RIMS Kôkyûroku Bessatsu
B66 (2017), 213–230

On 3–dimensional hyperbolic Coxeter pyramids

By

Yohei Komori* and Yuriko Umemoto**

Abstract

After classifying 3–dimensional hyperbolic Coxeter pyramids by means of elementary plane geometry, we calculate growth functions of corresponding Coxeter groups by using Steinberg formula and conclude that growth rates of them are always Perron numbers. We also calculate hyperbolic volumes of them and compare volumes with their growth rates. Finally we consider a geometric ordering of Coxeter pyramids compatible with their growth rates.

§ 1. Introduction

Let $\mathbb{H}^n$ denote hyperbolic n–space. A convex polyhedron $P \subset \mathbb{H}^n$ of finite volume is called a Coxeter polyhedron if all its dihedral angles are submultiples of π or 0.

Let G be a discrete group generated by finitely many reflections in hyperplanes containing facets of a Coxeter polyhedron $P \subset \mathbb{H}^n$ and denote by S the set of generating reflections. Then we call G a hyperbolic Coxeter group with respect to P. Note that the Coxeter polyhedron P itself is a fundamental domain of G. For each generator $s \in S$, one has $s^2 = 1$ while two distinct elements $s, s' \in S$ satisfy either no relation or provide the relation $(ss')^m = 1$ for some integer $m = m(s, s') \geq 2$. The first case arises if the corresponding mirrors admit a common perpendicular or intersect at the boundary of $\mathbb{H}^n$, while the second case arises if the mirrors intersect in $\mathbb{H}^n$.

For a pair (G, S) of a hyperbolic Coxeter group G and a finite set of generators S, we can define the word length $\ell_S(g)$ of $g \in G$ with respect to S by the smallest integer $n \geq 0$ for which there exist $s_1, s_2, \ldots, s_n \in S$ such that $g = s_1 s_2 \cdots s_n$. We assume

Received April 6, 2016. Revised May 3, 2017.
2010 Mathematics Subject Classification(s): Primary 20F55, Secondary 20F65.
Key Words: *Coxeter group, growth function, growth rate, Perron number.*
*Department of Mathematics, School of Education, Waseda University, Nishi-Waseda 1-6-1, Shinjuku, Tokyo 169-8050, Japan.
e-mail: ykomori@waseda.jp
**Department of Liberal Studies, National Institute of Technology, Nara College, 22 Yata-cho, Yamatokoriyama, Nara 639-1080, Japan.
e-mail: umemoto@libe.nara-k.ac.jp

that $\ell_S(1_G) = 0$ for the identity element $1_G \in G$. The growth series $f_S(t)$ of (G, S) is the formal power series $\sum_{k=0}^{\infty} a_k t^k$ where a_k is the number of elements $g \in G$ satisfying $\ell_S(g) = k$. Since G is an infinite group, the growth rate of (G, S), $\tau = \limsup_{k\to\infty} \sqrt[k]{a_k}$ satisfies $1 \leq \tau \leq |S|$ since $a_k \leq |S|^k$, where $|S|$ denotes the cardinality of S. By means of the Cauchy–Hadamard formula, it turns to the condition $1/|S| \leq R \leq 1$ for the radius of convergence R of $f_S(t)$. Therefore $f_S(t)$ is not only a formal power series but also an analytic function of $t \in \mathbb{C}$ on the disk $|t| < R$. It is known to be of exponential growth [5] and its growth rate $\tau > 1$ is an algebraic integer (cf. [3]).

In this paper, we classify Coxeter pyramids of finite volume in $\mathbb{H}^3$ by using the elementary plane geometry. Andreev classified all Coxeter polyhedra of finite volume in $\mathbb{H}^3$ [1]. Here we present his idea in terms of circles and rectangles in Euclidean plane by using the link of the apex $v = \infty$ of a pyramid in upper half space model $\mathbb{U}^3$ (see Section 2.3). Note that Tumarkin [13] already classified all Coxeter pyramids of finite volume in $\mathbb{H}^n$ whose combinatorial type is a pyramid over a product of two simplices. Then by means of our former result [8] essentially, we show that the growth rates of Coxeter pyramids in $\mathbb{H}^3$ are Perron numbers, that is, real algebraic integers $\tau > 1$ all of whose conjugates have strictly smaller absolute values. We also calculate hyperbolic volumes of Coxeter pyramids and check that volumes are not proportional to growth rates. Finally we introduce a partial ordering among Coxeter pyramids by using inclusion relation and show that this ordering is proportional to growth rates.

The paper is organized as follows. In Section 2 after reviewing the upper half space model of 3–dimensional hyperbolic space and the link of a vertex of a convex polyhedron, we classify Coxeter pyramids in $\mathbb{H}^3$. In Section 3 we collect some basic results of the growth functions and growth rates of Coxeter groups and show that growth rates of Coxeter pyramids are always Perron numbers. In Section 4 we calculate hyperbolic volumes of Coxeter pyramids and compare them with their growth rates. Finally, in Section 5 we introduce a geometric ordering of Coxeter pyramids compatible with their growth rates.

§ 2. Coxeter pyramids and their classification

§ 2.1. The hyperbolic 3–space $\mathbb{U}^3$ and its isometry group $I(\mathbb{U}^3)$

The Euclidean 3–space $(\mathbb{R}^3, |dx|)$ contains the Euclidean plane

$$\mathbb{E}^2 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$$

which defines the upper half space of $\mathbb{R}^3$

$$\mathbb{U}^3 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}.$$

Then $(\mathbb{U}^3, \frac{|dx|}{x_3})$ is a model of the hyperbolic 3–space, so called the *upper half space model.* Let us denote by $\partial\mathbb{U}^3 := \mathbb{E} \cup \{\infty\}$ the boundary at infinity of $\mathbb{U}^3$ in the extended space $\hat{\mathbb{R}}^3 := \mathbb{R}^3 \cup \{\infty\}$, and $\overline{\mathbb{U}^3} := \mathbb{U}^3 \cup \partial\mathbb{U}^3$ the closure of $\mathbb{U}^3$ in $\hat{\mathbb{R}}^3$.

A subset $S \subset \mathbb{U}^3$ is called a *hyperplane* of $\mathbb{U}^3$ if it is a Euclidean hemisphere or half plane orthogonal to $\mathbb{E}^2$. If we restrict the hyperbolic metric of $\mathbb{U}^3$ to S, then it is a model of the hyperbolic plane.

