

# A tensorial description of the Turaev cobracket on genus 0 compact surfaces

By

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## Abstract

We give a tensorial description of the Turaev cobracket on any genus 0 compact surface by the standard group-like expansion, where the Bernoulli numbers appear.

## Introduction

The free homotopy set of free loops on an oriented surface  $S$ ,  $\hat{\pi} = \hat{\pi}(S) = [S^1, S] = \pi_1(S)/(\text{conjugate})$ , has rich structures. In the classical theory of Riemann surfaces, the algebraic intersection number of two free loops plays a central role. As a non-commutative generalization of the intersection number, the Goldman bracket [4] of two free loops appears in the Weil-Petersson symplectic geometry [23], the Poisson structure on the moduli space of flat bundles [4] and the Skein algebra of links in the 3-manifold  $S \times [0, 1]$  [22]. In the case where  $S$  is a compact surface with connected boundary, Kuno and the author [8] gave a tensorial description of the Goldman bracket, and described Dehn twists on the surface  $S$  in terms of the Goldman Lie algebra. These results are generalized to any compact surfaces with non-empty boundary in [16] [9] [11].

On the other hand, the Turaev cobracket  $\delta$  is related to Turaev's earlier work [21], and was introduced by Turaev [22] in connection with the Skein algebra. It is a dual notion of the Goldman bracket, and measures the self-intersection of a single free loop. But little is known about the Turaev cobracket. As was discovered by Kuno and the author [10], the Turaev cobracket gives a geometric constraint of the images of the (higher) Johnson homomorphisms. In order to deduce some results from this fact, we

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need a tensorial description of the Turaev cobracket. In [10] and [17], the lowest degree term of the description was computed. When the preprint of this paper [7] was uploaded at the arXiv (June 10, 2015), there was no other full results on the tensorial description.

In this paper we will give the tensorial description of the Turaev cobracket for any genus 0 compact surface with respect to the standard group-like expansion  $\theta^{\text{std}}$ . Unfortunately the expansion  $\theta^{\text{std}}$  does not reflect the topology of the surface enough, so that we cannot deduce topological consequences from our result.

The description is stated in Theorem 1.2, where the Bernoulli numbers appear. In this paper, following the convention in [16], we agree that the function  $s(z)$  and the Bernoulli numbers  $B_{2m}$  are defined by

$$\begin{aligned} s(z) &= \frac{1}{e^{-z} - 1} + \frac{1}{z} = -\frac{1}{2} - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} z^{2m-1} \\ &= -\frac{1}{2} - \frac{1}{12}z + \frac{1}{720}z^3 - \frac{1}{30240}z^5 + \dots \end{aligned}$$

The appearance of the Bernoulli numbers comes from the tensorial description of the homotopy intersection form by Massuyeau-Turaev [16] (Theorem 2.3), and a formula for the coaction operation  $\mu$  by Fukuhara-Kawazumi-Kuno [3] (Theorem 2.1). The Kashiwara-Vergne problem in the formulation by Alekseev-Torossian [2] looks for a group-like expansion of the fundamental group of a pair of pants which is compatible with all the boundary components and satisfies some equation involved with the Bernoulli numbers and the divergence cocycle. As the author announced in [6], a regular homotopy version of the Turaev cobracket on genus 0 compact surfaces includes the divergence cocycle. Hence the result in this paper seems to suggest the following conjecture.

**Conjecture 0.1.** *The tensorial description of the Turaev cobracket with respect to any solution to the Kashiwara-Vergne problem is of simple expression. In particular, the description might be formal, namely, might equal its lowest degree term.*

It is our working hypothesis for studying the higher Johnson homomorphisms that there is a symplectic expansion for a compact surface with connected boundary whose description of the Turaev cobracket equals the lowest degree term, i.e., Schedler's cobracket [20]. In fact, Kuno [13] already found such an expansion for the surface of genus 1 with connected boundary up to degree 10 by a computer calculation. If Conjecture 0.1 would be true, our hypothesis should be a positive genus analogue of the Kashiwara-Vergne problem.

After the preprint of this paper was uploaded, Alekseev, Kuno, Naef and the author [1] obtained a formal description of the Turaev cobracket by regarding solutions of the

Kashiwara-Vergne problem as special expansions for genus 0 compact surfaces. This means that Conjecture 0.1 is true. Independently from our results, Massuyeau [15] obtained a formal description of the Turaev cobracket for genus 0 compact surfaces by the Kontsevich integral.

Theorem 2.3 in this paper is a modification of a theorem of Massuyeau and Turaev [16]. It says that the value of a group-like expansion at the boundary loop of a surface with connected boundary completely determines the tensorial description of the homotopy intersection form by the expansion. As is shown by Naef [18], this fact can be generalized in the light of a non-commutative Poisson geometry, which is one of the foundations of the work [1].

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## § 1. Statement of the Result

