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The geodesic growth series for pure Artin groups of dihedral type

By

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Abstract

We consider the pure Artin group of dihedral type, which is the kernel of the natural projection from the Artin group of dihedral type $I_2(k)$ to the associated Coxeter group. We present a rational function expression for the geodesic growth series of the pure Artin group of dihedral type with respect to a natural generating set, and we explicitly determine the denominator of this rational function expression. Moreover, we show that the growth rate of the series is a Pisot-Vijayaraghavan number.

§ 1. Introduction

For a finitely generated group G with a given generating set Γ, the corresponding *spherical growth series* is defined as

$$\mathcal{S}_{(G,\Gamma)}(t) := \sum_{n=0}^{\infty} \alpha_n t^n,$$

where α_n for $n \in \mathbf{N} \cup \{0\}$ is the number of elements in G whose lengths with respect to Γ are equal to n. The spherical growth series $\mathcal{S}_{(G,\Gamma)}(t)$ is a commonly employed measure of the rate of growth of G with respect to Γ, and has been explored for a number of

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interesting families of pairs (G,Γ). In particular, $\mathcal{S}_{(G,\Gamma)}(t)$ is known to be rational (i.e., it can be expressed as the quotient of two polynomials with integer-valued coefficients in the ring of formal power series $\mathbf{Z}[[t]]$) in a number of important cases (see, e.g., [3], [4], [5], [6], [7], [9], [10], [11], [12], [16], [18], [20], [22], [26], [27], [28] and [29]).

In the present paper, we are interested in the following series, defined analogously to $\mathcal{S}_{(G,\Gamma)}(t)$:

$$\mathcal{G}_{(G,\Gamma)}(t) := \sum_{n=0}^{\infty} \widetilde{\alpha}_n t^n,$$

where $\widetilde{\alpha}_n$ for $n \in \mathbf{N}\cup\{0\}$ is the number of geodesic words with respect to Γ whose lengths are equal to n. Recall that a word in the free monoid Γ^* generated by Γ is *geodesic* if the corresponding path in the Cayley graph of G with respect to Γ is a minimal length edge path joining its endpoints. The series $\mathcal{G}_{(G,\Gamma)}(t)$ is called the *geodesic growth series* for the pair (G,Γ). The growth rate for $\mathcal{G}_{(G,\Gamma)}(t)$ is defined as

$$\tau_{\mathcal{G}} := \limsup_{n\to\infty} \sqrt[n]{\widetilde{\alpha}_n}.$$

We also call $\tau_{\mathcal{G}}$ the *geodesic growth rate* for the pair (G,Γ). By the Cauchy-Hadamard theorem, the radius of convergence $R_{\mathcal{G}}$ of the series $\mathcal{G}_{(G,\Gamma)}(t)$ is the reciprocal of $\tau_{\mathcal{G}}$.

Although the geodesic growth series $\mathcal{G}_{(G,\Gamma)}(t)$ is not as well understood as the spherical growth series $\mathcal{S}_{(G,\Gamma)}(t)$, there are nonetheless many known pairs (G,Γ) for which the geodesic growth series is rational. The following are some examples: (1) any word-hyperbolic group with respect to an arbitrary generating set (see [5] and [10]); (2) any geometrically finite hyperbolic group with respect to a particular generating set (see [23]); (3) any irreducible affine Coxeter group with respect to the standard generating set (see [24]); (4) any right-angled Artin group with respect to the standard generating set (see [21] and [1]); (5) any Artin group of dihedral type, $G_{I_2(k)}$, with respect to the standard generating set (the so-called 'set of Artin generators') (see [26] and [22]); (6) any Artin group of large type with respect to the standard generating set (see [17]); (7) any Garside group with respect to a particular generating set (the so-called 'set of Garside generators') (see [8]). For each of the above examples, it has been shown that the set of all geodesic words of the group G with respect to the generating set Γ is a regular language over Γ, which implies the rationality of the geodesic growth series $\mathcal{G}_{(G,\Gamma)}(t)$. The pair (G,Γ) is said to be *strongly geodesic regular* or to form a *Cannon pair* if the set of all geodesic words of G with respect to Γ forms a regular language over Γ. In general, the regularity of a language consisting of geodesic words depends on the generating set Γ (see the example due to Cannon discussed in §4 of [23]).