Any isometry φ of $(\mathbb{U}^3, \frac{|dx|}{x_3})$ can be represented as a composition of reflections with respect to hyperplanes. Also φ extends to the boundary map $\hat{\varphi}$ of $\partial\mathbb{U}^3$ so that $\hat{\varphi}$ is a composition of reflections with respect to the boundaries of hyperplanes, that is, Euclidean circles or lines in $\mathbb{E}^2$. This correspondence induces the isomorphism between the isometry group $I(\mathbb{U}^3)$ of $\mathbb{U}^3$ and Möbius transformation group $M(\hat{\mathbb{C}})$ of the Riemann sphere $\hat{\mathbb{C}}$ after identifying $\mathbb{E}^2$ with $\mathbb{C}$ by $z = x_1 + ix_2$. $M(\hat{\mathbb{C}})$ is generated by $PSL(2, \mathbb{C})$ and J the reflection with respect to the real axis of $\mathbb{C}$, defined by $J(z) = \bar{z}$. In particular, $I(\mathbb{U}^3)$ contains isometries corresponding to rotations, dilations centered at $0 \in \mathbb{C}$, the reflection with respect to the real axis of $\mathbb{C}$ and parallel translations of $\mathbb{C}$.

§ 2.2. Convex polyhedra and their vertex links

A *closed half space* H_S is defined by the closed domain of $\mathbb{U}^3$ bounded by a hyperplane S. We define a *convex polyhedron* as a closed domain P of $\mathbb{U}^3$ which can be written as the intersection of finitely many closed half spaces; (it is also called a *finite-sided convex* 3*–polyhedron in* $\mathbb{U}^3$ in [10]):

$$P = \bigcap H_S.$$

In this presentation of P, we assume that $F_S := P \cap S$ is a hyperbolic polygon of S. F_S is called a *facet* of P and S is called the supporting plane of F_S. The *dihedral angle* $\angle ST$ between two facets F_S and F_T is defined as follows: if the intersection of two supporting planes S and T is nonempty, let us choose a point $x \in S \cap T$ and consider the outer-normal vectors $e_S, e_T \in \mathbb{R}^3$ of S and T with respect to P starting from x. Then the dihedral angle $\angle ST$ is defined by the real number $\theta \in [0, \pi)$ satisfying

$$\cos\theta = -\langle e_S, e_T\rangle$$

where $\langle\cdot,\cdot\rangle$ denote the Euclidean inner product of $\mathbb{R}^3$. It is easy to see that this does not depend on the choice of the base point $x \in S \cap T$. If S and T are parallel, that is, the closures of S and T in $\overline{\mathbb{U}^3}$ only intersect at a point on $\partial\mathbb{U}^3$, we define the dihedral angle $\angle ST$ is equal to zero, while if S and T are ultra-parallel, that is, the closures of S and T in $\overline{\mathbb{U}^3}$ never intersect, we do not define the dihedral angle.

If the intersection of two facets F_S and F_T of a convex polyhedron P consists of a geodesic segment, it is called an *edge* of P. If the intersection $\bigcap F_S$ of more than two facets is a point, it is called a *vertex* of P. If the closures of F_S and F_T in $\overline{\mathbb{U}^3}$ only intersect at a point on $\partial\mathbb{U}^3$, it is called an *ideal vertex* of P. It should be remarked that the hyperbolic volume of a non-compact convex polyhedron P is finite if and only if the closures of P in $\overline{\mathbb{U}^3}$ consists of P itself and ideal vertices.

Let $\Sigma = S(v;r)$ be the hyperbolic sphere of radius r centered at $v \in \mathbb{U}^3$. If we restrict the hyperbolic metric of $\mathbb{U}^3$ to Σ, then it is a model of the 2–dimensional spherical space. To analyse local geometry of P at a vertex v, it is useful to study an intersection of P with a hyperbolic sphere of small radius centered at v, which is called a *vertex link* $L(v)$ of P at v:

Theorem 2.1. (cf. [10, Theorem 6.4.1]) *Let $v \in \mathbb{U}^3$ be a vertex of a convex polyhedron P in $\mathbb{U}^3$ and Σ be a hyperbolic sphere of $\mathbb{U}^3$ based at v such that Σ meets just the facets of P incident with v. Then the link $L(v) := P \cap \Sigma$ of v in P is a spherical convex polygon in the hyperbolic sphere Σ. If F_S and F_T are two facets of P incident with v, then F_S and F_T are adjacent facets of P if and only if $F_S \cap \Sigma$ and $F_T \cap \Sigma$ are adjacent sides of $L(v)$. If F_S and F_T are adjacent facets of P incident with v, then the spherical dihedral angle between $F_S \cap \Sigma$ and $F_T \cap \Sigma$ in Σ is equal to the hyperbolic dihedral angle between the supporting hyperplanes S and T in $\mathbb{U}^3$.*

A *horosphere* Σ of $\mathbb{U}^3$ based at $v \in \partial\mathbb{U}^3$ is defined by a Euclidean sphere in $\mathbb{U}^3$ tangent to $\mathbb{E}^2$ at v when $v \in \mathbb{E}^2$, or a Euclidean plane in $\mathbb{U}^3$ parallel to $\mathbb{E}^2$ when $v = \infty$. If we restrict the hyperbolic metric of $\mathbb{U}^3$ to Σ, then it is a model of the Euclidean plane. Similar to the previous case, it is useful to study an intersection of P with a small horosphere centered at an ideal vertex v, which is called an *ideal vertex link* $L(v)$ of P at v:

Theorem 2.2. (cf. [10, Theorem 6.4.5]) *Let $v \in \partial\mathbb{U}^3$ be an ideal vertex of a convex polyhedron P in $\mathbb{U}^3$ and Σ be a horosphere of $\mathbb{U}^3$ based at v such that Σ meets just the facets of P incident with v. Then the link $L(v) := P \cap \Sigma$ of v in P is a Euclidean convex polygon in the horosphere Σ. If F_S and F_T are two facets of P incident with v, then F_S and F_T are adjacent facets of P if and only if $F_S \cap \Sigma$ and $F_T \cap \Sigma$ are adjacent sides of $L(v)$. If F_S and F_T are adjacent facets of P incident with v, then the Euclidean dihedral angle between $F_S \cap \Sigma$ and $F_T \cap \Sigma$ in Σ is equal to the hyperbolic dihedral angle between the supporting hyperplanes S and T in $\mathbb{U}^3$.*

By means of this theorem we can visualize $L(v)$ of P at v as follows: for an ideal vertex $v \in \mathbb{E}^2$, applying $\varphi \in I(\mathbb{U}^3)$ with $\hat{\varphi}(z) = \frac{1}{z-v}$ to P, we may assume that $v = \infty$. Since the vertical projection

$$\nu : \mathbb{U}^3 \to \mathbb{E}^2 \ ; \ (x_1, x_2, x_3) \mapsto (x_1, x_2)$$

maps a horosphere Σ isometrically onto the Euclidean plane $\mathbb{E}^2$, the vertex link $L(\infty)$ can be realized as a Euclidean polygon of $\mathbb{E}^2$, which will be a key idea for us to classify Coxeter pyramids in the next section.