Let  $S$  be a compact connected oriented surface with non-empty boundary. It is classified by its genus and the number of its boundary components, so that we may denote the surface  $S$  by the symbol  $\Sigma_{g,n+1}$  for some  $g, n \geq 0$ . Here the genus of  $S$  is  $g$ , and the number of the boundary components is  $n + 1$ . The fundamental group of the surface  $S$  is free of rank  $2g + n$ . In general, for a free group  $\pi$  of finite rank, we have the notion of group-like expansion. See [14]. In order to recall the definition of a group-like expansion, we need to prepare some tensor algebra. Let  $H$  be the first rational homology group of  $\pi$ , i.e.,  $H := (\pi/[\pi, \pi]) \otimes_{\mathbb{Z}} \mathbb{Q}$ . We denote  $[\gamma] := (\gamma \bmod [\pi, \pi]) \otimes 1 \in H$  for any  $\gamma \in \pi$ . The completed tensor algebra  $\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$  is endowed with the topology by the decreasing filtration  $\widehat{T}_{\geq p} := \prod_{m \geq p} H^{\otimes m}$ ,  $p \geq 1$ , and has the structure of a complete Hopf algebra with an augmentation  $\text{aug} : \widehat{T} \rightarrow \mathbb{Q}$ , a coproduct  $\Delta : \widehat{T} \rightarrow \widehat{T} \widehat{\otimes} \widehat{T}$  and an antipode  $\iota : \widehat{T} \rightarrow \widehat{T}$ . They are defined to be the unique continuous algebra (anti)-homomorphisms satisfying  $\text{aug}(X) = 0$ ,  $\Delta(X) = X \widehat{\otimes} 1 + 1 \widehat{\otimes} X$  and  $\iota(X) = -X$  for any  $X \in H$ , respectively. The group ring  $\mathbb{Q}\pi$  is also a Hopf algebra. The augmentation  $\text{aug} : \mathbb{Q}\pi \rightarrow \mathbb{Q}$ , the coproduct  $\mathbb{Q}\pi \rightarrow \mathbb{Q}\pi \otimes \mathbb{Q}\pi$  and the antipode  $\iota : \mathbb{Q}\pi \rightarrow \mathbb{Q}\pi$  are the unique algebra (anti)-homomorphisms satisfying  $\text{aug}(\gamma) = 1$ ,  $\Delta(\gamma) = \gamma \otimes \gamma$  and  $\iota(\gamma) = \gamma^{-1}$  for any  $\gamma \in \pi$ , respectively. The completion of  $\mathbb{Q}\pi$  with respect to the augmentation ideal  $I\pi := \text{Ker } \text{aug}$ ,  $\widehat{\mathbb{Q}\pi} := \varprojlim_{p \rightarrow \infty} \mathbb{Q}\pi / (I\pi)^p$ , is a complete Hopf algebra in a natural way.

**Definition 1.1** (See [16]). The map  $\theta : \pi \rightarrow \widehat{T}$  is a group-like expansion if the following three conditions hold:

1. The map  $\theta$  is multiplicative, i.e., we have  $\theta(\gamma_1\gamma_2) = \theta(\gamma_1)\theta(\gamma_2)$  for any  $\gamma_1$  and  $\gamma_2 \in \pi$ .
2. For any  $\gamma \in \pi$ ,  $\theta(\gamma) \equiv 1 + [\gamma] \pmod{\widehat{T}_{\geq 2}}$ .
3. For any  $\gamma \in \pi$ ,  $\theta(\gamma) \in \widehat{T}$  is group-like, i.e.,  $\Delta\theta(\gamma) = \theta(\gamma)\widehat{\otimes}\theta(\gamma) \in \widehat{T}\widehat{\otimes}\widehat{T}$ .

The linear extension of any group-like expansion induces an isomorphism of complete Hopf algebras  $\theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \widehat{T}$ ,  $\sum a_\gamma\gamma \mapsto \sum a_\gamma\theta(\gamma)$ .

The group-like expansion we study in this paper is defined as follows. Let  $S$  be the genus 0 compact surface  $\Sigma_{0,n+1}$  for some  $n \geq 0$ . Number the boundary components as  $\partial S = \coprod_{k=0}^n \partial_k S$ , and choose a basepoint  $*$   $\in \partial_0 S$ . The standard generators  $\gamma_k \in \pi_1(S, *)$ ,  $1 \leq k \leq n$ , are given such that each  $\gamma_k$  is a simple loop going around the  $k$ -th boundary  $\partial_k S$  in the positive direction, and the product  $\gamma_1\gamma_2 \cdots \gamma_n \in \pi_1(S, *)$  is homotopic to a simple loop around the 0-th boundary  $\partial_0 S$  in the negative direction. Here we read the product  $\gamma_1\gamma_2 \cdots \gamma_n$  as a loop going along first  $\gamma_1$ , next  $\gamma_2$ , and finally  $\gamma_n$ . Here we remark that the local intersection number  $\varepsilon_*(\dot{\gamma}_k(0), \dot{\gamma}_k(1))$  of  $\dot{\gamma}_k(0)$  and  $\dot{\gamma}_k(1)$  at the basepoint  $*$  equals  $+1$ . The fundamental group  $\pi_1(S, *)$  is a free group of rank  $n$  with free generators  $\gamma_k$ ,  $1 \leq k \leq n$ . We denote by  $x_k := [\gamma_k] \in H = H_1(S; \mathbb{Q})$ ,  $1 \leq k \leq n$ , the homology class of  $\gamma_k$ . Equivalently  $x_k$  is the homology class of the  $k$ -th boundary  $\partial_k S$ , so that we define  $x_0 := [\partial_0 S] = -[\gamma_1\gamma_2 \cdots \gamma_n] = -\sum_{k=1}^n x_k \in H = H_1(S; \mathbb{Q})$ . Then we can consider the exponential  $e^{x_k} = \exp(x_k) = \sum_{m=0}^{\infty} \frac{1}{m!} x_k^m \in \widehat{T} = \widehat{T}(H_1(S; \mathbb{Q}))$ . We define the *standard group-like expansion*  $\theta^{\text{std}} : \pi = \pi_1(S, *) \rightarrow \widehat{T} = \widehat{T}(H_1(S; \mathbb{Q}))$  as the unique group-expansion satisfying  $\theta^{\text{std}}(\gamma_k) = e^{x_k}$ ,  $1 \leq \forall k \leq n$ . Here we require these conditions only for  $k \geq 1$ , not for  $k = 0$ . The reason why one can compute the tensorial description of the Turaev cobracket with respect to the expansion  $\theta^{\text{std}}$  is that we can apply Theorem 2.1 to  $x_k = \theta^{\text{std}}(\log(\gamma_k))$ .

