RIMS Kôkyûroku Bessatsu
B65 (2017), 063–078

On applications of modulation spaces to dispersive equations

By

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§ 1. Introduction

In this report, we survey some recent results and [16] on applications of modulation spaces to nonlinear dispersive equations. We also give additional proofs skipped in [16] on the generalized 2D Zakharov-Kuznetsov equation.

1.1 Modulation spaces. Modulation spaces were introduced by H.G. Feichtinger [9] from the viewpoint of the time-frequency analysis. As one of time-frequency analysis techniques, we have the short-time Fourier transform of f

$$V_g f(x,\omega) = \int_{\mathbb{R}^n} e^{-it\cdot\omega} \overline{g(t-x)} f(t)dt,$$

where $g \in \mathcal{S}(\mathbb{R}^n)$ is said to be a window function. Here, we remark that, choosing a function g whose support is compact, the transform $V_g f$ enables us to understand the relation between time and frequency. By using this short-time Fourier transform, the Feichtinger's original definition of modulation spaces $M^s_{p,q}(\mathbb{R}^n)$ is given as follows. Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Then, the modulation space $M^s_{p,q}(\mathbb{R}^n)$ consists of all tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that

$$(1.1) \qquad \|f\|^{original}_{M^s_{p,q}} = \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \big| V_g f(x,\omega) \big|^p dx \Big)^{q/p} \langle \omega \rangle^{sq} d\omega \Big)^{1/q} < \infty,$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. In (1.1), the (quasi)-norm has to be read with natural modifications for $p = \infty$ or $q = \infty$. Moreover, in [9], it is proved that the (quasi)-norm

Received September 29, 2016. Revised January 26, 2017.
2010 Mathematics Subject Classification(s): 35Q53, 42B35
Key Words: *Well-posedness, Dispersive equation, Zakharov-Kuznetsov equation, Modulation space.*
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in (1.1) can be equivalently expressed by a Besov-type $\ell^q(L^p)$ one. Let $\sigma \in \mathcal{S}(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp}\sigma \subset [-1,1]^n \text{ and } \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1 \text{ for any } \xi \in \mathbb{R}^n,$$

and set

$$\|f\|_{M^s_{p,q}} = \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\square_k f\|^q_{L^p} \Big)^{1/q}, \tag{1.2}$$

with a usual modification for $q = \infty$. Here, $\sigma_k = \sigma(\cdot - k)$ and $\square_k = \mathcal{F}^{-1}\sigma_k\mathcal{F}$, and $\square_k$ is called as the frequency-uniform decomposition operator. Then, the (quasi)-norms in (1.1) and (1.2) are equivalent to each other (see [9, Corollary 4.2] and also [45, Proposition 2.1]). For $s = 0$, we write $M_{p,q}(\mathbb{R}^n) = M^0_{p,q}(\mathbb{R}^n)$ simply. For more precise properties, refer also to [10, 21, 22, 37, 38, 40].

After the first appearance of the modulation space by Feichtinger, the modulation space had not been investigated studiously for a long time. However, in the early twenty-first century, the modulation space was first applied to fields of partial differential equations, and then studies of the modulation space itself and its applications have been developed rapidly. We shall introduce two results which triggered off the opportunity.

The first result is an application to uniform boundedness of the unimodular Fourier multiplier $e^{i(-\Delta)^{\alpha/2}}$ by Bényi, Grochënig, Okoudjou and Rogers [2]. Here, $e^{i(-\Delta)^{\alpha/2}}$ is the multiplier defined by

$$\begin{aligned} e^{i(-\Delta)^{\alpha/2}} f(x) &= \mathcal{F}^{-1} e^{i|\xi|^\alpha} \mathcal{F} f(x) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{i|\xi|^\alpha} \widehat{f}(\xi) d\xi \end{aligned}$$

for $m > 0$. This multiplier arises naturally in the fundamental solution to the Cauchy problem for the linear dispersive equation:

$$\begin{cases} i\partial_t u = (-\Delta)^{\alpha/2} u, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

When $\alpha = 2$, the Cauchy problem (1.3) is that for the linear Schrödinger equation. In [2], they gave the surprising result which $e^{i(-\Delta)^{\alpha/2}}$ is bounded on $M_{p,q}(\mathbb{R}^n)$ for $1 \le p, q \le \infty$ and $\alpha \le 2$. In contrast with the modulation space, if we consider its boundedness in the frame of the Lebesgue space, it is well-known that, if $\alpha > 1$, then $e^{i(-\Delta)^{\alpha/2}}$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $p = 2$ (see [14]). Furthermore, Miyachi [29] studied that, if $1 < p < \infty$, $s \in \mathbb{R}$ and $\alpha > 1$, then $e^{i(-\Delta)^{\alpha/2}}$ is bounded from $L^p_s(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ if and only if $s \ge \alpha n|1/p - 1/2|$. Thus, we essentially require a loss of regularity to obtain the boundedness in the frame of the L^p space. This is a major difference between the modulation space and the L^p space. In addition to [2], Miyachi, Nicola, Rivetti, Tabacco and Tomita [30] proved that, if $1 \le p, q \le \infty$, $s \in \mathbb{R}$ and $\alpha > 2$, then $e^{i(-\Delta)^{m/2}}$ is bounded from $M^s_{p,q}(\mathbb{R}^n)$ into $M_{p,q}(\mathbb{R}^n)$ if and only if $s \ge (\alpha-2)n|1/p-1/2|$. Comparing the results [29] in the L^p space and [30] in the modulation space, we see that, for $\alpha > 2$, the required loss of regularity in the modulation space is smaller than that in the L^p space. This is also the advantage of the modulation space. We remark that the unimodular Fourier multiplier on $M^s_{p,q}$ for $0 < p < 1$ is also considered, for instance, in [1, 39]. Also, the boundedness of $e^{i(-\Delta)^{m/2}}$ from modulation spaces into Lebesgue spaces is given in [23], and that one from Wiener-amalgam spaces (roughly speaking, the space with the (quasi)-norm switched L^p and ℓ^q in (1.2)) into Lebesgue spaces is given in [6].

