

Localized L^p -estimates for eigenfunctions: II

By

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Abstract

If (M, g) is a compact Riemannian manifold of dimension $n \geq 2$ we give necessary and sufficient conditions for improved $L^p(M)$ -norms of eigenfunctions for all $2 < p \neq p_c = \frac{2(n+1)}{n-1}$, the critical exponent. Since improved $L^{p_c}(M)$ bounds imply improvement all other exponents, these conditions are necessary for improved bounds for the critical space. We also show that improved $L^{p_c}(M)$ bounds are valid if these conditions are met and if the half-wave operators, $U(t)$, have no caustics when $t \neq 0$. The problem of finding a necessary and sufficient condition for $L^{p_c}(M)$ improvement remains an interesting open problem.

§ 1. Local and Global Estimates of Eigenfunctions

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. If Δ_g is the associated Laplace-Beltrami operator, we shall consider the L^2 -normalized eigenfunctions satisfying

$$(1.1) \quad -\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x) \quad \text{and} \quad \int_M |e_\lambda|^2 dV_g = 1,$$

with dV_g denoting the Riemannian volume element. The purpose of this paper is to show that one has favorable $L^p(M)$ bounds for the e_λ if and only if there is not saturation of L^p or L^2 norms taken over very small sets that shrink as $\lambda \rightarrow \infty$. The sets depend on whether $p > 2$ is larger or smaller than the critical exponent $p = p_c = \frac{2(n+1)}{n-1}$. For $p > p_c$ there are improved $L^p(M)$ bounds if and only if L^p or L^2 norms over geodesic balls of radius λ^{-1} are not saturated, while for $2 < p < p_c$ one obtains improvement if and only if there is not saturation of these norms taken over $\lambda^{-1/2}$ tubular neighborhoods of

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unit length geodesics. We shall also show that we have improved $L^{p_c}(M)$ bounds if we have improved $L^p(M)$ estimates for all $p \in (2, \infty] \setminus \{p_c\}$ and if the half-wave operators $e^{-it\sqrt{-\Delta_g}}$ have no caustics for non-zero times (see §2).

Recall that in [11] we showed that for $p > 2$ one always has the universal bounds

$$(1.2) \quad \|e_\lambda\|_{L^p(M)} = O(\lambda^{\mu(p)}),$$

if we let

$$(1.3) \quad \mu(p) = \begin{cases} \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p}), & 2 < p \leq p_c, \\ n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}, & p_c \leq p \leq \infty. \end{cases}$$

These norms are saturated on the round sphere S^n ; however, for generic manifolds (M, g) it was shown by Zelditch and the author [16] that $\|e_\lambda\|_{L^p(M)} = o(\lambda^{\mu(p)})$ if $p > p_c$. Whether one generically has improvements for $2 < p < p_c$ or better yet for $p = p_c$ remains an open problem.

Let us now state two of our main results. If $B_r(x)$ denotes a geodesic ball of radius $0 < r < \text{Inj } M$ (the injectivity radius of (M, g)), then the first is the following.

Theorem 1.1. *The following are equivalent:*

$$(1.4) \quad \|e_\lambda\|_{L^p(M)} = o(\lambda^{\mu(p)}), \quad \text{for all } p > p_c,$$

$$(1.5) \quad \sup_{x \in M} \|e_\lambda\|_{L^p(B_{\lambda^{-1}}(x))} = o(\lambda^{\mu(p)}), \quad \text{for some } p > p_c,$$

$$(1.6) \quad \sup_{x \in M} \|e_\lambda\|_{L^2(B_{\lambda^{-1}}(x))} = o(\lambda^{-\frac{1}{2}}).$$

It was shown in [15] and [14] that on any (M, g) one has $\|e_\lambda\|_{L^2(B_r(x))} \leq Cr^{\frac{1}{2}}$ with $C = C_M$ for $\lambda^{-1} \leq r \leq \text{Inj } M$, and so (1.6) just involves improving this universal estimate in the extreme case where the radius $r = \lambda^{-1}$ is the frequency of the eigenfunction.

If Π denotes the space of all geodesics of unit length as measured by the metric on M and if $\mathcal{T}_\delta(\gamma)$ denotes the δ tubular neighborhood about a given $\gamma \in \Pi$, i.e.,

$$\mathcal{T}_\delta(\gamma) = \{x \in M : d_g(x, \gamma) < \delta\},$$

with $d_g(\cdot, \cdot)$ denoting the Riemannian distance function, then we also have the following complementary result.

