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Symmetry of symplectic derivation Lie algebras of free Lie algebras

By

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Abstract

We show that a certain symmetry exists in the stable irreducible decomposition of the Lie algebra consisting of symplectic derivations of the free Lie algebra generated by the first homology group of compact oriented surfaces.

§ 1. Introduction and statement of the main result

Let $\Sigma_{g,1}$ be a compact oriented surface of genus $g \geq 1$ with one boundary component and we denote $H_1(\Sigma_{g,1};\mathbb{Q})$ simply by H. Equipped with the skew symmetric bilinear form induced by the intersection pairing, the space H can be regarded as the standard symplectic vector space of dimension $2g$. Let $\mathcal{L}_H$ be the free Lie algebra generated by H and let $\mathcal{L}_H(k)$ be the degree k homogeneous part of it with respect to the natural grading. Let $\mathfrak{h}_{g,1}$ be the Lie algebra consisting of *symplectic* derivations of $\mathcal{L}_H$, which we call the symplectic derivation Lie algebra. It has a natural grading and the degree k part can be expressed as

$$\begin{aligned} \mathfrak{h}_{g,1}(k) &= \{D \in \mathrm{Hom}(H, \mathcal{L}_H(k+1)); D(\omega_0) = 0\} \\ &\cong \mathrm{Ker}\left(H \otimes \mathcal{L}_H(k+1) \xrightarrow{[\ ,\ ]} \mathcal{L}_H(k+2)\right), \end{aligned} \tag{1.1}$$

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where $\omega_0 = \sum_{i=1}^g [x_i, y_i] \in \mathcal{L}_H(2) \cong \wedge^2 H$ denotes the symplectic class (independent of a choice of a symplectic basis $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$ of H) and $D(\omega_0) = \sum_{i=1}^g([D(x_i), y_i] + [x_i, D(y_i)]) \in \mathcal{L}_H(k+2)$. The second isomorphism (see [8] for details) uses the Poincaré duality $H^* \cong H$ and the bracket operation $[\ ,\ ]$ on the Lie algebra $\mathcal{L}_H$. The Lie algebra $\mathfrak{h}_{g,1}$ plays a fundamental role first in the theory of Johnson homomorphisms and more recently in the cohomological study of the outer automorphism group $\mathrm{Out}F_n$ of free groups which was initiated by the Lie version of the theory of graph homology due to Kontsevich [6][7]. We refer to [9] for a recent survey and also to more recent papers including [3] where Enomoto and Satoh found a new series in the cokernel of the Johnson homomorphisms, our former paper [11], and [2] where Conant, Kassabov and Vogtmann present a remarkable new development.

Let $\mathrm{Sp}(2g, \mathbb{Q})$ be the symplectic group which we sometimes denote simply by Sp. If we fix a symplectic basis of H, then it can be considered as the standard representation of $\mathrm{Sp}(2g, \mathbb{Q})$. Each piece $\mathfrak{h}_{g,1}(k)$ is naturally an Sp-module so that it has an irreducible decomposition. It is well known that this decomposition stabilizes when g is sufficiently large.

The purpose of this note is to show that a certain symmetry exists in the structure of $\mathfrak{h}_{g,1}(k)$. To describe our symmetry, we have to consider $\mathfrak{h}_{g,1}(k)$ as a $\mathrm{GL}(2g, \mathbb{Q})$-module rather than an Sp-module. We define this structure by the identification of $\mathfrak{h}_{g,1}(k)$ with the *second* module in (1.1) which has a natural structure of a $\mathrm{GL}(2g, \mathbb{Q})$-module because the bracket operation is clearly a $\mathrm{GL}(2g, \mathbb{Q})$-morphism. It follows that each $\mathfrak{h}_{g,1}(k)$ has an irreducible decomposition as a $\mathrm{GL}(2g, \mathbb{Q})$-module and this decomposition also stabilizes when g is sufficiently large. This stable decomposition can be described as a linear combination of Young diagrams with $(k+2)$ boxes. The original decomposition as an Sp-module can be obtained by applying the restriction law associated with the pair $(\mathrm{GL}(2g, \mathbb{Q}), \mathrm{Sp}(2g, \mathbb{Q}))$.