The second result is an application to the following Cauchy problem for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^\kappa u, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

by Wang and Hudzik [45]. When we show the well-posedness for (1.4) by using a standard fixed point argument, some estimates for the Schrödinger semi-group $e^{it\Delta}$ play important roles. In [45], Wang and Hudzik focus on the Strichartz estimate:

$$\left\| e^{it\Delta} f \right\|_{L^\gamma_t L^p_x} \le C_{n,p} \left\| f \right\|_{L^2_x}, \tag{1.5}$$

where $\gamma = \gamma_p$, $2/\gamma_p = n(1/2 - 1/p)$ and $2 \le p \le \infty$ satisfying $\gamma_p \ge 2$ or $\gamma_p > 2$ if $n = 2$ (see [4, 18]). In order to establish the Strichartz estimate, we usually use the dispersive estimate:

$$\left\| e^{it\Delta} f \right\|_{L^p} \le C_{n,p} |t|^{-n(\frac{1}{2}-\frac{1}{p})} \left\| f \right\|_{L^{p'}}, \tag{1.6}$$

where $2 \le p \le \infty$ and $1/p + 1/p' = 1$. In [18], Keel and Tao suggested that, if estimate (1.6) with the time decay term $(1+|t|)$ instead of $|t|$ holds, then the Strichartz estimate in (1.5) follows for any $\gamma \ge \max(2, \gamma_p)$. From this suggestion, Wang and Hudzik established the dispersive estimate in the frame of the modulation space

$$\left\| e^{it\Delta} f \right\|_{M^s_{p,q}} \le C_{n,p} (1+|t|)^{-n(\frac{1}{2}-\frac{1}{p})} \left\| f \right\|_{M^s_{p',q}},$$

and so succeeded to obtain the refined Strichartz estimate in the modulation space. Then, by using the estimate, they proved the global (in time) well-posedness for (1.4) with small initial data in $M_{2,1}(\mathbb{R}^n)$ when $\kappa \in 2\mathbb{N}$ and $\kappa \ge 4/n$. Following the result on the modulation space, we recall some works on Sobolev spaces. Cazenave and Weissler [4] proved that the solution to the Cauchy problem (1.4) is locally well-posed in $H^{s_c}(\mathbb{R}^n)$

if the scaling critical exponent $s_c = n/2 - 2/\kappa \geq 0$, and also globally well-posed with small initial data in $\dot{H}^{s_c}(\mathbb{R}^n)$. We remark that, in the L^2-subcritical case, the global well-posedness for any data in $L^2(\mathbb{R}^n)$ was given in [41]. If we compare the global well-posedness result in the modulation space with that in the Sobolev space, we see that, in the modulation space, the regularity is uniformly zero independently of the nonlinear term and dimensions. We note that since $M_{2,1}(\mathbb{R}^n) \not\subset H^{\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$, the well-posedness in $M_{2,1}(\mathbb{R}^n)$ is the result treating a class of functions out of control of the scaling critical Sobolev space $H^{s_c}(\mathbb{R}^n)$. In addition to the result [45] on the nonlinear Schrödinger equation, the modulation space is applied to several partial differential equations in recent years. Bényi and Okoudjou [1] improved the result in [45] and proved the local well-posedness in $M_{p,1}(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$. In the paper, they obtained the same well-posedness results for the wave and the Klein-Gordon equations and also gave the blow-up properties for these three equations. In [44], Wang and Huang studied the well-posedness for the Korteweg-de Vries (KdV) equation, the Benjamin-Ono equation and the derivative nonlinear Schrödinger equation (DNLS) on one dimension in modulation spaces. Moreover, Wang [42] investigate the DNLS equation on higher dimensions and got the optimal global well-posedness with small initial data in a modulation space (this is the improvement of [43]). Quite recently, Ruzhansky, Wang and Zhang used the modulation space to prove the small data global well-posedness for the fourth order Schrödinger equation in [32]. Iwabuchi [15] obtained the the well-posedness for the Navier-Stokes equation and the nonlinear heat equation in modulation spaces with negative regularity (see also Wang, Zhao and Guo [46]).

