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Fixed-point property for affine actions on a Hilbert space

By

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Abstract

Gromov [7] showed that for fixed, arbitrarily large C, any uniformly C-Lipschitz affine action of a random group in his graph model on a Hilbert space has a fixed point. We announce a theorem stating that more general affine actions of the same random group on a Hilbert space have a fixed point. We discuss some aspects of the proof.

Introduction

In [10], Izeki, Kondo and the present author proved that a random group in the Gromov graph model had fixed-point property, meaning that any isometric action had a fixed point, for a large class of CAT(0) spaces, by using the method which concerns the n-step energy of maps. Naor and Silberman [17] proved a similar result for a class of p-uniformly convex geodesic metric spaces. (Note that CAT(0) spaces are 2-uniformly convex.) In these studies it seemed that the condition that actions are isometric was essential and without the condition the argument should break dwon. Gromov [7], however, had shown that any uniformly C-Lipschitz affine action of the same random group on a Hilbert space has a fixed point, where C may be arbitrarily large but should be specified in advance. The purpose of this article is to announce a fixed-point theorem for more general affine actions of the same random group, allowing the Lipschitz constants of the affine maps to have mild growth with respect to a certain length function on the group [11]. It is worth while to mention the following: if the Lipschitz constants of the affine maps are uniformly bounded, then the action reduces to an isometric one on a Banach space by replacing the Hilbert norm by an equivalent one.

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On the other hand, our case treats really non-isometric actions which cannot reduce to isometric ones.

A key of the proof is to verify the existence of a discrete harmonic map from the group into the Hilbert space which is equivariant with respect to the given action. In the case of isometric actions, the method of energy minimization coupled with scaling ultralimit argument was effective. In the general affine case, this method fails because a map minimizing local energy does not necessarily satisfy the condition of harmonicity. We therefore employ the method of discrete tension-contracting flow due to Gromov [7]. Indeed, we refine Gromov's method and derive the existence of a harmonic map still by coupling it with scaling ultralimit argument.

This article is organized as follows. In §1, we define affine action, discuss the rigidity of isometric actions and state Shalom's theorem on the rigidity and existence of uniformly Lipschitz affine actions. In §2, after discussing Nowak's fixed-point theorem for uniformly Lipschitz affine actions of a random group in the Gromov density model, we state our main fixed-point theorem. In §3, we discuss discrete harmonic maps and state an existence theorem for such maps. We also discuss the failure of the method of energy minimization. In §4, we introduce Gromov's discrete tension-contracting flow and outline the proof of the existence of harmonic maps. In §5, we outline the proof of the main theorem. In Appendix, we prove the existence of maps minimizing local energy which are equivariant with respect to a given affine action.

§ 1. Affine actions on a Hilbert space

Let $\mathcal{H}$ be a Hilbert space, and denote the algebra of all bounded linear operators of $\mathcal{H}$ by $\mathbb{B}(\mathcal{H})$. Let Γ be a finitely generated infinite group, and let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action. Thus, for $\gamma \in \Gamma$, $\rho(\gamma)\colon \mathcal{H} \to \mathcal{H}$ has the form

$$\rho(\gamma)(\mathbf{v}) = A(\gamma)(\mathbf{v}) + b(\gamma), \quad \mathbf{v} \in \mathcal{H},$$

where $A(\gamma) \in \mathbb{B}(\mathcal{H})$ and $b(\gamma) \in \mathcal{H}$. Since $\gamma \mapsto \rho(\gamma)$ is a homomorphism, we have

$$A(\gamma\gamma') = A(\gamma)A(\gamma'), \quad b(\gamma\gamma') = b(\gamma) + A(\gamma)b(\gamma'), \quad \gamma, \gamma' \in \Gamma.$$

Definition 1.1. An affine action $\rho\colon \Gamma \curvearrowright \mathcal{H}$ is called *uniformly C-Lipschitz* if $\rho(\gamma)\colon \mathcal{H} \to \mathcal{H}$ is a C-Lipschitz map, or equivalently $\|A(\gamma)\| \le C$, for all $\gamma \in \Gamma$.

Note that $C \ge 1$ necessarily.

Most basic example of a uniformly Lipschitz affine action is an isometric action. Recall that a σ-compact, locally compact topological group G is said to have property $F\mathcal{H}$ if any continuous isometric action $\rho\colon G \curvearrowright \mathcal{H}$ has a fixed point, that is, there exists

$\mathbf{v} \in \mathcal{H}$ such that $\rho(g)(\mathbf{v}) = \mathbf{v}$ for all $g \in G$. It is a celebrated result of Delorme [4] and Guichardet [8] that property $F\mathcal{H}$ is equivalent to Kazhdan's property (T). Kazhdan [13] defined this property for locally compact groups in terms of unitary representations, and proved that if Γ is a lattice in a Lie group G, then Γ has property (T) if and only if G has property (T). As examples, simple real Lie groups of real rank at least two have property (T). For $n \geq 2$, $\mathrm{Sp}(n,1)$ is a simple Lie group of real rank one that has property (T). Thus these Lie groups and their lattices have property $F\mathcal{H}$.

In his unpublished work, Shalom proved the following theorem which exhibits that higher-rank groups have stronger rigidity than $\mathrm{Sp}(n,1)$ (cf. [2, 19]).

Theorem 1.2 (Shalom). (i) *Any uniformly Lipschitz affine action of a simple real Lie group of real rank at least two (or its lattices) on $\mathcal{H}$ has a fixed point.*
(ii) $\mathrm{Sp}(n,1)$ *admits a uniformly Lipschitz affine action on $\mathcal{H}$ without fixed point.*

Mimura [16] observes that the action in the statement (ii) is indeed metrically proper. Hence, any infinite discrete subgroup of $\mathrm{Sp}(n,1)$ also admits a uniformly Lipschitz affine action on $\mathcal{H}$ without fixed point. This exhibits many infinite hyperbolic groups which admit such affine actions.

Shalom proposed the following (cf. [19])

Conjecture 1 (Shalom). Any non-elementary hyperbolic group admits a uniformly Lipschitz affine action on $\mathcal{H}$ without fixed point.

