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A remark on the unboundedness of the bilinear Hilbert transform on Hardy spaces

By

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Abstract

In this note, the unboundedness of the bilinear Hilbert transform from products of Hardy spaces $H^p \times H^q$ to L^r, $0 < p \le 1$, $0 < q \le \infty$, $1/p + 1/q = 1/r$, is considered.

§ 1. Introduction

The bilinear Hilbert transform H is defined by

$$\begin{aligned} H(f,g)(x) &= \lim_{\epsilon \to 0} \int_{|y|>\epsilon} f(x+y)g(x-y)\, \frac{dy}{y} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ix(\xi+\eta)} \, (i\pi \operatorname{sgn}(\xi - \eta))\, \widehat{f}(\xi) \widehat{g}(\eta)\, d\xi d\eta \end{aligned}$$

for $f, g \in \mathcal{S}$, where $\operatorname{sgn} \xi$ is the signum function. In the study of the Cauchy integral along Lipschitz curves, A.P. Calderón raised the problem whether the boundedness of H from $L^2 \times L^2$ to L^1 holds. After some 30 years, this problem was solved positively by Lacey-Thiele [4, 5]. More precisely, they proved that H is bounded from $L^p \times L^q$ to L^r for $1 < p, q \le \infty$ and $2/3 < r < \infty$ satisfying $1/p + 1/q = 1/r$. However, it is still open whether we can remove the restriction $r > 2/3$.

The purpose of this note is to consider the endpoint cases. In particular, we can prove the unboundedness of H from $H^1 \times H^1$ to $L^{1/2}$, even though we do not know whether H is bounded from $L^1 \times L^1$ to $L^{1/2,\infty}$, where H^1 is the Hardy space and $L^{1/2,\infty}$ is the weak $L^{1/2}$-space. More generally, we can prove

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Theorem 1. *Let $0 < p \le 1$, $0 < q \le \infty$ and $1/p + 1/q = 1/r$. Then the bilinear Hilbert transform H is not bounded from $H^p \times H^q$ to L^r.*

A counterexample for the boundedness of H from $L^1 \times L^q$ to $L^{q/(q+1)}$ by D. Bilyk can be found in [2, Exercise 7.1.9].

For $0 < \alpha < 1$, the bilinear fractional integral operator B_α is defined by

$$B_\alpha(f, g)(x) = \int_{\mathbb{R}} f(x+y)g(x-y)\, \frac{dy}{|y|^{1-\alpha}}.$$

It is known that B_α is bounded from $L^p \times L^q$ to L^r for $1 < p, q \le \infty$ and $r < \infty$, and from $L^p \times L^q$ to $L^{r,\infty}$ for $p = 1$ or $q = 1$, where $1/p + 1/q - \alpha = 1/r$ (Grafakos [1], Kenig-Stein [3]). We note that there is no restriction of r from below for this operator in contrast to the bilinear Hilbert transform. However, by the same argument as in the proof of Theorem 1, we can prove

Theorem 2. *Let $0 < \alpha < 1$, $0 < p \le 1$, $0 < q \le \infty$ and $1/p + 1/q - \alpha = 1/r$. Then the bilinear fractional integral operator B_α is not bounded from $H^p \times H^q$ to L^r.*

§ 2. Preliminaries

For two non-negative quantities A and B, the notation $A \lesssim B$ (resp. $A \gtrsim B$) means that $A \le CB$ (resp. $A \ge CB$) for some unspecified constant $C > 0$.

Let $\mathcal{S}$ and $\mathcal{S}'$ be the Schwartz spaces of all rapidly decreasing smooth functions on $\mathbb{R}$ and tempered distributions on $\mathbb{R}$, respectively. We define the Fourier transform $\widehat{f}$ of $f \in \mathcal{S}$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

Let $0 < p \le \infty$, and let $\phi \in \mathcal{S}$ be such that $\int_{\mathbb{R}} \phi(x)\, dx \neq 0$. Then the Hardy space H^p consists of all $f \in \mathcal{S}'$ such that

$$\|f\|_{H^p} = \| \sup_{0<t<\infty} |\phi_t * f| \|_{L^p} < \infty,$$

where $\phi_t(x) = t^{-1}\phi(x/t)$. It is known that H^p does not depend on the choice of the function ϕ, $H^1 \hookrightarrow L^1$ and $H^p = L^p$ for $1 < p \le \infty$. See Stein [6, Chapter 3] for more details on Hardy spaces.