1.2 Generalized Zakharov-Kuznetsov equations. In this subsection, we state recent works of the Cauchy problem for the generalized Zakharov-Kuznetsov (gZK) equation:

$$\begin{cases} \partial_t u + \partial_{x_1} \Delta u = \partial_{x_1}(u^{m+1}), \\ u(0) = u_0, \end{cases} \tag{1.7}$$

where $u = u(x,t)$ is a real valued function, $t > 0$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $m \in \mathbb{N}$ and the Laplacian $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$. The gZK equation can be seen as a multi-dimensional extension of the generalized KdV equation on one dimension:

$$\partial_t u + \partial_x^3 u = \partial_x(u^{m+1}), \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.8}$$

The Zakharov-Kuznetsov (ZK) equation ($m = 1$) on three space dimensions was first invented by Zakharov and Kuznetsov [47], and describes the propagation of ionic-acoustic waves in a magnetic field. Also, the two dimensional ZK equation arises from the hydrodynamic set of equations for ion density and velocity, which was studied in [24]. Here, we remark that the ZK equation is justified rigorously only on two and three dimensions (see [25]).

In this section, we consider the well-posedness for the Cauchy problem (1.7) with the initial data in modulation spaces. First, we introduce some known well-posedness results on the two dimensional case in Sobolev spaces. We note that the scaling critical exponent $s_c = n/2 - 2/m$.

1. Case $m = 1$. In [7], Faminskii showed the local and global well-posedness in $H^1(\mathbb{R}^2)$ (see also [3]). Then, Linares and Pastor [26] improved his result to the local well-posedness in $H^s(\mathbb{R}^2)$ for $s > 3/4$. Furthermore, Grünrock and Herr [13] and Molinet and Pilod [33] gave the local one in $H^s(\mathbb{R}^2)$ for $s > 1/2$, independently.
2. Case $m = 2$. Biagioni and Linares [3] studied the local well-posedness in $H^1(\mathbb{R}^2)$, and the local solution can be extended to the global one if the initial data in $L^2(\mathbb{R}^2)$ is sufficiently small. In [26] and [27], Linares and Pastor proved that the Cauchy problem (1.7) is locally well-posed in $H^s(\mathbb{R}^2)$ if $s > 3/4$ and globally well-posed in $H^s(\mathbb{R}^2)$ for $s > 53/63$, respectively. See also [8], where the global well-posedness in $H^1(\mathbb{R}^2)$ is given. Up to now, the local well-posedness in $H^s(\mathbb{R}^2)$ is pushed down to that for $s > 1/4$ by Ribaud and Vento [35].
3. Case $m \geq 3$. Linares and Pastor [27] investigated the local well-posedness in $H^s(\mathbb{R}^2)$ for $s > \max(\frac{3}{4}, 1 - \frac{3}{2(m-2)})$. They also studied the global well-posedness with the small initial data in $H^1(\mathbb{R}^2)$. Then, the regularity of the local problem was almost reached to the scaling critical exporent (i.e. $s > s_c$) for $m > 8$ by Fatah, Linares and Pastor [8], and for $m \geq 4$ by Ribaud and Vento [35]. Finally, Grünrock [12] brought the local and the small data global well-posedness results up the scaling critical exponent s_c, and obtained the results in the homogeneous Sobolev space $\dot{H}^{s_c}(\mathbb{R}^2)$ for $m \geq 3$.

We next state the three dimensional case. In comparison with the two dimensional case, the well-posedness results on the three dimensional case are quite few.

1. Case $m = 1$. Linares and Saut [28] studied the local well-posedness in $H^s(\mathbb{R}^n)$ for $s > 9/8$. Then, their result made progress to that for $s > 1$ in the paper [34] by Ribaud and Vento. Furthermore, Molinet and Pilod extended its local-in-time result to the global-in-time one in [33].
2. Case $m = 2$. Grünrock [11] proved that the Cauchy problem (1.7) is locally well-posed in $H^s(\mathbb{R}^n)$ if $s > 1/2$ and globally well-posed in $H^s(\mathbb{R}^n)$ for $s \geq 1$.
3. Case $m \geq 3$. Grünrock [12] showed the local and the small data global well-posedness in the scaling critical homogeneous Sobolev space $\dot{H}^{s_c}(\mathbb{R}^n)$ in one step.

Now, based on the above results on the Sobolev spaces, we consider the well-posedness in the modulation space. In [16], we have the following result. Let the auxiliary function space $\mathcal{X}_T$ be

$$\mathcal{X}_T = \left\{ u \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^n) : \sum_{k\in\mathbb{Z}^n} \|\Box_k u\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},T}} \le \rho \right\},$$

where $\bar{x} = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$ and $\|f\|_{L^p_{x_1} L^q_{\bar{x},T}} = \big\| \|f\|_{L^q_{\bar{x},T}} \big\|_{L^p_{x_1}}$. Then, we have the following well-posedness results (see Theorems 1.1 and 1.2 in [16]).