§ 2. Fixed-point property of random groups w.r.t. uniformly Lipschitz affine actions

In this section, we review two fixed-point theorems regarding uniformly Lipschitz affine actions of certain ramdom groups on a Hilbert space. Recall that in the Gromov density model $\mathcal{G}(m,l,d)$ of random groups, generators $s_1^{\pm 1}, \dots, s_m^{\pm 1}$ and a density $0 < d < 1$ are fixed, and choose $(2m-1)^{dl}$ words, each of them chosen uniformly and independently from the set of all reduced words of lenght l in $s_1^{\pm 1}, \dots, s_m^{\pm 1}$. The group Γ generated by $s_1^{\pm 1}, \dots, s_m^{\pm 1}$ and having those reduced words as relations is a constituent of the model $\mathcal{G}(m,l,d)$. Given a group property P (e.g. Kazhdan's property (T)), we say that a *random group in the Gromov density model has property P* if the probability of Γ having property P tends to one as $l \to \infty$.

Theorem 2.1 (Nowak [18]). *Fix $1 \leq C < \sqrt{2}$. Let Γ be a random group in the Gromov density model with density $1/3 < d < 1/2$. Then any uniformly C-Lipschitz affine action $\rho\colon \Gamma \curvearrowright \mathcal{H}$ has a fixed point.*

Note that the random group Γ of the theorem is non-elementary hyperbolic (hence infinite) [6, 20] and has property (T) [23, 15].

The proof of Theorem 2.1 is based on a fixed-point theorem for an isometric action of a deterministic group on a Banach space, which we shall review. Let Γ be a finitely generated group equipped with a finite, symmetric generating set S not containing the identity element. Modifying the construction as in [23], one constructs the link graph $\mathcal{L}(S)$; its vertices are the elements of S, generators s and t span an edge (written $s \sim t$) if $s^{-1}t$ is a generator, and the edges are suitably weighted. (For the account of the choice of weight, see [18, p. 703], [9, Proof of Lemma 3.1].)

Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$ and denote by $\kappa_p(S, \mathcal{B})$ the optimal constant in the p-Poincaré inequality for maps $f\colon S \to \mathcal{B}$:

$$\sum_{s\in S} \|f(s) - \overline{f}\|^p\, m(s) \leq \kappa^p \sum_{s\sim t} \|f(s) - f(t)\|^p\, m(s,t),$$

where $m(s,t)$ is the weight of the edge (s,t), $m(s) = \sum_{t\sim s} m(s,t)$, and $\overline{f} = \sum_{s\in S} m(s)f(s)/\sum_{s\in S} m(s)$, the mean value of f.

Theorem 2.2 (Nowak [18]). *Let $\mathcal{B}$ be a reflexive Banach space and let Γ and S be as above. If the link graph $\mathcal{L}(S)$ is connected and for some $1 < p < \infty$ and its adjoint index p^*, satisfying $1/p + 1/p^* = 1$, the corresponding Poincaré constants satisfy*

$$\max\{2^{-1/p}\kappa_p(S,\mathcal{B}), 2^{-1/p^*}\kappa_{p^*}(S,\mathcal{B}^*)\} < 1,$$

then any affine isometric action $\rho\colon \Gamma \curvearrowright \mathcal{B}$ has a fixed point.

Let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be a uniformly C-Lipschitz affine action, where $\mathcal{H}$ is a Hilbert space, and introduce a new norm on $\mathcal{H}$ by $|||\mathbf{v}||| = \sup_{\gamma\in\Gamma} \|A(\gamma)(\mathbf{v})\|$ for $\mathbf{v} \in \mathcal{H}$. Then $\mathcal{B} = (\mathcal{H}, |||\cdot|||)$ is a Banach space isomorphic to $\mathcal{H}$, thus reflexive, and $\rho\colon \Gamma \curvearrowright \mathcal{B}$ is an affine isometric action. Since the norms of $\mathcal{B}$ and $\mathcal{B}^*$ ($\cong \mathcal{H}$) satisfy $\|\cdot\| \leq |||\cdot||| \leq C\|\cdot\|$ and $C^{-1}\|\cdot\| \leq |||\cdot|||^* \leq \|\cdot\|$, respectively, it follows that

$$\kappa_2(S,\mathcal{B}), \kappa_2(S,\mathcal{B}^*) \leq C\,\kappa_2(S,\mathcal{H}) = C\,\kappa_2(S,\mathbb{R}).$$

Therefore, we obtain the following

Corollary 2.3. *Let Γ and S be as above, and suppose that the link graph $\mathcal{L}(S)$ is connected. Then any uniformly C-Lipschitz affine action $\rho\colon \Gamma \curvearrowright \mathcal{H}$ with $C\,\kappa_2(S,\mathbb{R}) < \sqrt{2}$ has a fixed point.*

Now let Γ be a random group in the Gromov density model with density $1/3 < d < 1/2$. By the argument due to Żuk [23], Kotowski and Kotowski [15], there is a random group Γ' in a different model so that Γ contains a quotient of Γ' as a finite index subgroup and the link graph $\mathcal{L}(S')$ of Γ' has $\kappa_2(S', \mathbb{R})$ arbitrarily close to one. Therefore, we may apply Corollary 2.3 to Γ' and the conclusion of Theorem 2.1 holds for Γ' and hence for Γ.

Gromov [7] claimed a result similar to Theorem 2.1 for a random group in the graph model which was also invented by him. To state Gromov's result, we first review the construction of this model. Let F_m denote the free group on m generators, and let S be the collection of these m elements and their inverses. Let $G = (V, E)$ be a finite connected graph, where V and E are the sets of vertices and undirected edges, respectively. We denote the set of directed edges by $\overrightarrow{E}$. A map $\alpha\colon \overrightarrow{E} \to S$ satisfying $\alpha((v,u)) = \alpha((u,v))^{-1}$ for all $(u,v) \in \overrightarrow{E}$ is called an *S-labelling* of G. Let $\mathcal{A}(m,G)$ denote the set of all S-labellings of G, consisting of $(2m)^{\#E}$ elements, and equip it with the uniform probability measure. For $\alpha \in \mathcal{A}(m,G)$ and a path $\overrightarrow{p} = (\overrightarrow{e}_1, \dots, \overrightarrow{e}_l)$ in G, where $\overrightarrow{e}_i \in \overrightarrow{E}$, define $\alpha(\overrightarrow{p}) = \alpha(\overrightarrow{e}_1) \cdot \dots \cdot \alpha(\overrightarrow{e}_l) \in F_m$. Then set

$$R_\alpha = \{\alpha(\overrightarrow{c}) \mid \overrightarrow{c} \text{ are cycles in } G\},$$
$$\Gamma_\alpha = F_m/\text{normal closure of } R_\alpha.$$

Let $\lambda_1(G, \mathbb{R})$ denote the second eigenvalue of the discrete Laplacian of G, acting on real-valued functions on V. The *girth* of G, denoted by $\mathrm{girth}(G)$, is the minimal length of a cycle (i.e. a closed path) in G.

