RIMS Kôkyûroku Bessatsu
B66 (2017), 021–036

Growth and cogrowth tightness of Kleinian and hyperbolic groups

By

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Abstract

Let G be a non-elementary discrete isometry group of the hyperbolic space or more generally a proper geodesic Gromov hyperbolic space X. We say that G is growth tight if for any non-trivial normal subgroup H the critical exponent $\kappa(H\backslash G)$ of the quotient group is strictly smaller than $\kappa(G)$. Moreover, G is cogrowth tight if the critical exponent $\delta(H)$ of any such H is strictly greater than $\delta(G)/2$. We review recent results on these properties of G with the addition of certain new observation. In particular, we see that a non-elementary quasi-convex cocompact discrete isometry group G of X is growth tight.

§ 1. Introduction

In this paper, we survey a recent progress on problems of the spectra of critical exponents defined by normal subgroups of a discrete isometry group of the hyperbolic space both in the classical sense and in the modern sense. More precisely, we first consider the hyperbolic space $\mathbb{H}^{n+1}$ of dimension $n+1$ and Kleinian groups acting on it isometrically. Then, we extend them to a proper geodesic metric space X hyperbolic in the sense of Gromov and discrete isometry groups acting on X. In particular, hyperbolic groups themselves can be our objects of study by considering their Cayley graphs.

For a discrete isometry group G acting on X, the critical exponent $\delta(G)$ of its Poincaré series is an important index which characterizes geometric properties of G. For any normal subgroup H of G, we can think of two kinds of critical exponents: $\delta(H)$

Received January 18, 2016. Revised December 21, 2016.
2010 Mathematics Subject Classification(s): Primary 37C35, 30F40, 53C23; Secondary 20F65, 57S30, 37D40.
Key Words: *growth tight, Kleinian group, critical exponent, volume entropy, Gromov hyperbolic space, quasi-convex cocompact.*
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for H itself acting on X and $\kappa(H\backslash G)$ for the quotient group acting on the quotient space $H\backslash X$. There are interesting researches on the spectra of these critical exponents taken over all normal subgroups. Among them, we focus on growth tightness in this paper, which is a concept introduced by Grigorchuk and de la Harpe [12]. It is said that G is growth tight if the spectra $\{\kappa(H\backslash G)\}_{H \triangleleft G}$ has multiplicity one at $\delta(G) = \kappa(G)$. Section 2 is devoted to basic definitions of these concepts in a general setting.

In Section 3, we summarize necessary facts for considering the problem on the critical exponents of Kleinian groups and introduce a recent result on the growth tightness of geometrically finite Kleinian groups by Dal'bo et al. [9]. In Section 5, we generalize this to the Gromov hyperbolic space. Especially, Theorem 5.2 is the corresponding assertion to the above result for Kleinian groups and we provide a proof for it (Section 7) following the arguments in [9]. This is one of new ingredients of this paper which contribute to our subject matter.

On the other hand, researches on the spectra $\{\delta(H)\}_{H \triangleleft G}$ have longer history, and the characterization of amenability of $H\backslash G$ by the equality $\delta(H) = \delta(G)$ is well known. However in this paper, we are interested in a different direction, namely, the lower bound of $\delta(H)$ for non-trivial normal subgroups H of G. In Section 4, we review several results on this problem for Kleinian groups and state that the lower bound is $\delta(G)/2$. Then, in analogy to the growth tightness, we define newly the cogrowth tightness of G, which means that $\delta(H) > \delta(G)/2$ for every non-trivial normal subgroup H of G. The introduction of this concept is another feature of this paper. The result of Roblin [19] implies that a Kleinian group of divergence type is cogrowth tight. In Section 6, we state our recent result on the Gromov hyperbolic space, which generalizes the cogrowth tightness for Kleinian groups.

Both for the growth and the cogrowth tightness, the existence of a sequence of non-trivial normal subgroups H of G whose critical exponents converge to the upper and the lower bounds, in other words, the non-isolation of $\kappa(G)$ and $\delta(G)/2$ in the spectra, is investigated in company with the original problems. In Sections 3–6, we also touch on this problem. In particular, we mention our expectation that there is a certain relation between $\kappa(H\backslash G)$ and $\delta(H)$, which might be useful for dealing with this kind of problems.

Acknowledgements. The author learnt problems on the growth tightness from lectures by Wenyuan Yang based on his paper [25] at Waseda University on March 16–20, 2015. The contents of the present paper is based on the author's talk in the conference "Geometry and Analysis of Discrete Groups and Hyperbolic Spaces" held at RIMS, Kyoto University on June 22–26, 2015. This research project is partially being developed jointly with Johannes Jaerisch.

§ 2. Basic concepts

Let (X, d) be a proper metric space. The group of all isometric automorphisms of (X, d) is denoted by $\mathrm{Isom}(X, d)$. A subgroup $G \subset \mathrm{Isom}(X, d)$ is *discrete* if it acts on X properly discontinuously.

The *critical exponent* $\delta(G)$ of a discrete group $G \subset \mathrm{Isom}(X, d)$ is defined by

$$\delta(G) = \limsup_{r \to \infty} \frac{1}{r} \log \#\{g \in G \mid d(x, g(x)) \le r\} \quad (x \in X).$$

This coincides with the infimum of exponents $s \ge 0$ such that the Poincaré series $P_G^s(x) = \sum_{g \in G} e^{-sd(x,g(x))}$ converges. We say that G is of *divergence type* if $\delta(G) < \infty$ and $P_G^{\delta(G)}(x) = \infty$. Otherwise, it is of convergence type.