Theorem 1.1. *Let space dimensions $n = 2$ and integers $m \ge 4$.*

(i) *For any $u_0 \in M_{2,1}(\mathbb{R}^2)$, there exists $T > 0$ such that the Cauchy problem (1.7) has a unique solution*

$$u \in C\big([0,T], M_{2,1}\big) \cap \mathcal{X}_T.$$

(ii) *There exists $\rho > 0$ such that if $u_0 \in M_{2,1}(\mathbb{R}^2)$ satisfies that $\|u_0\|_{M_{2,1}} \le \rho$, then the Cauchy problem (1.7) has a unique global solution*

$$u \in C\big([0,\infty), M_{2,1}\big) \cap \mathcal{X}_\infty.$$

We remark that the results in Theorem 1.1 correspond to that for the generalized KdV equation (1.8). Indeed, by Wang and Huang [44], it is proved that the Cauchy problem for the generalized KdV equation (1.8) is locally well-posed in $M_{2,1}(\mathbb{R})$ and globally well-posed with small initial data in $M_{2,1}(\mathbb{R})$ if $m \ge 4$. We note that, as stated above, the well-posedness in $M_{2,1}(\mathbb{R}^2)$ is the result treating a class of functions out of control of the scaling critical Sobolev spaces. In fact, we have for any $\varepsilon > 0$

$$\begin{aligned} &H^{s+n/2+\varepsilon} \subset B^{s+n/2}_{2,1} \subset M^s_{2,1} \subset H^s, \\ &H^{s+n/2} \not\subset M^s_{2,1} \not\subset H^{s+\varepsilon}. \end{aligned} \tag{1.9}$$

Here, it is known that the above inclusion relations are optimal (see [23, 45, 46]), and functions which belong to $M_{2,1}(\mathbb{R}^2)$ but not to $\dot{H}^{s_c}(\mathbb{R}^2)$ can be also established (refer to [42, Appendix B] or Remark 6). Moreover, from Theorem 1.1 and the sharp inclusion (1.9), we are able to obtain the local and the small data well-posedness for the Cauchy problem (1.7) with $m \ge 4$ in $H^{1+\varepsilon}(\mathbb{R}^2)$ for any $\varepsilon > 0$. However, the lower bound on the regularity is not reached to the scaling critical exponent $s_c = 1 - 2/m$, unfortunately. As a final remark on Theorem 1.1, we note that the global-in-time solution obtained in Theorem 1.1 scatters in the modulation space $M_{2,1}(\mathbb{R}^2)$. We will sketch the proof of this scattering theory in Remark 5.

In Theorem 1.1, we don't mention the quartic case when $m = 3$, and so we shall consider the case. Let

$$\mathcal{Y}^{1/4}_T = \{u \in \mathcal{S}' : \|u\|_{Y_{T,1}\cap Y_{T,2}} \le \rho\},$$

where

$$\|u\|_{Y_{T,1}} = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{1/4} \|\Box_k u\|_{L^\infty_T L^2_x} \text{ and } \|u\|_{Y_{T,2}} = \sum_{k\in\mathbb{Z}^n} \|\Box_k u\|_{L^4_{x_1} L^\infty_{\bar{x},T}}.$$

Then, we have the following (see Theorem 1.3 in [16]).

Theorem 1.2. *Let space dimensions $n = 2$ and $m = 3$. Then, for any $u_0 \in M^{1/4}_{2,1}(\mathbb{R}^2)$, there exists $T > 0$ such that the Cauchy problem (1.7) has a unique solution*

$$u \in C\big([0,T], M^{1/4}_{2,1}\big) \cap \mathcal{Y}^{1/4}_T.$$

Remark 1 (See [17]). We remark that Theorems 1.1 and 1.2 hold on arbitrary dimensions $n \geq 2$. Moreover, we note that, if $m = 3$, we obtain the small data global well-posedness in $M^{1/4}_{2,1}(\mathbb{R}^n)$ for $n \geq 3$, but not for $n = 2$. This comes from the fact that we don't know whether the L^4-based Strichartz-type estimate with the frequency-uniform decomposition:

$$\big\|\Box_k e^{-t\partial_{x_1}\Delta} u_0\big\|_{L^4_{x,t}} \leq C\big\|\Box_k u_0\big\|_{L^2_x} \text{ for } k \in \mathbb{Z}^n,$$

holds on two space dimensions $n = 2$, the estimate of which plays a crucial role when we prove the global well-posedness.

The plan of this report is as follows. In section 2, we collect some linear estimates. Then we sketch the proofs of our main theorems in Section 3.

§ 2. Linear estimates

We consider the Cauchy problem for the linear ZK equation

$$\begin{cases} \partial_t u + \partial_{x_1}\Delta u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

and collect some estimates for the associated linear operator $U(t) = \mathcal{F}^{-1} e^{it\phi(\xi)}\mathcal{F}$, where $\phi(\xi) = \xi_1(\xi_1^2 + \cdots + \xi_n^2)$. In this section, we only state two principal estimates, the Kato-type smoothing estimate and the maximal function estimate. In general, the Strichartz estimate are also required to prove the well-posedness. However, since we don't use the Strichartz estimate when we show Theorems 1.1 and 1.2, we omit the detail. In order to express these estimates, we use the following notations dividing n-variables into two parts. We write

$$x = (x_1, \bar{x}) \in \mathbb{R}^n \text{ with } x_1 \in \mathbb{R} \text{ and } \bar{x} = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}.$$

2.1 Kato-type smoothing estimates. Let us state the first linear estimate, a standard Kato-type smoothing estimate. The Kato-type estimate for (2.1) was first proved on two space dimensions by Faminskii [7]. The proof is similar to the corresponding estimate for the linear KdV equation, so that refer to [19, 20].

Proposition 2.1. *Let $n = 2$. Then we have*

$$\big\| |\nabla| U(t) u_0 \big\|_{L^\infty_{x_1} L^2_{\bar{x},t}} \le C \|u_0\|_{L^2_x},$$

where the Riesz potential $|\nabla| f = \mathcal{F}^{-1} |\xi| \mathcal{F} f$.

