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Central extensions of groups and adjoint groups of quandles

By

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Abstract

This paper develops an approach for describing centrally extended groups, as determining the adjoint groups associated with quandles. Furthermore, we explicitly describe such groups of some quandles. As a corollary, we determine some second quandle homology groups.

§ 1. Introduction

A quandle is a set with a binary operation $\lhd : X^2 \to X$ such that

1. The identity $a \lhd a = a$ holds for any $a \in X$.
2. The map $(\bullet \lhd a) : X \to X$ defined by $x \mapsto x \lhd a$ is bijective for any $a \in X$.
3. The identity $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ holds for any $a, b, c \in X$.

The axioms are partially motivated by knot theory and braidings. For example, any group is a quandle by the operation $a \lhd b := b^{-1}ab$; see §4 for other examples. Conversely, for a quandle X, we can define *the adjoint group* as the following group presentation:

$$\mathrm{As}(X) = \langle \ e_x \ \ (x \in X) \ \ | \ \ e_{x \lhd y}^{-1} \cdot e_y^{-1} \cdot e_x \cdot e_y \ \ (x, y \in X) \ \ \rangle.$$

It is known that the correspondence $X \mapsto \mathrm{As}(X)$ yields a functor from the category of quandles to that of groups with left-adjointness.

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Let us briefly explain some significance to determine $\mathrm{As}(X)$. First, when the adjoint functor $F : \mathcal{C} \rightleftharpoons \mathcal{C}' : G$ is given, the difference between $\mathrm{id}_{\mathcal{C}}$ and $G \circ F$ is important in some areas (see [Ger, N3] for the quantum representations, or see [AG, CJKLS, FRS, Kab, Joy, N1] and references therein for knot-invariants and pointed Hopf algebras), with a relation to centrally extended groups. Furthermore, as a result of Eisermann [Eis2] (see also Theorem 4.1), the second quandle homology $H_2^Q(X)$ can be computed from concrete expressions of $\mathrm{As}(X)$. Moreover, the main theorem in [N2] shows that certain universal knot-invariants from quandles turn out to be characterized by the third group homology $H_3^{\mathrm{gr}}(\mathrm{As}(X))$. However, as we see the definition of $\mathrm{As}(X)$ or some explicit computation [Cla2], it has been considered to be hard to deal with $\mathrm{As}(X)$ concretely.

In this paper, we develop a method for formulating practically $\mathrm{As}(X)$ in a purely algebraic way. This method is roughly summarized to 'universal central extensions of groups modulo type-torsion' (see §2–3); the main theorem 2.1 emphasizes importance of the concept of types. Furthermore, Section 4 demonstrates practical applications of the method. Actually, we succeed in determining $\mathrm{As}(X)$ and the associated homology $H_2^Q(X)$ of some quandles X (up to torsion). As a special case, Subsection 4.5 compares the main theorem with Howlett's theorem [How] concerning the Schur multipliers of Coxeter groups. Furthermore, in Section 5, we will see that the method is applicable to coverings in quandle theory.

Notation and convention. For a group G, we denote by $H_n^{\mathrm{gr}}(G)$ the usual group homology in trivial integral coefficients. Moreover, a homomorphism $f : A \to B$ between abelian groups is said to be *a* $[1/N]$*-isomorphism* and is denoted by $f : A \cong_{[1/N]} B$, if the localization of f at ℓ is an isomorphism for any prime ℓ that does not divide N. This paper does not need any basic knowledge in quandle theory, but assumes basic facts of group cohomology as in [Bro, Sections I, II and VII].

§ 2. Preliminaries and the main theorem

This section aims to state Theorem 2.1. We start by reviewing properties of quandles. A quandle X is said to be *of type* t_X, if $t_X > 0$ is the minimal N such that $x = x \lhd^N y$ for any $x, y \in X$, where we denote by $\bullet \lhd^N y$ the N-times on the right operation with y. Note that, if X is of finite order, it is of type t_X for some $t_X \in \mathbb{Z}$.

Next, let us study the adjoint group $\mathrm{As}(X)$ in some details. Define a right action $\mathrm{As}(X)$ on X by $x \cdot e_y := x \lhd y$ for $x, y \in X$. Notice the equality

$$e_{x \cdot g} = g^{-1} e_x g \in \mathrm{As}(X) \qquad (x \in X, \quad g \in \mathrm{As}(X)), \tag{2.1}$$

by definitions. The orbits of this action of $\mathrm{As}(X)$ on X are called *connected components of* X, denoted by $\mathrm{O}(X)$. For $i \in \mathrm{O}(X)$, we let $X_i \subset X$ be the orbit with respect to

i. If the action is transitive (i.e., $\mathrm{O}(X)$ is a singleton), X is said to be *connected*. Furthermore, with respect to $i \in \mathrm{O}(X)$, define a homomorphism

$$\varepsilon_i : \mathrm{As}(X) \longrightarrow \mathbb{Z} \quad \text{by} \begin{cases} \varepsilon_i(e_x) = 1 \in \mathbb{Z}, & \text{if } x \in X_i, \\ \varepsilon_i(e_x) = 0 \in \mathbb{Z}, & \text{if } x \in X \setminus X_i. \end{cases} \tag{2.2}$$

Note that the direct sum $\oplus_{i \in \mathrm{O}(X)} \varepsilon_i$ yields the abelianization $\mathrm{As}(X)_{\mathrm{ab}} \cong \mathbb{Z}^{\oplus \mathrm{O}(X)}$ by (2.1), which means that the group $\mathrm{As}(X)$ is of infinite order. Furthermore, if $O(X)$ is a singleton, we often omit writing the index i.

In addition, we briefly review *the inner automorphism group*, $\mathrm{Inn}(X)$, of a quandle X. Regard the action of $\mathrm{As}(X)$ as a group homomorphism ψ_X from $\mathrm{As}(X)$ to the symmetric group $\mathrm{Bij}(X, X)$. The group $\mathrm{Inn}(X)$ is defined as the image $\mathrm{Im}(\psi_X) \subset \mathrm{Bij}(X, X)$. Hence, we have a group extension

$$0 \longrightarrow \mathrm{Ker}(\psi_X) \longrightarrow \mathrm{As}(X) \xrightarrow{\psi_X} \mathrm{Inn}(X) \longrightarrow 0 \qquad \text{(exact)}. \tag{2.3}$$

By the equality (2.1), this kernel $\mathrm{Ker}(\psi_X)$ is contained in the center. Therefore, it is natural to focus on their second group homology; we show a theorem on $H_2^{\mathrm{gr}}(\mathrm{As}(X))$ as a useful estimate:

Theorem 2.1. *For any connected quandle X of type t_X (possibly, X could be of infinite order), the second group homology $H_2^{\mathrm{gr}}(\mathrm{As}(X))$ is annihilated by t_X. Furthermore, the abelian kernel* $\mathrm{Ker}(\psi_X)$ *in* (2.3) *is* $[1/t_X]$*-isomorphic to* $\mathbb{Z} \oplus H_2^{\mathrm{gr}}(\mathrm{Inn}(X))$.

The proof will appear in §6. In conclusion, metaphorically speaking, $\mathrm{As}(X)$ turns out to be the 'universal central extension' of $\mathrm{Inn}(X)$ up to t_X-torsion; hence, this theorem emphasizes importance of the concept of types; so as to investigate $\mathrm{As}(X)$, we shall study $\mathrm{Inn}(X)$ and $H_2^{\mathrm{gr}}(\mathrm{Inn}(X))$.

§ 3. Methods on inner automorphism groups.