We consider a normal subgroup H of a discrete group $G \subset \mathrm{Isom}(X, d)$. Clearly $\delta(H) \le \delta(G)$. The quotient group $H\backslash G$ acts on the quotient space $N_H = H\backslash X$ properly discontinuously. Then, we define the critical exponent of $H\backslash G$ in the same way as above and denote it by $\kappa(H\backslash G)$:

$$\kappa(H\backslash G) = \limsup_{r \to \infty} \frac{1}{r} \log \#\{[g] \in H\backslash G \mid d_{H\backslash X}([x], [g(x)]) \le r\} \quad ([x] \in N_H),$$

where $d_{H\backslash X}$ is the quotient distance. We call $\kappa(H\backslash G)$ the *growth* of H relative to G for the reason mentioned below.

Suppose that (X, d) admits a non-trivial Radon measure (volume) V invariant under every isometric automorphism of (X, d). If G is cocompact, then $\kappa(H\backslash G)$ gives the exponential growth rate of the volume of N_H. Actually, this exponential growth rate for a quotient space N_H is defined as the *volume entropy*

$$\omega(N_H) = \limsup_{r \to \infty} \frac{1}{r} \log V_{H\backslash X}(B([x], r)),$$

where $B([x], r)$ denotes the ball of center $[x] \in N_H$ and radius $r > 0$ and $V_{H\backslash X}$ is the volume on N_H induced from V. Then, $\kappa(H\backslash G) \le \omega(N_H)$ in general, and the equality holds if G is cocompact.

The following concept was introduced by Grigorchuk and de la Harpe [12] for a finitely generated group G equipped with the word metric d_A with respect to some generating system A.

Definition 2.1. A discrete group $G \subset \mathrm{Isom}(X, d)$ is said to be *growth tight* if $\kappa(H\backslash G) < \kappa(G)$ for every infinite normal subgroup H of G. Moreover, if $\sup \kappa(H\backslash G)$ taken over all such H is strictly less than $\delta(G)$, then G is called *uniformly growth tight*.

A recent paper by Arzhantseva, Cashen and Tao [1] shows that a large class of isometry groups possess the growth tightness. On the other hand, we expect that few of them have uniform growth tightness. Hence, there remain problems of finding a sequence of normal subgroups H_i such that $\lim_{i\to\infty} \kappa(H_i \backslash G) = \delta(G)$. We call such $\{H_i\}$ an *asymptotic sequence.*

§ 3. Growth tightness for Kleinian groups

We consider the hyperbolic space $X = \mathbb{H}^{n+1}$ of dimension $n+1$ with the hyperbolic metric d_h. We note here that several results cited in this paper can be extended to the spaces of variable negative curvature. We always assume an isometric automorphism to be orientation-preserving and denote the group of all such automorphisms by $\mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$. A discrete subgroup G of $\mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$ is called a Kleinian group.

The limit set $\Lambda(G)$ of G is the set of accumulation points of the orbit of G, which is located on the sphere at infinity $\partial X \cong \mathbb{S}^n$. If $\#\Lambda(G) \le 2$, then G is called *elementary.* Let $\mathrm{Hull}(\Lambda(G)) \subset \mathbb{H}^{n+1}$ be the convex hull of $\Lambda(G)$. If the hyperbolic volume of the ε-neighborhood ($\varepsilon > 0$) of the convex core $C_G = G \backslash \mathrm{Hull}(\Lambda(G))$ is finite, then G is called *geometrically finite.* Moreover, if C_G is compact, or equivalently, if G is geometrically finite without parabolic elements, then G is called *convex cocompact.* It is known that a geometrically finite Kleinian group is of divergence type.

We see that the critical exponent $\delta(G)$ of a non-elementary Kleinian group $G \subset \mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$ satisfies $0 < \delta(G) \le n$. When G is geometrically finite, we have that $\delta(G) = n$ if and only if G is of finite covolume.

The critical exponent can be interpreted by a geometric invariant of the hyperbolic manifold $N_G = G \backslash \mathbb{H}^{n+1}$. Let $\lambda_0(N_G)$ be the *bottom of spectra* of the hyperbolic Laplacian on N_G. Then, the Elstrodt-Patterson-Sullivan [23] theorem asserts the following. See Roblin and Tapie [20] for a recent account on this theorem.

Theorem 3.1. *For a Kleinian group $G \subset \mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$, if $\delta(G) \ge n/2$ then $\lambda_0(N_G) = (n - \delta(G))\delta(G)$, and if $\delta(G) \le n/2$ then $\lambda_0(N_G) = (n/2)^2$.*

On the other hand, the volume entropy $\omega(N_G)$ can be estimated by $\lambda_0(N_G)$ as Brooks [5] proved.

Theorem 3.2. *Any Kleinian group $G \subset \mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$ satisfies $\lambda_0(N_G) \le (\omega(N_G)/2)^2$.*

The combination of these two theorems immediately derives:

Corollary 3.3. *If a Kleinian group $H \subset \mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$ satisfies $\delta(H) \ge n/2$, then*

$$\delta(H) + \frac{1}{2}\omega(N_H) \ge n.$$

Note that when H is a normal subgroup of a Kleinian group $G \subset \mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$, the growth $\kappa(H\backslash G)$ does not necessarily coincide with the volume entropy $\omega(N_H)$. For instance, if G is geometrically finite with infinite covolume, then $\kappa(H\backslash G) < n$ whereas $\omega(N_H) = n$.