To describe our result, we need one more technical term from the theory of representations of symmetric groups (see [4] for example). Any Young diagram λ with k boxes defines an irreducible representation V_λ of the symmetric group $\mathfrak{S}_k$. The *conjugate* Young diagram of λ, denoted by λ', is the one obtained by interchanging the roles of rows and columns. For example $[4^2 1^4]' = [62^3]$. As is well known, the associated representation $V_{\lambda'}$ is isomorphic to the tensor product of V_λ with the alternating representation $V_{[1^k]}$.

Now we can state our main result.

Theorem 1.1. *Let $\mathfrak{h}_{g,1}$ be the symplectic derivation Lie algebra of the free Lie algebra $\mathcal{L}_H$ and let $\mathfrak{h}_{g,1}(k) = \mathrm{Ker}\big([\ ,\ ] : H \otimes \mathcal{L}_H(k+1) \to \mathcal{L}_H(k+2)\big)$ be the degree k part of it which we understand as a $\mathrm{GL}(2g, \mathbb{Q})$-module.*

Assume that $k \equiv 2$ or $3 \bmod 4$. Then the stable irreducible decomposition of

$\mathfrak{h}_{g,1}(k)$, *which is expressed as a linear combination of Young diagrams with* $(k+2)$ *boxes, is symmetric with respect to taking the conjugate Young diagrams.*

The first two cases are given by $\mathfrak{h}_{g,1}(2) = [2^2], \mathfrak{h}_{g,1}(3) = [31^2]$ (see [5] [1]) both of which are symmetric (i.e. self-conjugate) Young diagrams. Here, for simplicity, we use the same symbol of Young diagram for the corresponding irreducible $\mathrm{GL}(2g, \mathbb{Q})$-module. The summand $[2^2]$ plays a very important role in Hain's infinitesimal presentation of the Torelli groups while the summand $[31^2]$ is the first place where the trace maps introduced in [8] appear. In higher degrees, there occur plural irreducible components. In these cases, our result shows that the multiplicity m_λ corresponding to a Young diagram λ is equal to the multiplicity $m_{\lambda'}$ corresponding to the conjugate Young diagram λ'.

Remark 1. By combining the technique of this paper with that of [10], we obtain a structure theorem for the Lie subalgebra $\mathfrak{h}_{g,1}^{\mathrm{Sp}} \subset \mathfrak{h}_{g,1}$ consisting of Sp-invariant elements. More precisely, we obtain an orthogonal basis, with respect to a certain canonical metric, for each degree $2k$ part $\mathfrak{h}_{g,1}^{\mathrm{Sp}}(2k)$ which is parametrized by the set of partitions of k (equivalently the set of Young diagrams with k boxes) each element of which represents an eigenspace of mutually different eigenvalue. This gives, in particular, a complete description of the degeneration of $\mathfrak{h}_{g,1}^{\mathrm{Sp}}(2k)$ from the stable range of g one by one down to the case of genus one which corresponds to the eigenspace with the *largest* eigenvalue. This orthogonal direct sum decomposition should be useful in the study of the arithmetic mapping class group, cohomology of $\mathrm{Out}F_n$ and eventually the cohomology of the group of homology cobordism classes of homology cylinders. Details are discussed in [13].

Remark 2. Besides the theoretical meaning, our main theorem can be used to make explicit computer calculations concerning the structure of the Lie algebra $\mathfrak{h}_{g,1}$ as well as other related Lie algebras considerably easier. See our paper [12] for details. In fact, we noticed this symmetry when we tried to overcome memory problems which arose in our earlier computer calculations. With the help of this technique, we have determined the irreducible decomposition of $\mathfrak{h}_{g,1}(k)$ for all $k \leq 20$. For example, the dimensions of $\mathfrak{h}_{g,1}^{\mathrm{Sp}}(18)$ and $\mathfrak{h}_{g,1}^{\mathrm{Sp}}(20)$ are given by the following tables.

	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$
$\mathfrak{h}_{g,1}(18)^{\mathrm{Sp}}$	57	100908	888099	1548984	1710798
$\mathfrak{h}_{g,1}(20)^{\mathrm{Sp}}$	108	869798	12057806	25062360	29129790

	$g=6$	$g=7$	$g=8$	$g\geq 9$
$\mathfrak{h}_{g,1}(18)^{\mathrm{Sp}}$	1728591	1729620	1729656	1729657
$\mathfrak{h}_{g,1}(20)^{\mathrm{Sp}}$	29688027	29728348	29729957	29729988

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§ 2. Character formulae

In [6][7], Kontsevich described the following result.

