\pi(W(V))$ can be explicitly described by using ε-constants (like $\varepsilon_{\overline{\mathbb{Q}}_p}(\pi(W(V(-r)(\delta))), \zeta)$), which enables us to compare $f_{r,\delta}$ with $w(g_{r,\delta})$ by using $\varepsilon_{\overline{\mathbb{Q}}_p}(\pi(W(V(-r)(\delta))), \zeta)$, which proves (ii) since one has $\varepsilon_{\overline{\mathbb{Q}}_p}(\pi(W(V(-r)(\delta))), \zeta) = \varepsilon_{\overline{\mathbb{Q}}_p}(W(V(-r)(\delta)), \zeta)$ by Emerton's theorem.

§ 3. The global ε-conjecture for the rank two case

Throughout this section, we fix embeddings $\iota_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and fix an isomorphism $\iota\colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ such that $\iota \circ \iota_\infty = \iota_p$. Using this isomorphism, we identify $\Gamma(\mathbb{C}, \mathbb{Z}_l(1)) \xrightarrow{\sim} \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1)) =: \mathbb{Z}_l(1)$, and set $\zeta^{(l)} := \{\iota(\exp(\frac{2\pi i}{l^n}))\}_{n \geqq 1} \in \mathbb{Z}_l(1)$ for each prime l. Let S be a finite set of primes containing p. Let $\mathbb{Q}_S (\subseteq \overline{\mathbb{Q}})$ be the maximal Galois extension of $\mathbb{Q}$ which is unramified outside $S \cup \{\infty\}$, and set $G_{\mathbb{Q},S} := \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. Let $c \in G_{\mathbb{Q},S}$ be the restriction by ι_∞ of the complex conjugation.

§ 3.1. Review of the generalized Iwasawa main conjecture and the global ε-conjecture

In this subsection, we roughly recall the generalized Iwasawa main conjecture and the global ε-conjecture. See the original articles [Ka93a], [Ka93b] and [FK06] for the precise formulations.

Let T be an R-representation of $G_{\mathbb{Q},S}$. We set

$$\mathrm{H}^i(\mathbb{Z}[1/S], T) := \mathrm{H}^i(C^{\bullet}_{\mathrm{cont}}(G_{\mathbb{Q},S}, T))$$

for $i \geqq 0$. For each $l \in S$ and $i = 1, 2$, we set

$$\Delta^{(l)}_{R,i}(T) := \Delta_{R,i}(T|_{G_{\mathbb{Q}_l}}) \text{ and } \Delta^{(l)}_{R}(T) := \Delta_{R}(T|_{G_{\mathbb{Q}_l}})$$

which are defined in §2.1. For a $\mathbb{Z}[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module M, we set $M^{\pm} := \{x \in M \mid c(x) = \pm x\}$. Set $c_T := \mathrm{rank}_R T(-1)^+$ (note that $T(-1)^+$ is a finite projective R-module since we assume that $p \neq 2$). We define

$$\Delta^S_{R,1}(T) := \mathrm{Det}_R(C^{\bullet}_{\mathrm{cont}}(G_{\mathbb{Q},S}, T))^{-1}, \quad \Delta^S_{R,2}(T) := (\det_R(T(-1)^+), c_T)^{-1}$$

and

$$\Delta^S_R(T) := \Delta^S_{R,1}(T) \boxtimes \Delta^S_{R,2}(T).$$

We remark that $\Delta^S_R(T)$ is a graded invertible R-module of degree zero by the global Euler-Poincaré characteristic formula.

In [Ka93a], Kato proposed the following conjecture called the generalized Iwasawa main conjecture. See Conjecture 3.2.2 of [Ka93a] or Conjecture 2.3.2 of [FK06] for the precise formulation, in particular, for the interpolation condition of the zeta isomorphism.