Corollary 2.2. *Let $n = 2$. Then we have for any $k \in \mathbb{Z}^2$*

$$\big\| \Box_k |\nabla| U(t) u_0 \big\|_{L^\infty_{x_1} L^2_{\bar{x},t}} \le C \|\Box_k u_0\|_{L^2_x}.$$

Remark 2. Ribaud and Vento [34] established the Kato-type smoothing estimate on three space dimensions, and then the author [17] extended them to the higher dimensional cases $n \ge 4$.

2.2 Maximal function estimates. We next state a maximal function estimate. In [26], Linares and Pastor first constructed the estimate on two space dimensions

$$\|U(t)u_0\|_{L^4_{x_1} L^\infty_{\bar{x},T}} \le C \|u_0\|_{H^s_x} \tag{2.2}$$

for $s > 3/4$ and $0 \le T \le 1$. We remark that the above estimate holds only locally-in-time. After that, Grünrock [12] improved their local-in-time estimate (2.2) to the global-in-time one and obtained on $\mathbb{R}^2$

$$\|U(t)u_0\|_{L^4_{x_1} L^\infty_{\bar{x},t}} \le C \|u_0\|_{\dot{H}^s_x} \tag{2.3}$$

for $s > 3/4$ by using a symmetrizing transformation. As is seen in Section 1, we establish the solution in the modulation space with the uniform zero regularity independently of dimensions and a power of the nonlinear term. Thus, the required regularity $s > 3/4$ in the right hand side of (2.3) is too large to prove the well-posedness in $M_{2,1}$. However, if we re-construct the maximal function estimate with the frequency-uniform decomposition operator $\Box_k$, we have the following (see [16, Proposition 3.3]).

Proposition 2.3. *Let $n = 2$. Then we have for any $k \in \mathbb{Z}^2$*

$$\|\Box_k U(t) u_0\|_{L^4_{x_1} L^\infty_{\bar{x},t}} \le C \langle k \rangle^{1/4} \|\Box_k u_0\|_{L^2_x}.$$

Comparing the estimate in Proposition 2.3 with (2.3), the less regularity $1/4$ is required. Moreover, it is identical with the regularity for a maximal function estimate of the KdV equation by Kenig, Ponce and Vega [20]. Thus, referring to the result on the KdV equation by Wang and Huang [44], we are able to expect that the Cauchy problem for (1.7) is locally and globally well-posed in $M_{2,1}(\mathbb{R}^2)$.

Remark 3. Note that the maximal function estimate for the ZK equation in the Sobolev space has properly been established only on two dimensions. On the other hand, if we use the modulation space, we are able to construct the estimate on arbitrary dimensions. This means that Proposition 2.3 holds on $n \geq 2$ (see [17]).

§ 3. Proofs of theorems

In this section, we only prove Theorem 1.1. Before that we prepare some lemmas. The first one is given by the interpolation theorem between the Kato-type smoothing estimate and the maximal function estimate. One can find the proof in [20, Corollary 3.8], [44, Corollary 6.3] and [16, Proposition 3.5].

Lemma 3.1. *Let $n = 2$ and $4 \leq q \leq \infty$. Then we have for any $k \in \mathbb{Z}^2$*

$$\|\Box_k U(t) u_0\|_{L^{q+1}_{x_1} L^{2(q+1)}_{\bar{x},t}} \leq C \|\Box_k u_0\|_{L^2_x}.$$

Next, we collect estimates for the nonlinear term.

Lemma 3.2. *Let $n = 2$. Then we have for any $k \in \mathbb{Z}^2$*

$$\left\| \Box_k |\nabla| \int_0^t U(t-s) f(s) ds \right\|_{L^\infty_t L^2_x} \leq C \|\Box_k f\|_{L^1_x L^2_{\bar{x},t}},$$

where $|\nabla|$ is the Riesz potential.

Lemma 3.3. *Let $n = 2$ and $4 \leq q < \infty$. Then we have for any $k \in \mathbb{Z}^2$*

$$\left\| \Box_k \int_0^t U(t-s) f(s) ds \right\|_{L^\infty_t L^2_x} \leq C \|\Box_k f\|_{L^{\frac{q+1}{q}}_{x_1} L^{\frac{2(q+1)}{2q+1}}_{\bar{x},t}},$$

$$\left\| \Box_k |\nabla| \int_0^t U(t-s) f(s) ds \right\|_{L^{q+1}_{x_1} L^{2(q+1)}_{\bar{x},t}} \leq C \|\Box_k f\|_{L^1_{x_1} L^2_{\bar{x},t}}.$$

See [34, Proposition 3.5] and [16, Proposition 4.4] for the proofs of Lemmas 3.2 and 3.3, respectively. When we prove the above lemmas, we use a standard TT^* argument, so that we also refer to [5, 31, 43]. Finally, we state the following lemma (see [42, Lemma 4.2]).

Lemma 3.4. *Let $k, k^{(i)} \in \mathbb{Z}^n$ for $i = 1, \cdots, m+1$. Then we have*

$$\Box_k (\Box_{k^{(1)}} u \cdots \Box_{k^{(m+1)}} u) = 0$$

if $|k - k^{(1)} - \cdots - k^{(m+1)}| \geq C_n$.