§ 2.3. Coxeter pyramids and their classification

A convex polyhedron P of finite volume in $\mathbb{U}^3$ is called a *Coxeter polyhedron* if all its dihedral angles are submultiples of π or 0. Any Coxeter polyhedron can be described by a *Coxeter graph* defined as follows. The nodes of a Coxeter graph correspond to facets of P. Two nodes are joined by an edge labeled m if the corresponding dihedral angle is equal to π/m $(m \geqq 2)$ or $m = \infty$. By convention we omit labels $m = 3$ and delete edges for $m = 2$. Nodes are connected by a bold edge when the corresponding facets are parallel in $\mathbb{U}^3$.

Let $P \subset \mathbb{U}^3$ be a Coxeter pyramid, that is, a Coxeter polyhedron with five facets consisting of one quadrangular *base facet* and four triangular *side facets*. We call the vertex v an *apex* if it is opposite to a base facet. Note that the apex v should be an ideal vertex of P; in fact, if v is in $\mathbb{U}^3$, the vertex link $L(v)$ would be isometric to a spherical Coxeter quadrangle by Theorem 2.1. It contradicts to the absence of spherical Coxeter quadrangles by means of Gauss–Bonnet formula. By Theorem 2.2, the ideal vertex link of $L(v)$ is isometric to a Euclidean Coxeter quadrangle, that is, a Euclidean rectangle.

Applying isometries of $\mathbb{U}^3$, we now normalize a Coxeter pyramid P as follows: we may assume that the apex v is ∞ (see Figure 1), the support plane of the base facet is the hemisphere of radius one centered at the origin of $\mathbb{E}^2$, and side facets are parallel to x_1-axis or x_2-axis of $\mathbb{E}^2$. The vertex link $L(\infty)$ of the apex ∞ of P is projected on $\mathbb{E}^2$ by the vertical projection $\nu : \mathbb{U}^3 \to \mathbb{E}^2$, and we call its image $\nu(L(\infty))$ the *projected link*. It is a rectangle in the closed unit disk in $\mathbb{E}^2$, whose edges are parallel to x_1-axis or x_2-axis. Let A, B, C, D be the edges bounding the projected link $\nu(L(\infty))$ (see Figure 2), and denote by H_A, H_B, H_C and H_D the corresponding hyperplanes of $\mathbb{U}^3$. And let us denote by H_b the base facet. Then we have

$$\nu(L(\infty)) \subset \nu(\overline{H_b}). \tag{2.1}$$

Let us denote by $\theta_i = \angle H_i H_b$ the dihedral angle between the associated hyperplanes $H_i \subset \mathbb{U}^3$ $(i = A, B, C, D)$ supporting the side facets and the hyperplane H_b supporting the bace facet. Note that the Euclidean distance $d_{\mathbb{E}^2}(O, i)$ between the origin O and the edge i satisfies $d_{\mathbb{E}^2}(O, i) = \cos\theta_i$ (see Figure 2).

Applying isometries of $\mathbb{U}^3$ which induce a rotation of $\mathbb{E}^2$ centered at the origin or a reflection of $\mathbb{E}^2$ with respect to x_1-axis or x_2-axis, we may assume that the dihedral angles of a Coxeter pyramid, say $\theta_A = \pi/k$, $\theta_B = \pi/m$, $\theta_C = \pi/\ell$, $\theta_D = \pi/n$ satisfy

$$k \le \ell, \quad m \le n, \quad k \le m, \quad \text{and } \ \ell \le n \ \text{ when } \ k = m. \tag{2.2}$$

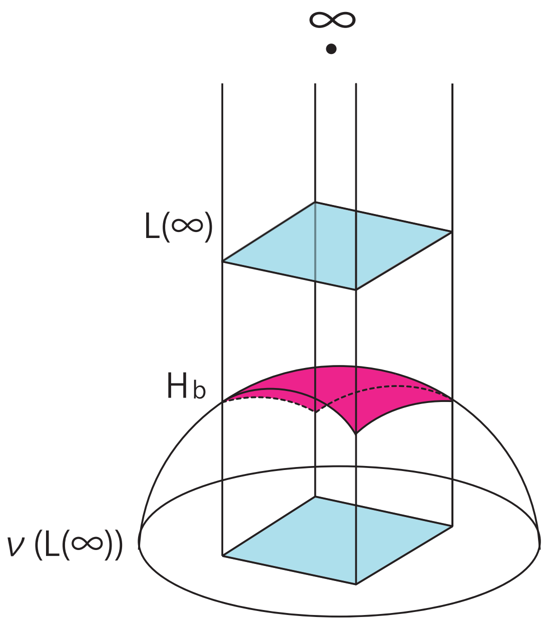


Figure 1. A pyramid in $\mathbb{U}^3$ with an apex ∞

To summarise,

Definition 2.3. A Coxeter pyramid P is called *normalized* if

- the apex of P is ∞.
- The support plane of the base facet H_b is the hemisphere of radius one centered at the origin of $\mathbb{E}^2$.
- Side facets H_A and H_C are parallel to x_1-axis while side facets H_B and H_D are parallel to x_2-axis of $\mathbb{E}^2$.

It is easy to see that there is a unique normalized Coxeter pyramid in its isometry class.

Now we can classify Coxeter pyramids in terms of Coxeter graphs by means of conditions (2.1) and (2.2). Andreev classified all Coxeter polyhedra of finite volume in $\mathbb{H}^3$ [1]. In the proof of the following theorem, we present his idea in terms of $\nu(L(\infty))$ and $\nu(\overline{H_b})$ in $\mathbb{E}^2$. It should be remarked that these graphs have first appeared in [13, Table 2].

Theorem 2.4. *Coxeter pyramids in* $\mathbb{U}^3$ *can be classified by Coxeter graphs in* Figure 3 *up to isometry.*

Proof. We determine all (k, ℓ, m, n) satisfying conditions (2.1) and (2.2). From Figure 4, the range of k for $\theta_A = \pi/k$ is $k = 2, 3, 4$. We can easily find (k, ℓ, m, n)