Let  $\delta : \mathbb{Z}\hat{\pi}' \rightarrow \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$  be the Turaev cobracket [22]. Here  $\mathbb{Z}\hat{\pi}' := \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbf{1}$  is the quotient of the  $\mathbb{Z}$ -free module over the set  $\hat{\pi}$ ,  $\mathbb{Z}\hat{\pi}$ , by the linear span of the constant loop  $\mathbf{1} \in \hat{\pi}$ . We denote by  $|\cdot|' : \mathbb{Z}\pi_1(S, p) \rightarrow \mathbb{Z}\hat{\pi} \rightarrow \mathbb{Z}\pi/\mathbb{Z}\mathbf{1} = \mathbb{Z}\hat{\pi}'$  the quotient map for any  $p \in S$ . The definition of the Turaev cobracket will be stated in §2. The Goldman bracket and the Turaev cobracket make  $\mathbb{Z}\hat{\pi}'$  a Lie bialgebra in the sense of Drinfel'd [22], so that we call it the Goldman-Turaev Lie bialgebra of the surface  $S$ . The bialgebra has a completion with respect to the augmentation ideal  $I\pi$ ,  $\widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{p \rightarrow \infty} \mathbb{Q}\hat{\pi}'/|(I\pi)^p|'$ . We have a natural continuous extension  $|\cdot|' : \widehat{\mathbb{Q}\pi} \rightarrow \widehat{\mathbb{Q}\hat{\pi}}$ . The Goldman bracket and the Turaev cobracket extend continuously to  $\widehat{\mathbb{Q}\hat{\pi}}$  [9][10]. In particular, the Turaev cobracket is a continuous map  $\delta : \widehat{\mathbb{Q}\hat{\pi}} \rightarrow \widehat{\mathbb{Q}\hat{\pi}} \widehat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$ .

On the tensor algebra side, we denote by  $N(\widehat{T})$  the quotient of  $\widehat{T}$  by the closure of  $\mathbb{Q}1 + [\widehat{T}, \widehat{T}]$ , where  $[\widehat{T}, \widehat{T}]$  is the  $\mathbb{Q}$ -linear subspace of  $\widehat{T}$  generated by the set  $\{uv - vu; u, v \in \widehat{T}\}$ . The vector space  $N(\widehat{T})$  is naturally isomorphic to the space of cyclic invariants  $\prod_{m=1}^{\infty} (H^{\otimes m})^{\mathbb{Z}/m}$ , where the cyclic group  $\mathbb{Z}/m$  acts on the space  $H^{\otimes m}$  by cyclic permutation. We denote by  $|\cdot|' : \widehat{T} \rightarrow N(\widehat{T})$  the quotient map. Any group-like expansion  $\theta$  induces a topological isomorphism  $\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\widehat{T})$  [9]. Thus we have the tensorial description  $\delta^\theta$  of the Turaev cobracket with respect to  $\theta$  defined by the diagram

$$\begin{array}{ccc} \widehat{\mathbb{Q}\hat{\pi}} & \xrightarrow{\delta} & \widehat{\mathbb{Q}\hat{\pi}} \widehat{\otimes} \widehat{\mathbb{Q}\hat{\pi}} \\ \theta \downarrow & & \theta \widehat{\otimes} \theta \downarrow \\ N(\widehat{T}) & \xrightarrow{\delta^\theta} & N(\widehat{T}) \widehat{\otimes} N(\widehat{T}). \end{array}$$

Now we can formulate our result.

**Theorem 1.2.** *Let  $\delta^{\text{std}} = \delta^{\theta^{\text{std}}}$  be the tensorial description of the Turaev cobracket with respect to the standard group-like expansion  $\theta^{\text{std}}$  for the surface  $S = \Sigma_{0,n+1}$ . Then, for any  $m \geq 1$  and any  $k_1, k_2, \dots, k_m \in \{1, 2, \dots, n\}$ , we have*

$$\begin{aligned} & \delta^{\text{std}}(x_{k_1} x_{k_2} \cdots x_{k_m}) \\ = & \text{alt}(|\cdot|' \widehat{\otimes} |\cdot|') \left( \sum_{1 \leq i < j \leq m} K_{k_i k_j} (x_{k_{j+1}} \cdots x_{k_m} x_{k_1} \cdots x_{k_{i-1}} \widehat{\otimes} x_{k_{i+1}} \cdots x_{k_{j-1}}) \right. \\ & - \frac{1}{2} \sum_{i=1}^m x_{k_1} \cdots x_{k_{i-1}} x_{k_{i+1}} \cdots x_{k_m} \widehat{\otimes} x_{k_i} \\ & \left. + \sum_{i=1}^m \sum_{q=1}^{\infty} \frac{B_{2q}}{(2q)!} \sum_{p=0}^{2q-1} (-1)^p \binom{2q}{p} x_{k_1} \cdots x_{k_{i-1}} x_{k_i}^p x_{k_{i+1}} \cdots x_{k_m} \widehat{\otimes} x_{k_i}^{2q-p} \right). \end{aligned}$$

Here, for  $1 \leq k, l \leq n$ , we denote

$$K_{k,l} := (1 \widehat{\otimes} l) \Delta \left( \epsilon_{kl} x_k x_l - \delta_{kl} \frac{x_k^2}{e^{-x_k} - 1} \right) \in \widehat{T} \widehat{\otimes} \widehat{T},$$

where  $\delta_{kl}$  is the Kronecker delta,  $\epsilon_{kl}$  is defined by

$$\epsilon_{kl} := \begin{cases} 1, & \text{if } k > l, \\ 0, & \text{if } k \leq l, \end{cases}$$

and  $\text{alt} : N(\widehat{T}) \widehat{\otimes} N(\widehat{T}) \rightarrow N(\widehat{T}) \widehat{\otimes} N(\widehat{T})$ ,  $u \widehat{\otimes} v \mapsto u \widehat{\otimes} v - v \widehat{\otimes} u$ , is the alternating operator.

## § 2. Preliminaries

Let  $S$  be a compact connected oriented surface with non-empty boundary. Choose a basepoint  $* \in \partial S$ , and denote  $\pi := \pi_1(S, *)$ . We begin by recalling the coaction