In this paper, for each integer $k \geq 3$, we consider the *pure Artin group* $P_{I_2(k)}$, which is the kernel of the projection from the Artin group of dihedral type, $G_{I_2(k)}$, to the associated Coxeter group, $\overline{G}_{I_2(k)}$. The group $P_{I_2(k)}$ is geometrically realized as the fundamental group of the complement of a torus link in the 3-dimensional sphere, which has a natural generating set A (cf. [25]). In particular, in the case $k = 3$, $P_{I_2(3)}$ is the pure braid group with three strands, and A is the standard generating set (cf. [2]). In [13], for any element g of $P_{I_2(k)}$, a particular geodesic representative of g is determined. Then, through analysis of the regularity of the language consisting of such geodesic representatives, a rational function expression for the spherical growth series $\mathcal{S}_{(P_{I_2(k)},A)}(t)$ of $P_{I_2(k)}$ is derived with respect to the generating set A. Moreover, in [13], all the geodesic representatives of any element of $P_{I_2(k)}$ are determined. From this, it is seen that all geodesic words of a particular type (Type 3 defined in [13]) form a regular language. In the present paper, by using arguments similar to those given in [13], we derive a rational function expression for the geodesic growth series $\mathcal{G}_{(P_{I_2(k)},A)}(t)$ of $P_{I_2(k)}$ with respect to the generating set A (see Theorem 3.2). Moreover, by using an algebraic argument, we explicitly determine the denominator of this rational function expression (see Theorem 3.3). From this theorem, we can detect a number-theoretic property concerning the geodesic growth rate $\tau_{\mathcal{G}}(k)$ for the pair $(P_{I_2(k)}, A)$: The geodesic growth rate $\tau_{\mathcal{G}}(k)$ is a Pisot-Vijayaraghavan number, i.e., a real algebraic integer $\tau > 1$ whose algebraic conjugates other than τ itself lie in the unit disk (see Corollary 3.5).

§ 2. Geodesic words of pure Artin groups of dihedral type

In this section, we present definitions and basic facts concerning pure Artin groups of dihedral type (see [13] and [15] for details). We inherit all of the notation used in [13].

Let k be an integer greater than 2, $G_{I_2(k)}$ be the *Artin group of dihedral type* $I_2(k)$ and $\overline{G}_{I_2(k)}$ be the *Coxeter group of dihedral type* $I_2(k)$. Then there is a natural surjective homomorphism

$$p : G_{I_2(k)} \to \overline{G}_{I_2(k)}.$$

We call the kernel of p the *pure Artin group of dihedral type* and write it $P_{I_2(k)}$. The group $P_{I_2(k)}$ has the following presentation:

$$P_{I_2(k)} = \langle\, a_1, \dots, a_k \mid a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1} \,\rangle.$$

In this paper, we consider the generating set

$$A = \{a_1, \dots, a_k, a_1^{-1}, \dots, a_k^{-1}\},$$

as in [13], and investigate the so-called 'geodesic growth series' of $P_{I_2(k)}$ with respect to A, whose definition is given in §3.

Let A^* and $\{a_1, \dots, a_k\}^*$ denote the free monoids generated by A and $\{a_1, \dots, a_k\}$, respectively. We refer to the finite set A as an *alphabet*, its elements as *letters*, and the elements of A^* (resp., $\{a_1, \dots, a_k\}^*$) as *words* (resp., *positive words*). Let ε denote the *null* word. A subset L of A^* is called a *language* over A. A language L is *regular* if L is recognized by some deterministic, finite-state automaton over A (see [10] or [19] for the definition of automata). The length of a word w is the number of letters it contains, which is denoted by $|w|$. The length of ε is zero. Since A generates the group $P_{I_2(k)}$, there exists a natural surjective monoid homomorphism $\pi : A^* \to P_{I_2(k)}$. If u and v are words, then $u = v$ means that $\pi(u) = \pi(v)$ and $u \equiv v$ means that u and v are identical letter by letter. A word $w \in \pi^{-1}(g)$ is called a *representative* of g. The length of a group element g is regarded as the quantity

