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A remark on the generalized Toscani metric in probability measures with moments

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Abstract

Motivated by a pioneer work of Hiroshi Tanaka[22](1978) by means of the probabilistic method, from the middle of 1990s, Toscani and his coauthors analytically studied the existence, the uniqueness and the asymptotic behavior of solutions to the Cauchy problem for the non cutoff spatially homogeneous Boltzmann equation of Maxwellian molecules, introducing the so-called Toscani metric defined in the space of the Fourier image of probability measures. By using the Toscani metric on probability measures with moments less than 2, Cannone-Karch[5] studied infinite energy solutions to the above Cauchy problem, which include self-similar solutions given by Bobylev-Cercignani[4]. The existence result of [5] for the mild singular cross section of the Boltzmann collision term was extended to the strong singular case by [15], and the smoothing effect for measure valued (finite and/or infinite energy) solutions has been completely solved in [19, 16, 17] (see also [18] for the non-Maxwellian molecules case). In [16, 17], the Toscani metric was generalized in order to characterize perfectly the Fourier image of probability measures with moments less than 2. Furthermore, in [9] authors have characterized the class of probability measures possessing finite moments of any positive order, in terms of the symmetric difference operators of their Fourier transforms, simplifying an earlier work[8] by the first author, where the forward difference operator and its iteration are used. This simple generalized Toscani metric was applied in [9] to show the continuity of the solution in L^1_α with respect to any positive time when the initial measure datum posses finite moment of order $\alpha > 2$, implicitly based on the equivalence between the generalized Toscani metric and the

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Monge-Kantorovich-Wasserstein metric. The purpose of this note is to give a supplementary proof of this equivalence, after a short review about the research on measure valued solutions to the spatially homogeneous non-cutoff Boltzmann equation of Maxwellian molecules.

§ 1. Introduction

Consider the spatially homogeneous Boltzmann equation,

$$\partial_t f(t,v) = Q(f,f)(t,v) \quad \text{for} \quad t>0,\, v\in\mathbb{R}^3, \tag{1.1}$$

which arises as a physical model for describing the behavior of a dilute gas by its density f under a simplified assumption that it depends only on the velocity v and time t. The most interesting and important part of this equation is the collision operator given on the right hand side that captures the change of rates of the density distribution through elastic binary collisions:

$$Q(f,g)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left[f(v')g(v'_*) - f(v)g(v_*)\right] d\sigma dv_* \tag{1.2}$$

for scalar-valued functions f, g on $\mathbb{R}^3$, where

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\,\sigma, \\ v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\,\sigma, \\ \mathbf{k} = \dfrac{v-v_*}{|v-v_*|}, \end{cases} \tag{1.3}$$

the collision kernel $B = B(|v-v_*|, \mathbf{k}\cdot\sigma)$ is a nonnegative measurable function and $d\sigma$ denotes the area measure on the unit sphere $\mathbb{S}^2$.

Of main interest are the inverse-power interacting potential models which specify the collision kernels in the form

$$B = |v-v_*|^\gamma\, b(\cos\theta) \tag{1.4}$$

with $\gamma > -3$. It is classified as soft potential if $-3<\gamma<0$, Maxwellian if $\gamma = 0$ and hard potential if $\gamma > 0$. The angular part b, defined implicitly, is known to be continuous or at least bounded away from $\theta = 0$ but develops a non-integrable singularity near $\theta = 0$ in the sense that

$$b(\cos\theta)\,\sin\theta \sim \theta^{-1-\nu} \quad \text{for some } 0<\nu<2 \text{ as } \theta\to 0. \tag{1.5}$$

In this note we shall only consider the Boltzmann equation (1.1) of Maxwellian molecules. As it is customary, replacing $b(\cos\theta)$ by

$$[b(\cos\theta) + b(\cos(\pi-\theta))]\,\mathbf{1}_{[0,\pi/2]}(\theta),$$

we shall assume $b(\cos\theta)$ is supported on $[0,\pi/2]$ (see [26]). In addition, it will be assumed to satisfy a weak integrability condition

$$\int_0^{\pi/2} b(\cos\theta)\sin^{\alpha_0}\left(\frac{\theta}{2}\right)\sin\theta\, d\theta < \infty \quad \text{for some} \quad 0<\alpha_0<2\,, \tag{1.6}$$

which is verified by the singular kernel b described as in (1.5) when $0<\nu<\alpha_0$. Therefore, the case $0<\nu<1$ is called the mild singularity, and another case $1\le\nu<2$ is called the strong singularity, in view of the critical value $\alpha_0=1$.