Now, we start considering growth tightness for Kleinian groups. For geometrically finite groups, the problem has been settled by Dal'bo et al. [9].

Theorem 3.4. *A non-elementary geometrically finite Kleinian group G is growth tight.*

One might expect that this can be generalized to any Kleinian group of divergence type. Towards this problem, the following theorem was proved in the same paper [9]. We remark that any non-trivial normal subgroup of a non-elementary Kleinian group is non-elementary; in particular, it is infinite.

Theorem 3.5. *Let H be a non-trivial normal subgroup of a non-elementary Kleinian group G. If $H\backslash G$ is of divergence type as acting on N_H, then $\kappa(H\backslash G) < \kappa(G)$.*

We note that if G is convex cocompact, then $H\backslash G$ is always of divergence type (which will be proved in Lemma 5.3 later in a more general setting), and hence Theorem 3.4 follows from Theorem 3.5 in this case. However, when G is geometrically finite with parabolic elements, the divergence of $H\backslash G$ does not necessarily hold true.

The existence of an asymptotic sequence of normal subgroups H_i of a geometrically finite Kleinian group G can be seen from a more general result by Yang [25], which is concerning relatively hyperbolic group action.

Theorem 3.6. *For a torsion-free geometrically finite Kleinian group G, there is a sequence of normal subgroups H of G such that $\lim_{i\to\infty} \kappa(H_i\backslash G) = \kappa(G)$.*

§ 4. Cogrowth tightness for Kleinian groups

Originally, the cogrowth of a normal subgroup $H \lhd G$ is defined by the ratio of $\delta(H)$ to $\delta(G)$ in relation to the growth $\kappa(H\backslash G)$. See Cohen [6]. Here, we refer to $\delta(H)$ itself as the cogrowth of H making the comparison with $\delta(G)$ in mind.

For the critical exponent $\delta(H)$ of a normal subgroup H of a Kleinian group $G \subset \mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$, Falk and Stratmann [10] obtained its lower bound as a consequence of a result in [15].

Theorem 4.1. *A non-trivial normal subgroup H of a non-elementary Kleinian group G satisfies $\delta(H) \geq \delta(G)/2$.*

Inspired by this result, we introduce the concept of cogrowth tightness as follows. Actually, for a free group G acting on its Cayley graph, the lower bound was previously known in [6] as well as in Grigorchuk [11]. Our definition is given in a more general setting for a discrete isometry group of a proper metric space (X, d).

Definition 4.2. Suppose that a discrete group $G \subset \mathrm{Isom}(X, d)$ satisfies

$$\inf_{1 \neq H \lhd G} \delta(H) \geq \delta(G)/2.$$

Under this assumption, we say that G is *cogrowth tight* if $\delta(H) > \delta(G)/2$ for every infinite normal subgroup H of G.

Returning to Kleinian groups, we find a criterion for the cogrowth tightness by the result of Roblin [19].

Theorem 4.3. *Let G be a non-elementary Kleinian group of divergence type and H its non-trivial normal subgroup. Then $\delta(H) > \delta(G)/2$. This implies that G is cogrowth tight.*

We note that for a special case where G is convex cocompact, the assertion of the above theorem was shown in [3] by a different proof. Recently, Jaerisch [13] found a simple argument for Theorems 4.1 and 4.3 by using the following result in [16].

Lemma 4.4. *If a normal subgroup H of a Kleinian group G is of divergence type, then $\delta(H) = \delta(G)$.*

When a Kleinian group G is of convergence type, we do not know whether it is cogrowth tight or not. In fact, we have no example of a Kleinian group G that is not cogrowth tight.

Problem 4.5. Find a non-elementary Kleinian group G and its non-trivial normal subgroup H such that $\delta(H) = \delta(G)/2$.

As in the case of growth tightness, we can also ask a problem of finding a sequence of non-trivial normal subgroups H_i such that $\lim_{i\to\infty} \delta(H_i) = \delta(G)/2$. We also call such $\{H_i\}$ an asymptotic sequence for cogrowth tightness.

The existence of an asymptotic sequence for cogrowth tightness is known only for the following special case as investigated in [3].

Theorem 4.6. *Let $G \subset \mathrm{Isom}_+(\mathbb{H}^2, d_h)$ be a cocompact Fuchsian group. Then, there is a sequence of non-trivial normal subgroups H_i of G such that* $\lim_{i\to\infty} \delta(H_i) = \delta(G)/2 = 1/2$.

An interesting fact is that this sequence also plays the role of an asymptotic sequence for growth tightness at the same time.