Conjecture 3.1. For any pair (R, T) such that T is an R-representation of $G_{\mathbb{Q},S}$, one can define an R-linear isomorphism (which we call the zeta isomorphism)

$$z^S_R(T) \colon \mathbf{1}_R \xrightarrow{\sim} \Delta^S_R(T)$$

which is compatible with any base change and any exact sequence and satisfies the following: for any pair $(L, V) = (R, T)$ which comes from a motive M over $\mathbb{Q}$ with coefficients in L, $z^S_L(V)$ can be described by the special value at $s = 0$ of the L-function $L(s, V^*)$ associated to V^* (under the validity of the meromorphic continuation of the L-function $L(s, V^*)$ and the Deligne-Beilinson conjecture for M).

Remark 3.2. In the rank one case, Kato (in §3.3 of [Ka93a]) defined the zeta isomorphism using the cyclotomic units and the Stickelberger elements. In this case, the existence of the zeta isomorphism is essentially equivalent to the Iwasawa main conjecture proved by Mazur-Wiles and Rubin, and the interpolation condition of the zeta isomorphism (i.e. the relation with the special values of Dirichlet L-functions) is essentially equivalent to the p-part of Bloch-Kato's Tamagawa number conjecture for Dirichlet motives over $\mathbb{Q}$ which was proved by Bloch-Kato [BK90] (in a special case), Burns-Greither [BG03] and Huber-Kings [HK03] (in the general case).

In §2.1, we defined $\mathbf{Dfm}(-)$ for representations of $G_{\mathbb{Q}_l}$. For an R-representation T of $G_{\mathbb{Q},S}$, we similarly define a $\Lambda_R(\Gamma)$-representation $\mathbf{Dfm}(T) := T \otimes_R \Lambda_R(\Gamma)$ of $G_{\mathbb{Q},S}$. We set $\Delta_R^{\mathrm{Iw},S}(T) := \Delta^S_{\Lambda_R(\Gamma)}(\mathbf{Dfm}(T))$, $\mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Z}[1/S], T) := \mathrm{H}^i(\mathbb{Z}[1/S], \mathbf{Dfm}(T))$, and (conjecturally) set $z_R^{\mathrm{Iw},S}(T) := z^S_{\Lambda_R(\Gamma)}(\mathbf{Dfm}(T))$.

We next recall the global ε-conjecture. Using the Poitou-Tate duality, one can define a canonical isomorphism

$$\Delta_R^S(T^*) \xrightarrow{\sim} \boxtimes_{l\in S}\Delta_R^{(l)}(T) \boxtimes \Delta_R^S(T) \tag{3.1}$$

(see [Ka93b] or §4.1 [Na2] for the definition).

In [Ka93b], Kato proposed the following conjecture called the global ε-conjecture (Conjecture 1.13 of [Ka93b] or Conjecture 3.5.5 of [FK06]).

Conjecture 3.3. For any pair (R,T) such that T is an R-representation of $G_{\mathbb{Q},S}$, the conjectural zeta isomorphism $z_R^S(T_0)\colon \mathbf{1}_R \xrightarrow{\sim} \Delta_R^S(T_0)$ for $T_0 = T, T^*$ and the conjectural p-adic local ε-isomorphism $\varepsilon^{(p)}_{R,\zeta^{(p)}}(T)\colon \mathbf{1}_R \xrightarrow{\sim} \Delta_R^{(p)}(T)$, and the l-adic local ε-isomorphism $\varepsilon^{(l)}_{R,\zeta^{(l)}}(T)\colon \mathbf{1}_R \xrightarrow{\sim} \Delta_R^{(l)}(T)$ defined by [Ya09] for $l \in S \setminus \{p\}$ satisfy the equality

$$z_R^S(T^*) = \boxtimes_{l\in S}\varepsilon^{(l)}_{R,\zeta^{(l)}}(T) \boxtimes z_R^S(T) \tag{3.2}$$

under the canonical isomorphism (3.1).

Remark 3.4. In the rank one case and when $S = \{p\}$, this conjecture was also proved by Kato in §4.22 of [Ka93b] (and it seems possible to prove it for the general S case in a similar way). More precisely, he proved that the zeta isomorphism defined in §3.3 of [Ka93a] and the local p-adic ε-isomorphism defined in [Ka93b] for the rank one case satisfy the equality in the conjecture above.