Following the preceding theorem to study the group $\mathrm{As}(X)$, we shall develop a method for describing the inner automorphism group $\mathrm{Inn}(X)$:

Theorem 3.1. *Let a group G act on a quandle X. Let a map $\kappa : X \to G$ satisfy the followings:*

1. *The identity $x \lhd y = x \cdot \kappa(y) \in X$ holds for any $x, y \in X$.*
2. *The image $\kappa(X) \subset G$ generates the group G, and the action $X \curvearrowleft G$ is effective.*

Then, there is an isomorphism $\mathrm{Inn}(X) \cong G$*, and the action $X \curvearrowleft G$ agrees with the natural action of* $\mathrm{Inn}(X)$.

Proof. Identify the action $X \curvearrowleft G$ with a group homomorphism $f : G \to \mathrm{Bij}(X, X)$. It follows from the first assumption that $f(\kappa(X)) \subset \mathrm{Inn}(X)$ and $f(\kappa(X))$ generates $\mathrm{Inn}(X)$; thus, f gives rise to an epimorphism $F : \langle\kappa(X)\rangle \to \mathrm{Inn}(X)$, where $\langle\kappa(X)\rangle$ is the subgroup of G generated by $\kappa(X)$. Then, the seccond assumption enssures that $\langle\kappa(X)\rangle = G$, and the bijectivity of F, i.e., $\mathrm{Inn}(X) \cong G$. Moreover, the agreement of the two actions follows by construction. □

This theorem is applicable to many quandles, in practice. Actually, as seen in Section 4, we can determine $\mathrm{Inn}(X)$ of many quandles X. However, we here explain that this theorem is inspired by the Cartan embeddings in symmetric space theory as follows:

Example 3.2. Let X be a symmetric space in differential geometry. Consider the group $\mathrm{Inn}(X) \subset \mathrm{Diff}(X)$ generated by the symmetries $\bullet \lhd y$ with compact-open topology. As is well known, $\mathrm{Inn}(X)$ has a Lie group structure, and the map $X \to \mathrm{Inn}(X)$ that sends y to $\bullet \lhd y$ is commonly called the Cartan embedding. As seen in textbooks on symmetric spaces, Theorem 3.1 had been used to determine $\mathrm{Inn}(X)$ concretely.

Furthermore, we suggest another computation when $\mathrm{Inn}(X)$ is perfect.

Proposition 3.3. *Let X be a quandle, and $O(X)$ be the set of orbits of the action $X \curvearrowleft \mathrm{As}(X)$. Set the epimorphism $\varepsilon_i : \mathrm{As}(X) \to \mathbb{Z}$ associated with $i \in O(X)$ defined in* (2.2). *If the group* $\mathrm{Inn}(X)$ *is perfect, i.e.,* $H_1^{\mathrm{gr}}(\mathrm{Inn}(X)) = 0$, *then we have an isomorphism*

$$\mathrm{As}(X) \cong \mathrm{Ker}(\oplus_{i \in O(X)} \varepsilon_i) \times \mathbb{Z}^{\oplus O(X)}, \tag{3.1}$$

and this $\mathrm{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ *is a central extension of* $\mathrm{Inn}(X)$ *and is perfect. In particular, if X is connected and the group homology* $H_2^{\mathrm{gr}}(\mathrm{Inn}(X))$ *vanishes, then* $\mathrm{As}(X) \cong \mathrm{Inn}(X) \times \mathbb{Z}$.

Proof. We will show the isomorphism (3.1). By the assumption $H_1^{\mathrm{gr}}(\mathrm{Inn}(X)) = 0$, the composite $\mathrm{Ker}(\psi_X) \hookrightarrow \mathrm{As}(X) \xrightarrow{\mathrm{proj.}} H_1^{\mathrm{gr}}(\mathrm{As}(X)) = \mathbb{Z}^{\oplus O(X)}$ obtained from (2.3) is surjective. Since $\mathbb{Z}^{\oplus O(X)}$ is free, we can choose a section $\mathfrak{s} : \mathbb{Z}^{\oplus O(X)} \to \mathrm{Ker}(\psi_X)$ of the composite. Hence, by the equality (2.1) and the inclusion $\mathrm{Ker}(\psi_X) \subset \mathrm{As}(X)$, the semi-direct product $\mathrm{As}(X) \cong \mathrm{Ker}(\oplus_{i \in O(X)} \varepsilon_i) \rtimes \mathbb{Z}^{\oplus O(X)}$ is trivial, leading to (3.1) as desired. Furthermore the kernel $\mathrm{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ is a central extension of $\mathrm{Inn}(X)$ by construction, and is perfect by the Kunneth theorem and $\mathrm{As}(X)_{\mathrm{ab}} \cong \mathbb{Z}^{\oplus O(X)}$. Hence we complete the proof. □

Remark. In general, the kernel $\mathrm{Ker}(\oplus_{i \in O(X)} \varepsilon_i)$ is not always the universal central extension of the perfect group $\mathrm{Inn}(X)$; see [N3, Theorem 4] with $g = 3$ as a counterexample such that the extension $\mathrm{Ker}(\oplus_{i \in O(X)} \varepsilon_i) \to \mathrm{Inn}(X)$ is not universal.

Finally, we conclude this section by giving two lemmas on $\mathrm{As}(X)$, which are used later.

Lemma 3.4. *Let X be a connected quandle of type $t < \infty$. Then, for any $x, y \in X$, we have the identity $(e_x)^t = (e_y)^t$ in the center of $\mathrm{As}(X)$.*

Proof. For every $b \in X$, note the equalities $(e_x)^{-t} e_b e_x^t = e_{(\cdots(b \lhd x)\cdots)\lhd x} = e_b$ in $\mathrm{As}(X)$. Namely $(e_x)^t$ lies in the center. Furthermore the connectivity admits $g \in \mathrm{As}(X)$ such that $x \cdot g = y$. Hence, it follows from (2.1) that $(e_x)^t = g^{-1}(e_x)^t g = (e_{x\cdot g})^t = (e_y)^t$ as desired. □

Lemma 3.5. *Let X be a connected quandle of finite order. Then $|\mathrm{Inn}(X)|/|X|$ is divisible by its type t_X .*

Proof. For $x, y \in X$, we define $m_{x,y}$ as the minimal n satisfying $x \lhd^n y = x$. Note that $(\bullet \lhd^{m_{x,y}} y)$ lies in the stabilizer $\mathrm{Stab}(x)$. Since $|\mathrm{Stab}(x)| = |\mathrm{Inn}(X)|/|X|$ by connectivity, any $m_{x,y}$ divides $|\mathrm{Inn}(X)|/|X|$; hence so does the type t_X. □

Furthermore, in some cases, we can calculate some torsion parts of their group homology:

Lemma 3.6. *Let X be a connected quandle of type t_X. If $H_2^{\mathrm{gr}}(\mathrm{Inn}(X))$ is annihilated by $t_X < \infty$, then there is a $[1/t_X]$-isomorphism $H_3^{\mathrm{gr}}(\mathrm{Adj}(X)) \cong H_3^{\mathrm{gr}}(\mathrm{Inn}(X))$.*

Proof. Consider the Lyndon-Hochschild spectral sequence of (2.3). It is sufficient for the proof to show that the differential

$$d_2 : E^2_{3,0} = H_3(\mathrm{Inn}(X); H_0(\mathrm{Ker}(\psi_X))) \longrightarrow E^2_{1,1} = H_1^{\mathrm{gr}}(\mathrm{Inn}(X); H_1^{\mathrm{gr}}(\mathrm{Ker}(\psi_X)))$$

is trivial modulo t_X. To show this, the inflation-restriction sequence of (2.3)

$$\mathrm{Ker}(\psi_X) \longrightarrow H_1^{\mathrm{gr}}(\mathrm{As}(X); \mathbb{Z}) \longrightarrow H_1^{\mathrm{gr}}(\mathrm{Inn}(X); \mathbb{Z}) \longrightarrow 0$$

and Lemma 3.4 imply that $H_1^{\mathrm{gr}}(\mathrm{Inn}(X); \mathbb{Z})$ is annihilated by t_X. Furthermore, we obtain the $[1/t_X]$-isomorphism $\mathrm{Ker}(\psi_X) \cong_{[1/t_X]} \mathbb{Z}$ from Theorem 2.1. Hence, the image of d_2 is trivial as required. □

§ 4. Six examples of $\mathrm{As}(X)$ and second quandle homology

Based on the preceding results on $\mathrm{As}(X)$, this section calculates $\mathrm{Inn}(X)$ and $\mathrm{As}(X)$ for six kinds of connected quandles X: Alexander, symplectic, spherical, Dehn, Coxeter and core quandles. These quandles are dealt with in six subsections in turn.

