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The geodesic growth series for pure Artin groups of dihedral type

By

Michihiko FUJI* and Takao SATOH**

Abstract

We consider the pure Artin group of dihedral type, which is the kernel of the natural projection from the Artin group of dihedral type $I_2(k)$ to the associated Coxeter group. We present a rational function expression for the geodesic growth series of the pure Artin group of dihedral type with respect to a natural generating set, and we explicitly determine the denominator of this rational function expression. Moreover, we show that the growth rate of the series is a Pisot-Vijayaraghavan number.

§1. Introduction

For a finitely generated group $G$ with a given generating set $\Gamma$, the corresponding spherical growth series is defined as

$$S_{(G,\Gamma)}(t) := \sum_{n=0}^{\infty} \alpha_n t^n,$$

where $\alpha_n$ for $n \in \mathbb{N} \cup \{0\}$ is the number of elements in $G$ whose lengths with respect to $\Gamma$ are equal to $n$. The spherical growth series $S_{(G,\Gamma)}(t)$ is a commonly employed measure of the rate of growth of $G$ with respect to $\Gamma$, and has been explored for a number of
interesting families of pairs \((G, \Gamma)\). In particular, \(S_{(G, \Gamma)}(t)\) is known to be rational (i.e., it can be expressed as the quotient of two polynomials with integer-valued coefficients in the ring of formal power series \(\mathbb{Z}[[t]]\)) in a number of important cases (see, e.g., [3], [4], [5], [6], [7], [9], [10], [11], [12], [16], [18], [20], [22], [26], [27], [28] and [29]).

In the present paper, we are interested in the following series, defined analogously to \(S_{(G, \Gamma)}(t)\):

\[
\mathcal{G}_{(G, \Gamma)}(t) := \sum_{n=0}^{\infty} \overline{\alpha}_n t^n,
\]

where \(\overline{\alpha}_n\) for \(n \in \mathbb{N} \cup \{0\}\) is the number of geodesic words with respect to \(\Gamma\) whose lengths are equal to \(n\). Recall that a word in the free monoid \(\Gamma^*\) generated by \(\Gamma\) is geodesic if the corresponding path in the Cayley graph of \(G\) with respect to \(\Gamma\) is a minimal length edge path joining its endpoints. The series \(\mathcal{G}_{(G, \Gamma)}(t)\) is called the geodesic growth series for the pair \((G, \Gamma)\). The growth rate for \(\mathcal{G}_{(G, \Gamma)}(t)\) is defined as

\[
\tau_{\mathcal{G}} := \lim_{n \to \infty} \sup n \overline{\alpha}_n.
\]

We also call \(\tau_{\mathcal{G}}\) the geodesic growth rate for the pair \((G, \Gamma)\). By the Cauchy-Hadamard theorem, the radius of convergence \(R_{\mathcal{G}}\) of the series \(\mathcal{G}_{(G, \Gamma)}(t)\) is the reciprocal of \(\tau_{\mathcal{G}}\).

Although the geodesic growth series \(\mathcal{G}_{(G, \Gamma)}(t)\) is not as well understood as the spherical growth series \(S_{(G, \Gamma)}(t)\), there are nonetheless many known pairs \((G, \Gamma)\) for which the geodesic growth series is rational. The following are some examples: (1) any word-hyperbolic group with respect to an arbitrary generating set (see [5] and [10]); (2) any geometrically finite hyperbolic group with respect to a particular generating set (see [23]); (3) any irreducible affine Coxeter group with respect to the standard generating set (see [24]); (4) any right-angled Artin group with respect to the standard generating set (see [21] and [1]); (5) any Artin group of dihedral type, \(G_{I_2(k)}\), with respect to the standard generating set (the so-called ‘set of Artin generators’) (see [26] and [22]); (6) any Artin group of large type with respect to the standard generating set (see [17]); (7) any Garside group with respect to a particular generating set (the so-called ‘set of Garside generators’) (see [8]). For each of the above examples, it has been shown that the set of all geodesic words of the group \(G\) with respect to the generating set \(\Gamma\) is a regular language over \(\Gamma\), which implies the rationality of the geodesic growth series \(\mathcal{G}_{(G, \Gamma)}(t)\). The pair \((G, \Gamma)\) is said to be strongly geodesic regular or to form a Cannon pair if the set of all geodesic words of \(G\) with respect to \(\Gamma\) forms a regular language over \(\Gamma\). In general, the regularity of a language consisting of geodesic words depends on the generating set \(\Gamma\) (see the example due to Cannon discussed in §4 of [23]).