Theorem 2.1 (Kontsevich [7, Theorem 3.2]). *Let W_k be the $\mathfrak{S}_{k+2}$-module with character*

$$\begin{aligned}
\chi(1^{k+2}) &= k!, \\
\chi(1^1 a^b) &= (b-1)! a^{b-1} \mu(a) \quad (a \geq 2, b \geq 1, ab = k+1), \\
\chi(a^b) &= -(b-1)! a^{b-1} \mu(a) \quad (a \geq 2, b \geq 1, ab = k+2),
\end{aligned}$$

and χ vanishes on all the other conjugacy classes, where μ denotes the Möbius function. Then there exists an isomorphism

$$\mathfrak{h}_{g,1}(k) \cong H^{\otimes(k+2)} \otimes_{A_{k+2}} W_k$$

of $\mathrm{Sp}(2g, \mathbb{Q})$*-modules, where $A_{k+2} = \mathbb{Q}\mathfrak{S}_{k+2}$ denotes the group algebra of $\mathfrak{S}_{k+2}$.*

Kontsevich mentioned that for the proof of the above theorem he developed a language of non-commutative geometry but explicit description was not given. Here we would like to present an argument, which we noticed some time ago, showing that the above theorem can be deduced directly from a classical result in the theory of free Lie algebras. We mention that a combinatorial treatment of the above theorem of Kontsevich is given in [3, Remark 7.5].

Let F_n be a free group of rank $n \geq 2$ and let $\mathcal{L}_n$ be the free Lie algebra generated by $H_1(F_n; \mathbb{Q}) \cong \mathbb{Q}^n$. We denote by $\mathcal{L}_n(k)$ the degree k part with respect to the natural grading. When $n = 2g$, we can identify $H_1(F_n; \mathbb{Q})$ and $\mathcal{L}_n(k)$ with H and $\mathcal{L}_H(k)$ as $\mathrm{GL}(n, \mathbb{Q})$-modules.

The classical character formula for the $\mathrm{GL}(n, \mathbb{Q})$-module $\mathcal{L}_n(k)$ ([14, Theorem 8-3]) can be phrased, somewhat differently from the usual expression, as follows.

Theorem 2.2 (Witt-Brandt, see [14]). *Let L_k be the $\mathfrak{S}_k$-module with character*

$$\begin{aligned}
\chi(1^k) &= (k-1)!, \\
\chi(a^b) &= (b-1)! a^{b-1} \mu(a) \quad (a \geq 2, b \geq 1, ab = k),
\end{aligned}$$

and χ *vanishes on all the other conjugacy classes. Then there exists an isomorphism*

$$\mathcal{L}_n(k) \cong H_1(F_n;\mathbb{Q})^{\otimes k} \otimes_{A_k} L_k$$

as a $\mathrm{GL}(n,\mathbb{Q})$*-module.*

Proof of Theorem 2.1. Recall that we identify $\mathfrak{h}_{g,1}(k)$ with the $\mathrm{GL}(2g,\mathbb{Q})$-module described in the second line of (1.1). It follows that we have a short exact sequence

$$0 \longrightarrow \mathfrak{h}_{g,1}(k) \longrightarrow H \otimes \mathcal{L}_H(k+1) \longrightarrow \mathcal{L}_H(k+2) \longrightarrow 0 \tag{2.1}$$

of $\mathrm{GL}(2g,\mathbb{Q})$-modules. By Theorem 2.2, we have an isomorphism

$$\mathcal{L}_H(k+2) \cong H^{\otimes(k+2)} \otimes_{A_{k+2}} L_{k+2}$$

and the character of L_{k+2}, denoted by χ_{k+2}, is given by

$$\begin{aligned} \chi_{k+2}(1^{k+2}) = (k+1)!, \quad \chi_{k+2}(a^b) = (b-1)!a^{b-1}\mu(a), \\ \chi_{k+2}(\text{other conjugacy class}) = 0. \end{aligned} \tag{2.2}$$