$\mu : \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi \otimes \mathbb{Z}\hat{\pi}'$ , which is introduced in [10] inspired by a construction of Turaev [21]. The alternating part of  $\mu$  is just the Turaev cobracket  $\delta$ , but  $\mu$  is of multiplicative nature as stated below. Choose another point  $*^+ \in \partial S$  near  $*$  in the positive direction. For any  $\gamma \in \pi$  we regard it as a path from  $*$  to  $*^+$ , and choose a representative of  $\gamma$  in general position. By abuse of notation, we also denote by  $\gamma$  the representative. Then the curve  $\gamma$  is an immersion, and its singularities are at worst transverse double points. For each double point  $p$  of  $\gamma$  we have a unique pair  $0 < t_1^p < t_2^p < 1$  of parameters such that  $\gamma(t_1^p) = \gamma(t_2^p) = p$ . Then  $\mu(\gamma) \in \mathbb{Z}\pi \otimes \mathbb{Z}\pi'$  is defined by

$$\mu(\gamma) := - \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) (\gamma_{0t_1^p} \gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}|',$$

where the sum runs over the set of self-intersection points of  $\gamma$ ,  $\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) \in \{\pm 1\}$  is the local intersection number with respect to the orientation of  $S$ , and  $\gamma_{s_1 s_2}$  is the restriction of  $\gamma$  to the interval  $[s_1, s_2] \subset [0, 1]$  for any  $0 \leq s_1 < s_2 \leq 1$ . The operation  $\mu$  is well-defined, i.e., independent of the choice of a representative [10]. The Turaev cobracket  $\delta : \mathbb{Z}\hat{\pi}' \rightarrow \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$  [22] can be defined to be the alternating part of  $\mu$

$$(2.1) \quad \delta \circ | \cdot |' = \text{alt} \circ (1 \otimes | \cdot |') \circ \mu : \mathbb{Z}\pi \rightarrow \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'.$$

Here  $\text{alt} : \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}' \rightarrow \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$  is the alternating operator as above. The map  $\mu$  extends continuously to the map  $\mu : \widehat{\mathbb{Q}}\pi \rightarrow \widehat{\mathbb{Q}}\pi \widehat{\otimes} \widehat{\mathbb{Q}}\hat{\pi}$ . For example, the extension  $\mu$  is computed as follows.

**Theorem 2.1** ([3]). *If  $\gamma \in \pi_1(S, *)$  is represented by a simple loop with  $\varepsilon_*(\dot{\gamma}(0), \dot{\gamma}(1)) = +1$ , then we have*

$$\mu(\log \gamma) = \frac{1}{2} 1 \widehat{\otimes} |\log \gamma|' + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \sum_{p=0}^{2m-1} \binom{2m}{p} (-1)^p (\log \gamma)^p \widehat{\otimes} |(\log \gamma)^{2m-p}|'.$$

We can define the tensorial description of the map  $\mu^\theta : \widehat{T} \rightarrow \widehat{T} \widehat{\otimes} N(\widehat{T})$  with respect to any group-like expansion  $\theta$  of the fundamental group  $\pi_1(S, *)$ . Theorem 1.2 follows immediately from the following.

**Theorem 2.2.** *Let  $\delta^{\text{std}} = \delta^{\theta^{\text{std}}}$  be the tensorial description of the Turaev cobracket with respect to the standard group-like expansion  $\theta^{\text{std}}$  for the surface  $S = \Sigma_{0, n+1}$ .*

Then, for any  $m \geq 1$  and any  $k_1, k_2, \dots, k_m \in \{1, 2, \dots, n\}$ , we have

$$\begin{aligned} & \mu^{\text{std}}(x_{k_1} x_{k_2} \cdots x_{k_m}) \\ &= (1 \otimes | \cdot |') \left( \sum_{1 \leq i < j \leq m} (x_{k_1} \cdots x_{k_{i-1}} \widehat{\otimes} 1) K_{k_i k_j} (x_{k_{j+1}} \cdots x_{k_m} \widehat{\otimes} x_{k_{i+1}} \cdots x_{k_{j-1}}) \right. \\ & \quad - \frac{1}{2} \sum_{i=1}^m x_{k_1} \cdots x_{k_{i-1}} x_{k_{i+1}} \cdots x_{k_m} \widehat{\otimes} x_{k_i} \\ & \quad \left. + \sum_{i=1}^m \sum_{q=1}^{\infty} \frac{B_{2q}}{(2q)!} \sum_{p=0}^{2q-1} (-1)^p \binom{2q}{p} x_{k_1} \cdots x_{k_{i-1}} x_{k_i}^p x_{k_{i+1}} \cdots x_{k_m} \widehat{\otimes} x_{k_i}^{2q-p} \right). \end{aligned}$$

Here it should be remarked  $|x_{k_1} \cdots x_{k_{i-1}} x_{k_{j+1}} \cdots x_{k_m}|' = |x_{k_{j+1}} \cdots x_{k_m} x_{k_1} \cdots x_{k_{i-1}}|' \in N(\widehat{T})$ . The rest of this paper is devoted to the proof of Theorem 2.2.

Our proof consists of Theorem 3.1, Theorem 2.1, and (a slight modification of) the tensorial description of the homotopy intersection form by Massuyeau-Turaev [16], which we will explain later in short. Let  $S$  be a (general) connected compact oriented surface with non-empty boundary. Choose basepoints  $*$  and  $*^+$  in  $\partial S$  as above. Then, using a short path along the boundary from  $*$  to  $*^+$ , we identify the fundamental groups  $\pi = \pi_1(S, *)$  and  $\pi_1(S, *^+)$  with the homotopy set of path from  $*$  to  $*^+$  and that from  $*^+$  to  $*$ . Then the homotopy intersection form  $\eta : \mathbb{Z}\pi_1(S, *) \otimes \mathbb{Z}\pi_1(S, *^+) \rightarrow \mathbb{Z}\pi$ , introduced by Papakyriakopoulos [19] and Turaev [21] independently, is defined as follows. For  $\gamma_1 \in \pi_1(S, *)$  and  $\gamma_2 \in \pi_1(S, *^+)$  we choose their representatives in general position. Then  $\eta(\gamma_1, \gamma_2) \in \mathbb{Z}\pi$  is defined by

$$\eta(\gamma_1, \gamma_2) := \sum_{p \in \gamma_1 \cap \gamma_2} \varepsilon_p(\gamma_1, \gamma_2) (\gamma_1)_{*p} (\gamma_2)_{p*^+},$$

where  $\varepsilon_p(\gamma_1, \gamma_2) \in \{\pm 1\}$  is the local intersection number of  $\gamma_1$  and  $\gamma_2$  at the intersection point  $p$ ,  $(\gamma_1)_{*p}$  the segment of  $\gamma_1$  from  $*$  to  $p$ , and  $(\gamma_2)_{p*^+}$  that of  $\gamma_2$  from  $p$  to  $*^+$ . We define a map  $\kappa : \mathbb{Z}\pi \otimes \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi \otimes \mathbb{Z}\pi$  by

$$\kappa(\gamma_1, \gamma_2) := -(1 \otimes \gamma_2) ((1 \otimes \iota) \Delta \eta(\gamma_1, \gamma_2)) (1 \otimes \gamma_1)$$

for  $\gamma_1, \gamma_2 \in \pi$ . In other words, if we denote  $\Delta u = \sum u' \otimes u''$  and  $\Delta v = \sum v' \otimes v''$  for  $u, v \in \mathbb{Q}\pi$ , we define

$$(2.2) \quad \kappa(u, v) = - \sum (1 \otimes v'') ((1 \otimes \iota) \Delta \eta(u', v')) (1 \otimes u'').$$