Furthermore, to determine the second quandle homology $H_2^Q(X)$ in trivial $\mathbb{Z}$-coefficients (see §6 for the definition), we will employ the following computation of Eisermann:

Theorem 4.1 ([Eis2, Theorem 1.15]). *Let X be a connected quandle. Fix an element $x_0 \in X$. Let* $\mathrm{Stab}(x_0) \subset \mathrm{As}(X)$ *be the stabilizer of x_0, and $\varepsilon : \mathrm{As}(X) \to \mathbb{Z}$ be the abelianization mentioned in* (2.2). *Then, $H_2^Q(X)$ is isomorphic to the abelianization of* $\mathrm{Stab}(x_0) \cap \mathrm{Ker}(\varepsilon)$.

§ 4.1. Alexander quandles

We start by discussing the class of Alexander quandles. Every $\mathbb{Z}[T^{\pm 1}]$-module X has a quandle structure with the operation $x \lhd y = y + T(x - y)$ for $x, y \in X$, and is called *the Alexander quandle.* This operation $\bullet \lhd y$ can be geometrically compared to the T-multiple with center y. The type is the minimal N such that $T^N = \mathrm{id}_X$ since $x \lhd^n y = y + T^n(x - y)$. Furthermore, it can be easily verified that an Alexander quandle X is connected if and only if $(1 - T)X = X$.

Let us review the concrete presentation of $\mathrm{As}(X)$, which is due to Clauwens [Cla2]. When X is connected, set up the homomorphism $\mu_X : X \otimes X \to X \otimes X$ defined by $\mu_X(x \otimes y) = x \otimes y - Ty \otimes x$. Further, he defined a group operation on $\mathbb{Z} \times X \times \mathrm{Coker}(\mu_X)$ by setting

$$(n, x, \alpha) \cdot (m, y, \beta) = (n + m, T^m x + y,\ \alpha + \beta + [T^m x \otimes y]),$$

and constructed a group isomorphism $\mathrm{As}(X) \to \mathbb{Z} \times X \times \mathrm{Coker}(\mu_X)$, which takes e_x to $(1, x, 0)$. As a result, the kernel of $\psi_X : \mathrm{As}(X) \to \mathrm{Inn}(X)$ equals $t_X\mathbb{Z} \times \mathrm{Coker}(\mu_X)$.

Thanks to his presentation of $\mathrm{As}(X)$, we can easily show a result of Clauwens that determines the homology $H_2^Q(X)$ of a connected Alexander quandle X. To be precise,

Proposition 4.2 (Clauwens [Cla2]). *Let X be a connected Alexander quandle. The homology $H_2^Q(X)$ is isomorphic to the quotient module* $\mathrm{Coker}(\mu_X) = X \otimes_{\mathbb{Z}} X/(x \otimes y - Ty \otimes x)_{x,y \in X}$.

Proof. By definition we can see that the $\mathrm{Ker}(\varepsilon) \cap \mathrm{Stab}(0)$ is the cokernel $\mathrm{Coker}(\mu_X)$. □

§ 4.2. Symplectic quandles

Let K be a commutative field, and let Σ_g be the closed surface of genus g. Consider the multiplicative group $K^\times$, and the quotient $K^\times/(K^\times)^2$ modulo 2. For $[r] \in K^\times/(K^\times)^2$, we fix a representative $r \in K^\times$, and consider the copy of $H^1(\Sigma_g; K) \setminus \{0\} =$

$K^{2g} \setminus \{0\}$, denoted by X_r. Let X be the union $\cup_{r \in K^\times/(K^\times)^2} X_r$ (here, we should notice that $X = X_r$ if K is an algebraically closed field.). Using the standard symplectic 2-form $\langle , \rangle : H^1(\Sigma_g; K) \times H^1(\Sigma_g; K) \to K$, the set X is made into a quandle by the operation $x \lhd y := r\langle x, y\rangle y + x \in X$ for $x \in X_r$ and $y \in X$, and is called *a symplectic quandle (over K)*. The operation $\bullet \lhd y : X \to X$ is commonly called *the transvection* of y. Note that the type of the quandle X is the characteristic of K since $x \lhd^N y = Nr\langle x, y\rangle y + x$.

We will determine $\mathrm{Inn}(X)$ and $\mathrm{As}(X)$ associated with the symplectic quandle X over K.

Lemma 4.3. $\mathrm{Inn}(X)$ *is isomorphic to the symplectic group $Sp(2g; K)$.*

Proof. Recall from the Cartan-Dieudoné theorem that the classical group $Sp(2g; K)$ is generated by transvections $(\bullet \lhd y)$.

We will show the desired isomorphism. For any $y \in X$, the map $(\bullet \lhd y) : X \to X$ is a restriction of a linear map $K^{2g} \to K^{2g}$. It thus yields a map $\kappa : X \to GL(2g; K)$, which factors through $Sp(2g; K)$ and satisfies the conditions in Theorem 3.1. Indeed, the condition (2) follows from the classical theorem and the effectivity of the standard action $K^{2g} \curvearrowleft Sp(2g; K)$. Therefore $\mathrm{Inn}(X) \cong Sp(2g; K)$ as desired. □

Proposition 4.4. *Take a field K of positive characteristic p and with $|K| > 10$. Assume the connectivity, that is, every $x \in K$ admits a square $\sqrt{x}$ in K. Let $X = K^2 \setminus \{0, 0\}$ be the symplectic quandle over K, and $\widetilde{Sp}(2g; K)$ be the universal central extension of $Sp(2g; K)$. Then $\mathrm{As}(X) \cong \mathbb{Z} \times \widetilde{Sp}(2g; K)$.*

Proof. Since X is connected and $\mathrm{Inn}(X) \cong Sp(2g; K)$ by Lemma 4.3, Proposition 3.3 implies $\mathrm{As}(X) \cong \mathrm{Ker}(\varepsilon) \times \mathbb{Z}$. Further, it follows from Theorem 2.1 that $H_2^{\mathrm{gr}}(\mathrm{As}(X))$ is annihilated by p. Hence, following the fact [Sus] that $H_2^{\mathrm{gr}}(Sp(2g; K))$ has no p-torsion, the kernel $\mathrm{Ker}(\varepsilon)$ must be the universal central extension of $Sp(2g; K)$, which completes the proof. □

Remark. This proposition holds even if the characteristic of K is zero and X is not connected; see [N2] for the proof. Furthermore, the paper [N2] also determines $H_2^Q(X)$ in the case where K is of infinite order.

Accordingly, hereafter, we will focus on finite fields $K = \mathbb{F}_q$ with $q > 10$:

Proposition 4.5. *Let X be the symplectic quandle over $\mathbb{F}_q$. If $q > 10$, then $\mathrm{As}(X) \cong \mathbb{Z}^{O(X)} \times Sp(2g; \mathbb{F}_q)$. Furthermore, $H_3^{\mathrm{gr}}(\mathrm{As}(X)) \cong \mathbb{Z}/(q^2 - 1)$.*

Proof. Since $(\mathbb{F}_q)^\times$ is cyclic, we first should notice that, if q is even $|O(X)| = 1$, and that, if q is odd, $|O(X)| = 2$ or 1 according to $q = 4r + 1$ or $q = 4r + 3$ for some $r \in \mathbb{Z}$.