Our primary concern here is to study measure-valued solutions to the Cauchy problem for the Boltzmann equation (1.1) with the initial data possessing finite moments:

$$f(0,v)=F_0\in P_\alpha(\mathbb{R}^3), \tag{1.7}$$

in view of the normalization, where we denote by $P_\alpha(\mathbb{R}^d)$, $\alpha\ge 0$, the set of all probability measures F on $\mathbb{R}^d$, $d\ge 1$, such that

$$\int_{\mathbb{R}^d}|v|^\alpha dF(v)<\infty\,. \tag{1.8}$$

In $P_\alpha(\mathbb{R}^d)$ with $\alpha>0$ we define the Monge-Kantorovich-Wasserstein distance $W_\alpha(F,G)$ as follows: For $F,G\in P_\alpha(\mathbb{R}^d)$,

$$\begin{aligned} W_\alpha(F,G) &= \Big(\inf_{L\in\Pi(F,G)}\int|v-w|^\alpha dL(v,w)\Big)^{1/\alpha}, \text{ if } \alpha\ge 1,\\ &= \inf_{L\in\Pi(F,G)}\int|v-w|^\alpha dL(v,w)\,, \text{ if } 0<\alpha<1, \end{aligned}$$

where $\Pi(F,G)$ denotes the set of all probability distributions L in $P_\alpha(\mathbb{R}^d\times\mathbb{R}^d)$ having F and G as marginal distributions, that is,

$$dF(v)=\int_{\mathbb{R}^d_w}dL(v,w),\;\; dG(w)=\int_{\mathbb{R}^d_v}dL(v,w).$$

We first recall a historical work given by the probabilist H. Tanaka[22].

Tanaka Theorem [Existence, Uniqueness and Asymptotic Behavior]. *If an initial datum $F_0\in P_2(\mathbb{R}^3)$ then there exists a unique solution $F_t\in P_2(\mathbb{R}^3)$ to the Cauchy problem* (1.1)-(1.7) *and we have*

$$W_2(F_t,\omega)\to 0\,,\;\; t\to\infty,$$

where $\omega(v)=(2\pi E)^{-3/2}e^{-|v|^2/2E}$ *for* $3E=\int|v|^2dF_0$.

This result has been analytically treated by Toscani and his coauthors[7, 12, 21, 24], based on the Toscani metric

$$\|\varphi-\tilde\varphi\|_2=\sup_{0\ne\xi\in\mathbb{R}^3}\frac{|\varphi(\xi)-\tilde\varphi(\xi)|}{|\xi|^2}, \quad \varphi,\tilde\varphi \text{ are Fourier images of } F,G\in P_2(\mathbb{R}^d),$$

and the Fourier transform representation of the Boltzmann equation given by Bobylev [2, 3];

Bobylev formula. If $\varphi(t,\xi)$ and $\varphi_0(\xi)$ are Fourier transforms of $f(t,v)$ and $f(0,v) = F_0$, respectively, then the Cauchy problem (1.1)-(1.7) is reduced to

$$\begin{cases} \partial_t \varphi(t,\xi) = \displaystyle\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Big(\varphi(t,\xi^+)\varphi(t,\xi^-) - \varphi(t,\xi)\varphi(t,0)\Big)d\sigma, \\ \\ \varphi(0,\xi) = \varphi_0(\xi), \ \text{ where } \xi^{\pm} = \dfrac{\xi}{2} \pm \dfrac{|\xi|}{2}\sigma\,. \end{cases} \tag{1.9}$$

