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Remark on the analytic smoothing effect for the Hartree equation

By

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Abstract

We give a review of [19], in which the author studied analytic solutions to the Cauchy problem for the d-dimensional Hartree equation under the assumption that the interaction potential V is in the weak $L^{d/2}$-space. Furthermore, we show some extended results. More precisely, we first give various smoothing effects for the equation. Next, an estimate for the radius of convergence of $\exp(-i|x|^2/(4t))u(t,x)$ is given.

§ 1. Introduction

In this paper, we give a review of the author's previous work [19], and show some extended results. We consider analytic solutions to the Cauchy problem for the nonlinear Schrödinger equation of the form

$$\begin{cases} iu_t + \Delta u = F(u), \\ u(0,x) = \phi(x). \end{cases} \tag{1.1}$$

Here, u is a complex-valued unknown function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, $d \geq 3$, $i = \sqrt{-1}$, Δ is the Laplacian in $\mathbb{R}^d$, $F(u)$ denotes the Hartree term $(V * |u|^2)u$ and $*$ is the convolution in $\mathbb{R}^d$. Throughout this paper, we assume that the interaction potential V is a complex-valued given function on $\mathbb{R}^d$ and belongs to the weak $L^{d/2}$ space. In other words, we assume that

$$\sup_{\lambda>0} \lambda \left| \left\{ x \in \mathbb{R}^d ; |V(x)| > \lambda \right\} \right|^{2/d} < \infty. \tag{1.2}$$

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There is a large literature on the Cauchy problem for nonlinear Schrödinger equations (see, e.g., [1, 11, 21] and references therein). In particular, Mochizuki [12] has proved that if the condition

$$\text{either} \quad |V(x)| \leq C|x|^{-2} \quad \text{or} \quad V \in L^{d/2},$$

which is stronger than (1.2), holds and ϕ is sufficiently small in the L^2-sense, then there exists a time-global solution u to the integral equation of the form

$$u(t) = U(t)\phi - i\int_0^t U(t-t')F(u(t'))dt', \quad t \in \mathbb{R} \tag{1.3}$$

such that $u(t)$ behaves like a free solution $U(t)\phi_+$ in the L^2-sense as $t \to \infty$, where $L^2 = L^2(\mathbb{R}^d)$ and $U(t) = e^{it\Delta}$. In particular, the inverse wave operator $\mathbf{V}_+ : \phi \mapsto \phi_+$ is well-defined on a neighborhood of 0 in L^2.

We now mention the analytic smoothing effect for Schrödiner equations. We first define linear operators

$$M(t) : \mathcal{S}'(\mathbb{R}^d) \ni \psi \mapsto \exp\left(\frac{i|x|^2}{4t}\right)\psi \in \mathcal{S}'(\mathbb{R}^d), \quad t \neq 0$$

and

$$J^\alpha = U(t)x^\alpha U(-t), \quad \alpha \in \mathbb{N}_0^d,\ t \in \mathbb{R}.$$

Then we have for any $t \neq 0$,

$$J^\alpha = M(t)(2it\partial_x)^\alpha M(-t), \quad \alpha \in \mathbb{N}_0^d. \tag{1.4}$$

As for the free Schrödiner equation $iu_t + \Delta u = 0$, it is easy to show that if the initial data ϕ satisfies $e^{\lambda|x|}\phi \in L^2$ for some $\lambda > 0$, then for any $t \neq 0$, the corresponding solution $U(t)\phi(x)$ becomes real-analytic in x. Indeed, since

$$\mathfrak{L}(x, \phi, L^2) := \limsup_{|\alpha|\to\infty} \left(\frac{\|x^\alpha \phi\|_2}{\alpha!}\right)^{1/|\alpha|} < \infty,$$

we see from the Sobolev embedding $W_2^d(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ and the identity (1.4) that

$$\begin{aligned} &\limsup_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha M(-t)U(t)\phi\|_\infty}{\alpha!}\right)^{1/|\alpha|} \leq \limsup_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha M(-t)U(t)\phi\|_2}{\alpha!}\right)^{1/|\alpha|} \\ &= \frac{1}{|2t|}\limsup_{|\alpha|\to\infty} \left(\frac{\|J^\alpha U(t)\phi\|_2}{\alpha!}\right)^{1/|\alpha|} = \frac{1}{|2t|}\limsup_{|\alpha|\to\infty} \left(\frac{\|x^\alpha \phi\|_2}{\alpha!}\right)^{1/|\alpha|} < \infty, \end{aligned} \tag{1.5}$$

and hence that the mapping $x \mapsto M(-t)U(t)\phi(x)$ can be extended a holomorphic function on the domain $\mathbb{R}^d + iP(|2t|/\mathfrak{L}(x,\phi,L^2))$ of $\mathbb{C}^d$ for any $t \neq 0$. Here, we have defined the polydisc $P(r) = (-r,r)^d$ $(0 < r \leq \infty)$.

The analytic smoothing effect still holds for some nonlinear Schrödiner equations and related equations (see, e.g., [2–10, 13–18, 22]). In particular, we can use methods in [8, 9, 15] to show the analyticity of the solutions to (1.1) and more detailed properties provided that $V(x)$ satisfies (1.2). In these methods, one has to assume that the initial data ϕ is small in the sense of some exponential weighted norm. On the other hand, as we mentioned above, when one shows only the global existence and asymptotics of solutions u to (1.1), one has only to assume that ϕ is small in the L^2-sense. Therefore, it is a natural question to ask whether we can show the analytic smoothing effect and related results even if we only to assume that ϕ is small in the L^2-sense and that $e^{\lambda|x|}\phi \in L^2$ for some $\lambda > 0$. The author [19] gave the following positive answers to this question:

(I) We can choose some η so that if $0 < \|\phi\| < \eta$ and

$$\mathfrak{L}(x,\phi,L^2) < \infty, \tag{1.6}$$

then the solution u to (1.1) is real-analytic for any $t \neq 0$, where $\|\cdot\| = \|\cdot\|_{L^2}$. More precisely, the mapping $x \mapsto M(-t)u(t,x)$ can be extended to a holomorphic function on the domain $\mathbb{R}^d + iP(|2t|/C(\phi))$ of $\mathbb{C}^d$. Here, we have defined

$$C(\phi) = \sup_{|\alpha|>0} \left(\frac{(1+|\alpha|)^p \, \|x^\alpha \phi\|}{\alpha! \, \|\phi\|} \right)^{1/|\alpha|} \tag{1.7}$$

and p is a positive constant dependent only on $\|\phi\|$, d and V.

(II) For any $\lambda > 0$ and $0 < \delta < \eta$ there exists some $\phi \in L^2$ such that $\|\phi\| = \delta$ and

$$\sup_{t\neq 0} \limsup_{|\alpha|\to\infty} |2t| \left(\frac{\|\partial_x^\alpha M(-t)u(t)\|}{\alpha!} \right)^{1/|\alpha|} \leq \mathfrak{L}(x,\phi,L^2) = \lambda.$$

Remark that if $V = 0$, then the above inequality becomes equality for any $\phi \in L^2$ satisfying (1.6).