Since $q > 10$, the first and second homology groups of $\mathrm{Inn}(X) \cong Sp(2g;\mathbb{F}_q)$ are known to be zero (see [FP, Fri]). Thus, $\widetilde{Sp}(2g;\mathbb{F}_q) = Sp(2g;\mathbb{F}_q)$, leading to $\mathrm{As}(X) \cong \mathbb{Z}^{O(X)} \times Sp(2g;\mathbb{F}_q)$ as stated. Furthermore, the latter part follows from the result $H_3^{\mathrm{gr}}(Sp(2g;\mathbb{F}_q)) \cong \mathbb{Z}/(q^2-1)$ in [FP, Fri]. □

As a result, we will determine the second homology $H_2^Q(X)$.

Proposition 4.6. *Let $q > 10$, and X be as above. If $g \geq 2$, the homology $H_2^Q(X)$ vanishes. If $g = 1$, then $H_2^Q(X) \cong (\mathbb{Z}/p)^{d|O(X)|}$, where $q = p^d$.*

Proof. Recall $\mathrm{As}(X) \cong \mathbb{Z}^{O(X)} \times Sp(2g;\mathbb{F}_q)$. Considering the standard action $X \curvearrowleft Sp(2g;\mathbb{F}_q)$, denote by G_X the stabilizer of $(1,0,\ldots,0) \in (\mathbb{F}_q)^{2g}$. Since Theorem 4.1 immediately means $H_2^Q(X) \cong H_1^{\mathrm{gr}}(G_X)^{|O(X)|}$, we will calculate $H_1^{\mathrm{gr}}(G_X)$ as follows. First, for $g = 1$, it can be verified that the stabilizer G_X is exactly the product $(\mathbb{Z}/p)^d$ as an abelian group; hence $H_2^Q(X) \cong (\mathbb{Z}/p)^{d|O(X)|}$ in the sequel. Next, for $g \geq 2$, the vanishing $H_2^Q(X) = H_1^{\mathrm{gr}}(G_X) = 0$ immediately follows from Lemma 4.7 below. □

Lemma 4.7. *Let $g \geq 2$ and $q > 10$. Let G_X denote the stabilizer of the action $X \curvearrowleft Sp(2g;\mathbb{F}_q)$ mentioned above. Then the homology groups $H_1^{\mathrm{gr}}(G_X)$ and $H_2^{\mathrm{gr}}(G_X)$ vanish.*

Proof. Since $q > 10$, recall from [FP, II. §6.3] the order of $Sp(2g;\mathbb{F}_q)$ as

$$|Sp(2g;\mathbb{F}_q)| = q^{g^2}(q^{2g}-1)(q^{2g-2}-1)\cdots(q^2-1).$$

Since $|X| = q^{2g}-1$, the order of G_X is equal to $q^{g^2} \cdot |Sp(2g-2;\mathbb{F}_q)|$. Thereby $H_1^{\mathrm{gr}}(G_X)$ and $H_2^{\mathrm{gr}}(G_X)$ are zero up to p-torsion, because of the inclusion $Sp(2g-2;\mathbb{F}_q) \subset G_X$ by definitions and the vanishing $H_1^{\mathrm{gr}} \oplus H_2^{\mathrm{gr}}(Sp(2g-2;\mathbb{F}_q)) \cong 0$ up to p torsion.

Finally, we may focus on the p-torsion of $H_1^{\mathrm{gr}} \oplus H_2^{\mathrm{gr}}(G_X)$. Following the proof of [Fri, Proposition 4.4], there is a certain subgroup "$\Delta(Sp(2g;\mathbb{F}_q))$" of G_X which contains a p-sylow group of $Sp(2g;\mathbb{F}_q)$ and this $\mathbb{Z}/p$-homology vanishes. Hence, $H_1^{\mathrm{gr}} \oplus H_2^{\mathrm{gr}}(G_X) = 0$ as required. □

§ 4.3. Spherical quandles

Let K be a field of characteristic not equal to 2, and fix $n \geq 2$ in this subsection. Take the standard symmetric bilinear form $\langle,\rangle : K^{n+1} \otimes K^{n+1} \to K$. Consider a set of the form

$$S_K^n := \{\ x \in K^{n+1} \mid \langle x, x\rangle = 1\ \}.$$

We define the operation $x \lhd y$ to be $2\langle x, y\rangle y - x \in S_K^n$. The pair $(S_K^n, \lhd)$ is a quandle of type 2, and is referred to as *a spherical quandle* (over K). This operation $\bullet \lhd y$ can be interpreted as a linear transformation which identically acts on y and $-\mathrm{Id}$ on the the subspace orthogonal tof y.

Then, similar to the proof of Lemma 4.3, one can readily determine $\mathrm{Inn}(X)$ as follows:

Lemma 4.8. *If n is odd, then $\mathrm{Inn}(S^n_K)$ is isomorphic to the orthogonal group $O(n+1;K)$. If n is even, $\mathrm{Inn}(S^n_K)$ is isomorphic to $SO(n+1;K)$.*

Next, we will focus on second homology group and $H_3^{\mathrm{gr}}(\mathrm{As}(X))$ of spherical quandles over $\mathbb{F}_q$. Here, the results are up to 2-torsion, whereas the 2-torsion part is the future problem.

Proposition 4.9. *Let X be a spherical quandle over $\mathbb{F}_q$. Let $q > 10$. For $n \geq 3$, the second homology $H_2^Q(X)$ is annihilated by 2. If $n = 1$, then the homology $H_2^Q(X)$ is $[1/2]$-isomorphic to the cyclic group $\mathbb{Z}/(q-\delta_q)$, where $\delta_q = \pm 1$ is according to $q \equiv \pm 1 (\mathrm{mod}\ 4)$.*

Proof. Assume n is odd. Under the standard action $X \curvearrowleft O(n+1;\mathbb{F}_q)$, the stabilizer of $(1,0,\dots,0) \in X$ is $O(n;\mathbb{F}_q)$. By a similar discussion to the proof of Proposition 4.6, $H_2^Q(X) \cong H_1^{\mathrm{gr}}(O(n;\mathbb{F}_q))$ modulo 2-torsion. For $n \geq 3$, the abelianization of $O(n;\mathbb{F}_q)$ is $(\mathbb{Z}/2)^2$; see [FP, II. §3]; hence the $H_2^Q(X)$ is annihilated by 2 as required. The same discussion in the even case of n works well, since the inclusion $SO(n) \to O(n)$ induces $H_*(SO(n;\mathbb{F}_q)) \cong_{[1/2]} H_*(O(n;\mathbb{F}_q))$ modulo 2-torsion.

Finally, when $n = 1$, the group $O(2;\mathbb{F}_q)$ is cyclic and of order $q - \delta_q$. Hence $H_2^Q(X) \cong H_1^{\mathrm{gr}}(O(2;\mathbb{F}_q)) \cong_{[1/2]} \mathbb{Z}/(q-\delta_q)$. □

Proposition 4.10. *Let $q > 10$. Then $H_3^{\mathrm{gr}}(\mathrm{As}(X)) \cong_{[1/2]} H_3^{\mathrm{gr}}(O(n+1;\mathbb{F}_q))$ up to 2-torsion.*

Proof. Since $q > 10$, $H_1^{\mathrm{gr}} \oplus H_2^{\mathrm{gr}}(O(n+1;\mathbb{F}_q))$ is known to be annihilated by 2; see [Fri, FP]. Hence, the conclusion readily results from Lemma 3.6. □

§ 4.4. Dehn quandle

Changing the subject, we now review Dehn quandle [Yet]. Denote by $\mathcal{M}_g$ the mapping class group of Σ_g, and consider the set, $\mathcal{D}_g$, defined by

$$\mathcal{D}_g := \{ \text{ isotopy classes of (unoriented) non-separating simple closed curves } \gamma \text{ in } \Sigma_g \ \}.$$