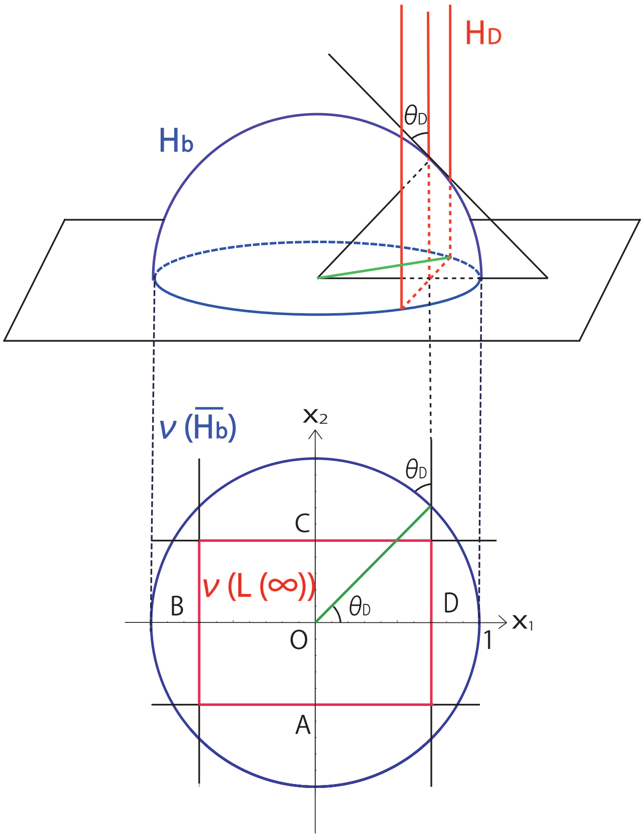
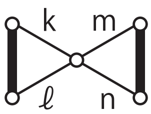


Figure 2. The image of the link $L(\infty)$ and the base facet H_b under the projection ν, and the dihedral angle $\theta_D = \angle H_D H_b$



$k = 2, 3, 4$; $m = 2, 3, 4$; $\ell = 3, 4$; $n = 3, 4$.

$k = 5, 6$; $m = 2, 3$; $\ell = 2, 3, 4, 5, 6$; $n = 3$.

Figure 3. Coxeter graphs of Coxeter pyramids in $\mathbb{H}^3$

satisfying conditions (2.1) and (2.2) from Figures 4 for $\theta_A = \pi/k$ with $k = 2, 3$ and 4 respectively. □

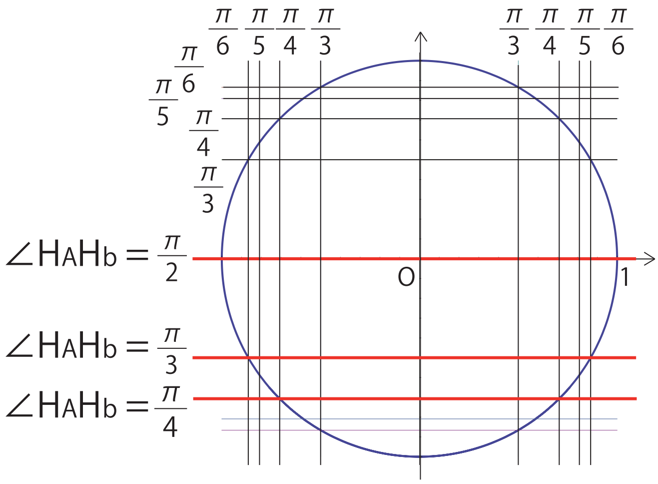


Figure 4. The case that $\theta_A = \pi/k$ with $k = 2, 3$ and 4

§ 3. The growth of Coxeter pyramids

§ 3.1. Growth functions of Coxeter groups

From now on, we consider the growth series of a Coxeter group (G, S) where S is its natural set of generating reflections. In practice, the growth series $f_S(t)$, which is analytic on $|t| < R$ where R is the radius of convergence of $f_S(t)$, extends to a rational function $P(t)/Q(t)$ on $\mathbb{C}$ by analytic continuation where $P(t), Q(t) \in \mathbb{Z}[t]$ are relatively prime. There are formulas due to Solomon [11, Corollary 2.3] and Steinberg [12, Corollary 1.29] to calculate the rational function $P(t)/Q(t)$ from the Coxeter graph of (G, S) (see also [3, Chapter 17] and [6, Chapter 3]), and we call this rational function the *growth function* of (G, S). Here we recall Steinberg's formula for the growth function of a Coxeter group.

Theorem 3.1. (Steinberg's formula)
For a Coxeter group (G, S), let us denote by (G_T, T) the Coxeter subgroup of (G, S) generated by a subset $T \subseteq S$, and let $f_T(t)$ be its growth function. Set $\mathcal{F} = \{T \subseteq S \mid G_T$ is finite$\}$. Then

$$\frac{1}{f_S(t^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(t)}.$$

§ 3.2. Growth rates of Coxeter groups

It is known that growth rates of hyperbolic Coxeter groups are bigger than 1 ([5]). The formulas of Solomon and Steinberg imply that $P(0) = 1$, and hence $a_0 = 1$ means that $Q(0) = 1$. Therefore the growth rate $\tau > 1$ becomes a real algebraic integer. If $t = R$ is the unique pole of $f_S(t) = P(t)/Q(t)$ on the circle $|t| = R$, then $\tau > 1$ is a real algebraic integer all of whose conjugates have strictly smaller absolute values. Such a real algebraic integer is called a *Perron number*.

For 2 and 3–dimensional cocompact hyperbolic Coxeter groups (having compact fundamental polyhedra), Cannon–Wagreich and Parry showed that their growth rates are Salem numbers ([2, 9]), where a real algebraic integer $\tau > 1$ is called a *Salem number* if τ^{-1} is an algebraic conjugate of τ and all algebraic conjugates of τ other than τ and τ^{-1} lie on the unit circle. It follows from the definition that a Salem number is a Perron number.

Kellerhals and Perren calculated the growth functions of all 4–dimensional cocompact hyperbolic Coxeter groups with at most 6 generators and checked numerically that their growth rates are not Salem numbers anymore while they are Perron numbers ([7]).

For the noncompact case, Floyd proved that the growth rates of 2–dimensional cofinite hyperbolic Coxeter groups are *Pisot–Vijayaraghavan numbers*, where a real algebraic integer $\tau > 1$ is called a Pisot–Vijayaraghavan number if all algebraic conjugates of τ other than τ lie in the open unit disk ([4]). A Pisot–Vijayaraghavan number is also a Perron number by definition.

From these results for low-dimensional cases, Kellerhals and Perren conjectured that the growth rates of hyperbolic Coxeter groups are Perron numbers in general.

§ 3.3. Calculation of the growth functions

Finding all finite Coxeter subgraphs of a Coxeter diagram for a Coxeter pyramid classified in the previous section, we can calculate growth functions of Coxeter pyramids by means of formulas due to Solomon and Steinberg. The denominator polynomials of growth functions of Coxeter pyramids are listed below, where (k, ℓ, m, n) represents the Coxeter graph in Figure 3. In Table 1 we also collect all growth rates of Coxeter pyramids (together with their volumes), which arise as the inverse of the smallest positive roots of denominator polynomials of the growth functions.