On the other hand, it can be seen that the tensor product $H \otimes \mathcal{L}_H(k+1)$ corresponds to the induced representation $\mathrm{Ind}_{\mathcal{S}_{k+1}}^{\mathcal{S}_{k+2}} L_{k+1}$. By the well-known formula for the character of induced representations (see e.g. [4]), we can deduce that the character of this induced representation, denoted by χ_{k+1}^{k+2}, is given by

$$\begin{aligned} \chi_{k+1}^{k+2}(1^{k+2}) = (k+2)k!, \quad \chi_{k+1}^{k+2}(1^1a^b) = (b-1)!a^{b-1}\mu(a), \\ \chi_{k+1}^{k+2}(\text{other conjugacy class}) = 0. \end{aligned} \tag{2.3}$$

Here the number $k+2$ appearing in the upper line corresponds to the index of the subgroup $\mathfrak{S}_{k+1} \subset \mathfrak{S}_{k+2}$. Now in view of the exact sequence (2.1), the module $\mathfrak{h}_{g,1}(k)$ corresponds to the virtual representation $\mathrm{Ind}_{\mathcal{S}_{k+1}}^{\mathcal{S}_{k+2}} L_{k+1} - L_{k+2}$ whose character is given by the difference (2.3)$-$(2.2). This is the required formula and the proof is complete. □

§ 3. Proof of the main result and an application

To prove the main theorem, we prepare the following simple technical lemma.

Lemma 3.1. *Let c be a positive integer which is not congruent to* $2 \bmod 4$. *Then for any decomposition*

$$c = ab$$

with positive integers a, b, the following condition holds:

$(*)$ *if a is even and $\mu(a) \neq 0$, then b is even.*

Proof. First of all, the equality $4m+2=2(2m+1)$ shows that the condition on c is necessary. Now assume the required condition. If c is odd, then c is not divisible by any even number so that $(*)$ clearly holds. Hence we can assume $c=4d$ for some d. If $c=ab$ such that a is even and $\mu(a)\neq 0$, then a is not divisible by 4 because otherwise $\mu(a)=0$. It follows that b is even and the proof is complete. □

Proof of Theorem 1.1. Let $\lambda=[\lambda_1,\cdots,\lambda_h]$ be a Young diagram with $(k+2)$ boxes and let V_λ be the corresponding irreducible representation of the symmetric group $\mathfrak{S}_{k+2}$. Then the multiplicity m_λ of V_λ in W_k is expressed as

$$m_\lambda=\frac{1}{(k+2)!}\sum_{\gamma\in\mathfrak{S}_{k+2}}\chi_k(\gamma)\chi_\lambda^V(\gamma)$$

where χ_k is the character of L_k given by (2.2) and χ_λ^V denotes the character of V_λ. On the other hand, by Theorem 2.1 the value $\chi_k(\gamma)$ may be non-zero only in the cases where the conjugacy class of $\gamma\in\mathfrak{S}_{k+2}$ is one of the following three types $1^{k+2}, 1^1a^b, a^b$. In the latter two types, two positive integers a,b satisfy the equality $ab=k+1$ or $ab=k+2$. If k satisfies the required condition, namely if $k\equiv 2$ or $3 \bmod 4$, then both $k+1$ and $k+2$ are *not* congruent to 2 mod 4. Therefore by Lemma 3.1, we conclude that the two numbers a,b satisfy one of the following three conditions

(i) a is odd

(ii) a is even and $\mu(a)=0$

(iii) a is even and b is even.

The sign of any odd cycle a is $+1$ while the sign of any even cycle a is -1. Therefore if γ satisfies (i) or (iii) of the above conditions, then the sign of γ is $+1$. Of course the sign of 1^{k+2} is $+1$. We can now conclude the following. If $\chi_k(\gamma)\neq 0$, then the sign of γ, which is the same as the character value $\chi^V_{[1^{k+2}]}(\gamma)$ of γ on the alternating representation $[1^{k+2}]$, is $+1$. It follows that we have the equality

$$\chi_\lambda^V(\gamma)=\chi_{\lambda'}^V(\gamma)$$

on such element γ where λ' denotes the conjugate Young diagram of λ. Summing up, we now conclude

$$m_\lambda=m_{\lambda'}$$

which finishes the proof. □

Remark 3. Under the same condition as in Theorem 1.1, we have an isomorphism

$$W_k\cong W_k\otimes V_{[1^{k+2}]}$$

of $\mathfrak{S}_{k+2}$-modules.

As for the case of the free Lie algebra, we have the following symmetry which was already proved in [15]. Here we describe a proof for the sake of self-containedness of this note.