$$\|g\| = \min\{|w| \mid w \in \pi^{-1}(g)\}.$$

A word $w \in A^*$ for which the relation $|w| = \|\pi(w)\|$ holds is termed *geodesic*. A word $w_1 \cdots w_m \in A^*$ is called a *reduced* word if $w_i \neq w_{i+1}^{-1}$ for all $i \in \{1, \dots, m-1\}$. A geodesic representative is a reduced word. Below, we consider some other alphabet, T, which is a subset of A^* (denoted by $\mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N}$), and we investigate the regularity of a language over T.

Let us first recall the notation and definitions concerning 'fundamental blocks' given in [13]. We begin by introducing the word

$$\nabla \equiv a_1 \cdots a_k.$$

It satisfies the following relation:

$$\nabla \equiv a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1}. \tag{2.1}$$

A *fundamental block* is a word with length smaller than k that appears as a subword in the terms of (2.1). There are $k(k-1)$ fundamental blocks. All of them are listed in [13]. Let $\mathrm{FB}^+ (\subset A^*)$ (resp., $\mathrm{FB}^- (\subset A^*)$) denote the set consisting of all the fundamental blocks (resp., all the inverses of fundamental blocks); for $I \in \{0, \dots, k-1\}$, $\mathrm{FB}^\pm_I$ (resp., $\mathrm{FB}^\pm_{\leq I}$) denotes the set consisting of all the elements of $\mathrm{FB}^\pm$ with length equal to I (resp., smaller than or equal to I); for $\mu \equiv a_i \cdots a_k a_1 \cdots a_j \in \mathrm{FB}^+$ (resp., $\mu^{-1} \equiv a_j^{-1} \cdots a_1^{-1} a_k^{-1} \cdots a_i^{-1} \in \mathrm{FB}^-$), we define $\mathcal{L}(\mu) := a_i$ and $\mathcal{R}(\mu) := a_j$ (resp., $\mathcal{L}(\mu^{-1}) := a_j$ and $\mathcal{R}(\mu^{-1}) := a_i$). For $\mu \equiv a_i \cdots a_k a_1 \cdots a_j \in \mathrm{FB}^+$, we call a_{j+1} the *letter subsequent to* μ. When $\mu \equiv a_i \cdots a_k$, we call a_1 the letter subsequent to μ. The letter subsequent to μ is denoted by $\mathcal{N}(\mu)$. For $\mu^{-1} \equiv a_j^{-1} \cdots a_1^{-1} a_k^{-1} \cdots a_i^{-1} \in \mathrm{FB}^-$, we call a_{i-1} the letter subsequent to μ^{-1}, which is denoted by $\mathcal{N}(\mu^{-1})$. When $a_i^{-1} \equiv a_1^{-1}$, we call a_k the letter subsequent to μ^{-1}.

In [13], we introduced the set

$$\widetilde{\mathbf{\Gamma}} := \{\xi \in A^* \mid |\xi| = \|\pi(\xi)\|\},$$

i.e., the set consisting of all of the geodesic words, and for each (P, N) satisfying $0 \leq P \leq k$ and $0 \leq N \leq k$, the sets

$$\begin{aligned} \widetilde{\mathbf{\Gamma}}_{P,N} &:= \left\{\xi \in \mathrm{WT} \,\middle|\, |\xi| = \|\pi(\xi)\|, (\mathrm{Pos}(\xi), \mathrm{Neg}(\xi)) = (P, N)\right\}, \\ G_{P,N} &:= \{g \in P_{I_2(k)} \mid (\mathrm{Pos}(g), \mathrm{Neg}(g)) = (P, N)\}, \end{aligned}$$

where $\mathrm{WT}_i = \{w \in A^* \mid w \text{ is a word of Type } i\ \}$ for each $i \in \{1, 2, 3\}$ (see §3 in [13] for the definition of Type i), $\mathrm{WT} = \mathrm{WT}_1 \cup \mathrm{WT}_2 \cup \mathrm{WT}_3$, and $\mathrm{Pos}(\xi), \mathrm{Neg}(\xi), \mathrm{Pos}(g)$ and $\mathrm{Neg}(g)$ are specific integers between 0 and k (defined in §§2 and 3 of [13]).