In this paper, for each integer \(k \geq 3\), we consider the pure Artin group \(P_{I_2(k)}\), which is the kernel of the projection from the Artin group of dihedral type, \(G_{I_2(k)}\), to the associated Coxeter group, \(\overline{G}_{I_2(k)}\). The group \(P_{I_2(k)}\) is geometrically realized as
the fundamental group of the complement of a torus link in the 3-dimensional sphere, which has a natural generating set $A$ (cf. [25]). In particular, in the case $k = 3$, $P_{I_2(3)}$ is the pure braid group with three strands, and $A$ is the standard generating set (cf. [2]). In [13], for any element $g$ of $P_{I_2(k)}$, a particular geodesic representative of $g$ is determined. Then, through analysis of the regularity of the language consisting of such geodesic representatives, a rational function expression for the spherical growth series $S_{(P_{I_2(k)}, A)}(t)$ of $P_{I_2(k)}$ is derived with respect to the generating set $A$. Moreover, in [13], all the geodesic representatives of any element of $P_{I_2(k)}$ are determined. From this, it is seen that all geodesic words of a particular type (Type 3 defined in [13]) form a regular language. In the present paper, by using arguments similar to those given in [13], we derive a rational function expression for the geodesic growth series $G_{(P_{I_2(k)}, A)}(t)$ of $P_{I_2(k)}$ with respect to the generating set $A$ (see Theorem 3.2). Moreover, by using an algebraic argument, we explicitly determine the denominator of this rational function expression (see Theorem 3.3). From this theorem, we can detect a number-theoretic property concerning the geodesic growth rate $\tau_G(k)$ for the pair $(P_{I_2(k)}, A)$: The geodesic growth rate $\tau_G(k)$ is a Pisot-Vijayaraghavan number, i.e., a real algebraic integer $\tau > 1$ whose algebraic conjugates other than $\tau$ itself lie in the unit disk (see Corollary 3.5).

§2. Geodesic words of pure Artin groups of dihedral type

In this section, we present definitions and basic facts concerning pure Artin groups of dihedral type (see [13] and [15] for details). We inherit all of the notation used in [13].

Let $k$ be an integer greater than 2, $G_{I_2(k)}$ be the Artin group of dihedral type $I_2(k)$ and $\overline{G}_{I_2(k)}$ be the Coxeter group of dihedral type $I_2(k)$. Then there is a natural surjective homomorphism

$$p : G_{I_2(k)} \rightarrow \overline{G}_{I_2(k)}.$$ 

We call the kernel of $p$ the pure Artin group of dihedral type and write it $P_{I_2(k)}$. The group $P_{I_2(k)}$ has the following presentation:

$$P_{I_2(k)} = \langle a_1, \ldots, a_k | a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1} \rangle.$$ 

In this paper, we consider the generating set $A = \{a_1, \ldots, a_k, a_1^{-1}, \ldots, a_k^{-1}\}$, as in [13], and investigate the so-called ‘geodesic growth series’ of $P_{I_2(k)}$ with respect to $A$, whose definition is given in §3.
Let $A^*$ and $\{a_1, \ldots, a_k\}^*$ denote the free monoids generated by $A$ and $\{a_1, \ldots, a_k\}$, respectively. We refer to the finite set $A$ as an \textit{alphabet}, its elements as \textit{letters}, and the elements of $A^*$ (resp., $\{a_1, \ldots, a_k\}^*$) as \textit{words} (resp., \textit{positive words}). Let $\epsilon$ denote the \textit{null word}. A subset $L$ of $A^*$ is called a \textit{language} over $A$. A language $L$ is \textit{regular} if $L$ is recognized by some deterministic, finite-state automaton over $A$ (see [10] or [19] for the definition of automata). The length of a word $w$ is the number of letters it contains, which is denoted by $|w|$. The length of $\epsilon$ is zero. Since $A$ generates the group $P_{I_2(k)}$, there exists a natural surjective monoid homomorphism $\pi: A^* \rightarrow P_{I_2(k)}$. If $u$ and $v$ are words, then $u = v$ means that $\pi(u) = \pi(v)$ and $u \equiv v$ means that $u$ and $v$ are identical letter by letter. A word $w \in \pi^{-1}(g)$ is called a \textit{representative} of $g$. The length of a group element $g$ is regarded as the quantity $\|g\| = \min\{|w| \mid w \in \pi^{-1}(g)\}$.