For $\alpha,\ \beta \in \mathcal{D}_g$, we define $\alpha \lhd \beta \in \mathcal{D}_g$ by $\tau_\beta(\alpha)$, where $\tau_\beta \in \mathcal{M}_g$ is the positive Dehn twist along β. The pair $(\mathcal{D}_g, \lhd)$ is a quandle, and called *(non-separating) Dehn quandle.* As is well-known, any two non-separating simple closed curves are conjugate by the product of some Dehn twists. Hence, the quandle $\mathcal{D}_g$ is connected, and is not of any type t. The Dehn quandle $\mathcal{D}_g$ is applicable to study 4-dimensional Lefschetz fibrations (see, e.g., [Yet, Zab, N3]). The natural inclusion $\kappa : \mathcal{D}_g \to \mathcal{M}_g$ implies $\mathrm{Inn}(\mathcal{D}_g) \cong \mathcal{M}_g$ by Theorem 3.1. Furthermore, if $g \geq 4$, there is an isomorphism $\mathrm{As}(\mathcal{D}_g) \cong \mathbb{Z} \times \mathcal{T}_g$ shown by [Ger], where $\mathcal{T}_g$ is the universal central extension of $\mathcal{M}_g$ associated with $H_2^{\mathrm{gr}}(\mathcal{M}_g) \cong \mathbb{Z}$.

The result of this subsection is the following:

Proposition 4.11. *If $g \geq 5$, then $H_2^Q(\mathcal{D}_g) \cong \mathbb{Z}/2$.*

Proof. We will use the facts that an epimorphism $G \to H$ between groups induces an epimorphism $G_{\mathrm{ab}} \to H_{\mathrm{ab}}$, and that $\mathcal{M}_{g,r}$ is perfect.

Fixing $\alpha \in \mathcal{D}_g$, we begin by observing the stabilizer $\mathrm{Stab}(\alpha) \subset \mathrm{As}(\mathcal{D}_g)$. Note that the map $\mathcal{D}_g \to \mathcal{M}_g$ sending β to τ_β yields a group epimorphism $\pi : \mathrm{As}(\mathcal{D}_g) \to \mathcal{M}_g$. Furthermore, by Proposition 3.3, the restriction of π to $\mathrm{Ker}(\varepsilon) \cong \mathcal{T}_g$ coincides with the projection $\mathcal{T}_g \to \mathcal{M}_g$. In particular, we thus have $\pi(\mathrm{Stab}(\alpha)) = \pi(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\varepsilon)) \subset \mathcal{M}_g$.

We will construct a surjection $H_2^Q(\mathcal{D}_g) \to \mathbb{Z}/2$. By the virtue of Theorem 4.1, it is enough to construct a surjection from the preceding $\pi(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\varepsilon))$ to $\mathbb{Z}/2$ for $g \geq 2$. As is shown [PR, Proposition 7.4], we have the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{g-1,2} \xrightarrow{\xi} \pi(\mathrm{Stab}(\alpha)) \xrightarrow{\lambda} \mathbb{Z}/2 \qquad \text{(exact)}. \tag{4.1}$$

Here ξ is the homomorphism induced from the gluing $(\Sigma_{g-1,2}, \partial(\Sigma_{g-1,2})) \to (\Sigma_g, \alpha)$, and λ is defined by the transposition of the connected components of boundaries of $\Sigma_g \setminus \alpha$. By considering a hyper-elliptic involution preserving the above α, the map λ is surjective. Hence $\pi(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\varepsilon))$ surjects onto $\mathbb{Z}/2$ as desired.

Finally, we will complete the proof. By Theorem 4.1 again, recall that $\big(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\varepsilon)\big)_{\mathrm{ab}} \cong H_2^Q(\mathcal{D}_g)$. To compute this, put the inclusion $\iota : \pi(\mathrm{Stab}(\alpha)) \to \mathcal{M}_g$. By the Harer-Ivanov stability theorem (see [Iva]), the composition $\iota \circ \xi : \mathcal{M}_{g-1,2} \to \mathcal{M}_g$ induces an epimorphism

$$(\iota \circ \xi)_* : H_2^{\mathrm{gr}}(\mathcal{M}_{g-1,2}; \mathbb{Z}) \longrightarrow H_2^{\mathrm{gr}}(\mathcal{M}_g; \mathbb{Z}) \qquad \text{for} \quad g \geq 5. \tag{4.2}$$

Since $H_2^{\mathrm{gr}}(\mathcal{M}_{g-1,2}; \mathbb{Z}) \cong H_2^{\mathrm{gr}}(\mathcal{M}_g; \mathbb{Z}) \cong \mathbb{Z}$ is known (see, e.g., [FM]), the epimorphism (4.2) is isomorphic. Let $(\iota \circ \xi)^*(\mathcal{T}_g)$ denote the central extension of $\mathcal{M}_{g-1,2}$ obtained by $\iota \circ \xi$. Since $\mathcal{M}_g$ and $\mathcal{M}_{g-1,2}$ are perfect, the group $(\iota \circ \xi)^*(\mathcal{T}_g)$ is also perfect by the isomorphism (4.2). Note that the group $\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\varepsilon)$ is isomorphic to $\iota^*(\mathcal{T}_g)$. Hence the abelianization $\big(\mathrm{Stab}(\alpha) \cap \mathrm{Ker}(\varepsilon)\big)_{\mathrm{ab}}$ never be bigger than $\mathbb{Z}/2$. In conclusion, we arrive at the conclusion. $\square$

§ 4.5. Coxeter quandles

We will focus on Coxeter quandles, and study the associated groups, and show Theorem 4.12.

This subsection assumes basic knowledge of Coxeter groups, as explained in [Aki1, How]. Given a Coxeter graph Γ, we can set the Coxeter group W. Let X_Γ be the set of the reflections in W, that is, the set of elements conjugate to the generators of W. Equipping X_Γ with conjugacy operation, X_Γ is made into a quandle of type 2. Denote the inclusion $X_\Gamma \hookrightarrow W$ by κ. Since W subject to the center Z_W effectivity acts on X_Γ, we have $\mathrm{Inn}(X_\Gamma) \cong W/Z_W$. Moreover, W is, by definition, isomorphic to the quotient of $\mathrm{As}(X_\Gamma)$ subject to the squared relations $(e_x)^2 = 1$ for any $x \in X_\Gamma$.

In this situation, we now give another easy proof of a part of the theorem shown by Howlett:

Theorem 4.12 (A connected result in [How, §2–4]). *Assume that the Coxeter quandle X_Γ is connected. Then, the second group homology $H_2^{\mathrm{gr}}(W)$ is annihilated by 2.*

Proof. Recall from Theorem 2.1 that $H_1^{\mathrm{gr}}(\mathrm{As}(X_\Gamma)) \cong \mathbb{Z}$ and $H_2^{\mathrm{gr}}(\mathrm{As}(X_\Gamma))$ is annihilated by 2. Therefore, the inflation-restriction exact sequence from the central extension $\mathrm{As}(X_\Gamma) \to W$ implies the desired 2-vanishing of $H_2^{\mathrm{gr}}(W)$. □

Finally, we will end this subsection by giving some comments. Recently, Akita [Aki2] determined the associated group $\mathrm{As}(X_\Gamma)$ as a $\mathbb{Z}^N$-central extended group of W. Furthermore, concerning the third homology $H_3(\mathrm{As}(X_\Gamma))$ in the case where X_Γ is connected, we obtain $H_3(\mathrm{As}(X_\Gamma)) \cong H_3(W)$ up to 2-torsion from Lemma 3.6. The odd torsion of $H_3^{\mathrm{gr}}(W)$ in a certain stable range is studied by Akita [Aki1].

§ 4.6. Core quandles

Given a group G, we let $X = G$ equipped with a quandle operation $g \lhd h := hg^{-1}h$. This quandle is called *core quandle* [Joy] and is of type 2. This last subsection will deal with core quandles, and show Proposition 4.13.