1. $(k, \ell, m, n) = (2, 3, 2, 3)$: $(t-1)(t^5 + 2t^4 + 2t^3 + t^2 - 1)$
2. $(2, 3, 2, 4)$: $(t-1)(t^7 + t^6 + 2t^5 + t^4 + 2t^3 + t - 1)$
3. $(2, 3, 2, 5)$: $(t-1)(t^{13} + t^{12} + 2t^{11} + 2t^{10} + 3t^9 + 2t^8 + 3t^7 + 2t^6 + 3t^5 + t^4 + 2t^3 + t - 1)$
4. $(2, 3, 3, 3)$: $(t-1)(t^4 + 2t^3 + t^2 + t - 1)$

5. $(2,4,2,4):\ (t-1)(t^4+2t^3+t^2+t-1)$
6. $(2,3,2,6):\ (t-1)(t^6+2t^5+t^4+t^3+t^2+t-1)$
7. $(2,3,3,4):\ (t-1)(t^7+2t^6+2t^5+2t^4+2t^3+t^2+t-1)$
8. $(2,4,3,3):\ (t-1)(t^7+2t^6+2t^5+3t^4+2t^3+t^2+t-1)$
9. $(2,3,3,5):\ (t-1)(t^{15}+2t^{14}+3t^{13}+5t^{12}+5t^{11}+7t^{10}+6t^9+7t^8+6t^7+6t^6+5t^5+3t^4+3t^3+t-1)$
10. $(2,3,3,6):\ (t-1)(t^8+2t^7+3t^6+3t^5+3t^4+2t^3+t^2+t-1)$
11. $(2,5,3,3):\ (t-1)(t^9+t^8+2t^6+t^4+t^3+2t-1)$
12. $(2,3,4,4):\ (t-1)(t^5+t^4+t^3+2t-1)$
13. $(2,6,3,3):\ (t-1)(2t^5+t^4+t^3+2t-1)$
14. $(2,3,4,5):\ (t-1)(t^{13}+t^{12}+2t^{11}+2t^{10}+3t^9+2t^8+3t^7+2t^6+3t^5+t^4+3t^3-t^2+2t-1)$
15. $(2,4,3,4):\ (t-1)(t^8+2t^7+3t^6+3t^5+3t^4+3t^3+t^2+t-1)$
16. $(2,3,4,6):\ (t-1)(t^8+2t^7+3t^6+4t^5+3t^4+3t^3+t^2+t-1)$
17. $(2,3,5,5):\ (t-1)(t^{11}+t^{10}+t^9+2t^8+t^7+2t^6+t^5+2t^4+t^3+2t-1)$
18. $(2,3,5,6):\ (t-1)(t^{14}+2t^{13}+3t^{12}+4t^{11}+5t^{10}+5t^9+5t^8+5t^7+5t^6+5t^5+3t^4+3t^3+t^2+t-1)$
19. $(3,3,3,3):\ (t-1)(t^2+2t-1)$
20. $(2,3,6,6):\ (t-1)(2t^6+3t^5+2t^4+2t^3+2t^2+t-1)$
21. $(2,4,4,4):\ (t-1)(2t^4+3t^3+2t^2+t-1)$
22. $(3,3,3,4):\ (t-1)(t^5+2t^4+t^2+2t-1)$
23. $(3,3,3,5):\ (t-1)(t^9+t^8-t^7+3t^6-t^5+t^4+2t^3-2t^2+3t-1)$
24. $(3,3,3,6):\ (t-1)(2t^7+t^6+4t^5+t^4+3t^3+2t-1)$
25. $(3,3,4,4):\ (t-1)(t^5+2t^4+t^3+t^2+2t-1)$
26. $(3,4,3,4):\ (t-1)(t^6+t^5+2t^4+t^3+t^2+2t-1)$
27. $(3,3,4,5):\ (t-1)(t^9+t^8+2t^6+3t^3-2t^2+3t-1)$
28. $(3,3,4,6):\ (t-1)(2t^8+3t^7+5t^6+6t^5+5t^4+4t^3+2t^2+t-1)$

29. $(3,3,5,5): \ (t-1)(t^7+t^6-t^5+2t^4-t^2+3t-1)$

30. $(3,3,5,6): \ (t-1)(2t^{10}+t^9+2t^8+t^7+2t^6+2t^5+t^4+2t^3+t^2+2t-1)$

31. $(3,3,6,6): \ (t-1)(4t^5+t^4+2t^3+t^2+2t-1)$

32. $(3,4,4,4): \ (t-1)(2t^6+t^5+2t^4+2t^3+t^2+2t-1)$

33. $(4,4,4,4): \ (t-1)(4t^3+t^2+2t-1)$

§ 3.4. Arithmetic property of growth rates

Now we state the main theorem for growth rates of Coxeter pyramids.

Theorem 3.2. *The growth rates of Coxeter pyramids are Perron numbers.*

Proof. Let us denote the denominator polynomial of $f_S(t)$ by $(t-1)g(t)$. Then except 4 cases $(2,3,4,5)$, $(3,3,3,5)$, $(3,3,4,5)$ and $(3,3,5,5)$ which are numbered 14, 23, 27 and 29 respectively in the list in Section 3.3, $g(t)$ has a form

$$\sum_{k=1}^{n} b_k t^k - 1$$

where b_k is a non-negative integer and the greatest common divisor of $\{k \in \mathbb{N} \mid b_k \neq 0\}$ is 1. After multiplying $g(t)$ by $(t+1)$ or $(t+1)^2$, 4 exceptional cases also have this form:

$$\begin{aligned}
(2,3,4,5):\ & (t+1)(t^{13}+t^{12}+2t^{11}+2t^{10}+3t^9+2t^8+3t^7+2t^6+3t^5+t^4+3t^3 \\
& -t^2+2t-1) \\
& = t^{14}+2t^{13}+3t^{12}+4t^{11}+5t^{10}+5t^9+5t^8+5t^7+5t^6+4t^5+4t^4 \\
& +2t^3+t^2+t-1 \\
(3,3,3,5):\ & (t+1)(t^9+t^8-t^7+3t^6-t^5+t^4+2t^3-2t^2+3t-1) \\
& = t^{10}+2t^9+2t^7+2t^6+3t^4+t^2+2t-1 \\
(3,3,4,5):\ & (t+1)(t^9+t^8+2t^6+3t^3-2t^2+3t-1) \\
& = t^{10}+2t^9+t^8+2t^7+2t^6+3t^4+t^3+t^2+2t-1 \\
(3,3,5,5):\ & (t+1)^2(t^7+t^6-t^5+2t^4-t^2+3t-1) \\
& = t^9+3t^8+2t^7+t^6+3t^5+t^4+t^3+4t^2+t-1.
\end{aligned}$$