A word $w \in A^*$ for which the relation $|w| = \|\pi(w)\|$ holds is termed \textit{geodesic}. A word $w_1 \cdots w_m \in A^*$ is called a \textit{reduced word} if $w_i \neq w_{i+1}^{-1}$ for all $i \in \{1, \ldots, m-1\}$. A geodesic representative is a reduced word. Below, we consider some other alphabet, $T$, which is a subset of $A^*$ (denoted by $FB_{\leq}^{+} \cup FB_{\overline{\leq}N}$), and we investigate the regularity of a language over $T$.

Let us first recall the notation and definitions concerning 'fundamental blocks' given in [13]. We begin by introducing the word $\nabla \equiv a_1 \cdots a_k$.

It satisfies the following relation:

\begin{equation}
\nabla \equiv a_1 \cdots a_k = a_2 \cdots a_k a_1 = a_3 \cdots a_k a_1 a_2 = \cdots = a_k a_1 \cdots a_{k-1}.
\end{equation}

A \textit{fundamental block} is a word with length smaller than $k$ that appears as a subword in the terms of (2.1). There are $k(k-1)$ fundamental blocks. All of them are listed in [13]. Let $FB^{+}(\subset A^{*})$ (resp., $FB^{-}(\subset A^{*})$) denote the set consisting of all the fundamental blocks (resp., all the inverses of fundamental blocks); for $I \in \{0, \ldots, k-1\}$, $FB_{I}^{\pm}$ (resp., $FB_{\leq I}^{\pm}$) denotes the set consisting of all the elements of $FB^{\pm}$ with length equal to $I$ (resp., smaller than or equal to $I$); for $\mu \equiv a_i \cdots a_k a_1 \cdots a_j \in FB^{+}$ (resp., $\mu^{-1} \equiv a_j^{-1} \cdots a_1^{-1} a_k^{-1} \cdots a_i^{-1} \in FB^{-}$), we define $L(\mu) := a_i$ and $R(\mu) := a_j$ (resp., $L(\mu^{-1}) := a_j$ and $R(\mu^{-1}) := a_i$). For $\mu \equiv a_i \cdots a_k a_1 \cdots a_j \in FB^{+}$, we call $a_{j+1}$ the \textit{letter subsequent to} $\mu$. When $\mu \equiv a_i \cdots a_k$, we call $a_1$ the letter subsequent to $\mu$. The letter subsequent to $\mu$ is denoted by $N(\mu)$. For $\mu^{-1} \equiv a_j^{-1} \cdots a_1^{-1} a_k^{-1} \cdots a_i^{-1} \in FB^{-}$, we call $a_{i-1}$ the letter subsequent to $\mu^{-1}$, which is denoted by $N(\mu^{-1})$. When $a_i^{-1} \equiv a_1^{-1}$, we call $a_k$ the letter subsequent to $\mu^{-1}$.
In [13], we introduced the set
\[ \overline{\Gamma} := \{ \xi \in A^* \mid |\xi| = \|\pi(\xi)\| \}, \]
i.e., the set consisting of all of the geodesic words, and for each \((P, N)\) satisfying \(0 \leq P \leq k\) and \(0 \leq N \leq k\), the sets
\[
\overline{\Gamma}_{P,N} := \{ \xi \in WT \mid |\xi| = \|\pi(\xi)\|, (\text{Pos}(\xi), \text{Neg}(\xi)) = (P, N) \},
\]
\[ G_{P,N} := \{ g \in P_{I_2(k)} \mid (\text{Pos}(g), \text{Neg}(g)) = (P, N) \}, \]
where \(WT_i = \{ w \in A^* \mid w \text{ is a word of Type } i \} \) for each \(i \in \{1, 2, 3\}\) (see §3 in [13] for the definition of Type \(i\)), \(WT = WT_1 \cup WT_2 \cup WT_3\), and \(\text{Pos}(\xi), \text{Neg}(\xi), \text{Pos}(g)\) and \(\text{Neg}(g)\) are specific integers between 0 and \(k\) (defined in §§2 and 3 of [13]).