By Propositions 3.3, 3.4 and 3.7 and Corollary 3.8 of [13], we have

$$\begin{cases} P + N \geq k + 1 \implies \widetilde{\mathbf{\Gamma}}_{P,N} = \emptyset, \quad G_{P,N} = \emptyset, \\ P + N \leq k \qquad\implies \pi^{-1}(G_{P,N}) \cap \widetilde{\mathbf{\Gamma}} = \widetilde{\mathbf{\Gamma}}_{P,N}, \end{cases} \tag{2.2}$$

$$\begin{cases} \widetilde{\mathbf{\Gamma}} = \displaystyle\bigcup_{P+N\leq k} \widetilde{\mathbf{\Gamma}}_{P,N} \quad \text{(disjoint union)}, \\ P_{I_2(k)} = \displaystyle\bigcup_{P+N\leq k} G_{P,N} \quad \text{(disjoint union)}, \end{cases} \tag{2.3}$$

and

$$\bigcup_{\substack{P+N\leq k \\ (P,N)\neq(k,0),(0,k)}} \widetilde{\mathbf{\Gamma}}_{P,N} = \{\xi \in \mathrm{WT}_3 \mid \mathrm{Pos}(\xi) + \mathrm{Neg}(\xi) \leq k\}.$$

We remark that an element $g \in P_{I_2(k)}$ has more than one geodesic representative if and only if $\mathrm{Pos}(g) + \mathrm{Neg}(g) = k$.

Now, choose any $P, N \in \mathbf{N} \cup \{0\}$ satisfying the conditions $P + N \leq k$ and $(P, N) \notin \{(k, 0), (0, k)\}$, and fix them. Then, any element $w \in \displaystyle\bigcup_{p\leq P,\ n\leq N} \widetilde{\mathbf{\Gamma}}_{p,n}$ can be expressed as

$$w \equiv v_1 \cdots v_m \in (\mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N})^*,$$

where $(\mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N})^*$ is the free monoid generated by the finite set $\mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N}$, and for each $j \in \{1, \ldots, m-1\}$, we have

$$\begin{cases} v_j, v_{j+1} \in \mathrm{FB}^{\pm} \Rightarrow \mathcal{N}(v_j) \neq \mathcal{L}(v_{j+1}), \\ v_j \in \mathrm{FB}^{\pm},\ v_{j+1} \in \mathrm{FB}^{\mp} \Rightarrow \mathcal{R}(v_j) \neq \mathcal{L}(v_{j+1}). \end{cases}$$

Thus, it is seen that the set $\bigcup_{p\leq P,\ n\leq N} \widetilde{\mathbf{\Gamma}}_{p,n}$ is a regular language over $\mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N}$. In fact, it is recognized by the deterministic, finite-state automaton $\mathbf{A}_{\leq P,\leq N}$ over $\mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N}$ defined as follows:

(i)**Set of states**: $\{\varepsilon\} \cup \mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N} \cup \{\mathbf{fail}\}$;
(ii)**Initial state**: $\{\varepsilon\}$;
(iii)**Set of accept states**: $\{\varepsilon\} \cup \mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N}$;
(iv)**Alphabet**: $\mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N}$;
(v)**Transitions**:
(v-1) $^\forall v \in \mathrm{FB}^+_{\leq P} \cup \mathrm{FB}^-_{\leq N}$, $\varepsilon \xrightarrow{v} v$;
(v-2) $^\forall u, v \in \mathrm{FB}^+_{\leq P}$,
if $\mathcal{N}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{N}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail}$;
(v-3) $^\forall u, v \in \mathrm{FB}^-_{\leq N}$,
if $\mathcal{N}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{N}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail}$;
(v-4) $^\forall u \in \mathrm{FB}^+_{\leq P}$, $^\forall v \in \mathrm{FB}^-_{\leq N}$,
if $\mathcal{R}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{R}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail}$;
(v-5) $^\forall u \in \mathrm{FB}^-_{\leq N}$, $^\forall v \in \mathrm{FB}^+_{\leq P}$,
if $\mathcal{R}(u) \neq \mathcal{L}(v)$, then $u \xrightarrow{v} v$, and if $\mathcal{R}(u) = \mathcal{L}(v)$, then $u \xrightarrow{v} \mathbf{fail}$.