(III) If ϕ and V satisfy some strong condition, then the mapping $x \mapsto M(-t)u(t,x)$ can be extended to an entire function on $\mathbb{C}^d$ for any $t \neq 0$.

(IV) In the case of the final value problem, we have some properties similar to (I)–(III).

The rest of this paper is organized as follows. In the next section, we state main results in [19] precisely. In Section 3, we introduce extended results. More precisely, we first give various smoothing effects for (1.1). Next, an estimate for the radius of convergence of $M(-t)u(t,x)$ is given. In Sections 4 and 5, we show the extended results.

§ 2. Main results in [19]

We first list some notation used in main results of [19]. For $1 \le a \le \infty$, we denote the Lebesgue space $L^a(\mathbb{R}^d)$ and its norm by L^a and $\|\cdot\|_a$, respectively. For $1 \le a \le \infty$ and $s \in \mathbb{R}$, H^s_a denotes the inhomogeneous Sobolev space $H^s_a(\mathbb{R}^d)$. For $\eta > 0$, by $B_\eta L^2$ we denote the closed ball in L^2 with radius η centered at origin. We put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a multi-index $\alpha \in \mathbb{N}_0^d$, we set $\langle\alpha\rangle = 1 + |\alpha|$. Put

$$r = \left(\frac{1}{2} - \frac{2}{3d}\right)^{-1}.$$

We denote $L^3(\mathbb{R}; L^r)$ and $(C \cap L^\infty)(\mathbb{R}; L^2) \cap L^3(\mathbb{R}; L^r)$ by Y and Z, respectively.

$$Z^\infty = \left\{v \in Z; \partial_x^\alpha v \in Z \ (\alpha \in \mathbb{N}_0^d)\right\}, \quad Z_\infty = \left\{v \in Z; J^\alpha v \in Z \ (\alpha \in \mathbb{N}_0^d)\right\},$$

and

$$H^\infty = \bigcap_{k=0}^{\infty} H^k, \quad H_\infty = \bigcap_{k=0}^{\infty} \mathcal{F}H^k,$$

where $\mathcal{F}$ is the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$. For a Banach space $\mathcal{X} \subset \mathcal{S}'(\mathbb{R}^d)$ and $\psi \in \mathcal{X}$, we put

$$\mathfrak{L}(\partial, \psi, \mathcal{X}) = \limsup_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha \psi\|_{\mathcal{X}}}{\alpha!}\right)^{1/|\alpha|}$$

and

$$\mathfrak{L}(J, \psi, \mathcal{X}) = \limsup_{|\alpha|\to\infty} \left(\frac{\|J^\alpha \psi\|_{\mathcal{X}}}{\alpha!}\right)^{1/|\alpha|}.$$

Remark that the embedding $H_q^{d+1}(\mathbb{R}^d) \hookrightarrow L^\infty$ implies that $\mathfrak{L}(\partial, \psi, L^\infty) \le \mathfrak{L}(\partial, \psi, L^q)$ for any $q \in [1, \infty]$. In particular, if $\mathfrak{L}(\partial, \psi, L^q) < \infty$ then ψ can be extended to a holomorphic function on the domain $\mathbb{R}^d + iP(1/\mathfrak{L}(\partial, \psi, L^q))$ of $\mathbb{C}^d$. For a Banach space $\mathcal{Y} \subset \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ and $v \in \mathcal{Y}$, we put

$$\mathfrak{L}(J, v, \mathcal{Y}) = \limsup_{|\alpha|\to\infty} \left(\frac{\|J^\alpha v\|_{\mathcal{Y}}}{\alpha!}\right)^{1/|\alpha|}.$$

For $\psi \in H^\infty$ and $p > 0$, we define $C^{\psi,p} = 0$ if $\psi = 0$, and

$$C^{\psi,p} = \sup_{|\alpha|>0} \left(\frac{\langle\alpha\rangle^p \|\partial_x^\alpha \psi\|}{\alpha! \|\psi\|}\right)^{1/|\alpha|}, \quad \psi \neq 0.$$

For $\psi \in H_\infty$ and $p > 0$, we define $C_{\psi,p} = 0$ if $\psi = 0$, and

$$C_{\psi,p} = \sup_{|\alpha|>0} \left(\frac{\langle\alpha\rangle^p \|x^\alpha \psi\|}{\alpha! \|\psi\|} \right)^{1/|\alpha|}, \quad \psi \neq 0.$$

We remark that $C^{\psi,p}$ (resp. $C_{\psi,p}$) is finite for any $p > 0$ provided that $\mathfrak{L}(\partial, \psi, L^2) < \infty$ (resp. $\mathfrak{L}(x, \psi, L^2) < \infty$).

We are ready to state main results of [19] precisely.

Theorem 2.1. *Assume (1.2). Then a positive number $\eta > 0$ satisfies the following properties:*

(1) For any $\phi \in B_\eta L^2$, there exists a unique solution $u \in Z$ to (1.3) and a function $\phi_+ \in L^2$ such that $U(-t)u(t) \to \phi_+$ as $t \to +\infty$ in L^2. Hence the inverse wave operator $\mathbf{V}_+ : B_\eta L^2 \ni \phi \mapsto \phi_+ \in L^2$ is well-defined.

(2) If $\phi \in B_\eta L^2 \cap H^\infty$ and $\mathfrak{L}(\partial, \phi, L^2) < \infty$, then $u \in Z^\infty$, $\mathbf{V}_+(\phi) \in H^\infty$ and

$$\mathfrak{L}(\partial, u, Z),\ \mathfrak{L}(\partial, \mathbf{V}_+(\phi), L^2) \leq C^{\phi,p}.$$

Here, p is a positive constant dependent only on $\|\phi\|$, d and V.

(3) If $\phi \in B_\eta L^2 \cap H_\infty$ and $\mathfrak{L}(x, \phi, L^2) < \infty$, then $u \in Z_\infty$, $\mathbf{V}_+(\phi) \in H_\infty$ and

$$\sup_{t\neq 0} |2t| \mathfrak{L}(\partial, M(-t)u(t), L^2),\ \mathfrak{L}(J, u, Z),\ \mathfrak{L}(x, \mathbf{V}_+(\phi), L^2) \leq C_{\phi,p}.$$

Here, p is a positive constant dependent only on $\|\phi\|$, d and V. In particular, it follows that $M(-t)u(t,x)$ $(t \neq 0)$ (resp. $\mathcal{F}\mathbf{V}_+(\phi)(x)$) can be extended to a holomorphic function on the domain $\mathbb{R}^d + iP(|2t|/C_{\phi,p})$ (resp. $\mathbb{R}^d + iP(1/C_{\phi,p})$).

Remark. Property (3) indicates that the analytic smoothing effect still holds even if we assume only that ϕ is small in the L^2-sense and that $e^{\lambda|x|}\phi \in L^2$ for some $\lambda > 0$.