Proposition 3.2 (Zhuravlev [15, Proposition 4.1]). *Assume that* $k \equiv 0,\ 1$ *or* $3 \mod 4$. *Then the stable irreducible decomposition of* $\mathcal{L}_n(k)$*, which is expressed as a linear combination of Young diagrams with* k *boxes, is symmetric with respect to taking the conjugate Young diagrams.*

Proof. The proof is given by replacing Theorem 2.1 with Theorem 2.2 in the above argument. □

As a sample application of our result, we show that a close relation exists between an earlier result of Asada and Nakamura in [1] and one particular result of Enomoto and Satoh in [3]. More precisely, it was proved in the former paper that the Sp-irreducible representation corresponding to the Young diagram $[2k, 2]$ occurs in $\mathfrak{h}_{g,1}(2k)$ with multiplicity one for any k. It is easy to see that this representation is the restriction of a unique $\mathrm{GL}(2g, \mathbb{Q})$ irreducible representation corresponding to the same Young diagram. On the other hand, it was proved in the latter paper that the $\mathrm{GL}(2g, \mathbb{Q})$ irreducible representation $[2^2, 1^{4m}]$ occurs in $\mathfrak{h}_{g,1}(4m+2)$ with multiplicity one for any m. If we set $k = 2m + 1$ in the former case, we see that both $\mathrm{GL}(2g, \mathbb{Q})$ irreducible representations corresponding to the Young diagrams $[4m + 2, 2]$ and $[2^2, 1^{4m}]$ occur in $\mathfrak{h}_{g,1}(4m + 2)$ with multiplicity one. Observe that these two Young diagrams are conjugate to each other. Therefore these two results are equivalent in the framework of our main result.

We have explicit general results concerning multiplicities of several types of Young diagrams which occur in $\mathfrak{h}_{g,1}$ other than those treated in [1][3]. However, here we omit them.

Finally we make a remark.

Remark 4. In the case of the Lie algebra $\mathfrak{a}_g = \bigoplus_k \mathfrak{a}_g(k)$ consisting of symplectic derivations of free associative algebra generated by H without unit, introduced by Kontsevich [6][7], it is easy to see that the $\mathrm{GL}(2g, \mathbb{Q})$ stable irreducible decomposition of degree k part $\mathfrak{a}_g(k)$ has the symmetry under taking conjugate Young diagrams for any odd k. This is because the sign of the cyclic permutation $(12 \cdots k + 2) \in \mathfrak{S}_{k+2}$ is

$+1$ for any odd k. For example, as $\mathrm{GL}(2g,\mathbb{Q})$-modules, we have

$$\begin{aligned}
\mathfrak{a}_g(1) &= [3] \oplus [1^3]\\
\mathfrak{a}_g(3) &= [5] \oplus [32] \oplus 2[31^2] \oplus [2^2 1] \oplus [1^5]\\
\mathfrak{a}_g(5) &= [7] \oplus 2[52] \oplus 3[51^2] \oplus 2[43] \oplus 5[421] \oplus 2[41^3] \oplus 3[3^2 1] \oplus 3[32^2] \oplus 5[321^2]\\
&\quad \oplus 3[31^4] \oplus 2[2^3 1] \oplus 2[2^2 1^3] \oplus [1^7]\\
\mathfrak{a}_g(7) &= [9] \oplus 3[72] \oplus 4[71^2] \oplus 6[63] \oplus 11[621] \oplus 6[61^3] \oplus 4[54] \oplus 18[531] \oplus 14[52^2]\\
&\quad \oplus 21[521^2] \oplus 8[51^4] \oplus 10[4^2 1] \oplus 18[432] \oplus 24[431^2] \oplus 24[42^2 1] \oplus 21[421^3]\\
&\quad \oplus 6[41^5] \oplus 6[3^3] \oplus 18[3^2 21] \oplus 14[3^3 1^3] \oplus 10[32^3] \oplus 18[32^2 1^2] \oplus 11[321^4]\\
&\quad \oplus 4[31^6] \oplus 4[2^4 1] \oplus 6[2^3 1^3] \oplus 3[2^2 1^5] \oplus [1^9]
\end{aligned}$$

and it is easy to see that these satisfy the required symmetry. In this case also, we have determined the irreducible decomposition of $\mathfrak{a}_g(k)$ for all $k \leq 20$.

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