§ 3. Geodesic growth series for $P_{I_2(k)}$

In this section, by considering the structure of the automaton $\mathbf{A}_{\leq P,\leq N}$ given in §2, we determine a rational function expression for the geodesic growth series of the group $P_{I_2(k)}$ with respect to the generating set A.

The *geodesic growth series* of the group $P_{I_2(k)}$ with respect to the generating set A is defined by the following formal power series:

$$\mathcal{G}_{(P_{I_2(k)},A)}(t) := \sum_{q=0}^{\infty} \widetilde{\alpha}_q t^q, \tag{3.1}$$

where for each $q \in \mathbf{N} \cup \{0\}$, we define

$$\widetilde{\alpha}_q := \sharp\{\xi \in A^* \mid |\xi| = \|\pi(\xi)\| = q\}. \tag{3.2}$$

Note that the radius of convergence of the growth series $\mathcal{G}_{(P_{I_2(k)},A)}(t)$ is greater than or equal to that of the growth series of the free group of rank k, which is equal to $\frac{1}{2k-1}$ (cf. Chapter VI of [9]). Thus, $\mathcal{G}_{(P_{I_2(k)},A)}(t)$ is a holomorphic function near the origin, $0 \in \mathbf{C}$.

For each pair (P,N), we define

$$\mathcal{G}_{P,N}(t) := \sum_{q=0}^{\infty} \sharp\{\xi \in \widetilde{\mathbf{\Gamma}}_{P,N} \mid |\xi| = q\}\ t^q.$$

Then, from the partition (2.3), we have

$$\mathcal{G}_{(P_{I_2(k)},A)}(t) = \mathcal{G}_{k,0}(t) + \mathcal{G}_{0,k}(t) + \sum_{\substack{P+N\le k \\ (P,N)\ne(k,0),(0,k)}} \mathcal{G}_{P,N}(t), \tag{3.3}$$

and from Proposition 3.7 of [13], we obtain

$$\mathcal{G}_{P,N}(t) = \mathcal{S}_{P,N}(t), \text{ if } P+N \le k-1, \tag{3.4}$$

where $\mathcal{S}_{P,N}(t)$ is the spherical growth series for the set $\mathbf{\Gamma}_{P,N}$ (see §§4 and 5 of [13] for their definitions).

In order to simplify the presentation of the growth series, for each $q \in \mathbf{N}\cup\{0\}$, we introduce the following:

$$\begin{cases} T_q := t + t^2 + \cdots + t^q, \text{ for } q \ge 1, \\ T_0 := 0. \end{cases}$$

First, let us consider the case in which $P+N \le k$ and $(P,N) \notin \{(k,0),(0,k)\}$. In this case, we have the following proposition.

Proposition 3.1. *For each P, N satisfying $P+N \le k$ and $(P,N) \notin \{(k,0),(0,k)\}$, we have*

$$\sum_{0\le p\le P,\ 0\le n\le N} \mathcal{G}_{p,n}(t) = \frac{1+T_P+T_N}{1-(k-1)(T_P+T_N)}.$$

Proof. From (3.4), if $P+N \le k-1$, the assertion is identical to that of Proposition 5.1 in [13]. Hence, we need only consider the case in which $P+N=k$ and $(P,N) \notin \{(k,0),(0,k)\}$, that is, $(P,N) \in \{(1,k-1),(2,k-2),\dots,(k-1,1)\}$. For $q \in \mathbf{N}\cup\{0\}$, we define

$$\widetilde{B}_q(P;N) := \{\xi \in \bigcup_{0\le p\le P,\ 0\le n\le N} \widetilde{\mathbf{\Gamma}}_{p,n} \mid |\xi| = q\}$$

and

$$\widetilde{\beta}_q(P;N) := \sharp \widetilde{B}_q(P;N).$$

Then, we have

$$\sum_{0\le p\le P,\ 0\le n\le N} \mathcal{G}_{p,n}(t) = \sum_{q=0}^{\infty} \widetilde{\beta}_q(P;N)\ t^q.$$