In this case we already proved our claim in Lemma 1 of [8]; for the sake of readability, we recall its proof. Let us put $h(t) = \sum_{k=1}^{n} b_k t^k$. Observe $h(0) = 0$, $h(1) > 1$, and $h(t)$ is strictly monotonously increasing on the open interval $(0,1)$. By the intermediate value theorem, there exists a unique real number r_1 in $(0,1)$ such that $h(r_1)=1$, that is, $g(r_1)=0$. Since all coefficients a_k of the growth series $f_S(t)$ are non-negative integers, r_1 is the radius of convergence of the growth series $f_S(t)$. Only we have to show is that $g(t)$ has no zeros on the circle $|t|=r_1$ other than $t=r_1$. Consider a complex number $r_1e^{i\theta}$ on the circle $|t|=r_1$ where θ is $0\le\theta<2\pi$. If we assume $g(r_1e^{i\theta})=0$, that is, $h(r_1e^{i\theta})=1$,

$$1=\sum_{k=1}^{n} b_k r_1^k \cos k\theta \le \sum_{k=1}^{n} b_k r_1^k = 1.$$

This implies that $\cos k\theta = 1$ for all $k\in\mathbb{N}$ with $b_k\neq 0$. Now the assumption that the greatest common divisor of $\{k\in\mathbb{N}\mid b_k\neq 0\}$ is equal to 1 concludes that $\theta=0$, hence $t=r_1$ is the unique pole of $f_S(t)$ on the circle $|t|=r_1$. □

§ 4. The hyperbolic volumes of Coxeter pyramids

In this section, we calculate hyperbolic volumes of Coxeter pyramids by decomposing them into orthotetrahedra (cf. [10, Section 10.4]). For this purpose, we use the projected link which we have introduced in the former section. We express a Coxeter orthotetrahedron by the symbol $[\pi/\alpha,\pi/\beta,\pi/\gamma]$ if its Coxeter graph is $\circ\overset{\alpha}{—}\circ\overset{\beta}{—}\circ\overset{\gamma}{—}\circ$, and also express an ordinary orthotetrahedron by the same symbol $[\theta,\eta,\zeta]$ where θ,η,ζ are not necessarily submultiples of π.

First, let us consider the *Coxeter orthopyramid* $P=(k,\ell,m,n)=(2,\ell,2,n)$ whose Coxeter graph is linear (see the Coxeter graph in Figure 5, where a bullet express the base facet). Note that its projected link is the rectangle on the first quadrant and its boundary (see the picture of a projected link in Figure 5). Then, by the hyperplane passing through the apex ∞, origin, and the vertex $u=\bigcap_{i=b,C,D}\overline{H_i}$, P is decomposed into two orthotetrahedra ①$[\pi/2-\alpha,\alpha,\pi/\ell]$ and ②$[\alpha,\pi/2-\alpha,\pi/n]$ with vertex $v=\infty$, where $\alpha=\arctan(\cos(\pi/\ell)/\cos(\pi/n))$ (see the orthotetrahedra with corresponding numbers ①, ② in Figure 5). Note that we omit to mention right angles in the pictures of Figure 5.

The hyperbolic volume of each orthotetrahedron is calculated as follows (cf. [10, Theorem 10.4.6]) :

$$\mathrm{Vol}_{\mathbb{H}^3}[\frac{\pi}{2}-\alpha,\alpha,\frac{\pi}{\ell}]=\frac{1}{4}\Big(\Pi(\frac{\pi}{2}-\alpha+\frac{\pi}{\ell})+\Pi(\frac{\pi}{2}-\alpha-\frac{\pi}{\ell})+2\Pi(\frac{\pi}{2}-(\frac{\pi}{2}-\alpha))\Big)$$
$$\mathrm{Vol}_{\mathbb{H}^3}[\alpha,\frac{\pi}{2}-\alpha,\frac{\pi}{n}]=\frac{1}{4}\Big(\Pi(\alpha+\frac{\pi}{n})+\Pi(\alpha-\frac{\pi}{n})+2\Pi(\frac{\pi}{2}-\alpha)\Big)$$

where Π denote *Lobachevsky function* defined by the formula of period π (cf. [10, Section 10.4]):

$$\Pi(\theta)=-\int_0^{\theta}\log|2\sin t|dt.$$

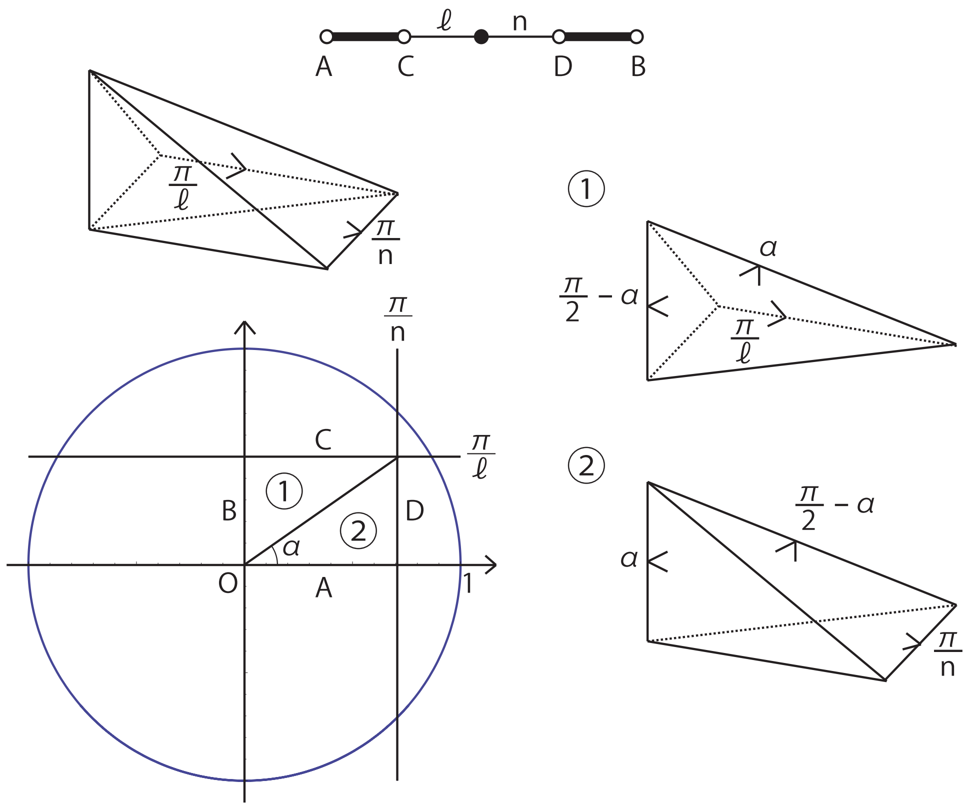


Figure 5. Decomposition of a Coxeter orthopyramid into orthotetrahedra

As a consequence, the hyperbolic volume of the orthopyramid is equal to the sum of volumes of two orthotetrahedra:

$$\mathrm{Vol}_{\mathbb{H}^3}(P) = \mathrm{Vol}_{\mathbb{H}^3}[\pi/2-\alpha, \alpha, \pi/\ell] + \mathrm{Vol}_{\mathbb{H}^3}[\alpha, \pi/2-\alpha, \pi/n].$$