Theorem 1.2. *The following are equivalent:*

$$(1.7) \quad \|e_\lambda\|_{L^p(M)} = o(\lambda^{\mu(p)}), \quad \text{for all } 2 < p < p_c,$$

$$(1.8) \quad \sup_{\gamma \in \Pi} \|e_\lambda\|_{L^p(\mathcal{T}_{\lambda^{-1/2}}(\gamma))} = o(\lambda^{\mu(p)}), \quad \text{for some } 2 < p < p_c,$$

$$(1.9) \quad \sup_{\gamma \in \Pi} \|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))} = o(1).$$

The scales in the two theorems are very natural. As far as the first one goes, recall that the L^2 -normalized zonal functions, Z_λ , on S^n saturate the L^p bounds in (1.2) for $p \geq p_c$ (see [10]). This is because, modulo lower order terms, the Z_λ behave like an oscillatory factor times $r^{-\frac{n-1}{2}}$ if r is larger than λ^{-1} , with r denoting the minimum of the distance to the two poles on S^n , and $|Z_\lambda| \approx \lambda^{\frac{n-1}{2}}$ if r is smaller than a fixed multiple of λ^{-1} . Using this fact, a simple calculation shows that the quantities in the left side of (1.4)–(1.5) and (1.6) are $\Omega(\lambda^{\mu(p)})$ and $\Omega(\lambda^{-\frac{1}{2}})$, respectively, assuming in the former case, that $p \geq p_c$. Similarly, if we write S^n as $\{x \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\}$ and if the highest weight spherical harmonics, Q_λ , are given by $\lambda^{\frac{n-1}{4}}(x_1 + ix_2)^\lambda$, where $\lambda = \lambda_k = \sqrt{k(k+n-1)}$, $k \in \mathbb{N}$, then these eigenfunctions have $L^2(S^n)$ norms which are comparable to one, and, moreover, the quantities in (1.7)–(1.8) and (1.9) are $\Omega(\lambda^{\mu(p)})$ and $\Omega(1)$, respectively, provided that $2 < p \leq p_c$. Thus, the Z_λ have the largest possible L^2 or L^p , $p \geq p_c$, mass in balls of radius λ^{-1} about either of the two poles, $\pm(0, \dots, 0, 1)$, on S^n , while the Q_λ have the largest possible L^2 or L^p , $2 < p \leq p_c$, mass in tubes of radius $\lambda^{-1/2}$ of the equator where $0 = x_3 = \dots = x_{n+1}$.

To verify Theorem 1.1, first notice that (1.4) trivially implies (1.5). Also, since $\mu(p) = n(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$ for $p \geq p_c$, by Hölder's inequality, (1.5) implies (1.6). To prove the nontrivial part of the theorem, which says that (1.6) implies (1.4), we recall the following recent result of the author from [15], which says that there is a uniform constant $C = C(M, g)$ so that for all $p > 2$

$$(1.10) \quad \|e_\lambda\|_{L^p(B_r(x))} \leq Cr^{-\frac{1}{2}} \lambda^{\mu(p)} \|e_\lambda\|_{L^2(B_{2r}(x))}, \quad \lambda \geq 1, \quad \lambda^{-1} \leq r < \text{Inj } M.$$

See Hezari and Rivière [8] for earlier related work.

We shall use the fact that the special case of (1.10) with $r = \lambda^{-1}$ and $p = \infty$ implies that

$$(1.11) \quad \|e_\lambda\|_{L^\infty(M)} \leq C \lambda^{\mu(\infty)} \left(\lambda^{\frac{1}{2}} \sup_{x \in M} \|e_\lambda\|_{L^2(B_{\lambda^{-1}}(x))} \right).$$