A sequence $\{G_j\}_{j\in\mathbb{N}}$ of finite graphs is called a *sequence of (bounded-degree) expanders* if it satisfies

(i) $\#V_j \to \infty$ as $j \to \infty$,

(ii) $\exists d,\ \forall j,\ \forall u \in V_j,\ 2 \le \deg(u) \le d$ (sparce),

(iii) $\exists \lambda > 0,\ \forall j,\ \lambda_1(G_j, \mathbb{R}) \ge \lambda$ (highly-connected).

Such a $\{G_j\}_{j\in\mathbb{N}}$ is said to have *diverging girth* if it further satisfies

(iv) $\mathrm{girth}(G_j) \to \infty$ as $j \to \infty$.

Now suppose that a sequence of expanders $\{G_j\}_{j\in\mathbb{N}}$ with diverging girth is given. Then the collection of groups $\mathcal{G}(m, G_j) = \{\Gamma_\alpha \mid \alpha \in \mathcal{A}(m, G_j)\}$ is the graph model of random groups. Given a group property P, we say that a *random group in the graph model has property P* if the probability of Γ_α having property P tends to one as $j \to \infty$. Gromov [7] and Silberman [21] proved that a random group in the graph model had fixed-point property for Hilbert spaces with respect to isometric actions, that is, it had property (T). This result was generalized to fixed-point property for CAT(0) spaces [10] (see also [7]) and for p-uniformly convex geodesic metric spaces [17]. In both generalizations the degrees of singularity of the relevant geodesic metric spaces should be suitably bounded.

We now state

Theorem 2.4 (Gromov [7]). *Fix $C > 0$. Let Γ be a random group in the graph model associated with a sequence of expanders with diverging girth. Then any uniformly C-Lipschitz affine action $\rho\colon \Gamma \curvearrowright \mathcal{H}$ has a fixed point.*

We can relax the condition that the Lipschitz constants of the relevant affine maps should be uniformly bounded. To state our result precisely, we introduce the following

Definition 2.5. Let Γ be a finitely generated group equipped with a finite, symmetric generating set S, and let $l\colon \Gamma \to \mathbb{Z}_{\geq 0}$ denote the word-length function with respect to S. For each conjugacy class c of Γ, we define

$$l_{\mathrm{conj}}(c) = \inf_{\gamma \in c} l(\gamma)$$

and call $l_{\mathrm{conj}}\colon \{\text{conjugacy classes of } \Gamma\} \to \mathbb{Z}_{\geq 0}$ the *conjugacy-length function* of Γ [3].

We now state

Theorem 2.6. *Fix $C > 0$ and $0 \leq \sigma < 1/10$. Let Γ be a random group in the graph model associated with a sequence of expanders with diverging girth and diameter growing at most linearly in girth. Then any affine action $\rho\colon \Gamma \curvearrowright \mathcal{H}$ satisfying*

$$\forall \gamma \in \Gamma,\ \|A(\gamma)\| \leq C\, l_{\mathrm{conj}}([\gamma])^{\sigma}, \tag{2.1}$$

where $[\gamma]$ denotes the conjugacy class containing γ, has a fixed point.

Remark 1. In order for a random group in the graph model to be non-elementary hyperbolic (hence infinite), the relevant sequence of expanders should satisfy some further conditions (cf. [7, 5, 1]). One of these conditions implies the diameter growth condition in Theorem 2.6, which is therefore essentially superficial.

§ 3. Discrete harmonic maps

Let Γ be a finitely generated group and fix a finite, symmetric generating set S. Let μ be the standard random walk of Γ associated with S, that is,

$$\mu(x \to x') \overset{\text{def}}{=} \begin{cases} 1/\#S \text{ if } \exists s \in S,\, x' = xs, \\ \quad 0 \quad \text{otherwise.} \end{cases}$$

The *barycenter map* of $\mathcal{H}$, bar: $\{$finite-support probability measures on $\mathcal{H}\} \to \mathcal{H}$, is given by

$$\mathrm{bar}\left(\sum_{i=1}^{m} t_i \mathrm{Dirac}(\mathbf{v}_i)\right) = \sum_{i=1}^{m} t_i \mathbf{v}_i. \tag{3.1}$$

Let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action.

Definition 3.1. A ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ is called *harmonic* if it satisfies

$$\mathrm{bar}(f_*\mu(x \to \bullet)) = f(x) \tag{3.2}$$

for all $x \in \Gamma$. Notice that

$$\mathrm{bar}(f_*\mu(x \to \bullet)) = \frac{1}{\#S} \sum_{s \in S} f(xs).$$

Remark 2. Since the action ρ is affine, we may conclude that a ρ-equivariant f is harmonic if it satisfies (3.2) for *some* $x \in \Gamma$. To see this, suppose that (3.2) holds for x, and write any $x' \in \Gamma$ as $x' = \gamma x$. Then

$$\begin{aligned}\mathrm{bar}(f_*\mu(x' \to \bullet)) &= \frac{1}{\#S} \sum_{s \in S} f(x's) = \frac{1}{\#S} \sum_{s \in S} \rho(\gamma)(f(xs)) \\ &= \rho(\gamma)\left(\frac{1}{\#S} \sum_{s \in S} f(xs)\right) = \rho(\gamma) f(x) \\ &= f(x'),\end{aligned}$$

and (3.2) holds for x', too.

The action ρ has a fixed point if and only if a ρ-equivariant constant map, which are trivially harmonic, exists. In contrast, we have the following existence result for nonconstant harmonic maps when ρ has no fixed point.

Theorem 3.2. *Let Γ be a finitely generated group equipped with a finite, symmetric generating set S, and let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action, where $\mathcal{H}$ is a Hilbert space, satisfying* (2.1) *for some $C > 0$ and $\sigma \geq 0$. Suppose that $\rho(\Gamma)$ has no fixed point. Then there exist a (possibly new) affine action $\rho'\colon \Gamma \curvearrowright \mathcal{H}'$, where $\mathcal{H}'$ is a (possibly new) Hilbert space, satisfying* (2.1) *for the same C, σ as above and a nonconstant harmonic ρ'-equivariant map $f\colon \Gamma \to \mathcal{H}'$.*

Before discussing the actual proof, we observe that the standard approach via energy minimization coupled with scaling ultralimit argument would fail.