Remark. The proof of Theorem 2.1 is quite similar to that of Theorem 3.1 shown in Section 4.

Under the assumption $\phi \in B_\eta L^2 \cap H^\infty$ and $\mathfrak{L}(\partial, \phi, L^2) < \infty$ (resp. $\phi \in B_\eta L^2 \cap H_\infty$ and $\mathfrak{L}(x, \phi, L^2) < \infty$), it is clear that $\mathfrak{L}(\partial, \phi, L^2) \leq \mathfrak{L}(\partial, u, Z)$ (resp. $\mathfrak{L}(x, \phi, L^2) \leq \mathfrak{L}(J, u, Z)$). For some ϕ, the inequality becomes the equality, which is natural in the case of the free Schrödinger equation.

Corollary 2.2. *Assume (1.2) and let η be the number appearing in Theorem 2.1. Fix $\delta \in (0, \eta)$ and $\lambda > 0$.*

(1) *There exists some $\phi \in H^\infty$ such that $\|\phi\| = \delta$ and the solution u to (1.3) and the function $\mathbf{V}_+(\phi)$ satisfy that*

$$\mathfrak{L}(\partial, \mathbf{V}_+(\phi), L^2) \leq \sup_{t\in\mathbb{R}} \mathfrak{L}(\partial, u(t), L^2) = \mathfrak{L}(\partial, u, Z) = \mathfrak{L}(\partial, \phi, L^2) = \lambda.$$

(2) *There exists some $\phi \in H_\infty$ such that $\|\phi\| = \delta$ and the solution u to (1.3) and the function $\mathbf{V}_+(\phi)$ satisfy that*

$$\mathfrak{L}(x, \mathbf{V}_+(\phi), L^2),\ \sup_{t\neq 0} |2t| \mathfrak{L}(\partial, M(-t)u(t), L^2) \leq \mathfrak{L}(J, u, Z) = \mathfrak{L}(x, \phi, L^2) = \lambda.$$

Remark. Applying known methods, one can obtain other estimates for $\mathfrak{L}(J, u, Z)$. For example, if we use the norm

$$\phi \mapsto \|\phi\|_{E(A)} := \sum_\alpha \frac{\|x^\alpha \phi\|}{\alpha!} A^{|\alpha|},$$

which was defined in [8, 15], then we can choose positive constants $\delta, C > 0$ so that for any $\phi \in L^2$ and $A > 0$ with $\|\phi\|_{E(A)} \leq \delta$, the solution u to (1.3) satisfies

$$\sum_\alpha \frac{\|J^\alpha u\|_Z}{\alpha!} A^{|\alpha|} \leq C\delta.$$

Hence we obtain

$$\mathfrak{L}(J, u, Z) \leq \inf \left\{ \frac{1}{A}; \|\phi\|_{E(A)} \leq \delta \right\}.$$

Unfortunately, it seems that such estimates are not applicable to prove Corollary 2.2.

Sketch of the proof of Corollary 2.2. We prove only (2) since (1) can be shown similarly. Let $p > 0$. Fix $m \in \mathbb{N}$ and $a \in (0, 1)$. Define the function

$$\varphi(y) = \chi_{[a,\infty)}(y) y^{-m-1/2} e^{-y}, \quad y \in \mathbb{R},$$

where $\chi_{[a,\infty)}$ is the indicator function of $[a, \infty)$. If $m > [2p] + 3$ and a is sufficiently small, then the function

$$\Phi(x) := \varphi(x_1) \cdots \varphi(x_d) \quad \big(x = (x_1, \cdots, x_d) \in \mathbb{R}^d\big)$$

satisfies $\Phi \in H_\infty$ and

$$\sup_{|\alpha|>0} \left(\frac{\langle \alpha \rangle^p \, \|x^\alpha \Phi\|}{\alpha! \, \|\Phi\|} \right)^{1/|\alpha|} = \mathfrak{L}(x, \Phi, L^2) = 1.$$

Set

$$\phi(x) = \delta\lambda^{-d/2}\Phi(\lambda^{-1}x)/\|\Phi\|, \quad x \in \mathbb{R}^d.$$

Then we obtain $\|\phi\| = \delta$ and $C_{\phi,p} = \mathfrak{L}(x, \phi, L^2) = \lambda$. By Theorem 2.1 and (1.4), we have the desired properties. □

It is a natural and interesting question to ask whether the solution $u(t)$ can be extended to an entire function on $\mathbb{C}^d$ provided that ϕ satisfies some strong condition. The following result is a partial answer:

Corollary 2.3. *Assume (1.2) and let η be the number appearing in Theorem 2.1. Assume, in addition, that $\phi \in B_\eta L^2$.*

(1) If $\phi \in H^\infty$, $\mathfrak{L}(\partial, \phi, L^2) = 0$, $\partial_x^\alpha V \in L^{d/2}$ ($\alpha \in \mathbb{N}_0^d$) and $\mathfrak{L}(\partial, V, L^{d/2}) = 0$, then the solution u to (1.3) and the function $\mathbf{V}_+(\phi)$ satisfy

$$\mathfrak{L}(\partial, u, Z), \ \mathfrak{L}(\partial, \mathbf{V}_+(\phi), L^2) = 0,$$

and for any $\varepsilon > 0$,

$$\lim_{t\to+\infty} \sup_\alpha \frac{\|\partial_x^\alpha (U(-t)u(t) - \mathbf{V}_+(\phi))\|}{\alpha!\varepsilon^{|\alpha|}} = 0.$$

(2) If $\phi \in H_\infty$, $\mathfrak{L}(x, \phi, L^2) = 0$, $\partial_x^\alpha V \in L^\infty$ ($\alpha \in \mathbb{N}_0^d$) and $\mathfrak{L}(\partial, V, L^\infty) = 0$, then for any $t \neq 0$, the solution u to (1.3) satisfies

$$\mathfrak{L}(\partial, M(-t)u(t), L^2) = 0.$$

In particular, $u(t,x)$ ($t \neq 0$) can be extended to the entire function

$$\sum_\alpha \frac{\partial_x^\alpha u(t,0)}{\alpha!} z^\alpha, \quad z \in \mathbb{C}^d.$$

Remark. The proof of Theorem 2.3 is similar to that of Theorem 3.3 shown in Section 5.

A result for the final value problem is obtained as in the proof of Theorem 2.1 and Corollary 2.3.

Theorem 2.4. *Assume (1.2) and let η be the number appearing in Theorem 2.1. Then a positive number $\eta' > 0$ satisfies the following properties:*

(1) For any $\phi \in B_{\eta'}L^2$, there exists a unique solution $u \in Z$ to

$$u(t) = U(t)\phi - i\int_{-\infty}^{t} U(t-t')F(u(t'))dt', \quad t \in \mathbb{R}$$

such that $u(0) \in B_\eta L^2$ and $U(-t)u(t) \to \phi$ as $t \to -\infty$ in L^2. Hence the wave operator $\mathbf{W}_- : B_{\eta'}L^2 \ni \phi \mapsto u(0) \in B_\eta L^2$ and the scattering operator $\mathbf{S} = \mathbf{V}_+ \circ \mathbf{W}_- : B_{\eta'}L^2 \to L^2$ are well-defined.