§ 3.2. Result on the global ε-conjecture for the rank two case

Let $k, N \geqq 1$ be positive integers. Set $S := \{l|N\}\cup\{p\}$. Let $f(\tau) = \sum_{n=1}^{\infty} a_n(f)q^n \in S_{k+1}(\Gamma_1(N))^{\mathrm{new}}$ be a normalized Hecke eigen new form of level N, weight $k+1$, where $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$, $q := \exp(2\pi i\tau)$ and $\Gamma_1(N) := \left\{ g \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}$.

Set $f^*(\tau) := \overline{f(-\overline{\tau})} = \sum_{n=1}^{\infty} \overline{a_n(f)} q^n$ (where $\overline{(-)}$ is the complex conjugation of $(-)$), which is also a Hecke eigen new form in $S_{k+1}(\Gamma_1(N))^{\mathrm{new}}$.

For each homomorphism $\delta\colon \mathbb{Z}_p^\times \to \mathbb{C}^\times$ with finite image (which we naturally regard as a Dirichlet character $\delta\colon (\mathbb{Z}/p^{n(\delta)})^\times \to \mathbb{C}^\times$ (where $n(\delta)$ is the exponent of the conductor of δ), or a Hecke character $\delta\colon \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ such that $\delta|_{\mathbb{R}_{>0}^\times} = 1$), define

$$L(f,\delta,s) := \sum_{n \geqq 1} \frac{a_n(f)\delta(n)}{n^s} \text{ and } L_{\{p\}}(f,\delta,s) := \sum_{n\geqq 1, (n,p)=1} \frac{a_n(f)\delta(n)}{n^s}$$

These L-functions absolutely converge when $\mathrm{Re}(s) > \frac{k}{2} + 1$. The L-function $L(f,\delta,s)$ is analytically continued to the whole $\mathbb{C}$. More precisely, if we denote by $\pi_f = \otimes'_{v\colon \text{place of } \mathbb{Q}} \pi_{f,v}$ the automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to f, then L-function $L(f,\delta,s)$ satisfies the following functional equation

$$\Gamma_{\mathbb{C}}(s) \cdot L(f,\delta,s) = \varepsilon(f,\delta,s) \cdot \Gamma_{\mathbb{C}}(k+1-s) \cdot L(f^*, \delta^{-1}, k+1-s) \ (s \in \mathbb{C}), \tag{3.3}$$

where we set $\Gamma_{\mathbb{C}}(s) := \frac{\Gamma(s)}{(2\pi)^s}$ and $\varepsilon(f,\delta,s)$ is the global ε-factor associated to $\pi_f \otimes (\delta \circ \det)$. The global ε-factor $\varepsilon(f,\delta,s)$ is defined as the product of the local ε-factors

$$\varepsilon(f,\delta,s) = \varepsilon_\infty(f,\delta,s) \prod_{l \in S} \varepsilon_l(f,\delta,s),$$

where, for $v \in S \cup \{\infty\}$, $\varepsilon_v(f,\delta,s)$ is the local ε-factor associated to the v-th component $\pi_{f,v} \otimes (\delta \circ \det)$ with respect to the additive character $\psi_v\colon \mathbb{Q}_v \to \mathbb{C}^\times$ and the Haar measure dx_v on $\mathbb{Q}_v$ which are uniquely characterized by $\psi_\infty(a) := \exp(-2\pi i \cdot a)$ $(a \in \mathbb{R})$, $\psi_l(\frac{1}{l^n}) = \exp(\frac{2\pi i}{l^n})$ $(n \in \mathbb{Z})$, $\int_{\mathbb{Z}_l} dx_l = 1$ and dx_∞ is the standard Lebesgue measure on $\mathbb{R}$. We remark that one has $\varepsilon_\infty(f,\delta,s) = i^{k+1}$.

