RIMS Kôkyûroku Bessatsu
B66 (2017), 015–019

On commensurability of quadratic differentials on surfaces

By

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Abstract

We consider commensurability of quadratic differentials on surfaces. Each commensurability class has a natural order by the covering relation. We show that each commensurability class contains a unique minimal (orbifold) element. We also discuss the relationship between commensurability of quadratic differentials and fibered commensurability, a notion introduced by Calegari-Sun-Wang.

§ 1. preface

Let S be an orientable surface of finite type and X a Riemann surface of finite analytic type which is homeomorphic to S. Throughout the paper, surfaces considered are orientable and of finite type with $3g(S)-3+p(S)>0$ where $g(S)$ is the genus and $p(S)$ is the number of punctures. A (holomorphic) *quadratic differential* $q=\{(U_j,z_j),q_j\}_j$ is a family of holomorphic functions $\{q_j\}_j$, one defined on each local chart (U_j,z_j) of X so that if $U_i\cap U_j\neq\emptyset$, then $q_j(z_j)=q_i(z_i)(dz_i/dz_j)^2$. Quadratic differentials are arrowed to have poles of degree one at the punctures. A quadratic differential on a topological surface S is a quadratic differential with respect to some complex structure on S. For a given homeomorphism $f:S\to S$ and a quadratic differential $q=\{(U_j,z_j),q_j\}_j$, we define push-forward by

$$f_*(q):=\{(f(U_j),z_j\circ f^{-1}),q_j\}_j.$$

Received November 24, 2015. Revised December 19, 2016.
2010 Mathematics Subject Classification(s): 32G15, 37E30.
Key Words: commensurability, quadratic differentials, pseudo-Anosov.
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Given a quadratic differential q, by integrating the square root of q, we get natural coordinates around nonzero points of q. This gives a singular Euclidean structure on S. Furthermore q has natural horizontal (resp. vertical) directions which defined to be $v \in T_zX$ so that $q(z)v^2$ is a positive (resp. negative) real number. In this paper, coverings considered are assumed to be unramified. Let $p : \widetilde{S} \to S$ be a finite covering. Since each quadratic differential is defined on charts, we can naturally lift quadratic differentials via any finite covering. For a quadratic differential q on S, its lift with respect to $p : \widetilde{S} \to S$ is denoted by p^*q. Conversely, a quadratic differential $\widetilde{q}$ on a Riemann surface $\widetilde{S}$ is said to be *symmetric* with respect to $p : \widetilde{S} \to S$ if $\widetilde{q}$ is a lift of a quadratic differential q on S. If $\widetilde{q}$ is symmetric, we use the notation of push-forward for coverings, i.e. $q = p_*(\widetilde{q})$. Note that if $\widetilde{q}$ is symmetric, then every quadratic differential on the Teichmüller geodesic determined by $\widetilde{q}$ is symmetric (c.f. [2]).

Definition 1.1. Let q_1 and q_2 be quadratic differentials on surfaces S_1 and S_2 respectively. The quadratic differential q_1 is said to *cover* q_2 if there exists a finite covering $p : S_1 \to S_2$ so that $p^*q_2 = q_1$.

Once we have a covering relation, we may consider commensurability.

Definition 1.2. Two quadratic differentials q_1 and q_2 on surfaces S_1 and S_2 respectively are *commensurable* (denoted $q_1 \sim q_2$) if there is a third quadratic differential $\widetilde{q}$ that covers both q_1 and q_2.

We also need to consider orbifolds. On a 2-dimensional orientable orbifold, quadratic differentials are defined on the surface that we get by puncturing every orbifold point. When we take coverings, we fill every puncture corresponding to an orbifold point once it is locally covered by a surface. It can be seen that this commensurability is an equivalence relation (Proposition 2.1). Two quadratic differentials q_1, q_2 are said to be *conjugate* if there exists a surface homeomorphism (or a orbifold automorphism) f such that $f_*q_1 = q_2$. By considering conjugacy classes, we endow each commensurability class with an order by the covering relation. The main theorem of this paper is the following.

Theorem 1.3. *Every commensurability class of quadratic differentials contains a unique minimal (orbifold) element.*

Remark. Theorem 1.3 can also be verified by the identity theorem if there is a zero of the quadratic differentials in the interior of the surface. The argument in this paper works even for the case where all the zeros are punctured.