Proposition 4.7. *Let $G \subset \mathrm{Isom}_+(\mathbb{H}^{n+1}, d_h)$ be a cocompact Kleinian group. If a sequence of non-trivial normal subgroups H_i of G satisfies $\lim_{i\to\infty} \delta(H_i) = \delta(G)/2 = n/2$, then $\lim_{i\to\infty} \kappa(H_i\backslash G) = \kappa(G) = n$.*

Proof. Since G is cocompact, the growth $\kappa(H\backslash G)$ coincides with the volume entropy $\omega(N_H)$ for any normal subgroup H of G. Then, we apply Corollary 3.3. Noting that $\delta(H_i) \geq n/2$ by Theorem 4.1, we have

$$\delta(H_i) + \frac{1}{2}\kappa(H_i\backslash G) \geq n.$$

Hence, $\lim_{i\to\infty} \delta(H_i) = n/2$ implies $\lim_{i\to\infty} \kappa(H_i\backslash G) = n$. □

§ 5. Growth tightness for hyperbolic groups

As a generalization of the hyperbolic space $(\mathbb{H}^{n+1}, d_h)$, we consider a proper geodesic Gromov hyperbolic space (X, d). It equips the boundary ∂X at infinity, and hence a lot of concepts on Kleinian groups can be similarly introduced to discrete subgroups of $\mathrm{Isom}(X, d)$.

A finitely generated group G is said to be a *hyperbolic group* if its Cayley graph $\mathrm{Cay}(G, A)$ for some generating system A equipped with the path metric d_A is a Gromov hyperbolic space. Since G acts on $(X, d) = \mathrm{Cay}(G, A)$ isometrically, properly discontinuously, and cocompactly, it can be regarded as a cocompact discrete group of $\mathrm{Isom}(X, d)$. Hence, the results on discrete groups acting on the Gromov hyperbolic space are also applicable to hyperbolic groups as special cases.

We consider growth tightness for a discrete group $G \subset \mathrm{Isom}(X, d)$. The following result was proved by Sabourau [21], generalizing the previous theorems for a non-elementary hyperbolic group by Arzhantseva and Lysënok [2] and for a cocompact discrete group $G \cong \pi_1(R) \subset \mathrm{Isom}(X, d)$ acting on a negatively curved Riemannian universal cover X of R by Sambusetti [22].

Proposition 5.1. *Let (X, d) be a proper geodesic Gromov hyperbolic space. If $G \subset \mathrm{Isom}(X, d)$ is cocompact, then G is growth tight.*

On the other hand, the proof in Dal'bo et al. [9] for Theorem 3.5 can be generalized to the Gromov hyperbolic space and the following theorem should be also obtained. There is no essential difference in the argument (see also the comment in the remark after [25, Theorem 1.4]), but we give a proof adapted to the Gromov hyperbolic space in the appendix of this paper. In particular, Proposition 5.1 follows from Corollary 5.4 to this theorem combined with Lemma 5.3 below.

Theorem 5.2. *Let (X, d) be a proper geodesic Gromov hyperbolic space. Let H be a non-trivial normal subgroup of a non-elementary discrete group $G \subset \mathrm{Isom}(X, d)$. If $H\backslash G$ is of divergence type as acting on N_H, then $\kappa(H\backslash G) < \kappa(G)$.*

A sufficient condition for $H\backslash G$ to be of divergence type was also given in [9]. Here, we only deal with the parabolic-free case for the sake of simplicity, and borrow the proof from [9] and adapt it to our situation. We say that a discrete group $G \subset \mathrm{Isom}(X, d)$ is *quasi-convex cocompact* if the union of all geodesic lines joining any two points of the limit set $\Lambda(G)$ modulo the action of G is compact. This condition is equivalent to that the orbit $G(x) \subset X$ of any $x \in X$ is *quasi-convex*, i.e., there is a constant $L \geq 0$ such that for every point z on any geodesic segment connecting any two points of $G(x)$, there exists $g \in G$ such that $d(z, g(x)) \leq L$. See Swenson [24] for equivalent conditions of quasi-convexity.

Lemma 5.3. *Let (X, d) be a proper geodesic Gromov hyperbolic space. Let H be a normal subgroup of a discrete group $G \subset \mathrm{Isom}(X, d)$. If G is quasi-convex cocompact, then $H\backslash G$ is of divergence type.*

Proof. The quotient group $H\backslash G$ acts on $N_H = H\backslash X$ properly discontinuously. For any $[x] \in N_H$, $r > 0$, and $L \geq 0$, we consider the following subset of the orbit of $[x]$ under $H\backslash G$:

$$A([x], L, r) = \{[g(x)] \in H\backslash G(x) \mid r - L < d_{H\backslash X}([x], [g(x)]) < r + L\}.$$

Since G is quasi-convex cocompact, the orbit $G(x)$ is quasi-convex for some constant $L \geq 0$. Then, we will show the following sub-multiplicativity of the number of the elements in the above set:

$$\#A([x], 2L, r_1 + r_2) \leq \#A([x], 2L, r_1) \cdot \#A([x], 2L, r_2) \quad (r_1, r_2 > 0).$$

Take any $[y] \in A([x], 2L, r_1 + r_2)$ and set $d_{H\backslash X}([x], [y]) = r_1 + r_2 + 2\ell$ with $|\ell| < L$. Take a point $[z]$ on a geodesic segment connecting $[x]$ and $[y]$ with $d_{H\backslash X}([x], [z]) = r_1 + \ell$ and $d_{H\backslash X}([y], [z]) = r_2 + \ell$. We lift them to X; there are $x \in X$, $y \in G(x)$, and z on the geodesic segment connecting x and y with $d(x, z) = r_1 + \ell$ and $d(y, z) = r_2 + \ell$. By the quasi-convexity of $G(x)$, there is some $g_0 \in G$ such that $d(z, g_0(x)) \leq L$. Then, we have $[g_0(x)] \in A([x], 2L, r_1)$ and $[y] \in A([g_0(x)], 2L, r_2)$. This implies that

$$A([x], 2L, r_1 + r_2) \subset \bigcup_{[g(x)] \in A([x], 2L, r_1)} A([g(x)], 2L, r_2).$$

Since $\#A([g(x)], 2L, r_2) = \#A([x], 2L, r_2)$, we obtain the required inequality.