The following lemma will be used in the proof of Theorem 1.

Lemma 3. *Let g_0 be a positive non-increasing function on $(0, \infty)$ satisfying $\int_0^\infty g_0(x)\, dx < \infty$, and set $g(x) = \mathrm{sgn}(x) e^{i|x|} g_0(|x|)$. Then g belongs to the Hardy space H^1.*

Proof. Our proof is based on the argument in [6, Chapter 4, Section 6.2]. Setting

$$a_k(x) = \begin{cases} \operatorname{sgn}(x)\frac{e^{i|x|}g_0(|x|)}{2^{k+2}g_0(2^k)}, & 2^k \le |x| < 2^{k+1}, \\ 0, & \text{otherwise}, \end{cases}$$

we can write $g(x) = \sum_{k\in\mathbb{Z}} \lambda_k a_k(x)$ with $\lambda_k = 2^{k+2}g_0(2^k)$ for $x \neq 0$. Since $\operatorname{supp} a_k \subset [-2^{k+1}, 2^{k+1}]$, $\|a_k\|_{L^\infty} \le 2^{-(k+2)}$ and $\int_{\mathbb{R}} a_k(x)\,dx = 0$, we see that a_k, $k \in \mathbb{Z}$, are H^1-atoms. On the other hand, by the non-increasing property of g_0,

$$\sum_{k\in\mathbb{Z}} \lambda_k = 2^3 \sum_{k\in\mathbb{Z}} \int_{2^{k-1}}^{2^k} g_0(2^k)\,dx \lesssim \int_0^\infty g_0(x)\,dx < \infty.$$

These facts imply that g has an atomic decomposition in H^1, and we have $g \in H^1$. $\square$

§ 3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let

$$\begin{aligned} A &= \{(1/p, 1/q) \,:\, 1 \le 1/p < \infty,\ 1 < 1/q < \infty\}, \\ B &= \{(1/p, 1/q) \,:\, 1 \le 1/p < \infty,\ 0 \le 1/q \le 1\}, \\ C &= \{(1/p, 1/q) \,:\, 0 \le 1/p \le 1,\ 1 \le 1/q < \infty\}, \\ D &= \{(1/p, 1/q) \,:\, 0 \le 1/p < 1,\ 0 \le 1/q < 1,\ 0 < 1/p + 1/q < 3/2\}. \end{aligned}$$

Our goal is to prove the unboundedness of the bilinear Hilbert transform H from $H^p \times H^q$ to L^r for $(1/p, 1/q) \in A \cup B$, where $1/p + 1/q = 1/r$. We first observe that it is sufficient to prove the unboundedness for $(1/p, 1/q) \in B$. In fact, by the work of Lacey-Thiele [4, 5], we know that the boundedness holds for $(1/p, 1/q) \in D$. If the boundedness holds for some $(1/p, 1/q) \in A$, then by interpolating between this point and an appropriate point in D, we have the boundedness for some $(1/p, 1/q) \in B \cup C$. But, since the boundedness for $(1/p, 1/q) \in B$ is equivalent to that for $(1/q, 1/p) \in C$ by the symmetry $H(f, g) = -H(g, f)$, we have the boundedness for some $(1/p, 1/q) \in B$. Hence, the unboundedness for all $(1/p, 1/q) \in B$ implies the unboundedness for all $(1/p, 1/q) \in A$.

We assume that $0 < p \le 1$, $1 \le q \le \infty$ and $1/p + 1/q = 1/r$. Let f be a smooth function on $\mathbb{R}$ such that $\widehat{f}(1) \neq 0$, $\operatorname{supp} f \subset [-1, 1]$ and $\int_{\mathbb{R}} x^k f(x)\,dx = 0$, $0 \le k \le [1/p - 1]$, where $[1/p - 1]$ is the integer part of $1/p - 1$. We note that f is a constant multiple of an H^p-atom. By a change of variable, for sufficiently large $x > 0$,

$$H(f, g)(x) = \lim_{\epsilon\to 0} \int_{|y|>\epsilon} f(x + y)g(x - y)\,\frac{dy}{y} = \int_{-1}^{1} f(y)g(2x - y)\,\frac{dy}{y - x}.$$