Further, note that for $q = 0$, we have

$$\widetilde{\beta}_0(P;N) = 1.$$

Then, by considering the structure of the automaton $\mathbf{A}_{\leq P, \leq N}$, we obtain the same recursive formula for $\widetilde{\beta}_q(P,N)$ as for $\beta_q(P,N)$ given in Lemma 5.2 of [13]. Moreover, we obtain the same equalities for $\widetilde{\beta}_q(P,N)$ as for $\beta_q(P,N)$ appearing in Lemma 5.3 of [13]. Therefore, we obtain the desired result. □

Next, consider the case in which $(P,N) \in \{(k,0),(0,k)\}$. Because all positive words are geodesic with respect to A (see Lemma 3.1 of [13]), the set of all positive words, i.e., $\{a_1,\dots,a_k\}^*$, is equal to $\bigcup_{0\leq p\leq k} \widetilde{\mathbf{\Gamma}}_{p,0}$. Hence, with $\sharp\{a_1,\dots,a_k\} = k$, we obtain

$$\sum_{p=0}^{k} \mathcal{G}_{p,0}(t) = \frac{1}{1-kt}. \tag{3.5}$$

Thus, from (2.3), (3.5) and Proposition 3.1, we have

$$\begin{aligned} \mathcal{G}_{k,0}(t) &= \sum_{p=0}^{k} \mathcal{G}_{p,0}(t) - \sum_{p=0}^{k-1} \mathcal{G}_{p,0}(t) \\ &= \frac{1}{1-kt} - \frac{1+T_{k-1}}{1-(k-1)T_{k-1}} \\ &= \frac{kt^k}{(1-kt)\{1-(k-1)T_{k-1}\}}. \end{aligned} \tag{3.6}$$

Then, by considering the inverses of positive words, we obtain

$$\mathcal{G}_{0,k}(t) = \mathcal{G}_{k,0}(t). \tag{3.7}$$

We are now ready to state the first main result of this paper. From (3.3), (3.6), (3.7) and Proposition 3.1, and employing the trick in Lemma 5.3 of [22], we obtain the following:

Theorem 3.2. *The geodesic growth series for the pure Artin group $P_{I_2(k)}$ of dihedral type with respect to the generating set A possesses the rational function expression*

$$\begin{aligned} \mathcal{G}_{(P_{I_2(k)},A)}(t) = &\frac{2kt^k}{(1-kt)\{1-(k-1)T_{k-1}\}} \\ &+ \sum_{p=1}^{k-1} \frac{1+T_p+T_{k-p}}{1-(k-1)(T_p+T_{k-p})} - \sum_{p=1}^{k-2} \frac{1+T_p+T_{k-1-p}}{1-(k-1)(T_p+T_{k-1-p})}. \end{aligned} \tag{3.8}$$

It is easy to verify that for each term on the right-hand side, the numerator and denominator have no common zero.

Next, we rewrite the right-hand side of (3.8) using the common denominator

$$G(t) := (1-kt) \prod_{\substack{a+b=k \\ b\geq a\geq 1}} \{1-(k-1)(T_a+T_b)\} \prod_{\substack{a+b=k-1 \\ b\geq a\geq 0}} \{1-(k-1)(T_a+T_b)\},$$

and sum the terms. Then we obtain a single fraction expression for $\mathcal{G}_{(P_{I_2(k)},A)}(t)$. Let $H(t)$ be its numerator. Then we have $\mathcal{G}_{(P_{I_2(k)},A)}(t) = \frac{H(t)}{G(t)}$. Now, we state the second main result of this paper:

Theorem 3.3. *The two polynomials $G(t)$ and $H(t)$ do not have a common zero.*

This theorem is proved in the next section.