Let us give some terminologies to state the proposition. Let $\mathbb{Z}/2$ be $\{\pm 1\}$. Take the wreath product $(G \times G) \rtimes \mathbb{Z}/2$, and the commutator subgroup $[G, G]$. Consider the epimorphism $(G \times G) \rtimes \mathbb{Z}/2 \to G/[G, G]$ which sends (g, h, σ) to $[gh]$. Then, the kernel is formed as

$$\mathcal{G}_1 := \{\ (g, h, \sigma)\ \in (G \times G) \rtimes \mathbb{Z}/2 \mid gh \in [G, G]\ \}.$$

Further, with respect to $x \in X$ and $(g, h, \sigma) \in \mathcal{G}_1$, we define $x \cdot (g, h, \sigma) := h^{-1}x^\sigma g$, which ensures an action of $\mathcal{G}_1$ on X. Further, consider a subgroup of the form

$$\mathcal{G}_2 := \big\{(z, z, \sigma) \in (G \times G) \rtimes \mathbb{Z}/2 \ \big|\ \ z^2 \in [G, G], \quad k^{-1}zk = z^\sigma \ \text{ for any } k \in G\ \big\},$$

which is contained in the center of $\mathcal{G}_1$. Then, the quotient action subject to $\mathcal{G}_2$ is effective.

Proposition 4.13. *There is a group isomorphism* $\mathrm{Inn}(X) \cong \mathcal{G}_1/\mathcal{G}_2$.

Proof. Consider the map $\kappa : X \to \mathcal{G}_1/\mathcal{G}_2$ which sends g to $[(g, g^{-1}, -1)]$. We claim that this $\mathcal{G}_1/\mathcal{G}_2$ is generated by the image $\Im(\kappa)$. Actually, we can easily verify that any element (g, h, σ) in $\mathcal{G}_1$ with $g_i,\ h_i \in G$ and $gh = g_1 h_1 g_1^{-1} h_1^{-1} \cdots g_m h_m g_m^{-1} h_m^{-1}$ is decomposed as

$$\kappa(1_G)^{\frac{\sigma+1}{2}} \cdot \kappa(gh^{-1}) \cdot \Big(\big(\kappa(g_1 h_1) \cdot \kappa(1_G) \cdot \kappa(g_1^{-1}) \cdot \kappa(h_1) \big) \cdots \big(\kappa(g_m h_m) \cdot \kappa(1_G) \cdot \kappa(g_m^{-1}) \cdot \kappa(h_m) \big) \Big).$$

Then, the routine discussion from Lemma 3.1 completes the proof. □

This proposition implies the difficulty to determine $\mathrm{Inn}(X)$, in general. Thus, it also seems hard to determine $\mathrm{As}(X)$. Actually, even if X is a connected core quandle, Proposition 4.13 implies that the kernel $\mathrm{Ker}(\psi)$ is complicated by the reason of the second homology $H_2^{\mathrm{gr}}(G)$ and $H_2^{\mathrm{gr}}(\mathrm{Inn}(X))$. For example, if X is the product of h-copies of the cyclic group $\mathbb{Z}/m$, i.e., X is the Alexander quandle of the form $(\mathbb{Z}/m)^h[T]/(T+1)$, then the kernel $\mathrm{Ker}(\psi)$ stated in Proposition 4.2 is not so simple.

§ 5. On quandle coverings

This section suggests that the results in Section 2 are applicable to quandle coverings.

Let us review coverings in the sense of Eisermann [Eis2, Eis1]. A map $f : Y \to Z$ between quandles is *a (quandle) homomorphism*, if $f(a \lhd b) = f(a) \lhd f(b)$ for any $a, b \in Y$. Furthermore, a quandle epimorphism $p : Y \to Z$ is a *(quandle) covering*, if the equality $p(\widetilde{x}) = p(\widetilde{y}) \in Z$ implies $\widetilde{a} \lhd \widetilde{x} = \widetilde{a} \lhd \widetilde{y} \in Y$ for any $\widetilde{a},\ \widetilde{x},\ \widetilde{y} \in Y$.

Let us mention a typical example. Given a connected quandle X with $a \in X$, recall the abelianization $\varepsilon_0 : \mathrm{As}(X) \to \mathbb{Z}$ in (2.2). Then, the kernel $\mathrm{Ker}(\varepsilon_0)$ has a quandle operation defined by setting

$$g \lhd h := e_a^{-1} g h^{-1} e_a h \qquad \text{for } g, h \in \mathrm{Ker}(\varepsilon_0).$$

We can easily see the independence of the choice of $a \in X$ up to quandle isomorphisms. Ones write $\widetilde{X}$ for the quandle $(\mathrm{Ker}(\varepsilon_0), \lhd)$, which is considered in [Joy, §7]. When X is of type t_X, so is the extended one $\widetilde{X}$ by Lemma 3.4. Furthermore, using the restricted action $X \curvearrowleft \mathrm{Ker}(\varepsilon_0) \subset \mathrm{As}(X)$, we see that the map $p : \widetilde{X} \to X$ sending g to $a \cdot g$ is a covering. This p is called *the universal (quandle) covering of* X, according to [Eis2, §5].

As a preliminary, we will explore some properties of quandle coverings.

Proposition 5.1. *For any quandle covering $p : Y \to Z$, the induced group surjection $p_* : \mathrm{As}(Y) \to \mathrm{As}(Z)$ is a central extension. Furthermore, if Y and Z are connected and Z is of type t_Z, then the abelian kernel* $\mathrm{Ker}(p_*)$ *is annihilated by t_Z.*

Proof. Fix a section $\mathfrak{s} : Y \to Z$. For any $y \in Z$, put arbitrary $y_i \in p^{-1}(y)$. Then,

$$e_{\mathfrak{s}(y)}^{-1} e_b e_{\mathfrak{s}(y)} = e_{b \lhd \mathfrak{s}(y)} = e_{b \lhd y_i} = e_{y_i}^{-1} e_b e_{y_i} \in \mathrm{As}(Y)$$

for any $b \in Y$. Here the second equality is due to the covering p. Denoting $e_{\mathfrak{s}(y)} e_{y_i}^{-1}$ by z_i, the equalities imply that z_i is central in $\mathrm{As}(Y)$. Since $e_{\mathfrak{s}(y)} = z_i e_{y_i}$, $\mathrm{As}(Y)$ is generated by $e_{\mathfrak{s}(y)}$ with $y \in Y$ and the central elements z_i associated with $y_i \in p^{-1}(y)$; consequently, the surjection p_* is a central extension.

We will show the latter part. Take the inflation-restriction exact sequence, i.e.,

$$H_2^{\mathrm{gr}}(\mathrm{As}(Z)) \longrightarrow \mathrm{Ker}(p_*) \longrightarrow H_1^{\mathrm{gr}}(\mathrm{As}(Y)) \longrightarrow H_1^{\mathrm{gr}}(\mathrm{As}(Z)) \longrightarrow 0 \quad \text{(exact)}.$$

By connectivities the third map from $H_1^{\mathrm{gr}}(\mathrm{As}(Y)) = \mathbb{Z}$ is an isomorphism. Since Theorem 2.1 says that $H_2^{\mathrm{gr}}(\mathrm{As}(Z))$ is annihilated by t_Z, so is the kernel $\mathrm{Ker}(p_*)$ as desired. □

Next, we will compute the second homology of $\widetilde{X}$ (Theorem 5.4) by showing propositions:

Proposition 5.2. *For any connected quandle X, the extended one $\widetilde{X}$ above is also connected.*