We also consider fibered commensurability and its relationship with the commensurability of quadratic differentials. In [1], Calegari-Sun-Wang introduced following commensurability on surface automorphisms. Let $\mathrm{Mod}(S)$ denote the mapping class group of S.

Definition 1.4 ([1]). Let S_1 and S_2 be orientable surfaces of finite type. A mapping class $\phi_1 \in \mathrm{Mod}(S_1)$ *covers* $\phi_2 \in \mathrm{Mod}(S_2)$ if there exists a finite covering $p : S_1 \to S_2$ and $k \in \mathbb{Z} \setminus \{0\}$ such that a lift φ of ϕ_2^k with respect to p satisfies $\varphi = \phi_1$. Two mapping classes are said to be *commensurable* if there exists a mapping class that covers both.

We call an orbifold automorphism pseudo-Anosov if it can be covered by a psuedo-Anosov mapping class on a surface. In [1] and [3], the following theorem was shown.

Theorem 1.5. *Every fibered commensurability class of pseudo-Anosov mapping classes contains a unique minimal (orbifold) element.*

We give a new proof of the above theorem by using Theorem 1.3.

§ 2. Unique minimal element

We first observe that commensurability of quadratic differentials is an equivalence relation. The reflectivity and symmetry are trivial.

Proposition 2.1. *Let q_i be a quadratic differential on a surface S_i for $i = 1, 2, 3$. Suppose $q_1 \sim q_2$ and $q_2 \sim q_3$. Then we have $q_1 \sim q_3$.*

Proof. Let $q_{j,j+1}$ be a quadratic differential on a surface $S_{j,j+1}$ which covers q_j and q_{j+1} for $j = 1, 2$. Then in the orbifold fundamental group $\pi_1(S_2)$, the images of $\pi_1(S_{1,2})$ and $\pi_1(S_{2,3})$ by the covering maps are of finite index. Hence the intersection $H := \pi_1(S_{1,2}) \cap \pi_1(S_{2,3})$ in $\pi_1(S_2)$ is also a finite index subgroup. Then the lift $\widetilde{q_2}$ of q_2 to the covering corresponding to H is also the lift of $q_{1,2}$ and $q_{2,3}$. Therefore $\widetilde{q_2}$ covers all q_1, q_2, and q_3, in particular $q_1 \sim q_3$. □

The main idea of the following proof of Theorem 1.3 is from [4, Lemma 4.11].

Proof of Theorem 1.3. We show that if $q_1 \sim q_2$, then both q_1 and q_2 cover the same quadratic differential q'. Recall that each quadratic differential q determines a singular Euclidean structure with horizontal and vertical foliation. Let $\mathrm{Sing}(q)$ denote the set of singular points of the singular Euclidean structure. This $\mathrm{Sing}(q)$ is finite and contains all punctures. Let $q_{1,2}$ be a quadratic differential on a surface $S_{1,2}$ which covers both q_1 and q_2 and let $p_i : S_{1,2} \to S_i$ denote the associated covering maps for $i = 1, 2$. Pick any $s \in \mathrm{Sing}(q_{1,2})$, then we define $\Sigma_1(s) := p_1^{-1}(p_1(s))$. Inductively define $\Sigma_{i+1}(s) := p_{[i+1]}^{-1}p_{[i+1]}(\Sigma_i)$ where $[k] = 1$ if k is odd and $[k] = 2$ if k is even. Since $\Sigma_i(s) \subset \Sigma_{i+1}(s) \subset \mathrm{Sing}(q_{1,2})$, we eventually have $\Sigma_i(s) = \Sigma_{i+1}(s)(=: \Sigma(s))$ for large enough i. Thus we have an equivalence relation on $\mathrm{Sing}(q_{1,2})$. Next, we pick any $x \in S_{1,2} \setminus \mathrm{Sing}(q_{1,2})$. There is a point $s' \in \mathrm{Sing}(q_{1,2})$ such that we can connect x and s' by a single Euclidean geodesic γ. The geodesic γ has well defined angle $\theta_\gamma \mod \pi$. Let $l_q(\gamma)$ denote the Euclidean length of γ. Since there are only finitely many points from $\Sigma(s')$ with angle θ_γ and Euclidean distance $l_q(\gamma)$, we get $\Sigma(x) \subset S \setminus \mathrm{Sing}(q)$ in the same way as above. Thus we get an equivalence relation $x \sim y :\iff y \in \Sigma(x)$ on $S_{1,2}$. Since this relation is defined by composing local homeomorphisms p_1 and p_2, the quotient map $p' : S_{1,2} \to S/{\sim}$ is a covering. Note that $S/{\sim}$ might be an orbifold. For each point $x \in S/{\sim}$, we may find a small open neighborhood U_x so that on all pre-images in $S_{1,2}$, the quadratic differentials can be identified via p_1 and p_2. Hence p' determines a quadratic differential q' on $S/{\sim}$. By construction, p' factors through $p_i : S_{1,2} \to S_i$ and hence both q_1 and q_2 cover q'. If there is another quadratic differential q_3 in the commensurability class that does not cover q', then we may apply the same argument as above to find a quadratic differential which is covered by q' and q_3. Since each time we get a new quadratic differential, the Euler characteristic of the underlying orbifold decreases, this process would terminate. Thus we get a unique minimal element. $\square$