It is known ([24, Theorem 1]) that $W_2(\cdot,\cdot)$ distance and Toscani metric are equivalent on the subset of $P_2(\mathbb{R}^d)$ with a fixed energy ($\int |v|^2 dF(v) = Ed$) and zero mean vector ($\int v_j dF(v) = 0$)[1]. Recently Cannone-Karch [5] extended the existence and uniqueness of solution for the initial datum in P_α with $\nu < (\alpha_0 \le)\alpha < 2$, motivated by the self-similar solution with infinite energy

$$f_{\alpha,K}(v,t) = e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t}) \tag{1.10}$$

given by Bobylev-Cercignani[4]. Here for any fixed $\alpha \in (\nu, 2)$ and $K > 0$,

$$\mu_\alpha = \frac{\lambda_\alpha}{\alpha}\ ,\ \ 0 < \lambda_\alpha = 2\pi\int_0^{\pi/2} b(\cos\theta)\left(\cos^\alpha\frac{\theta}{2} + \sin^\alpha\frac{\theta}{2} - 1\right)\sin\theta d\theta < \infty\,,$$

and $\Psi_{\alpha,K}(v)$ is a radially symmetric probability measure satisfying

$$\hat{\Psi}_{\alpha,K} \in \mathcal{K}^\alpha, \quad \lim_{|\xi|\to 0}\frac{1-\hat{\Psi}_{\alpha,K}(\xi)}{|\xi|^\alpha} = K.$$

Following [13, 5], we call the Fourier transform of a probability measure $F \in P_0(\mathbb{R}^d)$, that is,

$$\varphi(\xi) = \hat{F}(\xi) = \mathcal{F}(F)(\xi) = \int_{\mathbb{R}^d} e^{-iv\cdot\xi}dF(v)\,,$$

a characteristic function. Denote the set of all characteristic functions by $\mathcal{K}$. In [5], a subspace $\mathcal{K}^\alpha$ for $\alpha \ge 0$ was defined as follows:

$$\mathcal{K}^\alpha = \{\varphi \in \mathcal{K}\,;\ \|\varphi - 1\|_\alpha < \infty\}\,, \tag{1.11}$$

where

$$\|\varphi - 1\|_\alpha = \sup_{\xi\in\mathbb{R}^d}\frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}. \tag{1.12}$$

[1]zero mean vector condition is not restrictive because of Proposition 2.2.

The space $\mathcal{K}^\alpha$ endowed with the distance

$$\|\varphi-\tilde{\varphi}\|_\alpha=\sup_{\xi\in\mathbb{R}^d}\frac{|\varphi(\xi)-\tilde{\varphi}(\xi)|}{|\xi|^\alpha} \tag{1.13}$$

is a complete metric space (see [5, Proposition 3.10]). It follows that $\mathcal{K}^\alpha=\{1\}$ for all $\alpha>2$ and the following embeddings (Lemma 3.12 of [5]) hold

$$\{1\}\subset\mathcal{K}^\alpha\subset\mathcal{K}^\beta\subset\mathcal{K}^0=\mathcal{K}\quad\text{for all } 2\ge\alpha\ge\beta\ge 0.$$

However, even though the inclusion $\mathcal{F}(P_\alpha(\mathbb{R}^d))\subset\mathcal{K}^\alpha$ holds (under the zero mean condition when $\alpha>1$, see [5, Lemma 3.15] and Proposition 2.3 below), the space $\mathcal{K}^\alpha$ is strictly larger than $\mathcal{F}(P_\alpha(\mathbb{R}^d))$ for $\alpha\in(0,2)$, in other word, $\mathcal{F}^{-1}(\mathcal{K}^\alpha)\supsetneq P_\alpha(\mathbb{R}^d)$. Indeed, for each $\alpha\in(0,2)$, $\varphi_\alpha(\xi)=e^{-|\xi|^\alpha}$ belongs to $\mathcal{K}^\alpha$, which is the Fourier transform of the probability density $p_\alpha(v)$ of α-stable Lévy process. It is known ([5, Remark 3.16]) that $0<p_\alpha(v)\le C(1+|v|)^{-(\alpha+d)}$ and moreover

$$\frac{p_\alpha(v)}{|v|^{\alpha+d}}\to c_0\ \text{ when }\ |v|\to\infty,$$

where $c_0=\alpha 2^{\alpha-1}\pi^{-(d+2)/2}\sin\Big(\frac{\alpha\pi}{2}\Big)\Gamma\Big(\frac{\alpha+d}{2}\Big)\Gamma\Big(\frac{\alpha}{2}\Big)$.