Set $L := \mathbb{Q}_p(\{\iota_p(\iota_\infty^{-1}(a_n(f)))\}_{n \geqq 1}) \subseteq \overline{\mathbb{Q}}_p$. We denote by $\mathcal{O}_L$ the ring of integers of L. For $f_0 = f, f^*$, let T_{f_0} be the $\mathcal{O}_L$-representation of $G_{\mathbb{Q},S}$ of rank two associated to f_0 which is obtained as a quotient of the étale cohomology of a modular curve (this is denoted by $V_{\mathcal{O}_\lambda}(f_0)$ in §8.3 of [Ka04]). Set $V_{f_0} := T_{f_0}[1/p]$. By the Poincaré duality of the étale cohomology of a modular curve, one has canonical $G_{\mathbb{Q},S}$-equivariant isomorphisms

$$(T_f(k))^* \xrightarrow{\sim} T_{f^*}(1) \text{ and } (V_f(k))^* \xrightarrow{\sim} V_{f^*}(1),$$

which also induces a canonical isomorphism

$$\Delta_{\mathcal{O}_L}^{\mathrm{Iw},S}(T_{f^*}(1))^\eta \xrightarrow{\sim} \Delta_{\mathcal{O}_L}^{\mathrm{Iw},S}((T_f(k))^*)^\eta \xrightarrow{\sim} \Delta^S_{\Lambda_{\mathcal{O}_L}(\Gamma)}(\mathbf{Dfm}(T_f(k))^*).$$

We denote by $Q(\Lambda_{\mathcal{O}_L}(\Gamma))$ the total fraction ring of $\Lambda_{\mathcal{O}_L}(\Gamma)$. For a $\Lambda_{\mathcal{O}_L}(\Gamma)$-module or a graded invertible $\Lambda_{\mathcal{O}_L}(\Gamma)$-module M, we set

$$M_Q := M \otimes_{\Lambda_{\mathcal{O}_L}(\Gamma)} Q(\Lambda_{\mathcal{O}_L}(\Gamma))$$

to simplify the notation.

Now, we recall the definition of a (candidate of the) zeta-isomorphism

$$\widetilde{z}^{\mathrm{Iw},S}_{\mathcal{O}_L}(T_{f_0}(r)) : \mathbf{1}_{Q(\Lambda_{\mathcal{O}_L}(\Gamma))} \xrightarrow{\sim} \Delta^{\mathrm{Iw},S}_{\mathcal{O}_L}(T_{f_0}(r))_Q$$

for $f_0 = f, f^*$ and $r \in \mathbb{Z}$ given in §4.2 [Na2].

For an $\mathcal{O}_L$-representation T of $G_{\mathbb{Q},S}$ (which we also regard as a smooth $\mathcal{O}_L$-sheaf on the étale site over $\mathrm{Spec}(\mathbb{Z}[1/S])$), we define

$$\mathrm{H}^i(\mathbb{Z}[1/p,\zeta_{p^n}],T) := \mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}[1/p,\zeta_{p^n}]),(j_n)_*(T|_{\mathrm{Spec}(\mathbb{Z}[1/S,\zeta_{p^n}])}))$$

using the canonical inclusion $j_n\colon \mathrm{Spec}(\mathbb{Z}[1/S,\zeta_{p^n}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[1/p,\zeta_{p^n}])$, and define a $\Lambda_{\mathcal{O}_L}(\Gamma)$-module

$$\mathbf{H}^i(T) := \varprojlim_{n \geqq 0} \mathrm{H}^i(\mathbb{Z}[1/p,\zeta_{p^n}],T).$$

For the eigenform $f_0 = f, f^*$, by using the p-th layer of Kato's Euler system, he defined in Theorem 12.5 [Ka04] a non zero L-linear map

$$V_{f_0} \to \mathbf{H}^1(V_{f_0})\colon \gamma \mapsto \mathbf{z}^{(p)}_{\gamma}(f_0)$$

which interpolates the critical values of the L-functions $L(f_0^*,\delta,s)$ for all δ (see §12 of [Ka04]), and proved that $\mathbf{H}^1(V_{f_0})_Q \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/S],T_f)_Q$ is a free $Q(\Lambda_{\mathcal{O}_L}(\Gamma))$-module of rank one and $\mathbf{H}^2(V_{f_0})_Q = \mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/S],T_f)_Q = 0$.