Since for $p_c < p \leq \infty$,

$$\|e_\lambda\|_{L^p(M)} \leq \|e_\lambda\|_{L^\infty(M)}^{\theta(p)} \|e_\lambda\|_{L^{p_c}(M)}^{1-\theta(p)}, \quad \text{if } \theta(p) = \frac{p-p_c}{p},$$

we see that (1.2) and the special case of (1.11) with $p = p_c$ yield

$$(1.11') \quad \|e_\lambda\|_{L^p(M)} \leq C \left(\lambda^{\mu(\infty)} \sup_{x \in M} \lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(B_{\lambda^{-1}}(x))} \right)^{\theta(p)} \times \left(\lambda^{\mu(p_c)} \|e_\lambda\|_{L^2(M)} \right)^{1-\theta(p)} \\ = C \lambda^{\mu(p)} \left(\lambda^{\frac{1}{2}} \sup_{x \in M} \|e_\lambda\|_{L^2(B_{\lambda^{-1}}(x))} \right)^{\theta(p)}, \quad \text{if } p > p_c,$$

due to the fact that the eigenfunctions are L^2 -normalized, and, by (1.3),

$$\mu(p) = \theta(p)\mu(\infty) + (1 - \theta(p))\mu(p_c), \quad \text{if } p > p_c.$$

Clearly (1.11') shows that (1.6) implies (1.4), which completes the proof of Theorem 1.1.

To verify Theorem 1.2, we first note that of course (1.7) implies (1.8). Also, since $\mu(p) = \frac{n-1}{2}(\frac{1}{2} - \frac{1}{p})$ for $2 < p \leq p_c$, by Hölder's inequality, (1.9) follows from (1.8). The nontrivial part, which is that (1.9) implies (1.7) was first established in the two dimensional case by the author in [13] following earlier related partial results of Bourgain [7]. In this work we called the quantity in the left side of (1.9) the "Kakeya-Nikodym" norm of e_λ . For higher dimensions $n \geq 3$, the fact that (1.9) implies (1.7) is due to Blair and the author in [3], and refinements were made in [4], [5] and [2]. Zelditch and the author showed in [17] that when $n = 2$ if (M, g) has nonpositive curvature the Kakeya-Nikodym norms are $o(1)$ and hence one has (1.7), and this result was extended to higher dimensions by Blair and the author in [3].

§ 2. Improved Bounds for the Critical Space

The problem of showing that in certain cases one has $\|e_\lambda\|_{L^{p_c}(M)} = o(\lambda^{\mu(p_c)})$ is much more subtle than showing that either (1.4) or (1.7) is valid. Indeed, as is well known (see e.g. [15]) improvements for the critical space imply ones for all the other spaces. For $2 < p < p_c$ one just uses Hölder's inequality, while for $p > p_c$ one obtains improved L^p bounds from improved ones for the critical space via Sobolev/Bernstein inequalities. Indeed, if ρ as in (2.3) equals one at the origin and has compactly supported Fourier transform then $\rho(\lambda^{-1}(\lambda - \sqrt{-\Delta_g}))e_\lambda = e_\lambda$, and, by the arguments in §5.3, of [12], this operator has a kernel which, for every $N = 1, 2, 3, \dots$, is $O(\lambda^n(1 + \lambda d_g(x, y))^{-N})$ and so, by Young's inequality, $\|\rho(\lambda^{-1}(\lambda + \sqrt{-\Delta_g}))\|_{L^p(M) \rightarrow L^q(M)}$ is $O(\lambda^{n(\frac{1}{p} - \frac{1}{q})})$ for $p \leq q$. Using this fact and (1.3), one immediately sees that improved $L^{p_c}(M)$ bounds lead to improved $L^p(M)$ bounds for all $p > p_c$.

Thus, by Theorems 1.1 and 1.2, if $\|e_\lambda\|_{L^{p_c}(M)} = o(\lambda^{\mu(p_c)})$ one must have (1.6) and (1.9). An interesting question would be if these two necessary conditions for improved critical space bounds are sufficient. Although we cannot answer this question we can adapt arguments from [14] to obtain the following partial result.

Theorem 2.1. *Suppose that (1.6) and (1.9) are valid. Suppose further that if $P = \sqrt{-\Delta_g}$ the half-wave operators,*

$$(2.1) \quad U(t) = e^{-itP},$$

have no caustics when $t \neq 0$. We then have

$$(2.2) \quad \|e_\lambda\|_{L^{p_c}(M)} = o(\lambda^{\mu(p_c)}), \quad p_c = \frac{2(n+1)}{n-1}.$$

The assumption that the half-wave operators in (2.1) have no caustics for nonzero times is equivalent to the assumption that (M, g) has no conjugate points. This is always

the case if (M, g) has nonpositive curvature, and so Theorem 2.1 partly generalizes the results from [14] where it was shown that $\|e_\lambda\|_{L^{p_c}(M)} = O((\log \log \lambda)^{-\sigma_n})$ for some $\sigma_n > 0$ if (M, g) has nonpositive curvature.