The Poincaré series $P^s_{H\backslash G}([x])$ for $H\backslash G$ of exponent $s \geq 0$ satisfies

$$\frac{1}{4L}\sum_{n\geq 0} \#A([x], 2L, n)e^{-s(n+2L)} \leq P^s_{H\backslash G}([x]) \leq \sum_{n\geq 0} \#A([x], 2L, n)e^{-s(n-2L)}.$$

Hence, $P^s_{H\backslash G}([x])$ and $\sum_{n\geq 0} \#A([x], 2L, n)e^{-sn}$ have the same critical exponent $s = \kappa(H\backslash G)$ and either diverge or converge at this exponent simultaneously. Moreover, by setting $v_n = \#A([x], 2L, n)$, we have

$$\kappa(H\backslash G) = \limsup_{n\to\infty} \frac{1}{n}\log v_n.$$

By Fekete's lemma, the sub-multiplicativity $v_{n+m} \leq v_n v_m$ implies that $v_n \geq e^{\kappa(H\backslash G)n}$ for all n. Hence, $P^{\kappa(H\backslash G)}_{H\backslash G}([x])$ diverges. $\square$

Corollary 5.4. *Let (X, d) be a proper geodesic Gromov hyperbolic space. If $G \subset \mathrm{Isom}(X, d)$ is non-elementary and quasi-convex cocompact, then G is growth tight.*

Concerning the existence of an asymptotic sequence of non-trivial normal subgroups $H_i \lhd G$, Coulon [8] proved that for a non-elementary torsion-free hyperbolic group G, a sequence of normal subgroups $H_i = G^i$ generated by $\{g^i \mid g \in G\}$ satisfies $\lim_{i\to\infty} \kappa(H_i\backslash G) = \kappa(G)$. Recently, Yang [25] obtained an asymptotic sequence for a relatively hyperbolic group G. His results in particular include the following theorem, which can be generalized to any cusp-uniform relatively hyperbolic group action.

Theorem 5.5. *Let (X, d) be a proper geodesic Gromov hyperbolic space. For a quasi-convex cocompact discrete group $G \subset \mathrm{Isom}(X, d)$, there is a sequence of non-trivial normal subgroups H_i such that $\lim_{i\to\infty} \kappa(H_i\backslash G) = \kappa(G)$.*

§ 6. Cogrowth tightness for hyperbolic groups

We consider the lower bound of the critical exponents of non-trivial normal subgroups and the cogrowth tightness of a discrete group $G \subset \mathrm{Isom}(X, d)$ for a proper geodesic Gromov hyperbolic space (X, d). We can generalize the results for Kleinian groups as follows, which is proved in [17].

Theorem 6.1. *Let $G \subset \mathrm{Isom}(X, d)$ be a non-elementary discrete group for a proper geodesic Gromov hyperbolic space (X, d). Then, $\delta(H) \geq \delta(G)/2$ for a non-trivial normal subgroup H of G. Moreover, if G is of divergence type in addition, then $\delta(H) > \delta(G)/2$, which implies that G is cogrowth tight.*

The proof of this theorem in [17] is given by generalizing the argument of Jaerisch [13] for Kleinian groups. For the strict inequality, we use the generalization of Lemma 4.4 in this case.

Concerning the problem of asymptotic sequence, Grigorchuk [11] gave such a sequence of non-trivial normal subgroups H_i for the free group $G = \langle g_1, \ldots, g_n \rangle$ $(n \geq 2)$. Here, H_i is the normal closure generated by $\{g_1^i, \ldots, g_n^i\}$. Recently, a simple proof for this fact is given in [14] and this also shows that H_i is an asymptotic sequence for the growth tightness at the same time. Namely, $\lim_{i\to\infty} \kappa(H_i \backslash G) = \kappa(G)$. More generally, the following result is proved in [14].

Theorem 6.2. *Let G be a free group of rank ≥ 2. If a sequence of non-trivial normal subgroups H_i of G satisfies that each quotient graph $H_i \backslash \mathrm{Cay}(G, A)$ is planar and the girth of $H_i \backslash \mathrm{Cay}(G, A)$ tends to ∞ as $i \to \infty$, then both*

$$\lim_{i\to\infty} \delta(H_i) = \delta(G)/2 \quad \text{and} \quad \lim_{i\to\infty} \kappa(H_i \backslash G) = \kappa(G)$$

are satisfied. Here, A is the set of free generators of G and the girth of a graph means the minimal number of edges consisting of a non-trivial cycle.

Finally, we touch on a problem of whether or not an asymptotic sequence for the cogrowth tightness becomes that for the growth tightness. Intuitively speaking, an asymptotic sequence consists of "small" normal subgroups in some sense. Hence, there is no surprise even if one plays both roles. In Proposition 4.7, we have seen a certain condition under which this is the case for Kleinian groups by using the inequality of Corollary 3.3. We expect that the same inequality holds true for discrete isometry groups of the Gromov hyperbolic space.