(2) If $\phi \in B_{\eta'}L^2 \cap H^\infty$ and $\mathfrak{L}(\partial, \phi, L^2) < \infty$, then $u \in Z^\infty$, $\mathbf{W}_-(\phi), \mathbf{S}(\phi) \in H^\infty$ and

$$\mathfrak{L}(\partial, u, Z),\ \mathfrak{L}(\partial, \mathbf{W}_-(\phi), L^2),\ \mathfrak{L}(\partial, \mathbf{S}(\phi), L^2) \le C^{\phi,p}.$$

Here, p is a positive constant dependent only on $\|\phi\|$, d and V. Assume, in addition, that $\phi \in H^\infty$, $\mathfrak{L}(\partial, \phi, L^2) = 0$, $\partial_x^\alpha V \in L^{d/2}$ $(\alpha \in \mathbb{N}_0^d)$ and $\mathfrak{L}(\partial, V, L^{d/2}) = 0$, then

$$\mathfrak{L}(\partial, u, Z),\ \mathfrak{L}(\partial, \mathbf{W}_-(\phi), L^2),\ \mathfrak{L}(\partial, \mathbf{S}(\phi), L^2) = 0,$$

and for any $\varepsilon > 0$,

$$\lim_{t\to-\infty}\sup_\alpha \frac{\|\partial_x^\alpha (U(-t)u(t) - \mathbf{W}_-(\phi))\|}{\alpha!\varepsilon^{|\alpha|}} = \lim_{t\to+\infty}\sup_\alpha \frac{\|\partial_x^\alpha (U(-t)u(t) - \mathbf{S}(\phi))\|}{\alpha!\varepsilon^{|\alpha|}} = 0.$$

(3) If $\phi \in B_{\eta'}L^2 \cap H_\infty$ and $\mathfrak{L}(x, \phi, L^2) < \infty$, then $u \in Z_\infty$, $\mathbf{W}_-(\phi), \mathbf{S}(\phi) \in H_\infty$ and

$$\mathfrak{L}(J, u, Z),\ \mathfrak{L}(x, \mathbf{W}_-(\phi), L^2),\ \mathfrak{L}(x, \mathbf{S}(\phi), L^2) \le C_{\phi,p}.$$

Here, p is a positive constant dependent only on $\|\phi\|$, d and V.

§ 3. Extended results

In this section, we give some properties which are an extended version of Theorem 2.1 and Corollary 2.3. For this purpose, we list some notation. By $\mathbf{0}$, we denote the zero multi-index in d-dimensions. For $\mu \ge 1$, by $\mathcal{S}_\mu$ we denote the set of all functions ψ on $\mathbb{R}^d$ such that

$$\mathfrak{L}_\mu(x, \psi, L^2) := \limsup_{|\alpha|\to\infty} \left(\frac{\|x^\alpha\psi\|}{\alpha!^\mu}\right)^{1/|\alpha|} < \infty.$$

Remark that for any $\mu \ge 1$, The following three conditions are equivalent to each other:

(i) $\psi \in \mathcal{S}_\mu$.

(ii) $\psi \in H_\infty(\mathbb{R}^d)$ and some positive constants C and A satisfy that

$$\|x^\alpha\psi\| \le CA^{|\alpha|}\alpha!^\mu, \quad \alpha \in \mathbb{N}_0^d.$$

(iii) Some positive constant λ satisfies that

$$e^{\lambda|x|^{1/\mu}}\psi \in L^2.$$

Let $\mu \geq 1$ and $t \neq 0$. If $U(-t)u(t) \in \mathcal{S}_\mu$, then we see from the proof of (1.5) that

$$\limsup_{|\alpha|\to\infty} \left(\frac{\|\partial_x^\alpha M(-t)u(t)\|_\infty}{\alpha!^\mu} \right)^{1/|\alpha|} \leq \frac{1}{|2t|} \mathfrak{L}_\mu(x, U(-t)u(t), L^2),$$

and hence that $M(-t)u(t,\cdot) \in G^\mu(\mathbb{R}^d)$. Here, $G^\mu(\mathbb{R}^d)$ is the Gevrey space, which is the set of all C^∞ functions ψ on $\mathbb{R}^d$ such that for any compact subset K of $\mathbb{R}^d$ there exists some $A > 0$ satisfying

$$|\partial_x^\alpha \psi(x)| \leq A^{1+|\alpha|}\alpha!^\mu, \quad x \in K, \ \alpha \in \mathbb{N}_0^d.$$

For a function V on $\mathbb{R}^d$, we define

$$\mathfrak{L}_V = \begin{cases} \mathfrak{L}(\partial, V, L^\infty) \ \text{if } \partial_x^\alpha V \in L^\infty \ (\alpha \in \mathbb{N}_0^d), \\ \infty, \ \text{otherwise}. \end{cases}$$

For $a, b \in \mathbb{R}$, we set $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

The first theorem includes one of smoothing effects for (1.1).

Theorem 3.1. *Assume (1.2). Let $p > d$. Then a positive number $\kappa > 0$ and $C > 0$ satisfy the following properties:*

(1) For any $\phi \in B_\kappa L^2$, there exists a unique solution $u \in Z$ to (1.3) and a function $\phi_+ \in L^2$ such that $U(-t)u(t) \to \phi_+$ as $t \to +\infty$ in L^2. Hence the inverse wave operator $\mathbf{V}_+ : B_\kappa L^2 \ni \phi \mapsto \phi_+ \in L^2$ is well-defined.

(2) If $\phi \in B_\kappa L^2 \cap H^\infty \setminus \{0\}$, then the solution u to (1.3) and the function $\mathbf{V}_+(\phi)$ satisfy that $u \in Z^\infty$, $\mathbf{V}_+(\phi) \in H^\infty$ and

$$\|\partial_x^\alpha u\|_Z, \ \|\partial_x^\alpha \mathbf{V}_+(\phi)\| \leq \frac{C\|\phi\|\,\alpha!}{\langle\alpha\rangle^p} \left\{ \max_{\mathbf{0}\neq\beta\leq\alpha} \left(\frac{\langle\beta\rangle^p \|\partial_x^\beta \phi\|}{\beta!\,\|\phi\|} \right)^{1/|\beta|} \right\}^{|\alpha|}, \ \alpha \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}.$$

(3) If $\phi \in B_\kappa L^2 \cap H_\infty \setminus \{0\}$, then the solution u to (1.3) and the function $\mathbf{V}_+(\phi)$ satisfy that $u \in Z_\infty$, $\mathbf{V}_+(\phi) \in H_\infty$ and

$$\|J^\alpha u\|_Z, \ \|x^\alpha \mathbf{V}_+(\phi)\| \leq \frac{C\|\phi\|\,\alpha!}{\langle\alpha\rangle^p} \left\{ \max_{\mathbf{0}\neq\beta\leq\alpha} \left(\frac{\langle\beta\rangle^p \|x^\beta \phi\|}{\beta!\,\|\phi\|} \right)^{1/|\beta|} \right\}^{|\alpha|}, \ \alpha \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}.$$

In particular, for any $t \neq 0$, the mapping $x \mapsto M(-t)u(t,x)$ is in $C^\infty(\mathbb{R}^d)$.