On the other hand, we remark that $\mathcal{F}(P_2(\mathbb{R}^d))=\mathcal{K}^2$. Indeed, this can be proved by contradiction. If there exists a $\varphi(\xi)\in\mathcal{K}^2$ such that $F=\mathcal{F}^{-1}(\varphi)\notin P_2$, then we may assume there exist $\omega_0\in\mathbb{S}^{d-1}$and $A>0$ such that

$$\int_{\{|\frac{v}{|v|}-\omega_0|<10^{-10}\}\cap\{|v|\le A\}}|v|^2dF(v)\ge 100\|1-\varphi\|_2\,,$$

from which we have a contradiction because

$$\begin{aligned}
\|1-\varphi\|_2 &\ge \sup_\xi\frac{\mathrm{Re}(1-\varphi(\xi))}{|\xi|^2}\\
&\ge 2\int_{\{|\frac{v}{|v|}-\omega_0|<10^{-10}\}\cap\{|v|\le A\}}\frac{\sin^2\Big\{\frac{|v||\xi|}{2}\big(\frac{v}{|v|}\cdot\frac{\xi}{|\xi|}\big)\Big\}}{|v|^2|\xi|^2}|v|^2dF(v)\ \text{ for }\ \frac{\xi}{|\xi|}=\omega_0,\ |\xi|=\frac{\pi}{A}\\
&\ge\frac{2}{\pi^2}\int_{\{|\frac{v}{|v|}-\omega_0|<10^{-10}\}\cap\{|v|\le A\}}\big(\frac{v}{|v|}\cdot\omega_0\big)^2|v|^2dF(v)>50\|1-\varphi\|_2,
\end{aligned}$$

by using

$$\sin z\ge\frac{2z}{\pi}\ \text{ when }\ 0\le z\le\frac{\pi}{2}\,.$$

Since $\mathcal{K}^\alpha\supsetneq\mathcal{F}(P_\alpha(\mathbb{R}^3))$ for $\alpha\in(0,2)$, we introduce $\tilde{P}_\alpha=\mathcal{F}^{-1}(\mathcal{K}^\alpha)$ endowed also with distance (1.13). In [5], the existence and uniqueness in the mild singularity case are discussed only in the space $\tilde{P}_\alpha\supsetneq P_\alpha$ even when its initial datum belongs to P_α. The restriction of mild singularity was removed in [15], by rewriting the right hand side of the first equation of (1.9) as

$$\frac{1}{2}\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Big(\varphi(t,\xi^+)+\varphi(t,\tilde{\xi}^+)-2\varphi(t,\xi)\Big)\varphi(t,0)d\sigma \\ +\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\varphi(t,\xi^+)\Big(\varphi(t,\xi^-)-\varphi(t,0)\Big)d\sigma,$$

where $\tilde{\xi}^+$ is symmetric to ξ^+ on $\mathbb{S}^2$ (see [15, Figure 1]). The gap between $\tilde{P}_\alpha$ and P_α, $0<\alpha<2$, was filled in [16, 17] by introducing spaces $\mathcal{M}^\alpha$, $\widetilde{\mathcal{M}}^\alpha$, which are defined by

$$\mathcal{M}^\alpha=\left\{\varphi\in\mathcal{K};\ \|\varphi-1\|_{\mathcal{M}^\alpha}<\infty\right\},$$

$$\widetilde{\mathcal{M}}^\alpha=\left\{\varphi\in\mathcal{K};\ \|\operatorname{Re}\varphi-1\|_{\mathcal{M}^\alpha}+||\varphi-1||_\alpha<\infty\right\},$$

where

$$\|\varphi-1\|_{\mathcal{M}^\alpha}=\int_{\mathbb{R}^3}\frac{|\varphi(\xi)-1|}{|\xi|^{3+\alpha}}d\xi$$

and Re φ stands for the real part of $\varphi(\xi)$. Namely, we have

$$\mathcal{K}^\alpha\supsetneq\mathcal{F}(P_\alpha)=\mathcal{M}^\alpha=\widetilde{\mathcal{M}}^\alpha\supsetneq\mathcal{K}^{\alpha'}\text{ if }0<\alpha<\alpha'\le 2,\alpha\neq 1,\\ \mathcal{K}^1\supsetneq\mathcal{M}^1\supsetneq\mathcal{F}(P_1)=\widetilde{\mathcal{M}}^1\supsetneq\mathcal{K}^{\alpha'}\quad\text{if }1<\alpha'\le 2.$$