Conjecture 6.3. Let (X, d) be a proper geodesic Gromov hyperbolic space and $G \subset \mathrm{Isom}(X, d)$ a cocompact discrete group. Then, a non-trivial normal subgroup H of G satisfies

$$\delta(H) + \frac{1}{2}\kappa(H \backslash G) \geq \delta(G).$$

This has been proved in [14] for any free group G of finite rank.

§ 7. Appendix: Proof of Theorem 5.2

Let (X, d) be a proper geodesic Gromov hyperbolic space with hyperbolicity constant $\Delta \geq 0$. Hereafter, a multiple of Δ by a uniformly bounded positive integer is denoted by $\widetilde{\Delta}$, $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$, and so on without specifying the multiplicity. Let $G \subset \mathrm{Isom}(X, d)$ be a non-elementary discrete group and H a non-trivial normal subgroup of G. Since

H is non-elementary, it contains a hyperbolic element h. We fix this h and also fix a base point $o \in X$. Let ξ_+ and ξ_- be the attracting and the repelling fixed point of h, respectively.

Take a geodesic segment $[o, h(o)]$ connecting o and $h(o)$, and make a piecewise geodesic curve $\beta = \bigcup_{n \in \mathbb{Z}} h^n([o, h(o)])$ connecting ξ_+ and ξ_- with arc length parameter. Actually, $\beta : (-\infty, \infty) \to X$ is a quasi-geodesic line, i.e., there are constants $\mu \geq 1$ and $\nu \geq 0$ such that

$$|s - t| \leq \mu d(\beta(s), \beta(t)) + \nu$$

for all $s, t \in \mathbb{R}$ (see [7, Lemme 6.5]). Assuming that $\beta(0) = o$, we set $\beta_+ = \beta[0, \infty)$ and $\beta_- = \beta[0, -\infty)$.

For each $x \in X$, let $P_+(x) \subset \beta_+$ and $P_-(x) \subset \beta_-$ denote the sets of all nearest points from x to β_+ and β_-, respectively. We see that the diameter of $P_\pm(x)$ is uniformly bounded independent of x (cf. [4, III.Γ.3.11]). Indeed, choose y_1 and y_2 in $P_\pm(x)$ and consider a geodesic segment $[y_1, y_2]$. The Gromov product

$$(y_1 \,|\, y_2)_x = \frac{1}{2}(d(y_1, x) + d(y_2, x) - d(y_1, y_2))$$

satisfies $d([y_1, y_2], x) \leq (y_1 \,|\, y_2)_x + \widetilde{\Delta}$. Moreover, by stability of quasi-geodesics (see [4, III.H.1.7]), there is a constant $c = c(\Delta, \mu, \nu) \geq 0$ such that the geodesic segment $[y_1, y_2]$ is within distance c of the subarc of β with end points y_1 and y_2. Hence, we have

$$d(y_1, x) = d(y_2, x) = d(\beta_\pm, x) \leq d([y_1, y_2], x) + c \leq (y_1 \,|\, y_2)_x + \widetilde{\Delta} + c.$$

This inequality yields that $d(y_1, y_2) \leq 2\widetilde{\Delta} + 2c$. The uniform boundedness of the diameter of $P_\pm(x)$ also implies that of the set of parameters $\beta^{-1}(P_\pm(x)) \subset \mathbb{R}$.

The Gromov product can be extended to the boundary ∂X at infinity by taking the limit infimum of all convergent sequences. Concerning the Gromov product at infinity, we refer to Ohshika [18, §2.6.1]. For $\xi \in \partial X$ and $b > 0$, set

$$U(\xi, b) = \{x \in X \mid (x \,|\, \xi)_o > b\}.$$

It is known that $\{\overline{U(\xi, b)}\}_{b>0}$ is a neighborhood basis of $\overline{X} = X \cup \partial X$ at ξ. We also consider

$$V(\xi_\pm, r) = \{x \in X \mid P_\pm(x) \cap \beta(\pm r, \pm\infty) \neq \emptyset\}$$

for ξ_+ and ξ_- respectively and for $r > 0$.

Proposition 7.1. (i) *For any $r < \infty$, there exists some $b < \infty$ such that $U(\xi_\pm, b) \subset V(\xi_\pm, r)$;* (ii) *for any $b < \infty$, there exists some $r < \infty$ such that $V(\xi_\pm, r) \subset U(\xi_\pm, b)$.*

Proof. (i) We will show that $d(\beta[0,r],x) > d(\beta_+,x)$ for every $x \in U(\xi_+,b)$ by choosing a suitable b. This implies that $P_+(x) \subset \beta(r,\infty)$, which yields that $x \in V(\xi_+,r)$. The case for ξ_- is similarly treated. We use an inequality $d(o,x) \geq (x\,|\,\xi_+)_o + (\xi_+\,|\,o)_x$. As before (but $\widetilde{\Delta}$ might be different), the Gromov product at infinity also satisfies $(\xi_+\,|\,o)_x \geq d(\beta_+,x) - \widetilde{\Delta} - c$, where $c = c(\Delta,\mu,\nu)$ is the constant for stability of quasi-geodesics. Then, by the assumption $(x\,|\,\xi_+)_o > b$, we have

$$d(o,x) > d(\beta_+,x) + b - \widetilde{\Delta} - c.$$

Hence, $d(\beta[0,r],x) \geq d(o,x) - r > d(\beta_+,x)$ by taking $b = r + \widetilde{\Delta} + c$.