By these facts, we obtain a canonical $Q(\Lambda_{\mathcal{O}_L}(\Gamma))$-linear isomorphism

$$\Delta^{\mathrm{Iw},S}_{\mathcal{O}_L,1}(T_f(r))_Q \xrightarrow{\sim} (\mathbf{H}^1(T_f(r))_Q, 1) \tag{3.4}$$

for $r=0$. For general $r \in \mathbb{Z}$, we also define the isomorphism above induced by that for $r=0$ by using the canonical isomorphism

$$\mathbf{H}^i(T_f) \xrightarrow{\sim} \mathbf{H}^i(T_f(r))\colon \mathbf{z} \mapsto \mathbf{z}(r)$$

which is induced by the isomorphism

$$T_f \otimes_{\mathcal{O}_L} \Lambda_{\mathcal{O}_L}(\Gamma) \xrightarrow{\sim} T_f(r) \otimes_{\mathcal{O}_L} \Lambda_{\mathcal{O}_L}(\Gamma)\colon x \otimes y \mapsto (x \otimes \mathbf{e}_r) \otimes g_{\chi^r}(y)$$

where $g_{\chi^r}\colon \Lambda_{\mathcal{O}_L}(\Gamma) \xrightarrow{\sim} \Lambda_{\mathcal{O}_L}(\Gamma)$ is defined by $[\gamma] \mapsto \chi(\gamma)^{-r}[\gamma]$.

For each $l \in S \setminus \{p\}$ and $r \in \mathbb{Z}$, we set

$$L^{(l)}_{\mathrm{Iw}}(T_f(r)) := \det_{\Lambda_{\mathcal{O}_L}(\Gamma)}(1-\varphi_l|\mathbf{Dfm}(T_f(r))^{I_l}) = 1-a_l(f)\cdot l^{-r}\cdot[\sigma_l] \in \Lambda_{\mathcal{O}_L}(\Gamma) \cap Q(\Lambda_{\mathcal{O}_L}(\Gamma))^{\times}$$

(note that $\mathbf{Dfm}(T_f(r))^{I_l} = T_f^{I_l}(r) \otimes_{\mathcal{O}_L} \Lambda_{\mathcal{O}_L}(\Gamma)$ is free over $\Lambda_{\mathcal{O}_L}(\Gamma)$), where the second equality follows from the global-local compatibility of the Langlands correspondences proved by [La73], [Ca86]. Denote the sign of $(-1)^r$ by $\mathrm{sgn}(r) \in \{\pm\}$. Set $\Lambda^{\pm} := \{\lambda \in \Lambda_{\mathcal{O}_L}(\Gamma) | [\sigma_{-1}] \cdot \lambda = \pm\lambda\}$.

Using these preliminaries, we define the following $\Lambda_{\mathcal{O}_L}(\Gamma)$-linear morphism

(3.5)

$$\Theta_r(f) \colon (\mathbf{Dfm}(T_f(r))(-1))^+ = T_f^{\mathrm{sgn}(r-1)}(r-1)\otimes_{\mathcal{O}_L}\Lambda^+\oplus T_f^{\mathrm{sgn}(r)}(r-1)\otimes_{\mathcal{O}_L}\Lambda^- \to \mathbf{H}^1(V_f(r)) \colon$$
$$(\gamma\otimes\mathbf{e}_{r-1}\otimes\lambda^+, \gamma'\otimes\mathbf{e}_{r-1}\otimes\lambda^-) \mapsto \prod_{l\in S\setminus\{p\}} L_{\mathrm{Iw}}^{(l)}(T_{f^*}(1+k-r))^{\eta}\cdot(\lambda^+\cdot(\mathbf{z}_{\gamma}^{(p)}(f)(r))+\lambda^-\cdot(\mathbf{z}_{\gamma'}^{(p)}(f)(r))),$$