Definition 3.3. For a ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ and $x \in \Gamma$, define the *local energy* $E(f)(x)$ of f at x by

$$E(f)(x) \stackrel{\mathrm{def}}{=} \frac{1}{2} \sum_{x' \in \Gamma} \|f(x) - f(x')\|^2 \mu(x \to x'). \tag{3.3}$$

As will be verified in the appendix, under the assumption that $\rho(\Gamma)$ has no fixed point, one can always find a nonconstant ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ minimizing the local energy at x, though the Hilbert space $\mathcal{H}$ and the affine action $\rho\colon \Gamma \curvearrowright \mathcal{H}$ should possibly be renewed. We now focus on the question whether the map f satisfies (3.2) for x. For $\mathbf{v} \in \mathcal{H}$ and $t \in \mathbb{R}$, let $f_t\colon \Gamma \to \mathcal{H}$ be the ρ-equivariant map such that $f_t(e) = f(e) + t\mathbf{v}$. Then we have, taking $x = e$ for simplicity,

$$\begin{aligned}
E(f_t)(e) &= \frac{1}{2\#S}\sum_{s\in S}\|f_t(e) - f_t(s)\|^2 \\
&= \frac{1}{2\#S}\sum_{s\in S}\|f(e) + t\mathbf{v} - \rho(s)(f(e) + t\mathbf{v})\|^2 \\
&= \frac{1}{2\#S}\sum_{s\in S}\|(f(e) - f(s)) + t(\mathbf{v} - A(s)(\mathbf{v}))\|^2 \\
&= \frac{1}{2\#S}\sum_{s\in S}\left(\|(f(e) - f(s))\|^2 + 2t\,\langle f(e) - f(s), \mathbf{v} - A(s)(\mathbf{v})\rangle + O(t^2)\right),
\end{aligned}$$

and therefore

$$0 = \frac{d}{dt}E(f_t)(e)|_{t=0} = \frac{1}{\#S}\sum_{s\in S}\langle f(e) - f(s), \mathbf{v} - A(s)(\mathbf{v})\rangle\,.$$

If the action ρ is isometric, which means that $A(s)$ is orthogonal, then

$$\begin{aligned}
\text{R.H.S.} &= \frac{1}{\#S}\sum_{s\in S}\langle f(e) - f(s), \mathbf{v}\rangle - \frac{1}{\#S}\sum_{s\in S}\left\langle A(s^{-1})(f(e) - f(s)), \mathbf{v}\right\rangle \\
&= \frac{1}{\#S}\sum_{s\in S}\langle f(e) - f(s), \mathbf{v}\rangle - \frac{1}{\#S}\sum_{s\in S}\left\langle f(s^{-1}) - f(e), \mathbf{v}\right\rangle \\
&= \frac{2}{\#S}\sum_{s\in S}\langle f(e) - f(s), \mathbf{v}\rangle\,.
\end{aligned}$$

Since this vanishes for all $\mathbf{v} \in \mathcal{H}$, we conclude (3.2). However, if ρ is not isometric, the above computation fails and we would not be able to conclude (3.2), that is, that f is harmonic.

Instead, we use Gromov's discrete tension-contracting flow developed in [7, §3.6] and produce a harmonic f. Postponing the details to [11], we shall outline the argument for the proof of Theorem 3.2. In the remainder of this section, let Γ be a finitely generated group equipped with a finite, symmetric generating set S, and let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action, where $\mathcal{H}$ is a Hilbert space.

For a ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$, define new maps $Hf\colon \Gamma \to \mathcal{H}$ and $\Delta f\colon \Gamma \to \mathcal{H}$ by

$$\begin{aligned}
Hf(x) &\stackrel{\text{def}}{=} \frac{1}{2}\left(\sum_{x'\in\Gamma} f(x')\,\mu(x \to x') + f(x)\right) \\
&= \frac{1}{2}\left(\frac{1}{\#S}\sum_{s\in S} f(xs) + f(x)\right),
\end{aligned}$$

$$\begin{aligned}\Delta f(x) &\overset{\text{def}}{=} (1-H)f(x)\\ &= \frac{1}{2}\sum_{x'\in\Gamma}(f(x)-f(x'))\,\mu(x\to x')\\ &= \frac{1}{2\#S}\sum_{s\in S}(f(x)-f(xs)).\end{aligned}$$

The maps Hf and Δf are ρ-equivariant and A-equivariant, respectively. We call H (resp. Δ) the *averaging operator* (resp. *Laplacian*). Note that f is harmonic if and only if $\Delta f = 0$, or $Hf = f$.

Proposition 3.4 (cf. Gromov [7]). *We have*

$$\|\Delta Hf(x)\| \le \max_{x'\in x(S\cup\{e\})} \|\Delta f(x')\|$$

for all $x \in \Gamma$, and if the equality sign holds for some x then $\Delta f(x)$ is a constant vector independent of $x \in \Gamma$.

Motivated by this proposition, we introduce the following

Definition 3.5 (cf. Gromov [7]). Let $f\colon \Gamma \to \mathcal{H}$ be a ρ-equivariant map, and define $f_0 := f$ and $f_{i+1} := Hf_i$ inductively. We say that f is *(harmonically) stable* if

$$0 < \exists\lambda < 1,\ \exists i_0 \in \mathbb{N},\ \forall i \ge i_0,\ \forall x \in \Gamma,\ \|\Delta f_{i+1}(x)\| \le \lambda \max_{x'\in x(S\cup\{e\})} \|\Delta f_i(x')\|.$$

It should be mentioned that the above definition of harmonic stability is slightly modified from Gromov's original one and it is more suitable for our purpose.

Remark 3. Suppose f_{i_0} is harmonic, that is, $\Delta f_{i_0} = f_{i_0} - f_{i_0+1} = 0$ for some i_0. Then $f_i = f_{i_0}$ and thus $\Delta f_i = 0$ for all $i \ge i_0$. Therefore, f is stable.

Proposition 3.6. *Suppose that ρ satisfies* (2.1) *for some $C > 0$ and $\sigma \ge 0$.*
(i) *If a ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ is stable, then $\{f_i\}_{i\in\mathbb{N}}$ converges pointwise to a map $f_\infty\colon \Gamma \to \mathcal{H}$, and f_∞ is harmonic.*
(ii) *If a ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ is not stable, then there exist a Hilbert space $\mathcal{H}'$ and a nonconstant harmonic map $f'\colon \Gamma \to \mathcal{H}'$, equivariant with respect to an affine action $\rho'\colon \Gamma \curvearrowright \mathcal{H}'$ satisfying* (2.1) *for the same C, σ as above.*

This proposition implies Theorem 3.2. The proofs of the two propositions above will be given in [11].