To prove (2.2) we shall need to eventually use operators that reproduce eigenfunctions. To this end fix

$$(2.3) \quad \rho \in \mathcal{S}(\mathbb{R}) \quad \text{with} \quad \rho(0) = 1 \quad \text{and} \quad \hat{\rho}(t) = 0 \quad \text{if} \quad |t| \geq 1.$$

Then

$$(2.4) \quad \rho(T(\lambda - P))e_\lambda = e_\lambda \quad \text{if} \quad T \geq 1,$$

and

$$(2.5) \quad \rho(T(\lambda - P)) = \frac{1}{2\pi T} \int \hat{\rho}(t/T) e^{i\lambda t} e^{-itP} dt.$$

By the last part of (2.3) the integrand vanishes when $|t| \geq T$.

We shall require the following pointwise estimates for the kernels of these operators which make use of our assumption that e^{-itP} has no caustics when $t \neq 0$.

Lemma 2.2. *Fix ρ as in (2.3) and assume that $U(t)$ has no caustics for $t \neq 0$. Then the kernel of the operator in (2.5) satisfies*

$$(2.6) \quad |\rho(T(\lambda - P))(x, y)| \leq C(\lambda/d_g(x, y))^{\frac{n-1}{2}} + C_T \lambda^{\frac{n-1}{2}}, \quad \text{if} \quad T \geq 1,$$

where C is a uniform constant depending only on ρ and (M, g) , while C_T also depends on the parameter T .

Proof of Lemma 2.2. To prove this we may assume for simplicity that the injectivity radius of (M, g) is ten or more after possibly rescaling the metric which just has the effect of changing the eigenvalues of P by a fixed factor.

Fix $\beta \in C_0^\infty(\mathbb{R})$ satisfying $\beta(t) = 1$ if $|t| \leq 1$ and $\text{supp } \beta \subset (-2, 2)$. By (2.5) we can then write

$$(2.7) \quad \begin{aligned} \rho(T(\lambda - P))(x, y) &= \frac{1}{2\pi T} \int \beta(t) \hat{\rho}(t/T) e^{i\lambda t} U(t; x, y) dt \\ &\quad + \frac{1}{2\pi T} \int (1 - \beta(t)) \hat{\rho}(t/T) e^{i\lambda t} U(t; x, y) dt = I + II, \end{aligned}$$

with $U(t; x, y)$ denoting the kernel of $U(t) = e^{-itP}$.

Since we are assuming that $T \geq 1$ it is well known and not difficult to prove that the first term here satisfies the uniform bounds

$$(2.8) \quad |I| \leq C(\lambda/d_g(x, y))^{\frac{n-1}{2}}.$$

To prove this one uses the fact that our assumption about the injectivity radius means that we can obtain a parametrix for $U(t)$ for $|t| \leq 2$ as in [9] or in Theorem 4.1.2 in [12]. Using this fact it is not difficult to modify the proof of Lemma 5.1.3 in [12] and use this parametrix along with a simple stationary phase argument to obtain the uniform bounds (2.8).

Due to (2.8), the proof of (2.6) would be complete if we could show that the last term in (2.7) satisfies

$$(2.9) \quad |II| \leq C_T \lambda^{\frac{n-1}{2}}.$$

To prove this, let us recall some basic facts about the operators $U(t)$. First they are Fourier integral operators whose canonical relations are given by

$$(2.10) \quad \mathcal{C} = \{(t, x, \tau, \xi, y, \eta) : \tau = -p(y, \eta), (x, \xi) = \chi_t(y, \eta)\},$$

where $\chi_t : T^*M \setminus 0 \rightarrow T^*M \setminus 0$ denotes geodesic flow on the cotangent bundle. If we fix the time t then the canonical relation of $U(t) : \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$ is then

$$(2.11) \quad \mathcal{C}_t = \{(x, \xi, y, \eta) : (x, \xi) = \chi_t(y, \eta)\}.$$

The assumption that $U(t)$, $t \neq 0$, has no caustics means that the projection from \mathcal{C}_t to $M \times M$ has a differential with rank $2n - 1$ everywhere. The image of this projection then is an immersed hypersurface of codimension one. (See e.g., Proposition 6.1.5 and the discussion at the end of §6.1 in [12].)