We set χ_{Ω_k} as a characteristic function on $\Omega_k = \{k \in \mathbb{Z}^n : |k - k^{(1)} - \cdots - k^{(m+1)}| \le C_n\}$. Now, we shall begin with the proof of Theorem 1.1 (see also the proofs of [20, Theorems 2.8 and 2.10] and [44, Theorem 1.2] for the generalized KdV equation).

Proof of Theorem 1.1. We first show the global well-posedness with the small initial data in $M_{2,1}$. Set a mapping as

$$\mathcal{N}[u](t) = U(t)u_0 + \int_0^t U(t-s)\partial_{x_1}(u^{m+1})(s)ds, \tag{3.1}$$

where $U(t) = \mathcal{F}^{-1}e^{it\phi}\mathcal{F}$ with $\phi(\xi) = \xi_1(\xi_1^2 + \xi_2^2)$, and also the auxiliary function space $\mathcal{X}$ as

$$\mathcal{X} = \Big\{ u \in \mathcal{S}' : \sum_{k \in \mathbb{Z}^2} \|\Box_k u\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},t}} \le \rho \Big\}.$$

In order to obtain the well-poseness, we prove that the mapping $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is a contraction mapping.

We first consider the linear term in (3.1). From Proposition 3.1, we obtain

$$\big\|U(t)u_0\big\|_{\mathcal{X}} \le C\|u_0\|_{M_{2,1}}. \tag{3.2}$$

Next, we consider the Duhamel term in (3.1). By Lemmas 3.3 and 3.4 and the Hölder inequality, we have

$$\begin{aligned}
&\Big\| \int_0^t U(t-s)\partial_{x_1}(u^{m+1})ds \Big\|_{\mathcal{X}} \\
&\le C \sum_{k\in\mathbb{Z}^2} \sum_{k^{(1)},\cdots,k^{(m+1)}\in\mathbb{Z}^2} \chi_{\Omega_k} \big\|\Box_k(\Box_{k^{(1)}}u \cdots \Box_{k^{(m+1)}}u)\big\|_{L^1_{x_1}L^2_{\bar{x},t}} \\
&\le C \sum_{k^{(1)},\cdots,k^{(m+1)}\in\mathbb{Z}^2} \big\|\Box_{k^{(1)}}u\big\|_{L^{m+1}_{x_1}L^{2(m+1)}_{\bar{x},t}} \cdots \big\|\Box_{k^{(m+1)}}u\big\|_{L^{m+1}_{x_1}L^{2(m+1)}_{\bar{x},t}} \\
&= C\big(\|u\|_{\mathcal{X}}\big)^{m+1}.
\end{aligned} \tag{3.3}$$

Similarly, we have by Lemmas 3.2 and 3.4

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^2} \Big\|\Box_k \int_0^t U(t-s)\partial_{x_1}(u^{m+1})ds\Big\|_{L^\infty_t L^2_x} \\
&\le C \sum_{k\in\mathbb{Z}^2} \sum_{k^{(1)},\cdots,k^{(m+1)}\in\mathbb{Z}^2} \chi_{\Omega_k} \big\|\Box_k(\Box_{k^{(1)}}u \cdots \Box_{k^{(m+1)}}u)\big\|_{L^1_{x_1}L^2_{\bar{x},t}} \\
&\le C\big(\|u\|_{\mathcal{X}}\big)^{m+1}.
\end{aligned} \tag{3.4}$$

Note that, from Lemma 3.4, the summations on $k \in \mathbb{Z}^2$ in the above two expressions are finitely many.

Collecting (3.2) and (3.3), we have

$$\big\|\mathcal{N}[u]\big\|_{\mathcal{X}} \le C\big(\|u_0\|_{M_{2,1}} + \|u\|_{\mathcal{X}}^{m+1}\big)$$

and

$$\big\|\mathcal{N}[u] - \mathcal{N}[v]\big\|_{\mathcal{X}} \le C\big(\|u\|_{\mathcal{X}}^m + \|v\|_{\mathcal{X}}^m\big)\|u-v\|_{\mathcal{X}}.$$

Hence, if we assume that $\|u_0\|_{M_{2,1}} \le \rho/2$ with a sufficiently small $\rho > 0$, then $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is a strict contraction mapping. By using a fixed point argument, (1.7) has a unique solution in $\mathcal{X}$. Moreover, from (3.4), it is clear that $u \in C\big([0,\infty), M_{2,1}\big)$.

Next, we show the local well-posedness for any initial data in $M_{2,1}(\mathbb{R}^2)$. We put

$$\mathcal{X}_T = \Big\{u \in \mathcal{S}' : \sum_{k\in\mathbb{Z}^2} \|\Box_k u\|_{L_{x_1}^{m+1} L_{\bar{x},T}^{2(m+1)}} \le \rho\Big\},$$

and prove that the mapping $\mathcal{N} : \mathcal{X}_T \to \mathcal{X}_T$ is a contraction mapping. Suppose that $u_0 \in M_{2,1}$. From Lemma 3.1, we see that there exists $K = K(u_0,\rho) > 0$ such that

$$\sum_{|k|>K} \big\|\Box_k U(t)u_0\big\|_{L_{x_1}^{m+1} L_{\bar{x},T}^{2(m+1)}} \le C \sum_{|k|>K} \|\Box_k u_0\|_{L_x^2} \le \rho/4.$$