§ 4. Proof of Theorem 2.6

In this section, we prove Theorem 2.6. Let Γ be a finitely generated group equipped with a finite, symmetric generating set S. Let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action, where $\mathcal{H}$

is a Hilbert space, and suppose that ρ satisfies

$$\forall \gamma \in \Gamma,\ \|A(\gamma)\| \le C\, l(\gamma)^{\sigma} \tag{4.1}$$

for some $C > 0$ and $\sigma \ge 0$. (Note that this condition is weaker than (2.1).) For a ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ and $x \in \Gamma$, define the *local n-step energy* of f at x by

$$E^{(n)}(f)(x) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{x' \in \Gamma} \|f(x) - f(x')\|^2 \, \mu^n(x \to x'),$$

where μ^n is the n-th convolution of μ.

Lemma 4.1. *Suppose that $\sigma < 1/2$. Let $f\colon \Gamma \to \mathcal{H}$ be a harmonic ρ-equivariant map. Then we have*

$$E^{(n)}(f)(x) \gtrsim_{C,\sigma,x} n^{1-2\sigma} E(f)(x)$$

for all $x \in \Gamma$.

The proof of this lemma will be given in [11].

So far, the group Γ has been any finitely generated group. The following lemma, essentially due to Gromov and Silberman [7, 21], concerns a random Γ.

Lemma 4.2. *Let Γ be a random group in the graph model associated with a sequence of expanders with diverging girth and diameter growing at most linearly in girth, and let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action as above. Then for any ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ and any $x \in \Gamma$, we have*

$$E^{(n)}(f)(x) \lesssim_{C,\sigma,x,\lambda} n^{8\sigma} E(f)(x). \tag{4.2}$$

Here, n is a positive integer depending on f and x, and we may assume that n is arbitrarily large, and λ is the positive constant as in the definition of a sequence of expanders.

Proof. The proof of Proposition 2.14 in [21], which treats the case that the action is isometric, mostly works for the non-isometric case. For the readers' convenience we outline Silberman's argument, and explain how the term $n^{8\sigma}$ comes in.

The lemma is a consequence of the following statement. With probability tending to one as $j \to \infty$, the group Γ corresponding to $\alpha \in \mathcal{A}(m, G_j)$ has the following property: for any affine action $\rho\colon \Gamma \curvearrowright \mathcal{H}$ satisfying (4.1), any $n < \frac{\operatorname{girth}(G_j)}{2}$, any ρ-equivariant map $f\colon \Gamma \to \mathcal{H}$ and any $x \in \Gamma$, there exists $\sqrt{n} < l \le n$ such that

$$E^{(l)}(f)(x) \lesssim_{C,\sigma,x,\lambda} \operatorname{diam}(G_j)^{4\sigma}\, E(f)(x). \tag{4.3}$$

Indeed, choosing $n \simeq \frac{\mathrm{girth}(G_j)}{2}$, we have $\mathrm{diam}(G_j) \lesssim n \leq l^2$ by the assumption on $\mathrm{diam}(G_j)$, and therefore

$$E^{(l)}(f)(x) \lesssim_{C,\sigma,x,\lambda} l^{8\sigma} E(f)(x).$$

Note that $l \gtrsim \sqrt{\mathrm{girth}(G_j)}$; thus l diverges as $j \to \infty$.

For the time being, fix a member G_j of the expander sequence defining the graph model, and denote it by $G = (V, E)$. Let μ_G and ν_G denote the standard random walk on G and the standard probability measure on V given by

$$\mu_G(u,v) = \begin{cases} \frac{1}{\deg(u)} & \text{if } (u,v) \in \overrightarrow{E}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \nu_G(u) = \frac{\deg(u)}{2\#E},$$

respectively. For a map $\varphi\colon V \to \mathcal{H}$ and $n \in \mathbb{N}$, the *n-step energy* of φ is defined by

$$E_{\mu_G^n}(\varphi) = \frac{1}{2} \sum_{u \in V} \nu_G(u) \sum_{v \in V} \|\varphi(u) - \varphi(v)\|^2 \, \mu_G^n(u \to v).$$

Recall [21, Lemma 2.11] that we have

$$E_{\mu_G^n}(\varphi) \leq \frac{2}{\lambda_1(G, \mathbb{R})} E_{\mu_G}(\varphi) \tag{4.4}$$

for all maps $\varphi\colon V \to \mathcal{H}$ and all $n \in \mathbb{N}$.

Let $\alpha\colon \overrightarrow{E} \to S$ be an S-labelling of G, and Γ the corresponding group. Let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action, and $\widetilde{\rho}\colon F_m \curvearrowright \mathcal{H}$ its lift. The strategy in proving (4.3) is to transplant (4.4) onto Γ. In fact, we may work on F_m instead of Γ, and so we shall transplant (4.4) onto F_m. In order to do this, we 'push-forward', using α, the random walks μ_G and μ_G^n on G to those on F_m as follows.

If $u \in V$ and $x \in F_m$ are fixed, α induces a corresponding graph morphism $\beta_{u\to x}$ from G to $X = \mathrm{Cay}(F_m, S)$, the Cayley graph of F_m with respect to S, as follows: For $v \in V$, choose a path $\overrightarrow{p} = (\overrightarrow{e}_1, \ldots, \overrightarrow{e}_l)$ from u to v in G, and set

$$\beta_{u\to x}(v) \overset{\mathrm{def}}{=} x\,\alpha(\overrightarrow{p}) = x\,\alpha(\overrightarrow{e}_1) \cdot \cdots \cdot \alpha(\overrightarrow{e}_l).$$

To be precise, $\beta_{u\to x}$ is well-defined only on the set of vertices whose graph distance from u is less than $g/2$, where $g = \mathrm{girth}(G)$. We now define, for $n < g/2$, a random walk $\mu_{G,\alpha}^n$ on X by

$$\mu_{G,\alpha}^n(x \to \cdot) \overset{\mathrm{def}}{=} \sum_{u \in V} \nu_G(u)\, (\beta_{u\to x})_* \mu_G^n(u \to \cdot).$$

Note that the average over V is taken in order to produce a random walk independent of the individual vertices of G.