Next, let us consider a Coxeter pyramid P which is not an orthopyramid. By the hyperplanes each of which passes through the apex ∞, origin and the vertex $u = \bigcap_{i\in J} \overline{H_i}$, where J is one of the set $\{b, A, B\}$, $\{b, B, C\}$, $\{b, C, D\}$ and $\{b, D, A\}$, P is decomposed into orthopyramids (see Figure 6). Then, by decomposing each orthopyramid into two

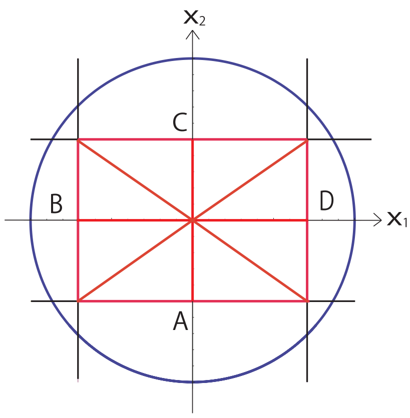


Figure 6. Decomposition of a Coxeter pyramid into orthotetrahedra

orthotetrahedra, we have a decomposition of P into orthotetrahedra, and we can calculate the hyperbolic volume of P. We collect hyperbolic volumes of all Coxeter pyramids in Table 1. We see that $(4, 4, 4, 4)$ has the maximal volume among all Coxeter pyramids in $\mathbb{H}^3$. But we can say much more; by considering the normalization defined in Definition 1, any pyramid P is contained in an ideal pyramid Q, that is, a pyramid all of whose vertices are ideal. Then by means of Theorem 10.4.12 of [10], we can calculate the volume of Q. Moreover the proof of Theorem 10.4.11 of [10] implies that $(4, 4, 4, 4)$, the pyramid obtained by taking the cone to ∞ from an ideal square on the hemisphere of radius 1 centered at the origin of $\mathbb{E}^2$ has the maximal volume among all ideal pyramids. Therefore

Theorem 4.1. *Among all hyperbolic pyramids in* $\mathbb{H}^3$*, the Coxeter pyramid of type* $(4, 4, 4, 4)$ *has maximal volume and as such the pyramid is unique.*

It should be remarked that the order of growths is not equal to that of volumes; for example the growth of $(2, 3, 2, 6)$ is smaller than that of $(2, 3, 3, 4)$ while the volume of $(2, 3, 2, 6)$ is bigger than that of $(2, 3, 3, 4)$.

(k,ℓ,m,n)	growth	volume	(k,ℓ,m,n)	growth	volume
$(2,3,2,3)$	1.73469	0.152661	$(2,3,5,6)$	2.40522	0.75522
$(2,3,2,4)$	1.90648	0.25096	$(3,3,3,3)$	2.41421	0.610644
$(2,3,2,5)$	1.9825	0.332327	$(2,3,6,6)$	2.42032	0.845785
$(2,3,3,3)$	2.06599	0.305322	$(2,4,4,4)$	2.45111	0.915966
$(2,4,2,4)$	2.06599	0.457983	$(3,3,3,4)$	2.53983	0.807242
$(2,3,2,6)$	2.01561	0.422892	$(3,3,3,5)$	2.58553	0.969976
$(2,3,3,4)$	2.1946	0.403621	$(3,3,3,6)$	2.60198	1.15111
$(2,4,3,3)$	2.23757	0.501921	$(3,3,4,4)$	2.64822	1.00384
$(2,3,3,5)$	2.24692	0.484988	$(3,4,3,4)$	2.65364	1.11256
$(2,3,3,6)$	2.26809	0.575553	$(3,3,4,5)$	2.68684	1.16657
$(2,5,3,3)$	2.30482	0.664654	$(3,3,4,6)$	2.70039	1.34771
$(2,3,4,4)$	2.30522	0.501921	$(3,3,5,5)$	2.72275	1.32931
$(2,6,3,3)$	2.33081	0.845785	$(3,3,5,6)$	2.73526	1.51044
$(2,3,4,5)$	2.34913	0.583287	$(3,3,6,6)$	2.74738	1.69157
$(2,4,3,4)$	2.35204	0.708943	$(3,4,4,4)$	2.75303	1.41789
$(2,3,4,6)$	2.36644	0.673853	$(4,4,4,4)$	2.84547	1.83193
$(2,3,5,5)$	2.38946	0.664654			

Table 1. Growth rates and hyperbolic volumes of all Coxeter pyramids in $\mathbb{H}^3$

§ 5. A geometric order of Coxeter pyramids compatible with their growth rates

We define a natural order of Coxeter pyramids in a geometric way. For Coxeter pyramids P_1 and P_2, we define $P_1 \preceq P_2$ if there exists $\varphi \in I(\mathbb{U}^3)$ such that $\varphi(P_1) \subset P_2$. It is easy to see that this binary relation is a partial order of the set of Coxeter pyramids. We also denote $G_1 \preceq G_2$ if corresponding Coxeter pyramids satisfy $P_1 \preceq P_2$. We present this geometric order in terms of (k,ℓ,m,n) which appeared in Section 2.3.

We study the relationship between this partial order and the growth rates of reflection groups. First, we shall state the following lemma which makes us to get back this order of reflection groups to that of the projected links of the apex ∞ of their fundamental Coxeter pyramids.

Lemma 5.1. *Let G_1 and G_2 be Coxeter groups defined by Coxeter pyramids. Then $G_1 \preceq G_2$ if and only if there exist corresponding Coxeter pyramids P_1 and P_2 with apex ∞ and bounded by a common hemisphere of radius 1 centered at the origin, such that $P_1 \subseteq P_2$. Furthermore, the projected links $\Delta_1 := \nu(L_1(\infty))$ and $\Delta_2 := \nu(L_2(\infty))$ on $\mathbb{E}^2$ of P_1 and P_2 satisfy $\Delta_1 \subseteq \Delta_2$ in this case.*

Proof. As we have already seen in Section 2, there exists a Coxeter pyramid P_i $(i = 1, 2)$ of G_i transferred to have the apex ∞ and bounded by a common hemisphere of radius 1 centered at the origin. Therefore since the side facets of P_1 and P_2 are parallel to each other, $P_1 \subseteq P_2$ if and only if $\Delta_1 \subseteq \Delta_2$. □

Proposition 5.2. *For Coxeter groups G_1 and G_2 defined by Coxeter pyramids, $G_1 \preceq G_2$ if and only if there exist corresponding Coxeter pyramids P_1 and P_2 with numbering (k', ℓ', m', n') and (k'', ℓ'', m'', n'') described in* Figure 3 *satisfying $k' \leq k''$, $\ell' \leq \ell''$, $m' \leq m''$, and $n' \leq n''$.*

Proof. It is obvious because the side facets of P_1 and P_2 are parallel to each other. □

We describe the order relations of Coxeter pyramids in Figure 7; each number (k, ℓ, m, n) represents the corresponding Coxeter pyramid and we attach its growth rate also. From this chart, we can see the following result.