Proof. It is enough to show that the identity $1_{\widetilde{X}} \in \widetilde{X} = \mathrm{Ker}(\varepsilon_0)$ is transitive to any element h in $\widetilde{X}$. Expand $h \in \widetilde{X} \subset \mathrm{As}(X)$ as $h = e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n}$ for some $x_i \in X$ and $\epsilon_i \in \mathbb{Z}$. Since $h \in \mathrm{Ker}(\varepsilon_0)$, note $\sum \epsilon_i = 0$. The connectivity of X ensures some $g_i \in \mathrm{As}(X)$ so that $a \cdot g_i^{\epsilon_i} = x_i$. Therefore $g_i^{-\epsilon_i} e_a g_i^{\epsilon_i} = e_{a \cdot g_i^{\epsilon_i}} = e_{x_i}^{\epsilon_i}$ by (2.1). In the sequel, we have

$$\bigl(\cdots (1_{\widetilde{X}} \lhd^{\epsilon_1} g_1) \cdots \lhd^{\epsilon_n} g_n\bigr) = e_a^{\sum \epsilon_i} 1_{\widetilde{X}} (g_1^{-\epsilon_1} e_a g_1^{\epsilon_1}) \cdots (g_n^{-\epsilon_n} e_a g_n^{\epsilon_n}) = e_{x_1}^{\epsilon_1} \cdots e_{x_n}^{\epsilon_n} = h.$$

These equalities in $\widetilde{X}$ imply the transitivity of $\widetilde{X}$ □

Proposition 5.3. *Let X be a connected quandle. Let $p_* : \mathrm{As}(\widetilde{X}) \to \mathrm{As}(X)$ be the epimorphism induced from the covering $p : \widetilde{X} \to X$. Then, under the canonical action of* $\mathrm{As}(\widetilde{X})$ *on $\widetilde{X}$, the stabilizer* $\mathrm{Stab}(1_{\widetilde{X}})$ *of* $1_{\widetilde{X}}$ *is equal to* $\mathbb{Z} \times \mathrm{Ker}(p_*)$ *in* $\mathrm{As}(\widetilde{X})$*. Furthermore, the summand $\mathbb{Z}$ is generated by $1_{\widetilde{X}}$.*

Proof. We can easily see that the stabilizer of $1_{\widetilde{X}}$ via the action $\mathrm{Ker}(\varepsilon_0) = \widetilde{X} \curvearrowleft \mathrm{As}(X)$ is $\underline{\mathrm{Stab}}(1_{\widetilde{X}}) = \{e_a^n\}_{n \in \mathbb{Z}} \subset \mathrm{As}(X)$ exactly. Notice that any central extension of $\mathbb{Z}$ is trivial; therefore, since p_* is a central extension (Proposition 5.1), the restriction $p_* : \mathrm{Stab}(1_{\widetilde{X}}) \to \underline{\mathrm{Stab}}(1_{\widetilde{X}}) = \mathbb{Z}$ implies the required identity $\mathrm{Stab}(1_{\widetilde{X}}) = \mathbb{Z} \times \mathrm{Ker}(p_*)$. □

Theorem 5.4. *The second quandle homology of the extended quandle* $\widetilde{X}$ *is isomorphic to the kernel of* $p_* : \mathrm{As}(\widetilde{X}) \to \mathrm{As}(X)$. *Namely,* $H_2^Q(\widetilde{X}) \cong \mathrm{Ker}(p_*)$. *In particular, it follows from Proposition 5.1 that, if* $t_X < \infty$, *then* $H_2^Q(\widetilde{X})$ *is annihilated by the type* t_X.

Proof. Note that $\widetilde{X}$ is connected (Proposition 5.2) and the kernel $\mathrm{Ker}(p_*)$ is abelian (Proposition 5.1). Accordingly, the desired isomorphism $H_2^Q(\widetilde{X}) \cong \big(\mathrm{Ker}(\varepsilon_{\widetilde{X}}) \cap \mathrm{Stab}(1_{\widetilde{X}})\big)_{\mathrm{ab}} = \mathrm{Ker}(p_*)$ follows immediately from Proposition 5.3 and Theorem 4.1. $\square$

Finally, we now discuss the third group homology.

Proposition 5.5. *The universal covering* $p : \widetilde{X} \to X$ *induces a* $[1/t_X]$*-isomorphism* $p_* : H_3^{\mathrm{gr}}(\mathrm{As}(\widetilde{X})) \cong H_3^{\mathrm{gr}}(\mathrm{As}(X))$.

Proof. By connectivity of $\widetilde{X}$ and Theorem 2.1, $H_2^{\mathrm{gr}}(\mathrm{As}(\widetilde{X}))$ and $H_2^{\mathrm{gr}}(\mathrm{As}(X))$ are annihilated by t_X. Furthermore, since the epimorphism $p_* : \mathrm{As}(\widetilde{X}) \to \mathrm{As}(X)$ is a central extension whose kernel is annihilated by t_X (Proposition 5.1), we readily obtain the $[1/t_X]$-isomorphism $p_* : H_3^{\mathrm{gr}}(\mathrm{As}(\widetilde{X})) \cong H_3^{\mathrm{gr}}(\mathrm{As}(X))$ from the Lyndon-Hochschild sequence of p_*. $\square$

These properties played a key role to prove the main theorem in [N1].

§ 6. Proof of Theorem 2.1.

The purpose of this section is to prove Theorem 2.1. Let us begin by reviewing the rack space introduced by Fenn-Rourke-Sanderson [FRS]. Let X be a quandle with discrete topology. We set up a disjoint union $\bigcup_{n\geq 0}([0,1] \times X)^n$, and consider the relations given by

$$(t_1, x_1, \dots, x_{j-1}, 0, x_j, t_{j+1}, \dots, t_n, x_n) \sim (t_1, x_1, \dots t_{j-1}, x_{j-1}, t_{j+1}, x_{j+1}, \dots, t_n, x_n).$$

$$(t_1, x_1, \dots, x_{j-1}, 1, x_j, t_{j+1}, \dots, t_n, x_n) \sim$$
$$(t_1, x_1 \lhd x_j, \dots, t_{j-1}, x_{j-1} \lhd x_j, t_{j+1}, x_{j+1}, \dots, t_n, x_n),$$

Then, the *rack space* BX is defined to be the quotient space. By construction, we have a cell decomposition of BX by regarding the projection $\bigcup_{n\geq 0}([0,1] \times X)^n \to BX$ as characteristic maps. From the 2-skeleton of BX, we have $\pi_1(BX) \cong \mathrm{As}(X)$. Considering the Eilenberg-MacLane space $K(\pi_1(BX),1))$, we have the classifying map $c : BX \hookrightarrow K(\pi_1(BX),1)$, i.e., an inclusion obtained by killing the higher homotopy groups of BX.

Theorem 6.1. *Let X be a connected quandle of type t, and let $t < \infty$. For $n = 2$ and 3, the induced map $c_* : H_n(BX) \to H_n^{\mathrm{gr}}(\mathrm{As}(X))$ is annihilated by t.*

Remark. This is still more powerful and general than a result of Clauwens [Cla1, Proposition 4.4], which stated that, if a quandle X of finite order satisfies a certain condition, then the composite $(\psi_X)_* \circ c_* : H_n(BX) \to H_n^{\mathrm{gr}}(\mathrm{As}(X)) \to H_n^{\mathrm{gr}}(\mathrm{Inn}(X))$ is annihilated by $|\mathrm{Inn}(X)|/|X|$ for any $n \in \mathbb{N}$. Here note from Lemma 3.5 that t is a divisor of the order $|\mathrm{Inn}(X)|/|X|$.

Since the induced map $c_* : H_2(BX) \to H_2^{\mathrm{gr}}(\mathrm{As}(X))$ with $n = 2$ is known to be surjective (cf. Hopf's theorem [Bro, II.5]), Theorem 2.1 is immediately obtained from Theorem 6.1 and the inflation-restriction exact sequence of (2.3). Hence, we may turn into proving Theorem 6.1.