We can now transplant (4.4) onto F_m, and it is here that something different occurs when the action ρ is non-isometric. Suppose $n < g/2$ and let $f\colon F_m \to \mathcal{H}$ be a $\widetilde{\rho}$-equivariant map.[1] When ρ is isometric,

$$E_{\mu^n_{G,\alpha}}(f)(x) = E_{\mu^n_G}(f \circ \beta_{u_0\to x})(x) \tag{4.5}$$

holds for a fixed $u_0 \in V$. Indeed,

$$\begin{aligned} E_{\mu^n_{G,\alpha}}(f)(x) &= \frac{1}{2}\sum_{u\in V}\nu_G(u)\sum_{x'\in F_m}\|f(x)-f(x')\|^2\,[(\beta_{u\to x})_*\mu^n_G(u\to\cdot)](x') \\ &= \frac{1}{2}\sum_{u\in V}\nu_G(u)\sum_{v\in V}\|f\circ\beta_{u\to x}(u)-f\circ\beta_{u\to x}(v)\|^2\,\mu^n_G(u\to v). \end{aligned}$$

If ρ is isometric, then we can replace $\beta_{u\to x}$ by $\beta_{u_0\to x}$ in the last expression and get the right-hand side of (4.5). Now consider the general case that ρ is not necessarily isometric. Let $\overrightarrow{p}$ and $\overrightarrow{r}$ be a path from u_0 to v and a shortest path from u to u_0, respectively, and let $\overrightarrow{q}$ denote the path from u to v traveling along $\overrightarrow{r}$ and $\overrightarrow{p}$ in this order. Then

$$\begin{aligned} f\circ\beta_{u\to x}(v) &= f(x\alpha(\overrightarrow{q})) \\ &= \widetilde{\rho}(x\alpha(\overrightarrow{r})x^{-1})f(x\alpha(\overrightarrow{p})) \\ &= \widetilde{\rho}(x\alpha(\overrightarrow{r})x^{-1})f\circ\beta_{u_0\to x}(v), \end{aligned}$$

and therefore

$$\|f\circ\beta_{u\to x}(u)-f\circ\beta_{u\to x}(v)\| \le \|\widetilde{A}(x\alpha(\overrightarrow{r})x^{-1})\|\|f\circ\beta_{u_0\to x}(u)-f\circ\beta_{u_0\to x}(v)\|,$$

where $\widetilde{A}$ is the linear part of $\widetilde{\rho}$. Since

$$\begin{aligned} \|\widetilde{A}(x\alpha(\overrightarrow{r})x^{-1})\| &\le \|\widetilde{A}(x)\|\|\widetilde{A}(\alpha(\overrightarrow{r}))\|\|\widetilde{A}(x^{-1})\| \\ &\le C^3\, l(x)^\sigma\, l(\alpha(\overrightarrow{r}))^\sigma\, l(x^{-1})^\sigma \\ &\le C^3\, D^\sigma\, l(x)^{2\sigma}, \end{aligned}$$

where $D = \operatorname{diam}(G)$, we obtain

$$E_{\mu^n_{G,\alpha}}(f)(x) \le C^6\, D^{2\sigma}\, l(x)^{4\sigma} E_{\mu^n_G}(f\circ\beta_{u_0\to x})(x),$$

and likewise,

$$E_{\mu_G}(f\circ\beta_{u_0\to x})(x) \le C^6\, D^{2\sigma}\, l(x)^{4\sigma} E_{\mu_{G,\alpha}}(f)(x).$$

[1] Equivalently, $f\colon F_m \to \mathcal{H}$ is the lift of a ρ-equivariant map $\Gamma \to \mathcal{H}$. In particular, the map $f\circ\beta_{u\to x}$ is well-defined on the whole vertex set V.

Together with (4.4), these imply

$$E_{\mu^n_{G,\alpha}}(f)(x) \le \frac{2\,C^{12}\,D^{4\sigma}\,l(x)^{8\sigma}}{\lambda_1(G,\mathbb{R})} E_{\mu_{G,\alpha}}(f)(x). \tag{4.6}$$

In order to conclude (4.3) (provisionally on F_m instead of Γ), we must show that with high probability the random walks $\mu_{G,\alpha}$ and $\mu^n_{G,\alpha}$ in (4.6) can be replaced by μ_X and μ^l_X, $\sqrt{n} < l \le n$, respectively, where μ_X is the standard random walk of X. This will be done by verifying that with high probability the random variables $\alpha \mapsto \mu_{G,\alpha}$ and $\alpha \mapsto \mu^n_{G,\alpha}$ concentrate on their expectations and that these expectations are computed in terms of μ_X and its convolutions.

We begin with the second issue. For $n < g/2$, the expectation $\overline{\mu}^n_{G,X}$ of the random variable $\alpha \mapsto \mu^n_{G,\alpha}$ can be computed and expressed as a convex combination of μ^l_X, $0 \le l \le n$:

$$\overline{\mu}^n_{G,X} = \sum_{l=0}^{n} w^{(n)}_l\, \mu^l_X, \tag{4.7}$$

where the weights $w^{(n)}_l$ satisfy

$$\sum_{\sqrt{n}<l\le n} w^{(n)}_l \ge C' \tag{4.8}$$

for a certain absolute constant $C' > 0$.

For the first issue, let j get large and observe that the random variables $\mu_{G_j,\bullet}$ and $\mu^n_{G_j,\bullet}$, where $n < g_j/2$, concentrate on their expectations $\overline{\mu}_{G_j,X}$ and $\overline{\mu}^n_{G_j,X}$, respectively. Indeed, one can verify that the map $\alpha \mapsto \mu^n_{G_j,\alpha}$ is Lipschitz with respect to the Hamming distance on $\mathcal{A}(m,G_j)$ with the Lipschitz constant depending only on the fixed parameters d, m. Using this fact, one deduces that with probability tending to one as $j \to \infty$,

$$\mu_{G_j,\alpha}(x \to x') \le 2\,\overline{\mu}_{G_j,X}(x \to x') \quad \text{and} \quad \mu^n_{G_j,\alpha}(x \to x') \ge \frac{1}{2}\,\overline{\mu}^n_{G_j,X}(x \to x')$$

hold for all $x, x' \in X$.