This all means that for t near a given $t_0 \neq 0$, modulo smooth errors, we can write the kernel of $U(t)$ as a finite sum of Fourier integrals which in local coordinates are of the form

$$(2.12) \quad \int_{\mathbb{R}^n} e^{i\varphi(x, y, t, \xi)} a(t, x, y, \xi) d\xi,$$

where $a \in S^0$ is a symbol of order zero and φ solves the eikonal equation

$$(2.13) \quad \varphi'_t = -p(x, \nabla_x \varphi),$$

on the support of a with

$$(2.14) \quad p(x, \xi) = \sqrt{\sum g^{jk}(x) \xi_j \xi_k}$$

being the principal symbol of $P = \sqrt{-\Delta_g}$. Here $(g^{jk}(x)) = (g_{jk}(x))^{-1}$ is the cometric written in our local coordinate system. The phase φ is real and smooth away from $\xi = 0$ and it is homogeneous of degree one in the ξ variable. Additionally, on $\text{supp } a$ we have

that $\nabla_x \varphi \neq 0$ if $\nabla_\xi \varphi = 0$ and $\xi \neq 0$. Consequently, by (2.13)–(2.14), we have on the support of a that

$$(2.15) \quad \left\langle \frac{\xi}{|\xi|}, \nabla_\xi \right\rangle \varphi'_t \neq 0, \quad \text{if } \varphi'_\xi = 0.$$

To use this we recall that since we are assuming that the projection from \mathcal{C}_t to $M \times M$ has a differential of rank $2n - 1$ everywhere, we must have that on $\text{supp } a$

$$(2.16) \quad \text{Rank } \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k} \equiv n - 1 \quad \text{if } \nabla_\xi \varphi = 0, \xi \neq 0.$$

(See e.g., Proposition 6.1.5 in [12].) Since φ is homogeneous of degree one in ξ we deduce from (2.15)–(2.16) that if we set

$$\Phi(t, x, y, \xi) = \varphi(x, y, t, \xi) + t$$

then on $\text{supp } a$ we must have that the mixed Hessian of Φ with respect to the $n + 1$ variables (t, ξ) satisfies

$$\det \left(\frac{\partial^2 \Phi}{\partial(t, \xi) \partial(t, \xi)} \right) \neq 0 \quad \text{if } \nabla_\xi \Phi = 0, \xi \neq 0.$$

Consequently, if a is as in (2.12) and if $b(t) \in C_0^\infty(\mathbb{R})$ vanishes outside of a small neighborhood of t_0 , we conclude from the method of stationary phase that

$$\begin{aligned} & \iint (1 - \beta(t)) \hat{\rho}(t/T) e^{it\lambda} e^{i\varphi(x, y, t, \xi)} b(t) a(t, x, y, \xi) d\xi dt \\ &= \lambda^n \int_{\mathbb{R}^{n+1}} (1 - \beta(t)) \hat{\rho}(t/T) b(t) a(t, x, y, \lambda\xi) e^{i\lambda\Phi(x, y, t, \xi)} d\xi dt \\ &= O(\lambda^n \lambda^{-\frac{n+1}{2}}) = O(\lambda^{\frac{n-1}{2}}). \end{aligned}$$

Since $2\pi T$ times the term II in (2.7) can be written as the sum of finitely many terms of this form (depending on T) (and an $O(\lambda^{-N})$ term coming from the smooth errors in the parametrix), we deduce that (2.9) must be valid, which completes the proof of Lemma 2.2. \square

We can now turn to the proof of Theorem 2.1. We shall adapt an argument from [14] which uses an idea of Bourgain [6] and Lorentz space estimates of Bak and Seeger [1] for the operators in (2.5) corresponding to $T = 1$.