Moreover, it is clear that there exists (sufficiently small) $T = T(u_0,\rho) > 0$ such that

$$\sum_{|k|\le K} \big\|\Box_k U(t)u_0\big\|_{L_{x_1}^{m+1} L_{\bar{x},T}^{2(m+1)}} \le \rho/4.$$

Collecting the above two expressions, we have for sufficiently small $T > 0$

$$\big\|U(t)u_0\big\|_{\mathcal{X}_T} = \sum_{k\in\mathbb{R}^2} \big\|\Box_k U(t)u_0\big\|_{L_{x_1}^{m+1} L_{\bar{x},T}^{2(m+1)}} \le \rho/2,$$

and hence we see that $U(t)u_0 \in \mathcal{X}_T$. Repeating the previous proof, we easily see that there exists $T = T(u_0,\rho) > 0$ such that

$$\big\|\mathcal{N}[u]\big\|_{\mathcal{X}_T} \le \rho/2 + C\|u\|_{\mathcal{X}_T}^{m+1}$$

and

$$\big\|\mathcal{N}[u] - \mathcal{N}[v]\big\|_{\mathcal{X}_T} \le C\big(\|u\|_{\mathcal{X}_T}^m + \|v\|_{\mathcal{X}_T}^m\big)\|u-v\|_{\mathcal{X}_T}.$$

Therefore, if we choose $\rho > 0$ properly, then $\mathcal{N} : \mathcal{X}_T \to \mathcal{X}_T$ is a strict contraction mapping. By using a fixed point argument, (1.7) has a unique solution in $\mathcal{X}_T$. Moreover, from the local-in-time version of estimate (3.4), it is clear that $u \in C\big([0,T], M_{2,1}\big)$. $\Box$

Remark 4. We remark that the time $T>0$ in the proof of the local well-posedness depends on the initial data u_0 itself and not on the norm of the data. Moreover, $\rho>0$ in $\mathcal{X}_T$ should be small (it is sufficient to choose ρ satisfies $C\rho^m \le 1/4$). These features are same as those for the generalized KdV equation in [20, 44].

Remark 5. Note that the solution obtained in Theorem 1.1 (i) scatters in $M_{2,1}(\mathbb{R}^2)$, that is, there exists $\rho>0$ such that for any $u_0 \in M_{2,1}(\mathbb{R}^2)$ with $\|u_0\|_{M_{2,1}} \le \rho$ there exists the unique functions $f_\pm \in M_{2,1}(\mathbb{R}^2)$ such that

$$\lim_{t\to\pm\infty} \left\|u(t) - U(t)f_\pm\right\|_{M_{2,1}} \to 0$$

(see also the proof of [20, Theorem 2.14] for the generalized KdV equation, or [36]).

To obtain the above statement, it suffices to prove that for $0<s<\tilde{s}$

$$\lim_{s,\tilde{s}\to\infty} \left\| \int_s^{\tilde{s}} U(-s)\partial_{x_1}(u^{m+1})ds \right\|_{M_{2,1}} = 0. \tag{3.5}$$

From Lemma 3.2, it follows that

$$\begin{aligned} &\left\| \int_s^{\tilde{s}} U(-s)\partial_{x_1}(u^{m+1})ds \right\|_{M_{2,1}} \\ &\le C \left(\sum_{k\in\mathbb{Z}^2} \left\|\Box_k u\right\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},t\in[s,\tilde{s}]}} \right)^{m+1}. \end{aligned}$$

Now, recalling the proof of the local well-posedness, we see that for any $\varepsilon>0$ there exists $K\in\mathbb{N}$ such that

$$\sum_{|k|>K} \left\|\Box_k u\right\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},t\in[s,\tilde{s}]}} \le \sum_{|k|>K} \left\|\Box_k u\right\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},t}} \le \varepsilon/2.$$

since $u\in\mathcal{X}$. Note that $\lim_{s,\tilde{s}\to\infty} \|\Box_k u\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},t\in[s,\tilde{s}]}} = 0$ for $|k|\le K$. Then, for any $\varepsilon>0$, there exists $S>0$ such that for any $s,\tilde{s}>S$

$$\sum_{|k|\le K} \left\|\Box_k u\right\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},t\in[s,\tilde{s}]}} \le \varepsilon/2,$$

since $u\in\mathcal{X}$. Collecting the above two statements, it follows that

$$\lim_{s,\tilde{s}\to\infty} \sum_{k\in\mathbb{Z}^2} \left\|\Box_k u\right\|_{L^{m+1}_{x_1} L^{2(m+1)}_{\bar{x},t\in[s,\tilde{s}]}} = 0,$$

and hence, we obtain (3.5). Then, setting as

$$f_+ = u_0 - \int_0^\infty U(-s)\partial_{x_1}(u^{m+1})ds,$$

and using (3.5), we easily see that the scattering result holds true.