To this end, we give a brief review of the rack and quandle homology. Let $C_n^R(X)$ be the free right $\mathbb{Z}$-module generated by X^n. Define a boundary $\partial_n^R : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial_n^R(x_1, \ldots, x_n) = \textstyle\sum_{1 \le i \le n} (-1)^i \big((x_1 \lhd x_i, \ldots, x_{i-1} \lhd x_i, x_{i+1}, \ldots, x_n) \\ -(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \big).$$

The composite $\partial_{n-1}^R \circ \partial_n^R$ is known to be zero. The homology is denoted by $H_n^R(X)$ and is called *the rack homology.* As is known, the cellular complex of the rack space BX is isomorphic to the complex $(C_*^R(X), \partial_*^R)$. In particular, we have the isomorphism $H_*(BX) \cong H_*^R(X)$. Furthermore, following [CJKLS], let $C_n^D(X)$ be a submodule of $C_n^R(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. It can be easily seen that the submodule $C_n^D(X)$ is a subcomplex of $C_n^R(X)$. Then the *quandle homology*, $H_n^Q(X)$, is defined to be the homology of the quotient complex $C_n^R(X)/C_n^D(X)$.

Furthermore, we now observe concretely the map $c_* : H_n(BX) \to H_n^{\mathrm{gr}}(\mathrm{As}(X))$ for $n \le 3$. Let us recall the (non-homogenous) standard complex $C_n^{\mathrm{gr}}(\mathrm{As}(X))$ of $\mathrm{As}(X)$; see e.g. [Bro, §I.5]. The map c_* can be described in terms of their complexes. In fact, Kabaya [Kab, §8.4] considered homomorphisms $c_n : C_n^R(X) \to C_n^{\mathrm{gr}}(\mathrm{As}(X))$, where the map c_n for $n \le 3$ are defined by setting

$$\begin{aligned} c_1(x) &= e_x, \\ c_2(x, y) &= (e_x, e_y) - (e_y, e_{x \lhd y}), \\ c_3(x, y, z) &= (e_x, e_y, e_z) - (e_x, e_z, e_{y \lhd z}) + (e_y, e_z, e_A) - (e_y, e_{x \lhd y}, e_z) + (e_z, e_{x \lhd z}, e_{y \lhd z}) \\ &\quad -(e_z, e_{y \lhd z}, e_A), \end{aligned}$$

where we denote $(x \lhd y) \lhd z \in X$ by A for short. As is shown (see [Kab, §8.4]), the induced map on homology coincides with the map above c_* up to homotopy.

We will construct a chain homotopy between $t \cdot c_n$ and zero, when X is connected and of type t. Define a homomorphism $h_i : C^R_i(X) \to C^{\mathrm{gr}}_{i+1}(\mathrm{As}(X))$ by setting

$$h_1(x) = \sum_{1 \le j \le t-1} (e_x, e_x^j),$$

$$h_2(x,y) = \sum_{1 \le j \le t-1} (e_x, e_y, e^j_{x\lhd y}) - (e_x, e^j_x, e_y) - (e_y, e_{x\lhd y}, e^j_{x\lhd y}) + (e_y, e^j_y, e_y),$$

$$h_3(x,y,z) = \sum_{1 \le j \le t-1} \big((e_x, e_y, e_z, e^j_A) - (e_x, e_z, e_{y\lhd z}, e^j_A) - (e_x, e_y, e^j_{x\lhd y}, e_z) - (e_y, e_{x\lhd y}, e_z, e^j_A)$$
$$+ (e_x, e_z, e^j_{x\lhd z}, e_{y\lhd z}) + (e_z, e_{x\lhd z}, e_{y\lhd z}, e^j_A) + (e_x, e^j_x, e_y, e_z) - (e_x, e^j_x, e_z, e_{y\lhd z})$$
$$+ (e_y, e_z, e_A, e^j_A) - (e_z, e_{y\lhd z}, e_A, e^j_A) - (e_z, e_{x\lhd z}, e^j_{x\lhd z}, e_{y\lhd z}) + (e_y, e_{x\lhd y}, e^j_{x\lhd y}, e_z)\big).$$

Lemma 6.2. *Let X be as above. Then we have the equality $h_1 \circ \partial^R_2 - \partial^{\mathrm{gr}}_3 \circ h_2 = t \cdot c_2$.*

Proof. Compute the both terms $h_1 \circ \partial^R_2$ and $\partial^{\mathrm{gr}}_3 \circ h_2$ in the left hand side as

$$h_1 \circ \partial^R_2(x,y) = \sum (e_x, e^j_x) - (e_{x\lhd y}, e^j_{x\lhd y}).$$

$$\partial^{\mathrm{gr}}_3 \circ h_2(x,y) = \partial^{\mathrm{gr}}_3 \sum \big((e_x, e_y, e^j_{x\lhd y}) - (e_x, e^j_x, e_y) - (e_y, e_{x\lhd y}, e^j_{x\lhd y}) + (e_y, e^j_y, e_y)\big)$$
$$= \Big(\sum (e_y, e^j_{x\lhd y}) - (e_x e_y, e^j_{x\lhd y}) + (e_x, e^j_x e_y) - (e_x, e_y) - (e^j_x, e_y) + (e^{j+1}_x, e_y) - (e_x, e^j_x e_y)$$
$$+ (e_x, e^j_x) - (e_{x\lhd y}, e^j_{x\lhd y}) + (e_x e_y, e^j_{x\lhd y}) - (e_y, e^{j+1}_{x\lhd y}) + (e_y, e_{x\lhd y})\Big) + (e_y, e^t_y) - (e^t_y, e_y)$$
$$= t\big((e_y, e_{x\lhd y}) - (e_x, e_y)\big) + (e^t_x, e_y) - (e_y, e^t_{x\lhd y}) - (e^t_y, e_y) + (e_y, e^t_y) + h_1 \circ \partial^R_2(x,y)$$
$$= -t \cdot c_2(x,y) + h_1 \circ \partial^R_2(x,y).$$

Here we use Lemma 3.4 for the last equality. Hence, the equalities complete the proof. □

Lemma 6.3. *Let X be as above. The difference $h_2 \circ \partial^R_3 - \partial^{\mathrm{gr}}_4 \circ h_3$ is chain homotopic to $t \cdot c_3$.*

Proof. This is similarly proved by a direct calculation. To this end, recalling the notation $A = (x \lhd y) \lhd z$, we remark two identities

$$e_z e_A = e_{x\lhd y} e_z, \quad e_{y\lhd z} e_A = e_{x\lhd z} e_{y\lhd z} \in \mathrm{As}(X).$$

Using them, a tedious calculation can show that the difference $(t \cdot c_3 - h_2 \circ \partial^R_3 - \partial^{\mathrm{gr}}_4 \circ h_3)(x,y,z)$ is equal to

$$(e_y, e_z, e^t_A) - (e^t_x, e_y, e_z) + (e^t_x, e_z, e_{e\lhd z}) - (e_y, e^t_{x\lhd y})$$

$$+(e_z, e^t_{x\lhd z}, e_{y\lhd z}) - (e_z, e_{y\lhd z}, e^t_A) + \sum_{1\leq j\leq t-1} (e_y, e^j_y, e_y) - (e_{y\lhd z}, e^j_{y\lhd z}, e_{y\lhd z}).$$

Note that this formula is independent of any $x \in X$ since the identity $(e_a)^t = (e_b)^t$ holds for any $a, b \in X$ by Lemma 3.4. However, the map $c_3(x, y, z)$ with $x = y$ is zero by definition. Hence, the map $t \cdot c_3$ is null-homotopic as desired. □

Proof of Theorem 6.1. The map $t \cdot c_*$ are obviously null-homotopic by Lemmas 6.2 and 6.3. □

The proof was an ad hoc computation in an algebraic way; however the theorem should be easily shown by a topological method:

Problem Does the t-vanishing of the map $c_* : H_n(BX) \to H^{\mathrm{gr}}_n(\mathrm{As}(X))$ hold for any $n \in \mathbb{N}$? Provide its topological proof. Further, how about non-connected quandles?

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