(ii) We will show that $x \notin U(\xi_+,b)$ implies $x \notin V(\xi_+,r)$ by choosing a suitable r. Take any $y \in P_+(x)$, which satisfies $d(\beta_+,x) = d(y,x)$. For this, we use the opposite inequalities

$$d(o,x) \leq (x\,|\,\xi_+)_o + (\xi_+\,|\,o)_x + \widetilde{\Delta}_1; \quad (\xi_+\,|\,o)_x \leq d(\beta_+,x) + c.$$

By the assumption $(x\,|\,\xi_+)_o \leq b$, we have that $d(o,x) \leq d(y,x) + b + \widetilde{\Delta}_1 + c$. On the other hand, $(o\,|\,y)_x \geq d([o,y],x) - \widetilde{\Delta}_2 \geq d(y,x) - \widetilde{\Delta}_2 - c$. From these estimates, we obtain that

$$\begin{aligned} d(y,x) - \widetilde{\Delta}_2 - c &\leq \frac{1}{2}(d(o,x) + d(y,x) - d(o,y)) \\ &\leq d(y,x) + \frac{1}{2}(b + \widetilde{\Delta}_1 + c - d(o,y)). \end{aligned}$$

Hence, $d(o,y) \leq b + \widetilde{\Delta}_1 + 2\widetilde{\Delta}_2 + 3c$. If we choose $r = \mu(b + \widetilde{\Delta}_1 + 2\widetilde{\Delta}_2 + 3c) + \nu$, then $y \notin \beta(r,\infty)$, and thus $x \notin V(\xi_+,r)$. □

By the uniform boundedness of the diameter of $P_\pm(x)$, we see that there is $r_0 > 0$ such that $V(\xi_+,r) \cap V(\xi_-,r) = \emptyset$ for every $r \geq r_0$. Moreover, there is $p = p(r) \in \mathbb{N}$ such that

$$h^n(X - V(\xi_-,r)) \subset V(\xi_+,r) \quad \text{and} \quad h^{-n}(X - V(\xi_+,r)) \subset V(\xi_-,r)$$

for every $n \geq p$.

Now, we take a family $G_H \subset G$ of minimal representatives of $H\backslash G$, which is a system of representatives consisting of $g \in [g] \in H\backslash G$ with $d(o,g(o)) = d_{H\backslash X}([o],[g(o)])$. Since H is normal in G, $g \in G_H$ if and only if $g^{-1} \in G_H$. Clearly $\mathrm{id} \in G_H$.

Lemma 7.2. *For a sufficiently large $r \geq r_0$, $V_+ = V(\xi_+,r)$ and $V_- = V(\xi_-,r)$ satisfy the following properties:*

(1) *The Gromov product $(g(o)\,|\,x)_o$ is uniformly bounded for every $x \in V_+ \cup V_-$ and for every $g \in G_H$;*

(2) *$g(x) \notin V_+ \cup V_-$ for every $x \in V_+ \cup V_-$ and for every $g \in G_H - \{\mathrm{id}\}$.*

Proof. We first show that the set $P_\pm(g(o))$ of the nearest points from $g(o)$ to $\beta_\pm$ respectively is within a uniformly bounded distance from o for all $g \in G_H$. Take any $y \in P_\pm(g(o))$, which satisfies $d(y, g(o)) \le d(o, g(o))$. We consider a geodesic segment $[o, y]$. The Gromov product satisfies $d([o, y], g(o)) \le (o \,|\, y)_{g(o)} + \widetilde{\Delta}$. Choosing some orbit point $h^m(o)$ ($m \in \mathbb{Z}$), we have that $d(h^m(o), g(o)) \le d([o, y], g(o)) + c + \ell$, where $c = c(\Delta, \mu, \nu) \ge 0$ is the constant for stability of quasi-geodesics and $\ell = d(o, h(o))$. Then, by the minimality of the elements of G_H and by summing up the above estimates, we obtain that

$$\begin{aligned} d(o, g(o)) &\le d(h^m(o), g(o)) \\ &\le (o \,|\, y)_{g(o)} + \widetilde{\Delta} + c + \ell \\ &= \frac{1}{2}(d(o, g(o)) + d(y, g(o)) - d(o, y)) + \widetilde{\Delta} + c + \ell \\ &\le d(o, g(o)) - \frac{1}{2} d(o, y) + \widetilde{\Delta} + c + \ell. \end{aligned}$$

This implies that $d(o, y) \le 2\widetilde{\Delta} + 2c + 2\ell$.

The above fact asserts that $g(o) \notin V(\xi_+, r) \cup V(\xi_-, r)$ for all sufficiently large $r > 0$. In particular, we see that $(g(o) \,|\, \xi_\pm)_o$ is uniformly bounded for every $g \in G_H$ by Proposition 7.1 (i).

(1) Suppose to the contrary that we cannot choose such an r. Then, by Proposition 7.1 (ii), there are sequences $\{x_n\} \subset X$ and $\{g_n\} \subset G_H$ such that either $(x_n \,|\, \xi_+)_o \to \infty$ or $(x_n \,|\, \xi_-)_o \to \infty$, and $(g_n(o) \,|\, x_n)_o \to \infty$ as $n \to \infty$. Since the Gromov products satisfy

$$(g_n(o) \,|\, \xi_\pm)_o \ge \min\{(g_n(o) \,|\, x_n)_o, (x_n \,|\, \xi_\pm)_o\} - \widetilde{\Delta}$$

(see [18, Lemma 2.57]), we have $(g_n(o) \,|\, \xi_\pm)_o \to \infty$. However, this contradicts the fact shown above.