By Propositions 3.3, 3.4 and 3.7 and Corollary 3.8 of [13], we have
\[
\begin{align*}
\{ P + N \geq k + 1 \Rightarrow \overline{\Gamma}_{P,N} = \emptyset, & \quad G_{P,N} = \emptyset, \\
P + N \leq k \Rightarrow \pi^{-1}(G_{P,N}) \cap \overline{\Gamma} = \overline{\Gamma}_{P,N},\end{align*}
\]
and
\[
\begin{align*}
\overline{\Gamma} &= \bigcup_{P+N\leq k} \overline{\Gamma}_{P,N} \quad \text{(disjoint union),} \\
P_{I_2(k)} &= \bigcup_{P+N\leq k} G_{P,N} \quad \text{(disjoint union),}
\end{align*}
\]
and
\[
\bigcup_{P+N\leq k, (P,N) \neq (k,0), (0,k)} \overline{\Gamma}_{P,N} = \{ \xi \in WT_3 \mid \text{Pos}(\xi) + \text{Neg}(\xi) \leq k \}.
\]
We remark that an element \(g \in P_{I_2(k)}\) has more than one geodesic representative if and only if \(\text{Pos}(g) + \text{Neg}(g) = k\).

Now, choose any \(P, N \in \mathbb{N} \cup \{0\}\) satisfying the conditions \(P + N \leq k\) and \((P, N) \notin \{(k,0), (0,k)\}\), and fix them. Then, any element \(w \in \bigcup_{p \leq P, n \leq N} \tilde{\Gamma}_{p,n}\) can be expressed as
\[
w \equiv v_1 \cdots v_m \in (FB_{\leq P}^+ \cup FB_{\leq N}^-)^*,
\]
where \((FB_{\leq P}^+ \cup FB_{\leq N}^-)^*\) is the free monoid generated by the finite set \(FB_{\leq P}^+ \cup FB_{\leq N}^-\), and for each \(j \in \{1, \ldots, m - 1\}\), we have
\[
\begin{align*}
\{ v_j, v_{j+1} \in FB^\pm \Rightarrow \mathcal{N}(v_j) \neq \mathcal{L}(v_{j+1}), \\
v_j \in FB^\pm, v_{j+1} \in FB^\pm \Rightarrow \mathcal{R}(v_j) \neq \mathcal{L}(v_{j+1}).\}
\end{align*}
\]
Thus, it is seen that the set \( \bigcup_{p \leq P, \, n \leq N} \overline{\Gamma}_{p, n} \) is a regular language over \( \mathbb{F}B_{\leq}^{+} \cup \mathbb{F}B_{\overline{\leq}N} \). In fact, it is recognized by the deterministic, finite-state automaton \( A_{\leq P, \leq N} \) over \( \mathbb{F}B_{\leq}^{+} \cup \mathbb{F}B_{\overline{\leq}N} \) defined as follows:

(i) Set of states: \( \{\varepsilon\} \cup \mathbb{F}B_{\leq}^{+} \cup \mathbb{F}B_{\overline{\leq}N} \cup \{\text{fail}\}; \)
(ii) Initial state: \( \{\varepsilon\} \);
(iii) Set of accept states: \( \{\varepsilon\} \cup \mathbb{F}B_{\leq}^{+} \cup \mathbb{F}B_{\overline{\leq}N} \);
(iv) Alphabet: \( \mathbb{F}B_{\leq}^{+} \cup \mathbb{F}B_{\overline{\leq}N} \);
(v) Transitions:

\( (v.1) \forall v \in \mathbb{F}B_{\leq}^{+} \cup \mathbb{F}B_{\overline{\leq}N}, \varepsilon \rightarrow^{v} v; \)
\( (v.2) \forall u, v \in \mathbb{F}B_{\leq}^{+}, \varepsilon \rightarrow^{u} v, \text{ if } \mathcal{N}(u) \neq \mathcal{L}(v), \text{ then } u \rightarrow^{v} v, \text{ and if } \mathcal{N}(u) = \mathcal{L}(v), \text{ then } u \rightarrow^{v} \text{ fail}; \)
\( (v.3) \forall u, v \in \mathbb{F}B_{\overline{\leq}N}, \varepsilon \rightarrow^{u} v, \text{ if } \mathcal{N}(u) \neq \mathcal{L}(v), \text{ then } u \rightarrow^{v} v, \text{ and if } \mathcal{N}(u) = \mathcal{L}(v), \text{ then } u \rightarrow^{v} \text{ fail}; \)
\( (v.4) \forall u \in \mathbb{F}B_{\leq}^{+}, \forall v \in \mathbb{F}B_{\overline{\leq}N}, \varepsilon \rightarrow^{u} v, \text{ if } \mathcal{R}(u) \neq \mathcal{L}(v), \text{ then } u \rightarrow^{v} v, \text{ and if } \mathcal{R}(u) = \mathcal{L}(v), \text{ then } u \rightarrow^{v} \text{ fail}; \)
\( (v.5) \forall u \in \mathbb{F}B_{\overline{\leq}N}, \forall v \in \mathbb{F}B_{\leq}^{+}, \varepsilon \rightarrow^{u} v, \text{ if } \mathcal{R}(u) \neq \mathcal{L}(v), \text{ then } u \rightarrow^{v} v, \text{ and if } \mathcal{R}(u) = \mathcal{L}(v), \text{ then } u \rightarrow^{v} \text{ fail}. \)

\[ \frac{\mathbb{F}B_{\leq}^{+} \cup \mathbb{F}B_{\overline{\leq}N}}{\mathbb{F}B_{\overline{\leq}N} \cup \mathbb{F}B_{\overline{\leq}N} \cup \{\text{fail}\}} \]

§ 3. Geodesic growth series for \( P_{I_2(k)} \)

In this section, by considering the structure of the automaton \( A_{\leq P, \leq N} \) given in §2, we determine a rational function expression for the geodesic growth series of the group \( P_{I_2(k)} \) with respect to the generating set \( A \).

The geodesic growth series of the group \( P_{I_2(k)} \) with respect to the generating set \( A \) is defined by the following formal power series:

\[ G_{(P_{I_2(k)}, A)}(t) := \sum_{q=0}^{\infty} \overline{\alpha}_q t^q, \]

where for each \( q \in \mathbb{N} \cup \{0\} \), we define

\[ \overline{\alpha}_q := \{ \xi \in A^* \mid |\xi| = \|\pi(\xi)\| = q \}. \]

Note that the radius of convergence of the growth series \( G_{(P_{I_2(k)}, A)}(t) \) is greater than or equal to that of the growth series of the free group of rank \( k \), which is equal to \( \frac{1}{2k-1} \) (cf. Chapter VI of [9]). Thus, \( G_{(P_{I_2(k)}, A)}(t) \) is a holomorphic function near the origin, \( 0 \in \mathbb{C} \).
For each pair \((P, N)\), we define
\[
G_{P,N}(t) := \sum_{q=0}^{\infty} \sharp \{\xi \in \overline{\Gamma}_{P,N} \mid |\xi| = q\} \ t^q.
\]

Then, from the partition (2.3), we have
\[
(3.3) \quad G_{(P_{I_{2}(k)},A)}(t) = G_{k,0}(t) + G_{0,k}(t) + \sum_{P+N \neq k,0,k} G_{P,N}(t),
\]
and from Proposition 3.7 of [13], we obtain
\[
(3.4) \quad G_{P,N}(t) = S_{P,N}(t), \text{ if } P+N \leq k-1,
\]
where \(S_{P,N}(t)\) is the spherical growth series for the set \(\Gamma_{P,N}\) (see §§4 and 5 of [13] for their definitions).

In order to simplify the presentation of the growth series, for each \(q \in \mathbb{N} \cup \{0\}\), we introduce the following:
\[
T_q := t + t^2 + \cdots + t^q, \text{ for } q \geq 1, \\
T_0 := 0.
\]

First, let us consider the case in which \(P+N \leq k\) and \((P, N) \notin \{(k,0), (0,k)\}\). In this case, we have the following proposition.

**Proposition 3.1.** For each \(P, N\) satisfying \(P+N \leq k\) and \((P, N) \notin \{(k,0), (0,k)\}\), we have
\[
\sum_{0 \leq p \leq P, 0 \leq n \leq N} G_{p,n}(t) = \frac{1 + T_P + T_N}{1 - (k-1)(T_P + T_N)}.
\]