Then we have a product formula

$$\mu(\gamma_1 \gamma_2) = \mu(\gamma_1)(\gamma_2 \otimes 1) + (\gamma_1 \otimes 1)\mu(\gamma_2) + (1 \otimes | \cdot |') \kappa(\gamma_1, \gamma_2).$$

More generally, we have

$$\begin{aligned}
& \mu(u_1 u_2 \cdots u_m) \\
&= \sum_{i=1}^m ((u_1 \cdots u_{i-1}) \otimes 1) \mu(u_i) ((u_{i+1} \cdots u_m) \otimes 1) \\
&+ \sum_{i < j} ((u_1 \cdots u_{i-1}) \otimes 1) (1 \otimes |' |) (\kappa(u_i, u_j) (u_{j+1} \cdots u_m \otimes u_{i+1} \cdots u_{j-1}))
\end{aligned} \tag{*}$$

for any  $m \geq 1$  and any  $u_1, u_2, \dots, u_m \in \mathbb{Z}\pi$  [10] (Corollary 4.3.4).

Massuyeau and Turaev [16] gave explicitly the tensorial description of the homotopy intersection form  $\eta$  with respect to any symplectic expansion [14] in the case  $S = \Sigma_{g,1}$ ,  $g \geq 1$ , i.e., the boundary  $\partial S$  is connected. In this case, we denote by  $\star \in \partial S$  a basepoint on the boundary, and by  $\zeta \in \pi_1(S, \star)$  the simple loop along the boundary in the negative orientation. The algebraic intersection number  $H \otimes H \rightarrow \mathbb{Q}$ ,  $X \otimes Y \mapsto X \cdot Y$ , is a non-degenerate pairing on  $H$ . The symplectic form  $\omega := \sum_{i=1}^g A_i B_i - B_i A_i \in H^{\otimes 2} \subset \widehat{T}$  is independent of the choice of a symplectic basis  $\{A_i, B_i\}_{i=1}^g \subset H = H_1(\Sigma_{g,1}; \mathbb{Q})$ . Throughout this paper we omit the symbol  $\otimes$  when it indicates the product in  $\widehat{T}$ . We have  $\theta(\zeta) \equiv 1 + \omega \pmod{\widehat{T}_{\geq 3}}$  for any group-like expansion  $\theta$ . Massuyeau [14] introduced the notion of a symplectic expansion: A group-like expansion  $\theta : \pi \rightarrow \widehat{T}$  is *symplectic* if  $\theta(\zeta) = \exp(\omega) (= \sum_{m=0}^{\infty} \frac{1}{m!} \omega^m) \in \widehat{T}$ , i.e.,  $\log \theta(\zeta) = \omega \in \widehat{T}$ . Symplectic expansions (in rational coefficients) exist [14] [12]. See also [5] for symplectic expansions in real coefficients. While their result deals only with symplectic expansions, but it is not hard to generalize it to any group-like expansion.

In order to give the tensorial description, Massuyeau and Turaev [16] introduced a continuous operation  $\overset{\bullet}{\rightsquigarrow} : \widehat{T}_{\geq 1} \times \widehat{T}_{\geq 1} \rightarrow \widehat{T}$  by

$$(X_1 \cdots X_{l-1} X_l) \overset{\bullet}{\rightsquigarrow} (Y_1 Y_2 \cdots Y_m) := (X_l \cdot Y_1) X_1 \cdots X_{l-1} Y_2 \cdots Y_m$$

for any  $l, m \geq 1$  and any  $X_i, Y_j \in H = H_1(\Sigma_{g,1}; \mathbb{Q})$ . Minus the symplectic form is the unit for the operation  $\overset{\bullet}{\rightsquigarrow}$ , i.e.,  $(-\omega) \overset{\bullet}{\rightsquigarrow} u = u \overset{\bullet}{\rightsquigarrow} (-\omega) = u$  for any  $u \in \widehat{T}_{\geq 1}$ . The restriction of  $\overset{\bullet}{\rightsquigarrow}$  to  $\widehat{T}_{\geq 2}$  is associative, and  $\overset{\bullet}{\rightsquigarrow}(\widehat{T}_{\geq l} \times \widehat{T}_{\geq m}) \subset \widehat{T}_{\geq (l+m-2)}$ . Hence, for any  $Z \in (-\omega) + \widehat{T}_{\geq 3}$ , there exists a unique  $Z^{-1} \in (-\omega) + \widehat{T}_{\geq 3}$  such that  $Z \overset{\bullet}{\rightsquigarrow} Z^{-1} = Z^{-1} \overset{\bullet}{\rightsquigarrow} Z = -\omega$ .

**Theorem 2.3** (Massuyeau-Turaev [16]). *Let  $\theta : \pi_1(\Sigma_{g,1}, \star) \rightarrow \widehat{T}$  be a group-like expansion. We denote  $\Omega = \Omega^\theta := \log \theta(\zeta) \in \omega + \widehat{T}_{\geq 3}$ . Then the tensorial description of the homotopy intersection form  $\eta : \widehat{\mathbb{Q}}\pi \times \widehat{\mathbb{Q}}\pi \rightarrow \widehat{\mathbb{Q}}\pi$  with respect to the expansion  $\theta$ ,  $\rho^\theta$ , is given by*

$$\rho^\theta(a, b) = (a - \text{aug}(a)) \overset{\bullet}{\rightsquigarrow} ((-\Omega)^{-1} + \omega s(\Omega) \omega) \overset{\bullet}{\rightsquigarrow} (b - \text{aug}(b))$$

for any  $a, b \in \widehat{T}$ .