Example 3.4.

$$\mathcal{G}_{(P_{I_2(3)},A)}(t) = \frac{(1+2t)(1-9t+28t^2-36t^3+16t^4+12t^5)}{(1-3t)(1-4t)(1-2t-2t^2)(1-4t-2t^2)},$$

$$\mathcal{G}_{(P_{I_2(4)},A)}(t) = \frac{(1-17t+87t^2-60t^3-432t^4-153t^5+2007t^6+1512t^7-297t^8-1026t^9-702t^{10}-216t^{11})}{(1-4t)(1-6t-3t^2)(1-6t-6t^2)(1-3t-3t^2-3t^3)(1-6t-3t^2-3t^3)}.$$

From Theorem 3.3, the radius of convergence of the series $\mathcal{G}_{(P_{I_2(k)},A)}(t)$ is realized as the absolute value of a zero of the polynomial $G(t)$. Hence, only from Lemma 3.1(i) and (ii) of [14], we obtain the following:

Corollary 3.5. *The geodesic growth rate $\tau_{\mathcal{G}}(k)$ for the pair $(P_{I_2(k)}, A)$ is a Pisot-Vijayaraghavan number.*

See Theorem 3.2 of [14]. Another demonstration of this corollary derived from Lemma 3.1(i) -(iv) of [14] is given there.

§ 4. Denominator of the geodesic growth series

In this section, we consider the denominators of the terms in the formula for $\mathcal{G}_{(P_{I_2(k)},A)}(t)$ given in Theorem 3.2, and through this consideration we demonstrate Theorem 3.3.

Let $k \geq 3$ be an integer. Define

$$f_i(t) := 1 - (k-1)\,(T_{i-1} + T_{k-i}) \qquad \text{for } i \in \{1, \cdots, k\},$$

and

$$\begin{aligned} g_0(t) &:= 1 - kt, \\ g_i(t) &:= 1 - (k-1)\,(T_i + T_{k-i}) \qquad \text{for } i \in \{1, \cdots, k-1\}. \end{aligned}$$

Then, the formula in Theorem 3.2 can be written as

$$\mathcal{G}_{(P_{I_2(k)},A)}(t) = \frac{2kt^k}{g_0(t) f_1(t)} + \sum_{p=1}^{k-1} \frac{1 + T_p + T_{k-p}}{g_p(t)} - \sum_{p=1}^{k-2} \frac{1 + T_p + T_{k-1-p}}{f_{p+1}(t)}. \tag{4.1}$$

Next, we prove the following lemma, from which Theorem 3.3 follows immediately.

Lemma 4.1.

1. *No two mutually different polynomials $f_i(t)$ and $f_j(t)$ have a common zero.*
2. *No two mutually different polynomials $g_i(t)$ and $g_j(t)$ have a common zero.*
3. *No two polynomials $f_i(t)$ and $g_j(t)$ have a common zero.*

Proof. First note that we have

$$\begin{aligned} &f_{\frac{k+1}{2}-j}(t) = f_{\frac{k+1}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \tfrac{k-1}{2}\}, \quad \text{if } k \text{ is odd}, \\ &f_{\frac{k}{2}+1-j}(t) = f_{\frac{k}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \tfrac{k}{2}\}, \quad \text{if } k \text{ is even}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &g_{\frac{k+1}{2}-j}(t) = g_{\frac{k-1}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \tfrac{k-1}{2}\}, \quad \text{if } k \text{ is odd}, \\ &g_{\frac{k}{2}-j}(t) = g_{\frac{k}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \tfrac{k}{2}-1\}, \quad \text{if } k \text{ is even}. \end{aligned} \tag{4.3}$$

Also, we know that

$$\begin{aligned} f_i(1) \neq 0, \quad f_i(0) \neq 0, \\ g_i(1) \neq 0, \quad g_i(0) \neq 0, \end{aligned} \tag{4.4}$$

for all $i \in \{1, \ldots, k-1\}$.

1. From (4.2), it is sufficient to consider the polynomials $f_1(t), \ldots, f_{\frac{k+1}{2}}(t)$ (resp., $f_1(t)$, $\ldots, f_{\frac{k}{2}}(t)$) if k is odd (resp., k is even).