§ 3. Fibered commensurability

In this section, we give a proof of Theorem 1.5.

Lemma 3.1. *Let $p : \widetilde{S} \to S$ be a finite covering of orientable 2-orbifolds. Let $f : \widetilde{S} \to \widetilde{S}$ be an orbifold automorphism. If there is a quadratic differential q such that $f_*^n(q)$ is symmetric with respect to $p : \widetilde{S} \to S$ for each $n \in \mathbb{Z}$. Then there is a finite covering $p' : \widetilde{S} \to S'$ which factors through p such that f is a lift of some homeomorphism $f' : S' \to S'$.*

Proof. Note that all $(p \circ f^n)_*(q)$ are commensurable to each other. We define $x \sim y$ if there exists n such that $p \circ f^n(x) = p \circ f^n(y)$. We take transitive closure of this $\sim$ to define an equivalence relation. Let $\Sigma(x)$ denote the equivalence class of x. Similarly to the proof of Theorem 1.3, we see that $\Sigma(x)$ is finite and defines a covering. Furthermore, by construction, we see that $f(\Sigma(x)) = \Sigma(f(x))$. Thus we have a homeomorphism $f' : S' \to S'$ so that f is a lift of f'. $\square$

Proof of Theorem 1.5. For $i = 1, 2$, let $f_i : S_i \to S_i$ be commensurable pseudo-Anosov homeomorphisms. We will find $f' : S' \to S'$ which is covered both by f_1 and f_2. Note that if two pseudo-Anosov maps are commensurable, then they give commensurable quadratic differentials whose horizontal and vertical foliations are stable and unstable foliations [2]. Let q_i be a quadratic differential on S_i associated to f_i for $i = 1, 2$ so that q_1 and q_2 are commensurable. Then by Theorem 1.3, there is a unique minimal quadratic differential q' on some orbifold S' in the commensurability class. Note that for all $i = 1, 2$ and $n \in \mathbb{Z}$, $(f_i^n)_* q_i$ is also symmetric with respect to the covering from S_i to S'. This is because f_i preserves projective classes of horizontal and vertical foliations of q_i. Hence if f_i is not any lift of a homeomorphism on S', we may find further covering from q' by Lemma 3.1. This contradicts the minimality of q', so f_i is a lift of some homeomorphism f_i' on S' for both $i = 1, 2$. The stable and unstable foliations of f_1' and f_2' agree. Note that there is a lower bound for the stretch factor of pseudo-Anosovs. Therefore there are integers a_i such that $a_1 \log \lambda(f_1) = a_2 \log \lambda(f_2)$ where $\lambda(f)$ is the stretch factor of f. Let n_i $(i = 1, 2)$ be integers such that $n_1 a_1 + n_2 a_2$ is the greatest common divisor of a_1 and a_2. Then by considering the stretch factor, we see that $f' := (f_1')^{n_1} \circ (f_2')^{n_2}$ satisfies that there are integers m_i such that $(f')^{m_i} \circ f_i'$ preserves q'. By appealing to the singular Euclidean structure with respect to q', it can readily be seen that $(f')^{m_i} \circ f_i'$ is of finite order. But if $(f')^{m_i} \circ f_i'$ is non-trivial, it contradicts the minimality of q'. Hence f' is a root of both f_1' and f_2'. Therefore f' is covered by both f_1 and f_2. Similarly to the proof of Theorem 1.3, after finite steps, we end up with a unique minimal element. □

Acknowledgements

The author would like to thank Michihiko Fujii for the invitation to the conference. He also thanks Hideki Miyachi for helpful conversations. Finally he thanks anonymous referee for useful comments.

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