where we set $\lambda^{\eta} := \eta(\lambda)$ for $\lambda \in Q(\Lambda_L(\Gamma))$. By Theorem 12.5 of [Ka04], the base change of $\Theta_r(f)$ to $Q(\Lambda_{\mathcal{O}_L}(\Gamma))$ is an isomorphism

$$\theta_r(f) \colon \Delta_{\mathcal{O}_L,2}^{\mathrm{Iw},S}(T_f(r))_Q^{-1} \xrightarrow{\sim} (\mathbf{H}^1(T_f(r))_Q, 1).$$

Finally, the isomorphism $\theta_r(f)$ naturally induces an isomorphism

$$\tilde{z}_{\mathcal{O}_L}^{\mathrm{Iw},S}(T_f(r)) \colon \mathbf{1}_{Q(\Lambda_{\mathcal{O}_L}(\Gamma))} \xrightarrow{\sim} \Delta_{\mathcal{O}_L}^{\mathrm{Iw},S}(T_f(r))_Q.$$

Our theorem concerning the global ε-conjecture is the following (Theorem 1.3 of [Na2]), which essentially proves the global ε-conjecture for $\mathbf{Dfm}(T_f)$.

Theorem 3.5. *Assume that $V_f|_{G_{\mathbb{Q}_p}}$ is absolutely irreducible and $\mathbf{D}_{\mathrm{cris}}(V_f(-r)(\delta))^{\varphi=1} = 0$ for any $0 \leqq r \leqq k-1$ and $\delta \colon \Gamma \to \overline{\mathbb{Q}}_p^{\times}$ with finite image. Then, one has the equality*

$$\tilde{z}_{\mathcal{O}_L}^{\mathrm{Iw},S}(T_{f^*}(1))^{\eta} = \boxtimes_{l\in S}\left(\varepsilon_{\mathcal{O}_L}^{\mathrm{Iw},(l)}(T_f(k)) \otimes \mathrm{id}_{Q(\Lambda_{\mathcal{O}_L}(\Gamma))}\right) \boxtimes \tilde{z}_{\mathcal{O}_L}^{\mathrm{Iw},S}(T_f(k))$$

under the isomorphism obtained by the base change to $Q(\Lambda_{\mathcal{O}_L}(\Gamma))$ of the canonical isomorphism

$$\Delta_{\mathcal{O}_L}^{\mathrm{Iw},S}(T_{f^*}(1))^{\eta} \xrightarrow{\sim} \boxtimes_{l\in S}\Delta_{\mathcal{O}_L}^{\mathrm{Iw},(l)}(T_f(k)) \boxtimes \Delta_{\mathcal{O}_L}^{\mathrm{Iw},S}(T_f(k))$$

defined in (3.1) *for $(R,T) = (\Lambda_{\mathcal{O}_L}(\Gamma), \mathbf{Dfm}(T_f(k)))$, where the isomorphism*

$$\varepsilon_{\mathcal{O}_L}^{\mathrm{Iw},(l)}(T_f(k)) := \varepsilon_{\mathcal{O}_L,\zeta^{(l)}}^{\mathrm{Iw}}(T_f(k)|_{G_{\mathbb{Q}_l}}) : \mathbf{1}_{\Lambda_{\mathcal{O}_L}(\Gamma)} \xrightarrow{\sim} \Delta_{\mathcal{O}_L}^{\mathrm{Iw},(l)}(T_f(k))$$

is the local ε-isomorphism defined in Theorem 2.3 *(resp.* [Ya09]*) for $l = p$ (resp. $l \neq p$) for the pair $(\Lambda_{\mathcal{O}_L}(\Gamma), \mathbf{Dfm}(T_f(k)|_{G_{\mathbb{Q}_l}}))$.*