Suppose that $f_i(t)$ and $f_j(t)$ $(1 \leq i < j)$ have a common zero ρ. Then we have

$$f_i(\rho) = f_j(\rho).$$

From this equality, we obtain

$$(\rho^i - \rho^{k-j+1})(\rho^{j-i-1} + \rho^{j-i-2} + \cdots + \rho + 1) = 0. \tag{4.5}$$

Also, from (4.4), we have $\rho \neq 0$. Hence, (4.5) implies that ρ is an algebraic integer over $\mathbf{Q}$. Next, note that from $f_i(\rho) = 0$, we also have

$$\rho + \cdots + \rho^{i-1} + \rho + \cdots + \rho^{k-i} = \frac{1}{k-1}. \tag{4.6}$$

The left-hand side of (4.6) is an algebraic integer over $\mathbf{Q}$. However, $\frac{1}{k-1}$ is not an algebraic integer for $k \geq 3$. Thus, we obtain a contradiction. Hence, $f_i(t)$ and $f_j(t)$ do not have a common zero.

2. From (4.3), it is sufficient to consider the polynomials $g_0(t), g_1(t), \ldots, g_{\frac{k-1}{2}}(t)$ (resp., $g_0(t), g_1(t), \ldots, g_{\frac{k}{2}}(t)$) if k is odd (resp., k is even).

Let $i \in \{1, \ldots, k-1\}$. Then, $g_i(t)$ and $g_0(t)$ do not have a common zero, because $g_i(\frac{1}{k}) \neq 0$. The result for $g_i(t)$ and $g_j(t)$ $(1 \leq i < j)$ is obtained by an argument similar to that given in Part 1.

3. From (4.2) and (4.3), we can assume that $i \leq \frac{k+1}{2}$ and $j \leq \frac{k-1}{2}$ (resp., $i \leq \frac{k}{2}$ and $j \leq \frac{k}{2}$) if k is odd (resp., k is even).

The fact that $f_i(\frac{1}{k}) \neq 0$ implies that no $f_i(t)$ has a common zero with $g_0(t)$. Next, suppose that $f_i(t)$ and $g_j(t)$ $(j \geq 1)$ have a common zero ρ. Then we have

$$f_i(\rho) = g_j(\rho). \tag{4.7}$$

Also, from (4.4), we know that $\rho \neq 0$. Thus, from (4.7), we obtain

$$\rho^{i-1} + \rho^{k-i} = \rho^j + \rho^{k-j}. \tag{4.8}$$

We now show that ρ is an algebraic integer over $\mathbf{Q}$. This is done by considering the following four cases.

Case 1: $i = j$**.** Here, from (4.8), we have

$$\rho^i - \rho^{i-1} = 0.$$

Case 2: $i < j$**.** Here, from (4.8), we have

$$\rho^{k-i} - \rho^{k-j} - \rho^j + \rho^{i-1} = 0.$$

If k is odd, then $j \leq \frac{k-1}{2}$. Hence, we have

$$k - i > k - j > j > i - 1.$$

If k is even, then $j \leq \frac{k}{2}$. Hence, we have

$$k - i > k - j \geq j > i - 1.$$

Case 3: $i > j$ and $j < \frac{k}{2}$. Here, from (4.8), we have

$$\rho^{k-j} + \rho^{j} - \rho^{k-i} - \rho^{i-1} = 0.$$

From $j < \frac{k}{2}$, we have

$$k - j > j.$$

Because $i \leq \frac{k+1}{2}$ (resp., $i \leq \frac{k}{2}$) if k is odd (resp., even), we have

$$k - j > k - i \geq i - 1.$$

Case 4: $i > j$ and $j = \frac{k}{2}$. Here, from (4.8), we have

$$2\rho^{j} - \rho^{i-1} - \rho^{2j-i} = 0.$$

If $j = i - 1$, we have

$$\rho^{i-1} - \rho^{i-2} = 0.$$

If $j < i - 1$, we have

$$\rho^{i-1} - 2\rho^{j} + \rho^{2j-i} = 0$$

and

$$i - 1 > j, \quad i - 1 > 2j - i.$$

Therefore, because ρ is not equal to zero, in each case, ρ is an algebraic integer over $\mathbf{Q}$. This implies a contradiction for the same reason as in Part 1. Hence, $f_i(t)$ and $g_j(t)$ do not have a common zero. $\square$

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