*Proof.* We modify the proof of Theorem 10.4 in Massuyeau-Turaev [16]. The tensorial description  $\rho^\theta$  is characterized by the condition

$$(2.3) \quad \forall X \in H, \quad \rho^\theta(X, e^{-\Omega}) = X.$$

Since  $s(z)z - 1 = z(e^{-z} - 1)^{-1}$ , we have

$$\rho^\theta(X, e^{-\Omega}) = \rho^\theta(X, \Omega) \frac{e^{-\Omega} - 1}{\Omega} = \rho^\theta(X, \Omega)(s(\Omega)\Omega - 1)^{-1}.$$

Hence the condition (2.3) is equivalent to

$$(2.4) \quad \forall X \in H, \quad \rho^\theta(X, \Omega) = Xs(\Omega)\Omega - X.$$

Now the map  $(a, b) \in \widehat{T} \times \widehat{T} \mapsto (a - \text{aug}(a))s(\Omega)(b - \text{aug}(b)) \in \widehat{T}$  is a Fox pairing in the sense of Massuyeau-Turaev [16]. Hence, if we introduce a unique Fox pairing  $\rho_\Omega : \widehat{T} \times \widehat{T} \rightarrow \widehat{T}$  characterized by the condition

$$(2.5) \quad \forall X \in H, \quad \rho_\Omega(X, \Omega) = -X,$$

then we have

$$\rho^\theta(a, b) = \rho_\Omega(a, b) + (a - \text{aug}(a))s(\Omega)(b - \text{aug}(b))$$

for any  $a$  and  $b \in \widehat{T}$ . Let  $\{A_i, B_i\}_{i=1}^g \subset H$  be a symplectic basis. The tensor

$$\begin{aligned} R_\Omega := \sum_{i,j=1}^g & (-B_i\rho_\Omega(A_i, A_j)B_j + B_i\rho_\Omega(A_i, B_j)A_j \\ & + A_i\rho_\Omega(B_i, A_j)B_j - A_i\rho_\Omega(B_i, B_j)A_j) \in \widehat{T}_{\geq 2} \end{aligned}$$

satisfies  $\rho_\Omega(a, b) = (a - \text{aug}(a)) \overset{\bullet}{\rightsquigarrow} R_\Omega \overset{\bullet}{\rightsquigarrow} (b - \text{aug}(b))$  for any  $a$  and  $b \in \widehat{T}$ . Then the condition (2.5) is equivalent to  $R_\Omega \overset{\bullet}{\rightsquigarrow} \Omega = \omega$ . This means  $R_\Omega = (-\Omega)^{-1}$ . Therefore we have

$$\begin{aligned} \rho^\theta(a, b) &= (a - \text{aug}(a)) \overset{\bullet}{\rightsquigarrow} R_\Omega \overset{\bullet}{\rightsquigarrow} (b - \text{aug}(b)) + (a - \text{aug}(a))s(\Omega)(b - \text{aug}(b)) \\ &= (a - \text{aug}(a)) \overset{\bullet}{\rightsquigarrow} ((-\Omega)^{-1} + \omega s(\Omega)\omega) \overset{\bullet}{\rightsquigarrow} (b - \text{aug}(b)). \end{aligned}$$

This proves the theorem. □

### § 3. Proof of the Result

Now we begin the proof of Theorem 2.2, from which Theorem 1.2 follows immediately by (2.1). Let  $S$  be the genus 0 compact surface  $\Sigma_{0,n+1}$  for some  $n \geq 0$ . We consider the standard group-like expansion  $\theta^{\text{std}} : \pi = \pi_1(S, *) \rightarrow \widehat{T} = \widehat{T}(H_1(\Sigma_{0,n+1}; \mathbb{Q}))$ . Choose

one point  $*_k \in \partial_k S$  for each component  $\partial_k S$  and let  $\xi_k \in \pi_1(S, *_k)$  be the simple positive boundary loop for  $1 \leq k \leq n$ . We can choose a simple path  $\chi_k$  from  $*$  to  $*_k$  such that  $\chi_k \xi_k \chi_k^{-1} = \gamma_k \in \pi_1(S, *)$ . We glue  $n$  copies of the surface  $\Sigma_{1,1}$  to the surface  $S = \Sigma_{0,n+1}$  along the boundary  $\partial_k S$ ,  $1 \leq k \leq n$ , such that the basepoints  $*$  and  $*_k$  are identified with each other. This gluing yields a surface  $\hat{S} \cong \Sigma_{n,1}$ . Let  $\{\alpha_k, \beta_k\}$  be a symplectic generator of the fundamental group of the  $k$ -th copy of  $\Sigma_{1,1}$  with basepoint  $*$ . Then the set  $\{\chi_k \alpha_k \chi_k^{-1}, \chi_k \beta_k \chi_k^{-1}\}_{k=1}^n$  is a symplectic generator of the fundamental group  $\pi_1(\hat{S}, *)$ . If we denote  $A_k := [\chi_k \alpha_k \chi_k^{-1}]$  and  $B_k := [\chi_k \beta_k \chi_k^{-1}] \in H_1(\hat{S}; \mathbb{Q})$ , then the set  $\{A_k, B_k\}_{k=1}^n$  is a symplectic basis of the homology group  $H_1(\hat{S}; \mathbb{Q})$ . The map  $\iota : \hat{T} = \hat{T}(H_1(S; \mathbb{Q})) \rightarrow \hat{T}(H_1(\hat{S}; \mathbb{Q}))$  defined by  $\iota(x_k) := A_k B_k - B_k A_k$  is an *injective* algebra homomorphism. See [9] §6.2.