**Proof.** From (3.4), if \(P+N \leq k-1\), the assertion is identical to that of Proposition 5.1 in [13]. Hence, we need only consider the case in which \(P+N = k\) and \((P, N) \notin \{(k,0), (0,k)\}\), that is, \((P, N) \in \{(1, k-1), (2, k-2), \ldots, (k-1,1)\}\). For \(q \in \mathbb{N} \cup \{0\}\), we define
\[
\overline{B}_q(P;N) := \{\xi \in \bigcup_{0 \leq p \leq P, 0 \leq n \leq N} \overline{\Gamma}_{p,n} \mid |\xi| = q\}
\]
and
\[
\overline{\beta}_q(P;N) := \# \overline{B}_q(P;N).
\]
Then, we have
\[
\sum_{0 \leq p \leq P, 0 \leq n \leq N} G_{p,n}(t) = \sum_{q=0}^{\infty} \overline{\beta}_q(P;N) \ t^q.
\]
Further, note that for \( q = 0 \), we have
\[
\overline{\beta}_0(P; N) = 1.
\]
Then, by considering the structure of the automaton \( A_{\leq P, \leq N} \), we obtain the same recursive formula for \( \overline{\beta}_q(P, N) \) as for \( \beta_q(P, N) \) given in Lemma 5.2 of [13]. Moreover, we obtain the same equalities for \( \overline{\beta}_q(P, N) \) as for \( \beta_q(P, N) \) appearing in Lemma 5.3 of [13]. Therefore, we obtain the desired result.

Next, consider the case in which \( (P, N) \in \{(k, 0), (0, k)\} \). Because all positive words are geodesic with respect to \( A \) (see Lemma 3.1 of [13]), the set of all positive words, i.e., \( \{a_1, \ldots, a_k\}^* \), is equal to \( \bigcup_{0 \leq p \leq k} \overline{\Gamma}_{p,0} \). Hence, with \( \# \{a_1, \ldots, a_k\} = k \), we obtain

\[
\sum_{p=0}^{k} \mathcal{G}_{p,0}(t) = \frac{1}{1-kt}.
\]
Thus, from (2.3), (3.5) and Proposition 3.1, we have

\[
\mathcal{G}_{k,0}(t) = \sum_{p=0}^{k} \mathcal{G}_{p,0}(t) - \sum_{p=0}^{k-1} \mathcal{G}_{p,0}(t)
= \frac{1}{1-kt} - \frac{1}{1-(k-1)T_{k-1}}
= \frac{kt^k}{(1-kt)(1-(k-1)T_{k-1})}.
\]
Then, by considering the inverses of positive words, we obtain

\[
\mathcal{G}_{0,k}(t) = \mathcal{G}_{k,0}(t).
\]

We are now ready to state the first main result of this paper. From (3.3), (3.6), (3.7) and Proposition 3.1, and employing the trick in Lemma 5.3 of [22], we obtain the following:

**Theorem 3.2.** The geodesic growth series for the pure Artin group \( P_{I_{2}(k)} \) of dihedral type with respect to the generating set \( A \) possesses the rational function expression

\[
\mathcal{G}_{(P_{I_{2}(k)}, A)}(t) = \frac{2kt^k}{(1-kt)(1-(k-1)T_{k-1})}
+ \sum_{p=1}^{k-1} \frac{1 + T_{p} + T_{k-p}}{1 - (k-1)(T_{p} + T_{k-p})} - \sum_{p=1}^{k-2} \frac{1 + T_{p} + T_{k-1-p}}{1 - (k-1)(T_{p} + T_{k-1-p})}.
\]
It is easy to verify that for each term on the right-hand side, the numerator and denominator have no common zero.

Next, we rewrite the right-hand side of (3.8) using the common denominator

\[
G(t) := (1 - kt) \prod_{\alpha + b = k \atop k \geq 1} \{1 - (k - 1)(T_a + T_b)\} \prod_{\alpha + b = k - 1 \atop b \geq 2} \{1 - (k - 1)(T_a + T_b)\},
\]

and sum the terms. Then we obtain a single fraction expression for \( \mathcal{G}_{(P_{I_2(k)}, A)}(t) \). Let \( H(t) \) be its numerator. Then we have \( \mathcal{G}_{(P_{I_2(k)}, A)}(t) = \frac{H(t)}{G(t)} \). Now, we state the second main result of this paper:

**Theorem 3.3.** The two polynomials \( G(t) \) and \( H(t) \) do not have a common zero.

This theorem is proved in the next section.