We establish the above theorem in Section 4. We now give a corollary.

Corollary 3.2. *Assume (1.2). Let $p > d$ and let κ be the number appearing in Theorem 3.1. If $\mu \geq 1$ and $\phi \in B_\kappa L^2 \cap \mathcal{S}_\mu \setminus \{0\}$, then the solution u to (1.3) and the function $\mathbf{V}_+(\phi)$ satisfy that $U(-t)u(t) \in \mathcal{S}_\mu$ $(t \in \mathbb{R})$ and $\mathbf{V}_+(\phi) \in \mathcal{S}_\mu$. Furthermore, for any $t \neq 0$, the mapping $x \mapsto M(-t)u(t,x)$ is in the Gevrey space $G^\mu(\mathbb{R}^d)$.*

Proof. Assume that $\mu \geq 1$ and $\phi \in B_\kappa L^2 \cap \mathcal{S}_\mu \setminus \{0\}$. We see from Theorem 3.1(2) that the solution u to (1.1) and the function $\mathbf{V}_+(\phi)$ satisfy that $u \in Z_\infty$, $\mathbf{V}_+(\phi) \in H_\infty$ and

(3.1)

$$\|J^\alpha u\|_Z\,,\ \|x^\alpha \mathbf{V}_+(\phi)\| \leq \frac{C\,\|\phi\|\,\alpha!}{\langle\alpha\rangle^p}\left\{\max_{\mathbf{0}\neq\beta\leq\alpha}\left(\frac{\langle\beta\rangle^p\,\|x^\beta\phi\|}{\beta!\,\|\phi\|}\right)^{1/|\beta|}\right\}^{|\alpha|},\ \ \alpha \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}.$$

By the inequality

$$\beta!^{1/|\beta|} \leq d\alpha!^{1/|\alpha|} \quad \text{if } \mathbf{0} \neq \beta \leq \alpha \tag{3.2}$$

and the existence of constants K and A such that

$$\|x^\alpha\phi\| \leq KA^{|\alpha|}\alpha!^\mu, \quad \alpha \in \mathbb{N}_0^d,$$

we obtain

$$\begin{aligned}\max_{\mathbf{0}\neq\beta\leq\alpha}\left(\frac{\langle\beta\rangle^p\,\|x^\beta\phi\|}{\beta!\,\|\phi\|}\right)^{1/|\beta|} &\leq \max_{\mathbf{0}\neq\beta\leq\alpha}\left(\frac{\langle\beta\rangle^p\,KA^{|\beta|}\beta!^\mu}{\beta!\,\|\phi\|}\right)^{1/|\beta|}\\ &\leq A\max_{\mathbf{0}\neq\beta\leq\alpha}\left(\frac{\langle\beta\rangle^p\,K}{\|\phi\|}\right)^{1/|\beta|}\max_{\mathbf{0}\neq\beta\leq\alpha}\left(\beta!^{\mu-1}\right)^{1/|\beta|}\\ &\leq \alpha!^{(\mu-1)/|\alpha|}Ad^{\mu-1}\max_{\mathbf{0}\neq\beta\leq\alpha}\left(\frac{\langle\beta\rangle^p\,K}{\|\phi\|}\right)^{1/|\beta|}.\end{aligned}$$

Therefore, it follows from (3.1) that

$$\|J^\alpha u\|_Z\,,\ \|x^\alpha \mathbf{V}_+(\phi)\| \leq \frac{C\,\|\phi\|}{\langle\alpha\rangle^p}\left\{Ad^{\mu-1}\sup_{|\beta|>0}\left(\frac{\langle\beta\rangle^p\,K}{\|\phi\|}\right)^{1/|\beta|}\right\}^{|\alpha|}\alpha!^\mu,\ \ \alpha \in \mathbb{N}_0^d \setminus \{\mathbf{0}\},$$

and hence that $\mathbf{V}_+(\phi) \in \mathcal{S}_\mu$ and $U(-t)u(t) \in \mathcal{S}_\mu$.

For the sake of completeness, we finally show (3.2). For any $n, m \in \mathbb{N}$, we have

$$n!^m \leq (n^m)^n \leq \left(\frac{(n+m)!}{n!}\right)^n,$$

which implies

$$n!^{1/n} \le (n+m)!^{1/(n+m)}.$$

Therefore, we obtain

$$\beta!^{1/|\beta|} \le |\beta|!^{1/|\beta|} \le |\alpha|!^{1/|\alpha|} \quad \text{if } \mathbf{0} \ne \beta \le \alpha.$$

Then (3.2) follows from the inequality

$$\frac{|\alpha|!}{\alpha!} \le \sum_{|\gamma|=|\alpha|} \frac{|\gamma|!}{\gamma!} 1^{\gamma_1} \cdots 1^{\gamma_d} = d^{|\alpha|}, \quad \alpha \in \mathbb{N}_0^d.$$

□

We next introduce an estimate for the radius of convergence of $M(-t)u(t,x)$.

Theorem 3.3. *Assume (1.2). Let $p > d$ and let κ be the number appearing in Theorem 3.1. For any $\phi \in B_\kappa L^2 \cap \mathcal{S}_1$, the solution u to (1.1) and the function $\mathbf{V}_+(\phi)$ satisfy*

$$\mathfrak{L}(\partial, M(-t)u(t), L^2) \le \left(\frac{\mathfrak{L}(x,\phi,L^2)}{|2t|} \vee \mathfrak{L}_V \right) \wedge \frac{C_{\phi,p}}{|2t|}, \quad t \ne 0.$$

Remark. We establish the above theorem in Section 5. Known results (Theorem 2.1 and Corollary 2.3) imply the following properties:

(a) For any $\phi \in B_\eta L^2 \cap \mathcal{S}_1$, it follows that

$$\mathfrak{L}(\partial, M(-t)u(t), L^2) \le \frac{C_{\phi,p}}{|2t|}, \quad t \ne 0$$

for some $p > d$.

(b) If $\mathfrak{L}(x,\phi,L^2) = \mathfrak{L}(\partial, V, L^\infty) = 0$, then

$$\mathfrak{L}(\partial, M(-t)u(t), L^2) = 0, \quad t \ne 0,$$

where v is the solution to (1.1) with $\phi = \psi$.

Hence Theorem 3.3 is strictly stronger than the above (a) and (b) provided that κ is sufficiently small.