Let  $\theta_k : \pi_1(\Sigma_{1,1}, *) \rightarrow \hat{T}(H_1(\Sigma_{1,1}; \mathbb{Q}))$  be a symplectic expansion for the  $k$ -th copy of  $\Sigma_{1,1}$ . We identify the target with the completed tensor algebra  $\hat{T}(\mathbb{Q}A_k \oplus \mathbb{Q}B_k) \subset \hat{T}(H_1(\hat{S}; \mathbb{Q}))$ , and define a group-like expansion  $\hat{\theta} : \pi_1(\hat{S}, *) \rightarrow \hat{T}(H_1(\hat{S}; \mathbb{Q}))$  by  $\hat{\theta}(\chi_k \alpha_k \chi_k^{-1}) := \theta_k(\alpha_k)$  and  $\hat{\theta}(\chi_k \beta_k \chi_k^{-1}) := \theta_k(\beta_k)$ . Then the diagram

$$\begin{array}{ccc} \pi_1(S, *) & \xrightarrow{\theta^{\text{std}}} & \hat{T}(H_1(S; \mathbb{Q})) \\ i_* \downarrow & & \downarrow \iota \\ \pi_1(\hat{S}, *) & \xrightarrow{\hat{\theta}} & \hat{T}(H_1(\hat{S}; \mathbb{Q})) \end{array}$$

commutes, where  $i : (S, *) \hookrightarrow (\hat{S}, *)$  is the inclusion. We have  $\hat{\theta}(\zeta) = \prod_{k=1}^n \exp(A_k B_k - B_k A_k) = \iota(\prod_{k=1}^n \exp(x_k))$ . Here we denote by  $u * v$  the Baker-Campbell-Hausdorff series of  $u$  and  $v \in \hat{T}_{\geq 1} = \hat{T}(H_1(S; \mathbb{Q}))_{\geq 1}$

$$u * v := \log((\exp u)(\exp v)) = u + v + \frac{1}{2}[u, v] + \frac{1}{12}[u, [u, v]] + \frac{1}{12}[v, [v, u]] + \cdots,$$

and consider the element  $\Xi := x_1 * x_2 * \cdots * x_n \in \hat{T}_{\geq 1}$ . Then we obtain  $\log \hat{\theta}(\zeta) = \iota(\Xi) \in \hat{T}(H_1(\hat{S}; \mathbb{Q}))$ , and, from the Massuyeau-Turaev theorem 2.3,

$$(3.1) \quad \rho^{\hat{\theta}}(a, b) = (a - \text{aug}(a)) \overset{\bullet}{\rightsquigarrow} ((-\iota(\Xi))^{-1} + \omega_S(\iota(\Xi))\omega) \overset{\bullet}{\rightsquigarrow} (b - \text{aug}(b))$$

for any  $a, b \in \hat{T}(H_1(\hat{S}; \mathbb{Q}))$ .

By the injective homomorphism  $\iota$ , the Massuyeau-Turaev operation  $\overset{\bullet}{\rightsquigarrow}$  on  $\hat{T}(H_1(\hat{S}; \mathbb{Q}))$  induces a continuous operation on  $\hat{T}_{\geq 1} = \hat{T}(H_1(\Sigma_{0,n+1}; \mathbb{Q}))_{\geq 1}$ ,  $\overset{\bullet}{\rightsquigarrow} : \hat{T}_{\geq 1} \times \hat{T}_{\geq 1} \longrightarrow \hat{T}_{\geq 1}$ , given by

$$x_{i_1} \cdots x_{i_{l-1}} x_{i_l} \overset{\bullet}{\rightsquigarrow} x_{j_1} x_{j_2} \cdots x_{j_m} = -\delta_{i_l j_1} x_{i_1} \cdots x_{i_{l-1}} x_{j_1} x_{j_2} \cdots x_{j_m}$$

for  $l, m \geq 1$  and  $1 \leq i_1, \dots, i_l, j_1, \dots, j_m \leq n$ . In fact, we have  $(A_k B_k - B_k A_k) \overset{\bullet}{\rightsquigarrow} (A_l B_l - B_l A_l) = -\delta_{kl} (A_k B_k - B_k A_k)$  for  $1 \leq k, l \leq n$ . The operation  $\overset{\bullet}{\rightsquigarrow}$  on  $\hat{T}_{\geq 1}$  is associative

with unit  $x_0 = -\sum_{k=1}^n x_k$ . Thus we can take the inverse element  $Z^{-1}$  of any  $Z \in x_0 + \widehat{T}_{\geq 2}$  with respect to the operation  $\overset{\bullet}{\rightsquigarrow}$ ,  $Z^{-1} \overset{\bullet}{\rightsquigarrow} Z = Z \overset{\bullet}{\rightsquigarrow} Z^{-1} = x_0$ .

Consider the inverse element  $-\Xi^{-1}$  of  $-\Xi = -x_1 * x_2 * \cdots * x_n$  with respect to the operation  $\overset{\bullet}{\rightsquigarrow}$ .

**Theorem 3.1.**

$$-\Xi^{-1} + x_0 s(\Xi)x_0 = x_0 - \sum_{k>l} x_k x_l + \sum_{k=1}^n s(x_k)x_k^2 = -\sum_{k>l} x_k x_l + \sum_{k=1}^n \frac{x_k^2}{e^{-x_k} - 1}.$$

*Proof.* We denote the left-hand side by

$$Y := -\Xi^{-1} + x_0 s(\Xi)x_0 = \sum_{m=1}^{\infty} Y_{(m)}, \quad Y_{(m)} \in H^{\otimes m}.$$

Since  $\Xi \equiv -x_0 + \frac{1}{2} \sum_{k<l} [x_k, x_l] \pmod{\widehat{T}_{\geq 3}}$ , we have  $Y_{(1)} = x_0$  and

$$\begin{aligned} Y_{(2)} &= \frac{1}{2} \sum_{k<l} [x_k, x_l] - \frac{1}{2} x_0^2 \\ &= \frac{1}{2} \sum_{k<l} (x_k x_l - x_l x_k) - \frac{1}{2} \sum_{k<l} (x_k x_l + x_l x_k) - \frac{1}{2} \sum_{k=1}^n x_k^2 \\ &= -\sum_{k>l} x_k x_l - \frac{1}{2} \sum_{k=1}^n x_k^2. \end{aligned}$$