Since we prove the unboundedness by using the behavior of $H(f,g)(x)$ for sufficiently large $x > 0$, this expression says that it is sufficient to consider $g(x)$ only for sufficiently large $x > 0$. Thus, we set

$$g(x) = \frac{e^{ix}}{x^{1/q}(\log x)^{(1+\epsilon)/q}}, \quad x \gg 1,$$

where $0 < \epsilon < 1$. Obviously, $g \in L^q$ for $1 \le q \le \infty$. Moreover, in the case $q = 1$, it follows from Lemma 3 that g can be extended to a function on $\mathbb{R}$ in H^1.

Let $x > 0$ be sufficiently large. Since

$$\begin{aligned} H(f,g)(x) &= \int_{-1}^{1} f(y) \frac{e^{i(2x-y)}}{(2x-y)^{1/q}(\log(2x-y))^{(1+\epsilon)/q}} \frac{dy}{y-x} \\ &= e^{2ix} \int_{-1}^{1} e^{-iy} f(y) \frac{1}{(2x)^{1/q}(\log(2x))^{(1+\epsilon)/q}(-x)}\, dy + e^{2ix} \int_{-1}^{1} e^{-iy} f(y) \\ &\quad \times \left(\frac{1}{(2x-y)^{1/q}(\log(2x-y))^{(1+\epsilon)/q}(y-x)} - \frac{1}{(2x)^{1/q}(\log(2x))^{(1+\epsilon)/q}(-x)} \right) dy \\ &= \frac{e^{2ix}}{(2x)^{1/q}(\log(2x))^{(1+\epsilon)/q}(-x)} \widehat{f}(1) + O\left(\frac{1}{x^{1/q+2}(\log x)^{(1+\epsilon)/q}} \right), \end{aligned}$$

we see that

$$|H(f,g)(x)| \gtrsim \frac{1}{x^{1/q+1}(\log x)^{(1+\epsilon)/q}}.$$

Note that $(1/q+1)r < 1$ if $p < 1$, and $(1/q+1)r = 1$ and $(1+\epsilon)r/q < 1$ if $p = 1$. Therefore,

$$\int_{x \gg 1} |H(f,g)(x)|^r \, dx \gtrsim \int_{x \gg 1} \left(\frac{1}{x^{1/q+1}(\log x)^{(1+\epsilon)/q}} \right)^r dx = \infty,$$

and we have the unboundedness of H from $H^p \times H^q$ to L^r. $\square$

Proof of Theorem 2. By the same reasoning as in Proof of Theorem 1, it is sufficient to show the unboundedness for $0 < p \le 1$ and $1 \le q \le \infty$. We assume that $0 < \alpha < 1$, $0 < p \le 1$, $1 \le q \le \infty$ and $1/p + 1/q - \alpha = 1/r$. Let f, g be the same functions as in the proof of Theorem 1. In the same way as for $H(f,g)$, we can prove

$$|B_\alpha(f,g)(x)| \gtrsim \frac{1}{x^{1/q+1-\alpha}(\log x)^{(1+\epsilon)/q}}.$$

for sufficiently large $x > 0$. Since $(1/q+1-\alpha)r < 1$ if $p < 1$, and $(1/q+1-\alpha)r = 1$ and $(1+\epsilon)r/q < 1$ if $p = 1$ and ϵ is sufficiently small, we have the unboundedness of B_α from $H^p \times H^q$ to L^r. $\square$

References

[1] L. Grafakos, On multilinear fractional integrals, *Studia Math.* **102** (1992), 49-56.

[2] L. Grafakos, Modern Fourier Analysis, Third edition, *Springer, New York*, 2014.

[3] C. Kenig and E. M. Stein, Multilinear estimates and fractional integration, *Math. Res. Lett.* **6** (1999), 1-15.

[4] M. Lacey and C. Thiele, L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, *Ann. of Math.* **146** (1997), 693-724.

[5] M. Lacey and C. Thiele, On Calderón's conjecture, *Ann. of Math.* **149** (1999), 475-496.

[6] E.M. Stein, Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals, *Princeton University Press, Princeton, NJ*, 1993.