§ 4. Proof of Theorem 3.1

Before proving Theorem 3.1, we first mention some inequalities (for the proof, see [19]).

Proposition 4.1 (Strichartz type estimates)**.** *For any $\psi \in L^2$ and $G \in L^1(\mathbb{R}; L^2)$, we have*

$$U(t)\psi, \ \int_0^t U(t-t')G(t')dt' \in Z$$

and

$$\|U(t)\psi\|_Z \le C\|\psi\|, \quad f \in L^2,$$
$$\left\|\int_0^t U(t-t')G(t')dt'\right\|_Z \le C\|G\|_{L^1(\mathbb{R};L^2)}, \quad G \in L^1(\mathbb{R}; L^2).$$

Here, C is a positive constant independent of f and G.

Proposition 4.2 (Estimates for the nonlinearity)**.**

(1) Assume (1.2). Then

$$\|(V * (\psi_1\psi_2))\psi_3\| \le C\|\psi_1\|_r\|\psi_2\|_r\|\psi_3\|_r, \quad \psi_1, \psi_2, \psi_3 \in L^r.$$

Here, C is a positive constant independent of ψ_1, ψ_2 and ψ_3.

(2) If $W \in L^\infty$, then

$$\|(W * (\psi_1\psi_2))\psi_3\| \le \|W\|_\infty\|\psi_1\|\|\psi_2\|\|\psi_3\|, \quad \psi_1, \psi_2, \psi_3 \in L^2.$$

Proposition 4.3. *If $p > d$, then*

$$\sup_\alpha \sum_{\beta+\gamma=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle}\right)^p \le \sup_\alpha \sum_{\beta+\gamma+\delta=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle\langle\delta\rangle}\right)^p < \infty.$$

We next define a function space. For $\phi \in H_\infty \setminus \{0\}$ and $p > 0$, we define $g_\phi(\mathbf{0}) = 1$,

$$g_\phi(\alpha) = \left\{\max_{\substack{\beta\le\alpha \\ |\beta|>0}}\left(\frac{\langle\beta\rangle^p\|x^\beta\phi\|}{\beta!\|\phi\|}\right)^{1/|\beta|}\right\}^{|\alpha|}, \quad \alpha \in \mathbb{N}_0^d \setminus \{\mathbf{0}\}$$

and

$$Z_\phi = \left\{v \in Z_\infty; \|v\|_{Z_\phi} := \sup_{\alpha\in\mathbb{N}_0^d}\frac{\langle\alpha\rangle^p\|J^\alpha v\|_Z}{\alpha! g_\phi(\alpha)} < \infty\right\}.$$

Remark that we obtain for any $\phi \in H_\infty$,

$$\frac{\langle\alpha\rangle^p\|x^\alpha\phi\|}{\alpha! g_\phi(\alpha)} \le \|\phi\|, \quad \alpha \in \mathbb{N}_0^d \tag{4.1}$$

and

$$g_\phi(\beta+\gamma+\delta) \geq g_\phi(\beta)g_\phi(\gamma)g_\phi(\delta), \quad \beta,\gamma,\delta \in \mathbb{N}_0^d. \tag{4.2}$$

The reason why we put the term $\langle\alpha\rangle^p$ in the definition of the Z_ϕ-norm is that we establish the inequality

$$\frac{\langle\alpha\rangle^p \|J^\alpha F(v)\|_{L^1(\mathbb{R};L^2)}}{\alpha! g_\phi(\alpha)} \leq C\left(\max_{\beta\leq\alpha} \frac{\langle\beta\rangle^p \left\|J^\beta v\right\|_Z}{\beta! g_\phi(\beta)}\right)^3, \quad v \in Z_\infty,\ \alpha \in \mathbb{N}_0^d.$$

Proof of Theorem 3.1. Assume (1.2). We have only to show (1) and (3) in the case $\phi \in H_\infty \setminus \{0\}$. Put $p > d$ and $v \in Z_\infty$. By C, we denote a positive constant independent of $\phi, v, \alpha, \beta, \gamma$ and δ. By (1.4), the Leibniz rule and Proposition 4.2(1), we obtain

$$\begin{aligned}
&\|J^\alpha F(v)\|_{L^1(I;L^2)} = \left\|M(t)(2it\partial_x)^\alpha \left\{\left(V * \left(M(-t)v\,\overline{M(-t)v}\right)\right) M(-t)v\right\}\right\|_{L^1(I;L^2)} \\
&\leq \sum_{\beta+\gamma+\delta=\alpha} \left\{\frac{\alpha!}{\beta!\gamma!\delta!}\right. \\
&\quad \times \left.\left\|\left(V * \left((2it\partial_x)^\beta M(-t)v\,\overline{(2it\partial_x)^\gamma M(-t)v}\right)\right) M(t)(2it\partial_x)^\delta M(-t)v\right\|_{L^1(I;L^2)}\right\} \\
&\leq \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!}{\beta!\gamma!\delta!} \left\|\left(V * \left(J^\beta v\,\overline{J^\gamma v}\right)\right) J^\delta v\right\|_{L^1(I;L^2)}
\end{aligned}$$

and

$$\frac{\|J^\alpha F(v)\|_{L^1(I;L^2)}}{\alpha!} \leq C \sum_{\beta+\gamma+\delta=\alpha} \frac{\left\|J^\beta v\right\|_Z}{\beta!} \frac{\|J^\gamma v\|_Z}{\gamma!} \frac{\left\|J^\delta v\right\|_Z}{\delta!}, \quad \alpha \in \mathbb{N}_0^d.$$

By (4.2) and Proposition 4.3, we have

$$\begin{aligned}
&\frac{\langle\alpha\rangle^p \|J^\alpha F(v)\|_{L^1(I;L^2)}}{\alpha! g_\phi(\alpha)} \\
&\leq C \sum_{\beta+\gamma+\delta=\alpha} \left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle\langle\delta\rangle}\right)^p \frac{\langle\beta\rangle^p \left\|J^\beta v\right\|_Z}{\beta! g_\phi(\beta)} \frac{\langle\gamma\rangle^p \|J^\gamma v\|_Z}{\gamma! g_\phi(\gamma)} \frac{\langle\delta\rangle^p \left\|J^\delta v\right\|_Z}{\delta! g_\phi(\delta)} \\
&\leq C\left(\max_{\beta\leq\alpha} \frac{\langle\beta\rangle^p \left\|J^\beta v\right\|_Z}{\beta! g_\phi(\beta)}\right)^3, \quad \alpha \in \mathbb{N}_0^d.
\end{aligned}$$

Therefore, it follows from Proposition 4.1 and (4.1) that the mapping

$$Z_\phi \ni v \mapsto \widetilde{v} := U(t)\phi - i\int_0^t U(t-t')F(v(t'))dt' \in Z_\phi$$

is well-defined,

$$\|\widetilde{v}\|_Z \le C\|\phi\| + C\|v\|_Z^3, \quad v \in Z$$

and

$$\|\widetilde{v}\|_{Z_\phi} \le C\|\phi\| + C\|v\|_{Z_\phi}^3, \quad v \in Z_\phi. \tag{4.3}$$