**Example 3.4.**

\[
\mathcal{G}_{(P_{I_2(3)}, A)}(t) = \frac{(1+2t)(1-9t+28t^2-36t^3+16t^4+12t^5)}{(1-3t)(1-4t)(1-2t-2t^2)(1-4t-2t^2)},
\]

\[
\mathcal{G}_{(P_{I_2(4)}, A)}(t) = \frac{(1-17t+87t^2-60t^3-432t^4-153t^5+2007t^6+1512t^7-297t^8-1026t^9-702t^{10}-216t^{11})}{(1-4t)(1-6t-3t^2)(1-6t-6t^2)(1-3t-3t^2-3t^3)(1-6t-3t^2-3t^3)}.
\]

From Theorem 3.3, the radius of convergence of the series \( \mathcal{G}_{(P_{I_2(k)}, A)}(t) \) is realized as the absolute value of a zero of the polynomial \( G(t) \). Hence, only from Lemma 3.1(i) and (ii) of [14], we obtain the following:

**Corollary 3.5.** The geodesic growth rate \( \tau_{\mathcal{G}}(k) \) for the pair \( (P_{I_2(k)}, A) \) is a Pisot-Vijayaraghavan number.

See Theorem 3.2 of [14]. Another demonstration of this corollary derived from Lemma 3.1(i)-(iv) of [14] is given there.

§ 4. Denominator of the geodesic growth series

In this section, we consider the denominators of the terms in the formula for \( \mathcal{G}_{(P_{I_2(k)}, A)}(t) \) given in Theorem 3.2, and through this consideration we demonstrate Theorem 3.3.
Let \( k \geq 3 \) be an integer. Define
\[
f_i(t) := 1 - (k-1)(T_{i-1} + T_{k-i}) \quad \text{for } i \in \{1, \cdots, k\},
\]
and
\[
g_0(t) := 1 - kt,
g_i(t) := 1 - (k-1)(T_i + T_{k-i}) \quad \text{for } i \in \{1, \cdots, k-1\}.
\]

Then, the formula in Theorem 3.2 can be written as
\[
\mathcal{G}_{(P_{I_2(k)},A)}(t) = \frac{2kt^k}{g_0(t)f_1(t)} + \sum_{p=1}^{k-1} \frac{1 + T_p + T_{k-p}}{g_p(t)} - \sum_{p=1}^{k-2} \frac{1 + T_p + T_{k-1-p}}{f_{p+1}(t)}.
\]

Next, we prove the following lemma, from which Theorem 3.3 follows immediately.

**Lemma 4.1.**

1. No two mutually different polynomials \( f_i(t) \) and \( f_j(t) \) have a common zero.
2. No two mutually different polynomials \( g_i(t) \) and \( g_j(t) \) have a common zero.
3. No two polynomials \( f_i(t) \) and \( g_j(t) \) have a common zero.

Proof. First note that we have
\[
f_{\frac{k+1}{2}-j}(t) = f_{\frac{k+1}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \frac{k-1}{2}\}, \quad \text{if } k \text{ is odd},
\]
\[
f_{\frac{k}{2}+1-j}(t) = f_{\frac{k}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \frac{k}{2}\}, \quad \text{if } k \text{ is even},
\]
and
\[
g_{\frac{k+1}{2}-j}(t) = g_{\frac{k-1}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \frac{k-1}{2}\}, \quad \text{if } k \text{ is odd},
\]
\[
g_{\frac{k}{2}-j}(t) = g_{\frac{k}{2}+j}(t) \quad \text{for } j \in \{1, \cdots, \frac{k}{2}-1\}, \quad \text{if } k \text{ is even}.
\]

Also, we know that
\[
f_i(1) \neq 0, \quad f_i(0) \neq 0, \quad g_i(1) \neq 0, \quad g_i(0) \neq 0,
\]
for all \( i \in \{1, \ldots, k-1\} \).

1. From (4.2), it is sufficient to consider the polynomials \( f_1(t), \ldots, f_{\frac{k+1}{2}}(t) \) (resp., \( f_1(t), \ldots, f_{\frac{k}{2}}(t) \)) if \( k \) is odd (resp., \( k \) is even).