Proof of Theorem 2.1. Since $\mu(p_c) = 1/p_c$, proving (2.2) is equivalent to showing that if $E_\lambda : L^2(M) \rightarrow L^2(M)$ is the projection onto the eigenspace with eigenvalue λ then

$$(2.2') \quad \|E_\lambda\|_{L^2(M) \rightarrow L^{p_c}(M)} = o(\lambda^{\frac{1}{p_c}}).$$

To do this we shall use an estimate of Bak and Seeger [1] that will allow us to deduce (2.2') from the easier weak-type estimates

$$(2.2'') \quad \|E_\lambda\|_{L^2(M) \rightarrow L^{p_c, \infty}(M)} = o(\lambda^{\frac{1}{p_c}}).$$

Indeed Bak and Seeger showed that if χ_λ denote the standard spectral projection operators

$$\chi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1)} E_{\lambda_j} f,$$

then on any (M, g) one has the Lorentz space estimates

$$\|\chi_\lambda\|_{L^2(M) \rightarrow L^{p_c, 2}(M)} = O(\lambda^{\frac{1}{p_c}}).$$

Since E_λ is a projection operator and $E_\lambda = \chi_\lambda \circ E_\lambda$ this implies that

$$(2.17) \quad \|E_\lambda\|_{L^2(M) \rightarrow L^{p_c, 2}(M)} = O(\lambda^{\frac{1}{p_c}}).$$

If one interpolates between (2.17) and (2.2'') one obtains (2.2') (see e.g. Chapter V in Stein and Weiss [18] or §4 in [14]).

Let us rewrite (2.2''). Given f with $L^2(M)$ norm one, we shall let

$$\omega_{E_\lambda f} = |\{x \in M : |E_\lambda f(x)| > \alpha\}|, \quad \alpha > 0,$$

denote the distribution function of $E_\lambda f$, with $|U|$ denoting the dV_g measure of $U \subset M$. Then (2.2'') is just the statement that for any fixed $\varepsilon > 0$ we can find a $\Lambda_\varepsilon < \infty$ so that

$$(2.18) \quad \omega_{E_\lambda f}(\alpha) \leq \varepsilon \lambda \alpha^{-\frac{2(n+1)}{n-1}}, \quad \text{if } \|f\|_{L^2(M)} = 1 \text{ and } \lambda \geq \Lambda_\varepsilon.$$

To prove this, we first note that because we are assuming (1.9), by Theorem 1.2 since $2 < \frac{2n}{n-1} < p_c$ and since $\mu(\frac{2n}{n-1}) = \frac{1}{2} \cdot \frac{n-1}{2n}$, we have

$$\|E_\lambda\|_{L^2(M) \rightarrow L^{\frac{2n}{n-1}}(M)} = o(\lambda^{\frac{1}{2} \cdot \frac{n-1}{2n}}).$$

Therefore, by Chebyshev's inequality, given $\delta > 0$ we can find a $\Lambda_\delta < \infty$ so that

$$(2.19) \quad \omega_{E_\lambda f}(\alpha) \leq \delta \lambda^{\frac{1}{2}} \alpha^{-\frac{2n}{n-1}}, \quad \text{if } \|f\|_{L^2(M)} = 1 \text{ and } \lambda \geq \Lambda_\delta.$$

A calculation shows that these bounds yield (2.18) if α satisfies

$$(2.20) \quad \alpha \leq \lambda^{\frac{n-1}{4}} (\varepsilon/\delta)^{n-1}.$$

We shall specify $\delta = \delta(\varepsilon)$ in a moment.

We are also assuming that (1.6) is valid and hence, by Theorem 1.1,

$$\|E_\lambda\|_{L^2(M) \rightarrow L^\infty(M)} = o(\lambda^{\frac{n-1}{2}}).$$

Thus, given $\delta > 0$ as above we have that there must be a $\Lambda_\delta < \infty$ so that

$$\|E_\lambda f\|_{L^\infty(M)} \leq \delta \lambda^{\frac{n-1}{2}}, \quad \text{if } \|f\|_{L^2(M)} = 1 \text{ and } \lambda \geq \Lambda_\delta.$$

This means that for such λ we have

$$(2.21) \quad \omega_{E_\lambda f}(\alpha) = 0 \quad \text{if } \alpha \geq \delta \lambda^{\frac{n-1}{2}}.$$