Similarly, we have

$$\|\widetilde{v_1} - \widetilde{v_2}\|_Z \le C \max_{j=1,2} \|v_j\|_Z^2 \|v_1 - v_2\|_Z, \quad v_1, v_2 \in Z$$

and

$$\|\widetilde{v_1} - \widetilde{v_2}\|_{Z_\phi} \le C \max_{j=1,2} \|v_j\|_{Z_\phi}^2 \|v_1 - v_2\|_{Z_\phi}, \quad v_1, v_2 \in Z_\phi.$$

We see from the standard contraction argument that if $\kappa > 0$ is sufficiently small and $\phi \in B_\kappa L^2$, then (1.3) has the unique solution u in the sense of Z, and that $u \in Z_\phi$ and

$$\|u\|_{Z_\phi} \le C\|\phi\|. \tag{4.4}$$

By (4.4), we obtain for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| > 0$,

$$\|J^\alpha u\|_Z \le \|u\|_{Z_\phi} \frac{\alpha! g_\phi(\alpha)}{\langle\alpha\rangle^p} \le \frac{C\|\phi\|\,\alpha!}{\langle\alpha\rangle^p} \left\{ \max_{\mathbf{0}\ne\beta\le\alpha} \left(\frac{\langle\beta\rangle^p \left\|x^\beta\phi\right\|}{\beta!\,\|\phi\|} \right)^{1/|\beta|} \right\}^{|\alpha|}.$$

As in the proof of (4.3), we see from the formula

$$\mathbf{V}_+(\phi) = \phi - i\int_0^\infty U(-t')F(u(t'))dt'$$

that $\mathbf{V}_+(\phi) \in H_\infty$ and

$$\|x^\alpha \mathbf{V}_+(\phi)\| \le \frac{C\|\phi\|\,\alpha!}{\langle\alpha\rangle^p} \left\{ \max_{\mathbf{0}\ne\beta\le\alpha} \left(\frac{\langle\beta\rangle^p \left\|x^\beta\phi\right\|}{\beta!\,\|\phi\|} \right)^{1/|\beta|} \right\}^{|\alpha|}.$$

Hence Theorem 3.1 holds true. □

§ 5. Proof of Theorem 3.3

In this section, we show Theorem 3.3. Assume (1.2). Let $p > d$ and let κ be the number appearing in Theorem 3.1. Fix $\varepsilon > 0$ and $t \ne 0$. Define $\mathfrak{L}_{\phi,\varepsilon} = \mathfrak{L}(x, \phi, L^2) + \varepsilon$, $\mathfrak{L}_{V,\varepsilon} = \mathfrak{L}_V + \varepsilon$ and $\mathfrak{L}_\varepsilon(t) = \mathfrak{L}_{\phi,\varepsilon} \vee |2t|\mathfrak{L}_{V,\varepsilon}$. Assume in addition that $\phi \in B_\kappa L^2 \cap \mathcal{S}_1$.

Let u be the solution to (1.3). We see from Theorem 3.1(3) that $\mathfrak{L}(\partial, M(-t)u(t), L^2) \leq C_{\phi,p}/|2t|$. Therefore, it suffices to show that $\mathfrak{L}(\partial, M(-t)u(t), L^2) \leq \mathfrak{L}_\varepsilon(t)/|2t|$ in the case $\phi, V \neq 0$ and $\mathfrak{L}_V < \infty$. By C, we denote a positive constant independent of $\phi, u, \alpha, \beta, \gamma$ and ε.

We can choose some $N \in \mathbb{N}$ so that

$$\left(\frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|}{\alpha!\,\|\phi\|}\right)^{1/|\alpha|} \leq \mathfrak{L}_{\phi,\varepsilon} \quad \text{and} \quad \left(\frac{\langle\alpha\rangle^{d+1}\|\partial_x^\alpha V\|_\infty}{\alpha!\,\|V\|_\infty}\right)^{1/|\alpha|} \leq \mathfrak{L}_{V,\varepsilon}$$

for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| > N$. Therefore, we have

$$\sup_{|\alpha|>N} \frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|}{\alpha!\mathfrak{L}_{\phi,\varepsilon}^{|\alpha|}} \leq \|\phi\| \quad \text{and} \quad \sup_{|\alpha|>N} \frac{\langle\alpha\rangle^{d+1}\|\partial_x^\alpha V\|_\infty}{\alpha!\mathfrak{L}_{V,\varepsilon}^{|\alpha|}} \leq \|V\|_\infty .$$

Put

$$K_\phi = \max_{|\alpha|\leq N} \frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|}{\alpha!} \quad \text{and} \quad K_V = \max_{|\alpha|\leq N} \frac{\langle\alpha\rangle^{d+1}\|\partial_x^\alpha V\|_\infty}{\alpha!}.$$

Hence we obtain

$$\sup_\alpha \frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|}{\alpha!\mathfrak{L}_{\phi,\varepsilon}^{|\alpha|}} \leq (1 \vee \mathfrak{L}_{\phi,\varepsilon}^{-N})K_\phi \tag{5.1}$$

and

$$\sup_\alpha \frac{\langle\alpha\rangle^{d+1}\|\partial_x^\alpha V\|_\infty}{\alpha!\mathfrak{L}_{V,\varepsilon}^{|\alpha|}} \leq (1 \vee \mathfrak{L}_{V,\varepsilon}^{-N})K_V. \tag{5.2}$$

Fix $\alpha \in \mathbb{N}_0^d$. By (1.4), the Leibniz rule and Proposition 4.2(2), we obtain

$$\begin{aligned}\|J^\alpha F(u(t))\| &= \left\|M(t)(2it\partial_x)^\alpha \left\{\left(V * |u(t)|^2\right) M(-t)u(t)\right\|\right\} \\ &\leq \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \left\|\left((2it\partial_x)^\beta V * |u(t)|^2\right) M(t)(2it\partial_x)^\gamma M(-t)u(t)\right\| \\ &\leq \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}|2t|^{|\beta|} \left\|\left(\partial_x^\beta V * |u(t)|^2\right) J^\gamma u(t)\right\|\end{aligned}$$

and

$$\frac{\|J^\alpha F(u(t))\|}{\alpha!} \leq \|u\|_Z^2 \sum_{\beta+\gamma=\alpha} |2t|^{|\beta|} \frac{\left\|\partial_x^\beta V\right\|_\infty}{\beta!} \frac{\|J^\gamma u(t)\|}{\gamma!}.$$