To compute the higher degree term  $Y_{(m)}$  for each  $m \geq 3$ , we introduce a topological algebra automorphism  $Q$  of  $\widehat{T}$  defined by

$$Q(x_k) = -x_{n-k}, \quad 1 \leq k \leq n,$$

inspired by Kuno's work [13]. See also [12] Example 5.3. It is clear to see  $Q(\Xi) = -\Xi$  and  $Qx_0 = -x_0$ . Here we have

$$Q(u \overset{\bullet}{\rightsquigarrow} v) = -(Qu) \overset{\bullet}{\rightsquigarrow} (Qv)$$

for any  $u$  and  $v \in \widehat{T}_{\geq 1}$ . In fact, we compute  $(Qx_k) \overset{\bullet}{\rightsquigarrow} (Qx_l) = (-x_{n-k}) \overset{\bullet}{\rightsquigarrow} (-x_{n-l}) = -\delta_{kl} x_{n-k} = Q(\delta_{kl} x_k) = -Q(x_k \overset{\bullet}{\rightsquigarrow} x_l)$  for any  $1 \leq k, l \leq n$ . In particular, for any  $Z \in x_0 + \widehat{T}_{\geq 2}$ , we have  $x_0 = -Qx_0 = -Q(Z \overset{\bullet}{\rightsquigarrow} Z^{-1}) = (QZ) \overset{\bullet}{\rightsquigarrow} (QZ^{-1})$ , and so  $Q(Z^{-1}) = (QZ)^{-1}$ . Moreover we have  $s(-z) = -1 - s(z)$ . Therefore

$$(3.2) \quad QY = -(Q\Xi)^{-1} + x_0 s(Q\Xi)x_0 = \Xi^{-1} - x_0^2 - x_0 s(\Xi)x_0 = -Y - x_0^2.$$

On the other hand, we have

$$(3.3) \quad Y^{-1} = -1 + e^{-\Xi} = -1 + e^{-x_n} \cdots e^{-x_2} e^{-x_1}.$$

In fact,  $\Xi = \Xi \overset{\bullet}{\rightsquigarrow} Y \overset{\bullet}{\rightsquigarrow} Y^{-1} = -\Xi \overset{\bullet}{\rightsquigarrow} \Xi^{-1} \overset{\bullet}{\rightsquigarrow} Y^{-1} + \Xi s(\Xi) Y^{-1} = -Y^{-1} + \Xi s(\Xi) Y^{-1} = \frac{\Xi}{e^{-\Xi} - 1} Y^{-1}$ . Since the algebra  $\widehat{T}$  has no zero divisor, we obtain (3.3).

Let  $W$  (resp.  $I$ ) be the closed linear subspace in  $\widehat{T}_{\geq 1}$  generated by the set  $\{x_{k_1} x_{k_2} \cdots x_{k_m}; k_1 \geq k_2 \geq \cdots \geq k_m\}$  (resp.  $\{x_{k_1} x_{k_2} \cdots x_{k_m}; \#\{k_1, k_2, \dots, k_m\} \geq 2\}$ ). The subspace  $W$  (resp.  $I$ ) is a subalgebra (resp. a two-sided ideal) of  $\widehat{T}_{\geq 1}$  with respect to the multiplication  $\overset{\bullet}{\rightsquigarrow}$ . Since

$$Y = x_0 + \sum_{m=1}^{\infty} \overbrace{(x_0 - Y^{-1}) \overset{\bullet}{\rightsquigarrow} (x_0 - Y^{-1}) \overset{\bullet}{\rightsquigarrow} \cdots \overset{\bullet}{\rightsquigarrow} (x_0 - Y^{-1})}^{m \text{ times}}.$$

and  $x_0 - Y^{-1} \in W$  from (3.3), we have  $Y \in W$ . It is clear that the direct sum decomposition  $W = (W \cap I) \oplus \bigoplus_{k=1}^n x_k \mathbb{Q}[[x_k]]$  holds, and so  $W \cap \text{Ker}(Q + 1) \subset \bigoplus_{k=1}^n x_k \mathbb{Q}[[x_k]]$ , while we have  $Q(Y - Y_{(2)}) = -(Y - Y_{(2)})$  from (3.2). Hence we have  $Y - Y_{(2)} \in \bigoplus_{k=1}^n x_k \mathbb{Q}[[x_k]]$ . This implies that it suffices to show the theorem modulo the ideal  $I$ . From (3.3) we have

$$\begin{aligned} Y^{-1} \overset{\bullet}{\rightsquigarrow} (x_0 + \sum_{k=1}^n x_k^2 s(x_k)) &= Y^{-1} \overset{\bullet}{\rightsquigarrow} \left( \sum_{k=1}^n x_k \frac{x_k}{e^{-x_k} - 1} \right) \\ &\equiv \left( \sum_{k=1}^n e^{-x_k} - 1 \right) \overset{\bullet}{\rightsquigarrow} \left( \sum_{k=1}^n x_k \frac{x_k}{e^{-x_k} - 1} \right) = - \sum_{k=1}^n (e^{-x_k} - 1) \frac{x_k}{e^{-x_k} - 1} = x_0. \end{aligned}$$

Hence we have  $Y \equiv x_0 + \sum_{k=1}^n x_k^2 s(x_k) \pmod{I}$ , as was to be shown.  $\square$

As a corollary, we conclude

$$(3.4) \quad \rho^{\theta^{\text{std}}}(a, b) = (a - \text{aug}(a)) \overset{\bullet}{\rightsquigarrow} \left( - \sum_{k>l} x_k x_l + \sum_{k=1}^n \frac{x_k^2}{e^{-x_k} - 1} \right) \overset{\bullet}{\rightsquigarrow} (b - \text{aug}(b))$$

for any  $a, b \in \widehat{T} = \widehat{T}(H_1(S; \mathbb{Q}))$ . In particular, by (2.2), we have

$$(3.5) \quad \kappa^{\text{std}}(x_k, x_l) = -(1 \widehat{\otimes} l) \Delta \left( \epsilon_{kl} x_k x_l - \delta_{kl} \frac{x_k^2}{e^{-x_k} - 1} \right) = -K_{k,l} \in \widehat{T} \widehat{\otimes} \widehat{T},$$

where  $\kappa^{\text{std}}$  is the tensorial description of  $\kappa$  with respect to the standard exponential expansion  $\theta^{\text{std}}$ . Recall  $x_k = \log \theta^{\text{std}}(\gamma_k)$ . Consequently, substituting (3.5) and Theorem 2.1 to the product formula (\*), we obtain Theorem 2.2. This completes the proof.  $\square$

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