Now for any $\widetilde{\rho}$-equivariant map $f\colon F_m \to \mathcal{H}$, we obtain

$$E_{\mu_{G_j,\alpha}}(f)(x) \le 2\,E_{\overline{\mu}_{G_j,X}}(f)(x) = 2\,E_{\mu_X}(f)(x)$$

and

$$\begin{aligned} E_{\mu^n_{G_j,\alpha}}(f)(x) &\ge \frac{1}{2}\,E_{\overline{\mu}^n_{G_j,X}}(f)(x) \ge \frac{1}{2} \sum_{\sqrt{n}<l\le n} w^{(n)}_l\, E_{\mu^l_X}(f)(x) \\ &\ge \frac{C'}{2} \min_{\sqrt{n}<l\le n} E_{\mu^l_X}(f)(x). \end{aligned}$$

Together with (4.6), these imply that there exists $\sqrt{n} < l \le n$ (which depends on f and x) such that

$$E_{\mu_X^l}(f)(x) \le \frac{8\,C^{12}\,D_j^{4\sigma}\,l(x)^{8\sigma}}{C'\lambda} E_{\mu_X}(f)(x).$$

Now let $f\colon \Gamma \to \mathcal{H}$ be a ρ-equivariant map and set $\widetilde{f} = f \circ \pi$. Let $x \in \Gamma$ and choose $\widetilde{x} \in \pi^{-1}(x) \subset F_m$ so that $l(\widetilde{x}) = l(x)$. Since the ball of radius less than $g_j/2$ with center $\widetilde{x}$ in X is isometrically isomorphic to that of the same radius with center x in $\mathrm{Cay}(\Gamma, S)$, the above inequality (for $\widetilde{f}$, $\widetilde{x}$) implies

$$E^{(l)}(f)(x) \le \frac{8\,C^{12}\,D_j^{4\sigma}\,l(x)^{8\sigma}}{C'\lambda} E(f)(x),$$

that is, (4.3). □

Theorem 2.6 now follows by combining Theorem 3.2, Lemma 4.1 and Lemma 4.2.

Appendix

Let Γ be a finitely generated group equipped with a finite, symmetric generating set S, and let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action, where $\mathcal{H}$ is a Hilbert space. In §3, we referred to the following fact: if $\rho(\Gamma)$ has no fixed point, then energy minimization coupled with scaling ultralimit argument produces a nonconstant map from Γ to a (possibly new) Hilbert space $\mathcal{H}'$ which is equivariant with respect to a (possibly new) affine action $\rho'\colon \Gamma \curvearrowright \mathcal{H}'$ and minimizes the local energy at a point. While this fact would not be useful for our purpose of proving Theorem 3.2 as we observed that we would not be able to conclude the resulting map is harmonic, it might be so in other circumstances. Therefore, we shall verify the above fact by proving the following

Proposition 4.3. *Let Γ be a finitely generated group equipped with a finite, symmetric generating set S, and let $\rho\colon \Gamma \curvearrowright \mathcal{H}$ be an affine action, where $\mathcal{H}$ is a Hilbert space. Suppose that $\rho(\Gamma)$ has no fixed point. Fix $x \in \Gamma$. Then there exist a (possibly new) affine action $\rho'\colon \Gamma \curvearrowright \mathcal{H}'$, where $\mathcal{H}'$ is a (possibly new) Hilbert space, and a nonconstant ρ'-equivariant map $f\colon \Gamma \to \mathcal{H}'$ minimizing the local energy at x. If ρ satisfies* (4.1) *for some $C > 0$ and $\sigma \ge 0$, then ρ' also satisfies* (4.1) *for the same C, σ.*

Before proceeding to the proof, we review the definitions of ultrafilter and the ultralimit of a sequence of metric spaces.

A nonempty subset $\omega \subset 2^{\mathbb{N}}$ is called an *ultrafilter* on $\mathbb{N}$ if it satisfies the following conditions:

(i) $\emptyset \notin \omega$.

(ii) $A \in \omega,\ A \subset B \quad \Rightarrow \quad B \in \omega.$

(iii) $A, B \in \omega \quad \Rightarrow \quad A \cap B \in \omega$.

(iv) For any subset $A \subset \mathbb{N}$, $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.

An ultrafilter ω on $\mathbb{N}$ is called *non-principal* if it satisfies also

(v) For any finite subset $F \subset \mathbb{N}$, $F \notin \omega$ (hence, $\mathbb{N} \setminus F \in \omega$).

Let ω be a non-principal ultrafilter on $\mathbb{N}$. Let $(a_j)_{j=1}^{\infty} \subset \mathbb{R}$ be a sequence of real numbers. We call $\alpha \in \mathbb{R}$ an *ω-limit* of (a_j) and write $\omega\text{-}\lim_j a_j = \alpha$ if $\{j \in \mathbb{N} \mid |a_j - \alpha| < \varepsilon\} \in \omega$ holds for any $\varepsilon > 0$. Let (Y_j, d_j, o_j) be a sequence of metric spaces with base point. On the set of sequences (y_j), where $y_j \in Y_j$ and $d_j(o_j, y_j)$ is bounded independent of j, consider the equivalence relation $[(y_j) \sim (z_j) \;\Leftrightarrow\; \omega\text{-}\lim_j d_j(y_j, z_j) = 0]$, and denote the equivalence class of (y_j) by $y_\infty = \omega\text{-}\lim_j y_j$. Let Y_∞ denote the set of equivalence classes, and endow it with the metric $d_\infty(y_\infty, z_\infty) = \omega\text{-}\lim_j d_j(y_j, z_j)$. One writes $(Y_\infty, d_\infty, o_\infty) = \omega\text{-}\lim_j (Y_j, d_j, o_j)$, called the *$\omega$-limit* of (Y_j, d_j, o_j). It is known that the metric space (Y_∞, d_∞) is necessarily complete.

Proof of Proposition 4.3 We shall follow [22] and [14] which treat the case that the action is isometric.

Fix a non-principal ultrafilter ω on $\mathbb{N}$. We divide the proof into two cases, according to whether $E_0 := \inf E(f)(x)$ is strictly positive or not, where the infimum is taken over all ρ-equivariant maps $f\colon \Gamma \to \mathcal{H}$.

Case 1. The case that $E_0 > 0$.

This is a simpler case, and we only outline the argument. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of ρ-equivariant maps $\Gamma \to \mathcal{H}$ such that $E(f_j)(x) \searrow E_0$. Set $\mathbf{v}_j = f_j(x)$ and define $(\mathcal{H}_\infty, \|\cdot\|_\infty, \mathbf{v}_\infty) = \omega\text{-}\lim_j (\mathcal{H}, \|\cdot\|, \mathbf{v}_j)$. Then an affine action $\rho_\infty\colon \Gamma \curvearrowright \mathcal{H}_\infty$ is induced and satisfies (4.1). Define a map $f_\infty\colon \Gamma \to \mathcal{H}_\infty$ by $f_\infty(y) = \omega\text{-}\lim_j f_j(y)$ for $y \in \Gamma$. Then f_∞ is ρ_∞-equivariant, and

$$E(f_\infty)(x) = \omega\text{-}\lim_j E(f_j)(x) = E_0;$$

in particular, f_∞ is nonconstant. On the other hand, one can verify that $E(g)(x) \geq E_0$ for all ρ_∞-equivariant maps $g\colon \Gamma \to \mathcal{H}_\infty$. Thus, f_∞ minimizes the local energy at x.