By (2.20) and (2.21), we have reduced matters to showing that for large λ we have

$$(2.18') \quad |\{|E_\lambda f(x)| > \alpha\}| = \omega_{E_\lambda f}(\alpha) \leq \varepsilon \lambda \alpha^{-\frac{2(n+1)}{n-1}},$$

$$\text{if } \|f\|_{L^2(M)} = 1 \text{ and } \alpha \in I_{\varepsilon, \delta} = \left(\lambda^{\frac{n-1}{4}} (\varepsilon/\delta)^{n-1}, \delta \lambda^{\frac{n-1}{2}}\right).$$

To prove this, we note that if ρ is as in (2.3) then for any $T \geq 1$ we have

$$E_\lambda f = \rho(T(\lambda - P))E_\lambda f \quad \text{and} \quad \|E_\lambda f\|_{L^2(M)} \leq \|f\|_{L^2(M)}.$$

As a result, we would have (2.18') if we could choose $T = T_\varepsilon \gg 1$ so that for large λ we have

$$(2.18'') \quad |\{x : |\rho(T_\varepsilon(\lambda - P))h(x)| > \alpha\}| \leq \varepsilon \lambda \alpha^{-\frac{2(n+1)}{n-1}},$$

$$\text{if } \|h\|_{L^2(M)} = 1 \text{ and } \alpha \in I_{\varepsilon, \delta}.$$

To prove this, as in [14] we shall adapt an argument of Bourgain [6] which exploits the bounds in Lemma 2.2 that are based on our other assumption that $U(t)$, $t \neq 0$, has no caustics. To do so put

$$(2.22) \quad r = \lambda \alpha^{-\frac{4}{n-1}} T_\varepsilon^{\frac{2}{n-1}}.$$

Note then that

$$(2.23) \quad r \geq \lambda^{-1} \quad \text{if } \delta \leq T_\varepsilon^{-\frac{1}{2}} \text{ and } \alpha \in I_{\varepsilon, \delta},$$

due to the fact that $\alpha \leq \lambda^{\frac{n-1}{2}} \delta$ if $\alpha \in I_{\varepsilon, \delta}$. Since the δ in (2.19) and (2.22) can be made arbitrarily small, we shall assume that we have this condition after we specify T_ε in a bit.

Let $A = A_\alpha = |\{|\rho(T_\varepsilon(\lambda - P))h(x)| > \alpha\}|$ denote the set in (2.18''). Then we are trying to show that $|A|$ satisfies the bounds there assuming that $\alpha \in I_{\varepsilon, \delta}$. At the expense of replacing A by a set of proportional measure, we may assume that

$$(2.24) \quad A = \bigcup_j A_j \quad \text{where } \text{diam } A_j \leq r \text{ and } d_g(A_j, A_k) \geq C_0 r, \quad j \neq k,$$

where r is as in (2.22)–(2.23) and C_0 will be specified shortly. Here $\text{diam } U$ denotes the diameter of $U \subset M$ as measured by the Riemannian distance function.

In addition to (2.6) we shall also require the following simple estimate from [15] and [14], which says that for $T, \lambda \geq 1$ we have the uniform bounds

$$(2.25) \quad \|\rho(T(\lambda - P))f\|_{L^2(B_r(x))} \leq Cr^{\frac{1}{2}}\|f\|_{L^2(M)} \quad \text{if } \lambda^{-1} \leq r \leq \text{Inj } M.$$

For later use, we note that the dual version of this inequality says that we have the bounds

$$\|\rho(T(\lambda - P))h\|_{L^2(M)} \leq Cr^{\frac{1}{2}}\|h\|_{L^2}, \quad \text{if } \lambda^{-1} \leq r \leq \text{Inj } M, \text{ and } \text{supp } h \subset B_r(x),$$

where the constant is independent of $x \in M$.

If

$$\psi_\lambda(x) = \begin{cases} \rho(T_\varepsilon(\lambda - P))h(x)/|\rho(T_\varepsilon(\lambda - P))h(x)|, & \text{if } \rho(T_\varepsilon(\lambda - P))h(x) \neq 0 \\ 1, & \text{otherwise} \end{cases}$$

denotes the signum function of $\rho(T_\varepsilon(\lambda - P))h$ and if we let a_j be ψ_λ times the indicator function, $\mathbb{1}_{A_j}$, of the set A_j as above, then since we are assuming that $\|h\|_2 = 1$, by Chebyshev's inequality and the Cauchy-Schwarz inequality

$$\begin{aligned} \alpha|A| &\leq \left| \int \sum_j \rho(T_\varepsilon(\lambda - P))h \overline{\psi_\lambda \mathbb{1}_{A_j}} dV_g \right| \\ &\leq \left(\int \left| \sum_j \rho(T_\varepsilon(\lambda - P))a_j \right|^2 dV_g \right)^{1/2}. \end{aligned}$$