Suppose that \( f_i(t) \) and \( f_j(t) \) (\( 1 \leq i < j \)) have a common zero \( \rho \). Then we have
\[
f_i(\rho) = f_j(\rho).
\]
From this equality, we obtain

\[(4.5) \quad (\rho^i - \rho^{k-i+1})(\rho^{j-i-1} + \rho^{j-i-2} + \cdots + \rho + 1) = 0.\]

Also, from (4.4), we have \(\rho \neq 0\). Hence, (4.5) implies that \(\rho\) is an algebraic integer over \(\mathbb{Q}\). Next, note that from \(f_i(\rho) = 0\), we also have

\[(4.6) \quad \rho + \cdots + \rho^{i-1} + \rho + \cdots + \rho^{k-i} = \frac{1}{k-1}.\]

The left-hand side of (4.6) is an algebraic integer over \(\mathbb{Q}\). However, \(\frac{1}{k-1}\) is not an algebraic integer for \(k \geq 3\). Thus, we obtain a contradiction. Hence, \(f_i(t)\) and \(f_j(t)\) do not have a common zero.

2. From (4.3), it is sufficient to consider the polynomials \(g_0(t), g_1(t), \ldots, g_{\frac{k-1}{2}}(t)\) (resp., \(g_0(t), g_1(t), \ldots, g_{\frac{k}{2}}(t)\)) if \(k\) is odd (resp., \(k\) is even).

Let \(i \in \{1, \ldots, k-1\}\). Then, \(g_i(t)\) and \(g_0(t)\) do not have a common zero, because \(g_i(\frac{1}{k}) \neq 0\). The result for \(g_i(t)\) and \(g_j(t)\) \((1 \leq i < j)\) is obtained by an argument similar to that given in Part 1.

3. From (4.2) and (4.3), we can assume that \(i \leq \frac{k+1}{2}\) and \(j \leq \frac{k-1}{2}\) (resp., \(i \leq \frac{k}{2}\) and \(j \leq \frac{k}{2}\)) if \(k\) is odd (resp., \(k\) is even).

The fact that \(f_i(\frac{1}{k}) \neq 0\) implies that no \(f_i(t)\) has a common zero with \(g_0(t)\). Next, suppose that \(f_i(t)\) and \(g_j(t)\) \((j \geq 1)\) have a common zero \(\rho\). Then we have

\[(4.7) \quad f_i(\rho) = g_j(\rho).\]

Also, from (4.4), we know that \(\rho \neq 0\). Thus, from (4.7), we obtain

\[(4.8) \quad \rho^{i-1} + \rho^{k-i} = \rho^j + \rho^{k-j}.\]

We now show that \(\rho\) is an algebraic integer over \(\mathbb{Q}\). This is done by considering the following four cases.

**Case 1:** \(i = j\). Here, from (4.8), we have

\[\rho^i - \rho^{i-1} = 0.\]

**Case 2:** \(i < j\). Here, from (4.8), we have

\[\rho^{k-i} - \rho^{k-j} - \rho^j + \rho^{i-1} = 0.\]

If \(k\) is odd, then \(j \leq \frac{k-1}{2}\). Hence, we have

\[k - i > k - j > j > i - 1.\]
If \( k \) is even, then \( j \leq \frac{k}{2} \). Hence, we have
\[
k - i > k - j \geq j > i - 1.
\]

**Case 3:** \( i > j \) and \( j < \frac{k}{2} \). Here, from (4.8), we have
\[
\rho^{k-j} + \rho^j - \rho^{k-i} - \rho^{i-1} = 0.
\]
From \( j < \frac{k}{2} \), we have
\[
k - j > j.
\]
Because \( i \leq \frac{k+1}{2} \) (resp., \( i \leq \frac{k}{2} \)) if \( k \) is odd (resp., even), we have
\[
k - j > k - i \geq i - 1.
\]

**Case 4:** \( i > j \) and \( j = \frac{k}{2} \). Here, from (4.8), we have
\[
2\rho^j - \rho^{i-1} - \rho^{2j-i} = 0.
\]
If \( j = i - 1 \), we have
\[
\rho^{i-1} - \rho^{i-2} = 0.
\]
If \( j < i - 1 \), we have
\[
\rho^{i-1} - 2\rho^j + \rho^{2j-i} = 0
\]
and
\[
i - 1 > j, \quad i - 1 > 2j - i.
\]

Therefore, because \( \rho \) is not equal to zero, in each case, \( \rho \) is an algebraic integer over \( \mathbb{Q} \). This implies a contradiction for the same reason as in Part 1. Hence, \( f_i(t) \) and \( g_j(t) \) do not have a common zero. \( \square \)

**References**

Growth series of pure Artin groups