Define $I=[0,t]$ if $t>0$, and $I=[t,0]$ if $t<0$. By Proposition 4.3 and (5.2), we have for any $t'\in I$,

$$\begin{aligned}&\frac{\langle\alpha\rangle^{d+1}\|J^\alpha F(u(t'))\|}{\alpha!\mathfrak{L}_\varepsilon(t)^{|\alpha|}}\\ &\le \|u\|_Z^2\sum_{\beta+\gamma=\alpha}\left(\frac{\langle\alpha\rangle}{\langle\beta\rangle\langle\gamma\rangle}\right)^{d+1}|2t'|^{|\beta|}\frac{\langle\beta\rangle^{d+1}\|\partial_x^\beta V\|_\infty}{\beta!|2t|^{|\beta|}\mathfrak{L}_{V,\varepsilon}^{|\beta|}}\frac{\langle\gamma\rangle^{d+1}\|J^\gamma u(t')\|}{\gamma!\mathfrak{L}_\varepsilon(t)^{|\gamma|}}\\ &\le C\|u\|_Z^2(1\vee\mathfrak{L}_{V,\varepsilon}^{-N})K_V\max_{\gamma\le\alpha}\frac{\langle\gamma\rangle^{d+1}\|J^\gamma u(t')\|}{\gamma!\mathfrak{L}_\varepsilon(t')^{|\gamma|}}.\end{aligned}$$

It follows from (1.3) and (5.1) that

$$\begin{aligned}\frac{\langle\alpha\rangle^{d+1}\|J^\alpha u(t)\|}{\alpha!\mathfrak{L}_\varepsilon(t)^{|\alpha|}}&\le\frac{\langle\alpha\rangle^{d+1}\|x^\alpha\phi\|}{\alpha!\mathfrak{L}_{\phi,\varepsilon}^{|\alpha|}}+\int_I\frac{\langle\alpha\rangle^{d+1}\|J^\alpha F(u(t'))\|}{\alpha!\mathfrak{L}_\varepsilon(t)^{|\alpha|}}dt'\\ &\le(1\vee\mathfrak{L}_{\phi,\varepsilon}^{-N})K_\phi+C\|u\|_Z^2(1\vee\mathfrak{L}_{V,\varepsilon}^{-N})K_V\int_I\max_{\gamma\le\alpha}\frac{\langle\gamma\rangle^{d+1}\|J^\gamma u(t')\|}{\gamma!\mathfrak{L}_\varepsilon(t')^{|\gamma|}}dt'.\end{aligned}$$

Using the Gronwall inequality, we have

$$\max_{\gamma\le\alpha}\frac{\langle\gamma\rangle^{d+1}\|J^\gamma u(t)\|}{\gamma!\mathfrak{L}_\varepsilon(t)^{|\gamma|}}\le(1\vee\mathfrak{L}_{\phi,\varepsilon}^{-N})K_\phi\exp\Big(C\|u\|_Z^2(1\vee\mathfrak{L}_{V,\varepsilon}^{-N})K_V|t|\Big).$$

Therefore, we obtain

$$\mathfrak{L}(J,u(t),L^2)\le\mathfrak{L}_\varepsilon(t).$$

The desired inequality $\mathfrak{L}(\partial,M(-t)u(t),L^2)\le\mathfrak{L}_\varepsilon(t)/|2t|$ follows from (1.4).

References

[1] Cazenave, T., Semilinear Schrodinger Equations, *Amer-ican Mathematical Society, Providence, RI,* 2003.

[2] Chihara, H., Gain of analyticity for semilinear Schrödinger equations, *J. Differential Equations*, **246** (2009), 681–723.

[3] Hayashi, N. and Kato, K., Analyticity in time and smoothing effect of solutions to nonlinear Schrödinger equations, *Comm. Math. Phys.*, **184** (1997), 273–300.

[4] Hayashi, N., Naumkin, P.I. and Pipolo, P.-N., Analytic smoothing effects for some derivative nonlinear Schrödinger equations, *Tsukuba J. Math.*, **24** (2000), 21–34.

[5] Hayashi, N. and Saitoh, S., Analyticity and global existence of small solutions to some nonlinear Schrödinger equations, *Comm. Math. Phys.*, **129** (1990), 27–41.

[6] Hayashi, N. and Saitoh, S., Analyticity and smoothing effect for the Schrödinger equation, *Ann. Inst. H. Poincare Phys. Theor.*, **52** (1990), 163–173.

[7] Hoshino, G. and Ozawa, T., Analytic smoothing effect for a system of nonlinear Schrödinger equations, *Differ. Equ. Appl.*, **5** (2013), 395–408.

[8] Hoshino, G. and Ozawa, T., Analytic smoothing effect for nonlinear Schrödinger equation in two space dimensions, *Osaka J. Math.*, **51** (2014), 609–618.

[9] Hoshino, G. and Ozawa, T., Analytic smoothing effect for nonlinear Schrödinger equation with quintic nonlinearity, *J. Math. Anal. Appl.*, **419** (2014), 285–297.

[10] Kato, K. and Taniguchi, K., Gevrey regularizing effect for nonlinear Schrödinger equations, *Osaka J. Math.*, **33** (1996), 863–880.

[11] Linares, F. and Ponce, G., Introduction to Nonlinear Dispersive Equations, *Springer, NewYork,* 2009.

[12] Mochizuki, K., On small data scattering with cubic convolution nonlinearity, *J. Math. Soc. Japan*, **41** (1989), 143–160.

[13] Nakamitsu, K., Analytic finite energy solutions of the nonlinear Schrödinger equation, *Comm. Math. Phys.*, **260** (2005), 117–130.

[14] Ozawa, T. and Yamauchi, K., Remarks on analytic smoothing effect for the Schrödinger equation, *Math. Z.*, **261** (2009), 511–524.

[15] Ozawa, T. and Yamauchi, K., Analytic smoothing effect for global solutions to nonlinear Schrödinger equation, *J. Math. Anal. Appl.*, **364** (2010), 492–497.

[16] Ozawa, T., Yamauchi, K. and Yamazaki, Y., Analytic smoothing effect for solutions to Schrödinger equations with nonlinearity of integral type, *Osaka J. Math.*, **42** (2005), 737–750.

[17] Robbiano, L. and Zuily, C., Microlocal analytic smoothing effect for the Schrödinger equation, *Duke Math. J.*, **100** (1999), 93–129.

[18] Robbiano, L. and Zuily, C., Effect régularisant microlocal analytique pour léquation de Schrödinger: le cas donneés oscillantes, *Comm. Partial Differential Equations*, **25** (2000), 1891–1906.

[19] Sasaki, H., Small analytic solutions to the Hartree equation, *J. Funct. Anal.*, **270** (2016), 1064–1090.

[20] Simon, J.C.H. and Taflin, E., Wave operators and analytic solutions of nonlinear Klein-Gordon equations and of nonlinear Schrödinger equations, *Comm. Math. Phys.*, **99** (1985), 541–562.

[21] Sulem, C. and Sulem, P.-L., The Nonlinear Schrodinger Equation. Self-Focusing and Wave Collapse, *Springer,* 1999.

[22] Takuwa, H., Analytic smoothing effects for a class of dispersive equations, *Tsukuba J. Math.*, **28** (2004), 1–34.