Finally, we roughly explain the idea of the proof of this theorem by using (the proof of) Theorem 2.3. We freely use notations used in the explanation of the proof of Theorem 2.3. For simplicity, we assume that $S = \{p\}$ and $V_f(k)|_{G_{\mathbb{Q}_p}}$ satisfies the condition in (ii) of Theorem 2.3. Then, we see $\mathbf{z}_{\gamma}^{(p)}(f)(k)$ as an element of $(\Pi(V_f(k))^*)^{g_p=1}$

by the canonical map $\mathbf{H}^1(V_f(k)) \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f(k)) \xrightarrow{\sim} (\Pi(V_f(k))^*)^{g_p=1}$, and similarly see $\mathbf{z}^{(p)}_{\gamma'}(f^*)(1)$ as an element of $(\Pi(V_f(k))^*)^{g_p=\delta_{V_f(k)}(p)}$ by the canonical map $\mathbf{H}^1(V_{f^*}(1)) \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_{f^*}(1)) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, (V_f(k))^*) \xrightarrow{\sim} (\Pi(V_f(k))^*)^{g_p=\delta_{V_f(k)}(p)}$ for any $\gamma \in V_f, \gamma' \in V_{f^*}$. By definitions of Poitou-Tate exact sequence and $\varepsilon^{\mathrm{Iw},(p)}_{L,\zeta^{(p)}}(V_f(k))$, it suffices to compare $\mathbf{z}^{(p)}_{\gamma}(f)(k)$ with $w(\mathbf{z}^{(p)}_{\gamma'}(f^*)(1))$ in $(\Pi(V_f(k))^*)^{g_p=1}$. Since $\Pi(V_f(k))^{\mathrm{alg}}$ is dense in $\Pi(V_f(k))$ and $\{f_{r,\delta}\}_{r,\delta}$ as in the proof of Theorem 2.3 generates $\Pi(V_f(k))^{\mathrm{alg}}$, it suffices to compare $[\mathbf{z}^{(p)}_{\gamma}(f)(k), f_{r,\delta}]$ with $[w(\mathbf{z}^{(p)}_{\gamma'}(f^*)(1)), f_{r,\delta}] = [\mathbf{z}^{(p)}_{\gamma'}(f^*)(1), w(f_{r,\delta})]$ for any (r,δ). By the formulae (Proposition 3.17, Proposition 3.19 [Na2]), $[\mathbf{z}^{(p)}_{\gamma}(f)(k), f_{r,\delta}]$ can be described by using $\exp^*(\mathbf{z}^{(p)}_{\gamma}(f)(k)_{-r,\delta})$. Since $w(f_{r,\delta})$ can be described by using $g_{r,\delta}$ and $\varepsilon_{\overline{\mathbb{Q}}_p}(W(V_f(k-r)(\delta)), \zeta^{(p)})$, $[\mathbf{z}^{(p)}_{\gamma'}(f^*)(1), w(f_{r,\delta})]$ can be described by using $\exp^*(\mathbf{z}^{(p)}_{\gamma'}(f^*)(1)_{r,\delta^{-1}})$ and $\varepsilon_{\overline{\mathbb{Q}}_p}(W(V_f(k-r)(\delta)), \zeta^{(p)})$. By Kato's explicit reciprocity law (Theorem 12.5(1) [Ka04]), $\exp^*(\mathbf{z}^{(p)}_{\gamma}(f)(k)_{-r,\delta})$ (resp. $\exp^*(\mathbf{z}^{(p)}_{\gamma'}(f^*)(1)_{r,\delta^{-1}})$) can be described by using $L(f^*, \delta^{-1}, r+1)$ (resp. $L(f, \delta, k-r)$). Therefore, we can compare $\exp^*(\mathbf{z}^{(p)}_{\gamma}(f)(k)_{-r,\delta})$ with $\exp^*(\mathbf{z}^{(p)}_{\gamma'}(f^*)(1)_{r,\delta^{-1}})$ by using $\varepsilon_{\overline{\mathbb{Q}}_p}(W(V_f(k-r)(\delta)), \zeta^{(p)})$ by the classical functional equation (3.3) of $L(f, \delta, s)$, which proves Theorem 3.5.