As a result, if

$$S_\lambda = \rho(T_\varepsilon(\lambda - P))^* \circ \rho(T_\varepsilon(\lambda - P)) = |\rho|^2(T_\varepsilon(\lambda - P)),$$

then

$$\begin{aligned} (2.26) \quad \alpha^2|A|^2 &\leq \sum_j \int |\rho(T_\varepsilon(\lambda - P))a_j|^2 dV_g + \sum_{j \neq k} \int \rho(T_\varepsilon(\lambda - P))a_j \overline{\rho(T_\varepsilon(\lambda - P))a_k} dV_g \\ &= \sum_j \int |\rho(T_\varepsilon(\lambda - P))a_j|^2 dV_g + \sum_{j \neq k} \int S_\lambda a_j \overline{a_k} dV_g \\ &= I + II. \end{aligned}$$

By (2.22)–(2.24) and the dual version of (2.25)

$$I \leq Cr \sum_j \int |a_j|^2 dV_g = Cr|A| = C\lambda\alpha^{-\frac{4}{n-1}}T_\varepsilon^{-\frac{2}{n-1}}|A|.$$

Thus if we let

$$T_\varepsilon = (\varepsilon/2C)^{\frac{n-1}{2}},$$

we have

$$(2.27) \quad I \leq \frac{1}{2}\varepsilon\lambda\alpha^{-\frac{4}{n-1}}|A|.$$

To estimate the other term, II , in (2.26), we note that since we have finally specified T_ε , by Lemma 2.2 we have that the kernel $S_\lambda(x, y)$ of S_λ satisfies

$$|S_\lambda(x, y)| \leq CT_\varepsilon^{-1}(\lambda/d_g(x, y))^{\frac{n-1}{2}} + C_\varepsilon\lambda^{\frac{n-1}{2}},$$

due to the fact that the Schwartz class function $|\rho|^2$ equals one at the origin and has compactly supported Fourier transform by (2.3).

Therefore, by (2.22) and (2.24),

$$(2.28) \quad \begin{aligned} II &\leq [CT_\varepsilon^{-1}(\lambda/C_0r)^{\frac{n-1}{2}} + C_\varepsilon\lambda^{\frac{n-1}{2}}] \sum_{j \neq k} \|a_j\|_{L^1} \|a_k\|_{L^1} \\ &\leq CC_0^{-\frac{n-1}{2}}\alpha^2|A|^2 + C_\varepsilon\lambda^{\frac{n-1}{2}}|A|^2. \end{aligned}$$

Since we are assuming in (2.18'') that $\alpha \in I_{\varepsilon, \delta}$ and hence $\alpha \geq \lambda^{\frac{n-1}{4}}(\varepsilon/\delta)^{n-1}$, we can control the last term as follows

$$C_\varepsilon\lambda^{\frac{n-1}{2}}|A|^2 \leq C_\varepsilon(\delta/\varepsilon)^{2(n-1)}\alpha^2|A|^2 \leq \frac{1}{4}\alpha^2|A|^2 \quad \text{if } C_\varepsilon(\delta/\varepsilon)^{2(n-1)} \leq \frac{1}{4}.$$

Since the δ in (2.20) and (2.21) can be taken to be as small as we like, because we are assuming (1.6) and (1.9), we can fix such a δ which also satisfies the condition in (2.23) and obtain this bound for the last term in (2.28). As a result, if we choose the constant C_0 in (2.24) so that $CC_0^{-\frac{n-1}{2}} = \frac{1}{4}$, then by (2.28) we have

$$II \leq \frac{1}{2}\alpha^2|A|^2.$$

If we combine this with (2.26) and (2.27) we deduce that for large enough λ we have

$$\alpha^2|A|^2 \leq \varepsilon\lambda\alpha^{-\frac{4}{n-1}}|A|.$$

Since this is equivalent to the statement that

$$|A| \leq \varepsilon\lambda\alpha^{-\frac{2(n+1)}{n-1}},$$

the proof of (2.18'') and hence that of Theorem 2.1 is complete. \square

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