Corollary 5.3. *Let G_1 and G_2 be two Coxeter groups defined by Coxeter pyramids, and τ_1 and τ_2 be their growth rates respectively. Then $\tau_1 \leq \tau_2$ if $G_1 \preceq G_2$.*

Finally it should be remarked that the hyperbolic volume of the defining Coxeter pyramid of G_1 is smaller than or equal to that of G_2 if $G_1 \preceq G_2$ from the definition, while the converse is not true in general.

Acknowledgement

The authors thank Professor Ruth Kellerhals for telling them the reference [13]. They are also grateful to Professor Michihiko Fujii for his generosity to their submission of this article. The first author was partially supported by Grant-in-Aid for Scientific Research(C) (26400151), Ministry of Education, Science and Culture of Japan.

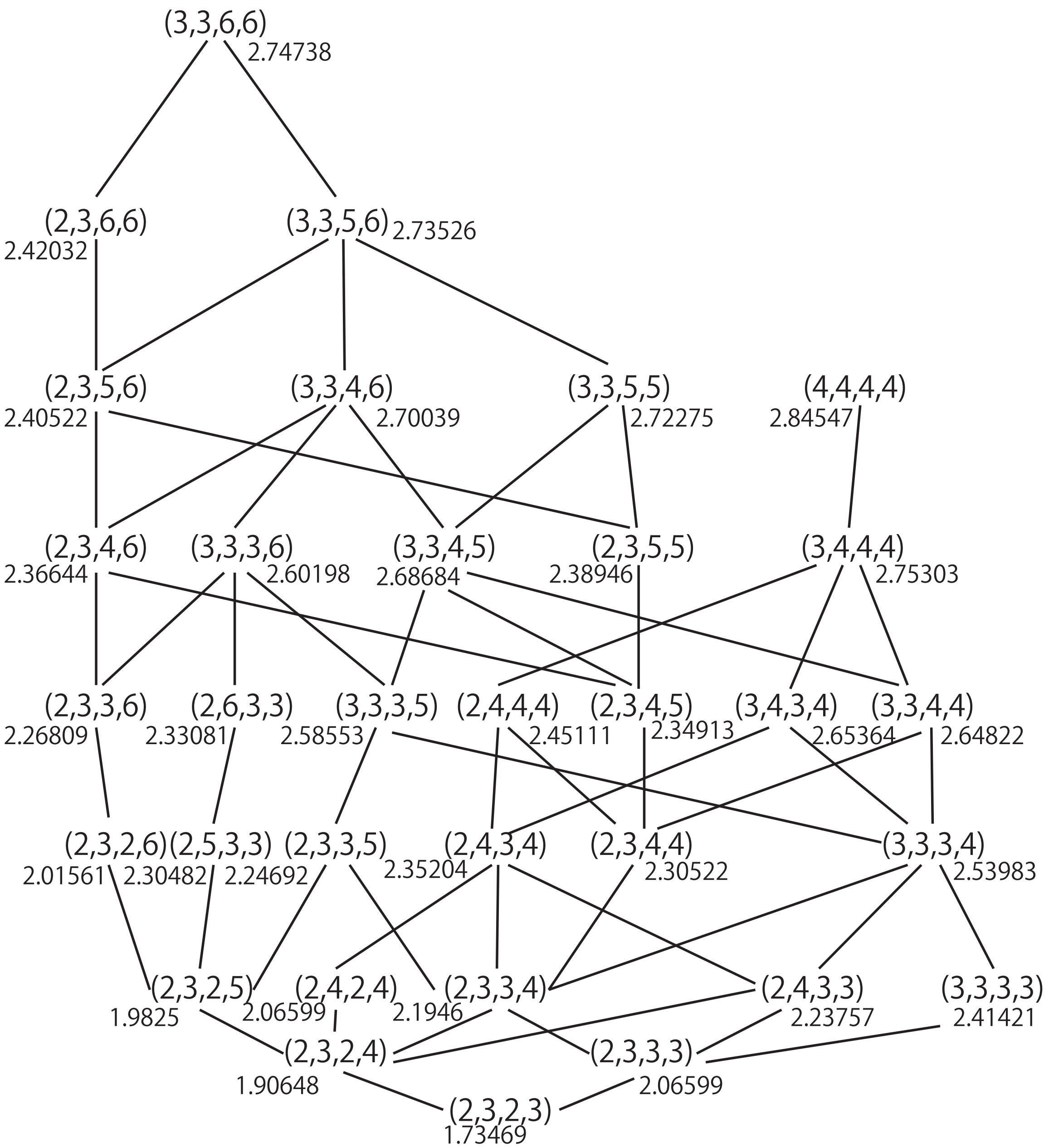


Figure 7. A partial order of Coxeter pyramids compatible with their growth rates

References

[1] E. M. Andreev, On convex polyhedra of finite volume in Lobačevskiĭ space, *Mat. Sb.*, Nov. Ser. **83** (1970), 256–260. English transl.: Math. USSR, Sb. **12** (1971), 255–259.

[2] J. W. Cannon and Ph. Wagreich, Growth functions of surface groups, *Math. Ann.* **293** (1992), no. 2, 239–257.

[3] M. W. Davis, *The Geometry and Topology of Coxeter Groups*, in: London Mathematical Society Monographs, Princeton University Press, 2008.

[4] W. J. Floyd, Growth of planar Coxeter groups, P.V. numbers, and Salem numbers, *Math. Ann.* **293** (1992), no. 3, 475–483.

[5] P. de la Harpe, Groupes de Coxeter infinis non affines, *Exposition. Math* **5** (1987), 91–96.

[6] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge Univ. Press, Cambridge, 1990.

[7] R. Kellerhals and G. Perren, On the growth of cocompact hyperbolic Coxeter groups, *European J. Combin.* **32** (2011), no. 8, 1299–1316.

[8] Y. Komori and Y. Umemoto, On the growth of hyperbolic 3-dimensional generalized simplex reflection groups, *Proc. Japan Acad. Ser. A Math. Sci.* **88** (2012), no. 4, 62–65.

[9] W. Parry, Growth series of Coxeter groups and Salem numbers, *J. Algebra* **154** (1993), no. 2, 406–415.

[10] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Grad. Texts in Math. **149**, Springer, New York, 1994.

[11] L. Solomon, The orders of the finite Chevalley groups, *J. Algebra* **3** (1966), 376–393.

[12] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society **80**, Amer. Math. Soc., Providence, RI, 1968.

[13] P. V. Tumarkin, Hyperbolic Coxeter n-polytopes with $n+2$ facets, *Math. Notes* **75** (2004), no. 5-6, 848–854.