Remark 6. We establish an function belonging to $M_{2,1}(\mathbb{R}^2)$, but not to $\dot{H}^{s_c}(\mathbb{R}^2)$, where, $s_c = 1 - 2/m$. One can refer to [42, Appendix B]. Set a function f as

$$\widehat{f}(\xi) = \varepsilon \sum_{j\in\mathbb{N}} j^{-2}\chi_{\Omega_j}(\xi)$$

for any $\varepsilon > 0$, where χ_{Ω_j} is a characteristic function on $\Omega_j = (2^j, 0) + \big[-\frac{1}{2}, \frac{1}{2}\big]^2$ (the unit cube centered at $\big(2^j, 0\big)$). Also, we can easily justify that there exists such a function f. Then, it follows that $\|f\|_{M_{2,1}} = C\varepsilon$ and $\|f\|_{\dot{H}^{s_c}} = \infty$ for any $\varepsilon > 0$. In fact, from the Plancherel theorem,

$$\|f\|_{M_{2,1}} = C\varepsilon \cdot \sum_{k\in\mathbb{Z}^2} \Big\|\sigma_k(\xi) \cdot \sum_{j\in\mathbb{N}} j^{-2}\chi_{\Omega_j}(\xi)\Big\|_{L^2}.$$

Observe that the support of χ_{Ω_j} is pairwise disjoint. Then,

$$\begin{aligned}\|f\|_{M_{2,1}} &= C\varepsilon \cdot \sum_{j\in\mathbb{N}} j^{-2} \sum_{\ell\in\Lambda} \Big\|\sigma_{(2^j,0)+\ell}(\xi) \cdot \chi_{\Omega_j}(\xi)\Big\|_{L^2} \\ &= C'\varepsilon \cdot \sum_{j\in\mathbb{N}} j^{-2} \\ &= C''\varepsilon,\end{aligned}$$

where the constants C, C' and C'' depend only on dimensions and $\Lambda = \{\ell \in \mathbb{Z}^2 : \max\big(|\ell_1|, |\ell_2|\big) \le 1 \text{ for } \ell = (\ell_1, \ell_2)\}$. On the other hand, we similarly have

$$\begin{aligned}\|f\|^2_{\dot{H}^{s_c}} &= C\varepsilon^2 \cdot \Big\||\xi|^{1-2/m} \sum_{j\in\mathbb{N}} j^{-2}\chi_{\Omega_j}(\xi)\Big\|^2_{L^2} \\ &= C\varepsilon^2 \cdot \sum_{j\in\mathbb{N}} j^{-4} \Big\||\xi|^{1-2/m}\chi_{\Omega_j}(\xi)\Big\|^2_{L^2} \\ &\ge C'\varepsilon^2 \cdot \sum_{j\in\mathbb{N}} j^{-4} 2^{2j(1-\frac{2}{m})} \Big\|\chi_{\Omega_j}(\xi)\Big\|^2_{L^2} \\ &= C'\varepsilon^2 \cdot \sum_{j\in\mathbb{N}} j^{-4} 2^{2j(1-\frac{2}{m})},\end{aligned}$$

where the constants C and C' depend only on dimensions. Recalling that $1 - \frac{2}{m} > 0$ since we assume $m \ge 4$, then we see that $\|f\|_{\dot{H}^{s_c}} = \infty$ for any $\varepsilon > 0$. We also remark that we have also $\|f\|_{\dot{B}^{s_c}_{2,\infty}} = \infty$ by the similar argument. Indeed, it follows that for the Littlewood-Paley decomposition φ_ℓ $(\ell \in \mathbb{Z})$

$$\begin{aligned}\|f\|_{\dot{B}^{s_c}_{2,\infty}} &= C\varepsilon \cdot \sup_{\ell\in\mathbb{Z}} 2^{\ell(1-2/m)} \Big\|\varphi_\ell(\xi) \sum_{j\in\mathbb{N}} j^{-2}\chi_{\Omega_j}(\xi)\Big\|_{L^2} \\ &= C\varepsilon \cdot \sup_{\ell\ge 0} 2^{\ell(1-2/m)} \Big\|\varphi_\ell(\xi) \sum_{j=\ell-1}^{\ell+1} j^{-2}\chi_{\Omega_j}(\xi)\Big\|_{L^2},\end{aligned}$$

where we regard as $\chi_{\Omega_j}(\xi) \equiv 0$ for $j = -1, 0$. As above, since $\chi_{\Omega_j}(\xi)$ is pairwise disjoint, we have

$$\begin{aligned}
\|f\|_{\dot{B}^{s_c}_{2,\infty}} &= C\varepsilon \cdot \sup_{\ell \geq 0} 2^{\ell(1-2/m)} \left(\sum_{j=\ell-1}^{\ell+1} j^{-4} \big\| \varphi_\ell(\xi)\chi_{\Omega_j}(\xi) \big\|_{L^2}^2 \right)^{1/2} \\
&\geq C\varepsilon \cdot \sup_{\ell \geq 1} 2^{\ell(1-2/m)} \ell^{-2} \big\| \varphi_\ell(\xi)\chi_{\Omega_\ell}(\xi) \big\|_{L^2} \\
&= C\varepsilon \cdot \sup_{\ell \geq 1} 2^{\ell(1-2/m)} \ell^{-2} \\
&= \infty
\end{aligned}$$

for any $\varepsilon > 0$. Here, the above constants C depends only on dimensions. Thus, we see that this function f don't also belong to $\dot{B}^{s_c}_{2,\infty}(\mathbb{R}^2)$.

Acknowledgement

The author would like to express his sincere gratitude to Professor Mitsuru Sugimoto for his helpful advice and comments. This work is supported by Grant-in-aid for Scientific Research of JSPS Fellow No. 15J07897.

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