References

[BB08] D. Benois, L. Berger, Théorie d'Iwasawa des représentations cristallines. II, Comment. Math. Helv. **83** (2008), no. 3, 603-677.

[BG03] D. Burns, C. Greither, On the equivariant Tamagawa number conjecture for Tate motives, Invent. Math. **153** (2003), 303-359.

[BK90] S. Bloch, K. Kato, L-functions and Tamagawa numbers of motives. The Grothendieck Festschrift, Vol. I, 333-400, Progr. Math. **86**, Birkhäuser Boston, Boston, MA 1990.

[Ca86] H. Carayol, Sur les représentations l-adiques associées aux formes modulaires de Hilbert, Ann. Sci. ENS. **19** (1986), 409-468.

[Co10] P. Colmez, Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ,Γ)-modules, Astérisque **330** (2010), 281-509.

[CDP14] P. Colmez, G. Dospinescu, V. Paskunas, The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, Camb. J. Math. **2** (2014), no.1, 1-47.

[De73] P. Deligne, Les constantes des équations fonctionelles des fonctions L, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501-597. Lecture Notes in Math., vol. 349, Springer, Berlin, 1973.

[Em] M. Emerton, Local-global compatibility in the p-adic Langlands programme for $\mathrm{GL}_{2/\mathbb{Q}}$, preprint.

[FK06] T. Fukaya, K. Kato, A formulation of conjectures on p-adic zeta functions in non commutative Iwasawa theory, Proceedings of the St. Petersburg Mathematical Society. Vol. XII (Providence, RI), Amer. Math. Soc. Transl. Ser. 2, vol. 219, Amer. Math. Soc., 2006, pp. 1-85.

[HK03] A. Huber, G. Kings, Bloch-Kato conjecture and Main conjecture of Iwasawa theory for Dirichlet characters, Duke Math. J. **119** (2003), 393-464.

[Ka93a] K. Kato, Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. I. Arithmetic algebraic geometry (Trento, 1991), 50-163, Lecture Notes in Math., vol. 1553, Springer, Berlin, 1993.

[Ka93b] K. Kato, Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. Part II. Local main conjecture, unpublished preprint.

[Ka04] K. Kato, p-adic Hodge theory and values of zeta functions of modular forms, in Cohomologies p-adiques et applications arithmétiques. III, Astérisque **295**, Soc. Math. France, Paris, 2004, pp. ix, 117-290.

[KM76] F. Knudsen, D. Mumford, The projectivity of the moduli space of stable curves I, Math. Scand. **39** (1976), no. 1, 19-55.

[La73] R. P. Langlands, Modular forms and l-adic representations, Modular functions of one variable II, Lecture Notes in Math. **349**, Springer-Verlag, New York, 1973, pp. 361-500.

[LVZ13] D. Loeffler, O. Venjakob, S. L. Zerbes, Local ε-isomorphism, Kyoto J. Math. **55** (2015), no.1, 63-127.

[Na14a] K. Nakamura, Zariski density of crystalline representations for any p-adic field, J. Math. Sci. Univ. Tokyo **21** (2014), 79-127.

[Na14b] K. Nakamura, Iwasawa theory of de Rham (φ, Γ)-modules over the Robba ring, J. Inst. Math. Jussieu **13**, (2014), no.1, 65-118.

[Na1] K. Nakamura, A generalization of Kato's local ε-conjecture for (φ, Γ)-modules over the Robba ring, to appear in Algebra Number Theory.

[Na2] K. Nakamura, Local ε-isomorphisms for rank two p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a functional equation of Kato's Euler system, preprint in arXiv:1502.04924.

[Pa13] V. Paskunas, The image of Colmez's Montreal functor, Publ. Math. Inst. Hautes Études Sci. **118** (2013), 1-191.

[Ve13] O. Venjakob, On Kato's local ε-isomorphism Conjecture for rank one Iwasawa modules, Algebra Number Theory **7** (2013), 2369-2416.

[Ya09] S. Yasuda, Local constants in torsion rings, J. Math. Sci. Univ. Tokyo **16** (2009), no. 2, 125-197.