Case 2. The case that $E_0 = 0$.

Define $\delta\colon \mathcal{H} \to \mathbb{R}_{\geq 0}$ by $\delta(\mathbf{v}) = \max_{s \in S} \|\rho(s)(\mathbf{v}) - \mathbf{v}\|$. While $\delta > 0$ since $\rho(\Gamma)$ has no fixed-point, we have $\inf_{\mathbf{v} \in \mathcal{H}} \delta(\mathbf{v}) = 0$; indeed,

$$\begin{aligned} E(f)(x) &= \frac{1}{2\#S} \sum_{s \in S} \|f(xs) - f(x)\|^2 \\ &= \frac{1}{2\#S} \sum_{s \in S} \|\rho(x)\{\rho(sx^{-1})(f(x)) - \rho(x^{-1})(f(x))\}\|^2, \end{aligned}$$

which is clearly comparable to $\delta(\rho(x^{-1})(f(x)))^2$.

In order to proceed, we need the following elementary fact: let Y be a complete metric space and $\varphi\colon Y \to \mathbb{R}$ a strictly positive continuous function. Then there exists $y \in Y$ such that $d_Y(z,y) \le \varphi(y) \ \Rightarrow\ \varphi(z) \ge \frac{1}{2}\varphi(y)$. Let $j \in \mathbb{N}$ and apply this fact to the function $j\delta\colon \mathcal{H} \to \mathbb{R}$. Then we get $\mathbf{v}_j \in \mathcal{H}$ such that $\|\mathbf{w} - \mathbf{v}_j\| \le j\delta(\mathbf{v}_j) \ \Rightarrow\ \delta(\mathbf{w}) \ge \frac{1}{2}\delta(\mathbf{v}_j)$. Now let $(\mathcal{H}_\infty, \|\cdot\|_\infty, \mathbf{v}_\infty) = \omega\text{-}\lim_j \left(\mathcal{H}, \frac{1}{\delta(\mathbf{v}_j)}\|\cdot\|, \mathbf{v}_j\right)$.

We shall define an affine action $\rho_\infty\colon \Gamma \curvearrowright \mathcal{H}_\infty$. Let $\mathbf{w}_\infty \in \mathcal{H}_\infty$ and write $\mathbf{w}_\infty = \omega\text{-}\lim_j \mathbf{w}_j$. By definition, there exists $M > 0$ such that $\|\mathbf{w}_j - \mathbf{v}_j\| \le M\delta(\mathbf{v}_j)$ for all $j \in \mathbb{N}$. Then

$$\begin{aligned}\|\rho(s)(\mathbf{w}_j) - \mathbf{v}_j\| &\le \|\rho(s)(\mathbf{w}_j) - \rho(s)(\mathbf{v}_j)\| + \|\rho(s)(\mathbf{v}_j) - (\mathbf{v}_j)\| \\ &\le C\,\|\mathbf{w}_j - \mathbf{v}_j\| + \delta(\mathbf{v}_j) \\ &\le (CM+1)\,\delta(\mathbf{v}_j),\end{aligned}$$

where $C = \|A(s)\|$. It follows that $\omega\text{-}\lim_j \rho(s)(\mathbf{w}_j)$ exists, and it is easy to verify that this limit is independent of the choice of $\mathbf{w}_j$. Hence, by defining $\rho_\infty(s)(\mathbf{w}_\infty) = \omega\text{-}\lim_j \rho(s)(\mathbf{w}_j)$, we obtain a well-defined map $\rho_\infty(s)\colon \mathcal{H}_\infty \to \mathcal{H}_\infty$, which is clearly C-Lipschitz. It is also easy to see that the affineness, that is, the property of preserving internally dividing points, of $\rho(s)$ is inherited by $\rho_\infty(s)$. Let $\gamma \in \Gamma$ and write $\gamma = s_1 \dots s_l$, where $s_1, \dots, s_l \in S$. Let $\mathbf{w}_\infty = \omega\text{-}\lim_j \mathbf{w}_j \in \mathcal{H}_\infty$. Then the ultralimit of $\rho(\gamma)(\mathbf{w}_j) = \rho(s_1)\dots\rho(s_l)(\mathbf{w}_j)$ exists and equals to $\rho_\infty(s_1)\dots\rho_\infty(s_l)(\mathbf{w}_\infty)$. Thus, defining $\rho_\infty(\gamma)(\mathbf{w}_\infty) = \omega\text{-}\lim_j \rho(\gamma)(\mathbf{w}_j)$, we have $\rho_\infty(\gamma) = \rho_\infty(s_1)\dots\rho_\infty(s_l)$ and obtain an affine action $\rho_\infty\colon \Gamma \curvearrowright \mathcal{H}_\infty$. It is clear that if ρ satisfies (4.1), then ρ_∞ also satisfies (4.1) with the same constants.

We now verify that $\delta_\infty \ge \frac{1}{2}$, where δ_∞ is the function δ with respect to ρ_∞. To do so, take any $\mathbf{w}_\infty = \omega\text{-}\lim_j \mathbf{w}_j \in \mathcal{H}_\infty$, so that $\|\mathbf{w}_j - \mathbf{v}_j\| \le M\,\delta(\mathbf{v}_j)$ for some $M > 0$, and set $A_s := \{j \in \mathbb{N} \mid \|\rho(s)(\mathbf{w}_j) - \mathbf{w}_j\| \ge \frac{1}{2}\delta(\mathbf{v}_j)\}$ for $s \in S$. For $j > M$, $\|\mathbf{w}_j - \mathbf{v}_j\| \le j\,\delta(\mathbf{v}_j)$, and therefore $\delta(\mathbf{w}_j) \ge \frac{1}{2}\delta(\mathbf{v}_j)$, that is, $j \in \cup_{s\in S} A_s$. Thus $\cup_{s\in S} A_s \in \omega$. But this means $A_s \in \omega$ for some $s \in S$. Therefore, $\|\rho_\infty(s)(\mathbf{w}_\infty) - \mathbf{w}_\infty\|_\infty \ge \frac{1}{2}$, and $\delta_\infty \ge \frac{1}{2}$. We thus recover the situation of Case 1. □

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