(2) Suppose to the contrary that we cannot choose such an r. Then, Proposition 7.1 (ii), there are sequences $\{x_n\} \subset X$ and $\{g_n\} \subset G_H$ such that either $(x_n \,|\, \xi_+)_o \to \infty$ or $(x_n \,|\, \xi_-)_o \to \infty$, and either $(g_n(x_n) \,|\, \xi_+)_o \to \infty$ or $(g_n(x_n) \,|\, \xi_-)_o \to \infty$ as $n \to \infty$. Moreover, we may assume that $d(o, g_n^{-1}(o)) \to \infty$. By statement (1), we see that both $(g_n(o) \,|\, g_n(x_n))_o$ and $(g_n^{-1}(o) \,|\, x_n)_o$ are uniformly bounded. However, by the equality

$$(g_n(o) \,|\, g_n(x_n))_o + (g_n^{-1}(o) \,|\, x_n)_o = (o \,|\, x_n)_{g_n^{-1}(o)} + (g_n^{-1}(o) \,|\, x_n)_o = d(o, g_n^{-1}(o)),$$

this is impossible. □

The remainder of the proof is essentially the same as that for [9, Theorem 1.1]. For the sake of completeness, we extract it from the original paper. We take a sufficiently large constant $r > 0$ as in Lemma 7.2, and then set $p = p(r) \in \mathbb{N}$. By [9, Proposition 3.2] which can be verified by the above arguments also in our case, there is $\lambda > 0$ such that for any $l, l' \geq 1$ and $g_1, \ldots, g_l, g'_1, \ldots, g'_{l'}$ in G_H satisfying

$$g_1 h^p \cdots h^p g_l = g'_1 h^p \cdots h^p g'_{l'},$$

we have that $d(g_1(o), g'_1(o)) \leq \lambda$.

We consider a maximal λ-separated net of G_H, which is a subset $A \subset G_H$ including id that satisfies (i) $d(a(o), a'(o)) > \lambda$ for any distinct $a, a' \in A$; (ii) for every $g \in G_H$, there is some $a \in A$ such that $d(g(o), a(o)) \leq \lambda$. The existence of such a subset A is guaranteed by the Zorn lemma. A repeated application of the fact in the previous paragraph implies that if

$$g_1 h^p \cdots h^p g_l = g'_1 h^p \cdots h^p g'_{l'}$$

for $g_1, \ldots, g_l, g'_1, \ldots, g'_{l'} \in A$, then $l = l'$ and $g_i = g'_i$ $(1 \leq i \leq l)$. We call this property the injectivity of reduced words generated by $A^* = A - \{\mathrm{id}\}$ and $\{\tilde{h}\}$ for $\tilde{h} = h^p$.

By considering the Poincaré series for G_H and A, we can similarly define the critical exponents $\delta(G_H)$ and $\delta(A)$, though they are not groups. Then, we have $\delta(G_H) = \delta(A)$. Indeed, $\delta(G_H) \geq \delta(A)$ is clear by $G_H \supset A$, and the inverse inequality can be seen by

$$\sum_{g \in G_H} e^{-sd(o,g(o))} \leq \sum_{a \in A} \sum_{\substack{g \in G_H \\ d(g(o),a(o)) \leq \lambda}} e^{-sd(o,g(o))} \leq M e^{s\lambda} \sum_{a \in A} e^{-sd(o,a(o))},$$

where M is the number of the orbit $G(o)$ within distance λ of o. By the assumption that $H\backslash G$ is of divergence type, we see that

$$\sum_{g \in G_H} e^{-sd(o,g(o))} = \sum_{[g] \in H\backslash G} e^{-sd_{H\backslash X}([o],[g(o)])}$$

diverges at $s = \delta(G_H) = \kappa(H\backslash G)$. Hence, $\sum_{a \in A} e^{-sd(o,a(o))}$ diverges at $s = \delta(A)$.

Finally, we estimate the Poincaré series $P^s_G(o)$ by using the injectivity of reduced words generated by A^* and $\{\tilde{h}\}$. It follows that

$$\begin{aligned} P^s_G(o) &\geq \sum_{k \geq 1} \sum_{a_1,\ldots,a_k \in A^*} \exp(-sd(o, a_1\tilde{h}a_2\tilde{h}\cdots a_k\tilde{h}(o)) \\ &\geq \sum_{k \geq 1} \sum_{a_1,\ldots,a_k \in A^*} \exp(-s(d(o,a_1\tilde{h}(o)) + \cdots + d(o,a_k\tilde{h}(o)))) \\ &\geq \exp(-sd(o,\tilde{h}(o))) \sum_{k \geq 1} \left(\sum_{a \in A^*} \exp(-sd(o,a(o))) \right)^k. \end{aligned}$$

Since the Poincaré series for A diverges at the critical exponent $s = \delta(A)$, we can find $s_0 > \delta(A)$ such that $\sum_{a \in A^*} e^{-s_0 d(o,a(o))} \geq 1$. By the above estimate, this implies that

$P_G^{s_0}(o)$ diverges. Therefore, we have $\delta(G) \geq s_0 > \delta(A) = \kappa(H\backslash G)$. This completes the proof of Theorem 5.2.

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