The characterization of probability measures by their Fourier transforms in [16, 17], together with the smoothing effect of measure valued solutions proved in [19], lead us to the following;

Theorem 1.1. *Assume that $b(\cos\theta)$ satisfies (1.6) for some $\alpha_0\in(0,2)$ and let $\alpha\in[\alpha_0,2]$. If $F_0\in P_\alpha(\mathbb{R}^3)$, then there exists a unique measure valued solution $F_t\in C([0,\infty),P_\alpha(\mathbb{R}^3))$ to the Cauchy problem (1.1)-(1.7), where the continuity with respect to t is according to the weak topology deduced by one of equivalent conditions stated in Theorem 2.1 below. If $b(\cos\theta)$ satisfies (1.5) and if $F_0\in P_\alpha(\mathbb{R}^3)$ for $\alpha\in(\nu,2]$ is not a single Dirac mass then F_t admits the probability density $dF_t(v)=f(t,v)$ satisfying*

$$f(t,v)\in C((0,\infty);L^1_\alpha(\mathbb{R}^3)\cap H^\infty(\mathbb{R}^3)).$$

As pointed out in [6], the smoothing effect of this theorem solves the Bobylev-Cercignani conjecture in [4, Remark on page 1055];

Corollary 1.2. *If $b(\cos\theta)$ satisfies (1.5) and if $\nu<\alpha<2$, then the Bobylev-Cericignani self-similar solution (1.10) is defined by a radially symmetric function $\Psi_{\alpha,K}(v)$ belonging to $L^1_\beta(\mathbb{R}^3)\cap H^\infty(\mathbb{R}^3)$, $\forall\beta<\alpha$.*

Theorem 1.1 was extended in [9] to the case for initial data $F_0 \in P_\alpha(\mathbb{R}^3)$ for arbitrary $\alpha > 2$, by characterizing the class of probability measures possessing finite moments of any positive order, in terms of the symmetric difference operators of their Fourier transforms, which will be detailed in the next section (see the space $\mathcal{M}^\delta_k(\mathbb{R}^d)$ defined by (2.11) and Theorem 2.4). At the end of this introduction, we remark that the asymptotic behavior of the Bobylev-Cercignani self-similar solution is studied by [6] and [20], in different topologies each other. Furthermore, as for the existence and the smoothing effect of measure valued solutions for the spatially homogeneous non-cutoff Boltzmann equation of non-Maxwellian molecules, we refer [18] and references therein. As it is customary, we shall write $\langle v\rangle = \sqrt{1+|v|^2}$ for $v \in \mathbb{R}^d$ and use the notation $A \lesssim B$ to indicate the inequality $A \le cB$ for a generic constant c.

§ 2. Generalized Toscani metric

First we recall the equivalence between the weak convergence in $P_\alpha(\mathbb{R}^d)$ and the Monge-Kantorovich-Wasserstein metric (see [27, Theorem 7.12], [28, Theorem 6.9]). The notation $F_n \rightharpoonup F$ in $P_0(\mathbb{R}^d)$ means that F_n converges weakly to F, that is, $\int \psi dF \to \int \psi dF$ for any bounded continuous ψ.

Theorem 2.1 ([27, Theorem 7.12]). *Let $\alpha > 0$, let $\{F_n\} \subset P_\alpha(\mathbb{R}^d)$, and let $F \in P_0(\mathbb{R}^d)$. Then the following four statements are equivalent:*
(i) *For $\{F_n\} \subset P_\alpha(\mathbb{R}^d)$, $F \in P_0(\mathbb{R}^d)$ we have*

$$W_\alpha(F_n, F) \to 0. \tag{2.1}$$

(ii) *We have $F_n \rightharpoonup F$ in $P_0(\mathbb{R}^d)$ and*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|v|\ge R} |v|^\alpha dF_n(v) = 0, \ \ \textbf{(tightness } \text{condition)}. \tag{2.2}$$

(iii) *We have $F_n \rightharpoonup F$ in $P_0(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} |v|^\alpha dF_n(v) \to \int_{\mathbb{R}^d} |v|^\alpha dF. \tag{2.3}$$

(iv) *Whenever $\psi \in C(\mathbb{R}^d)$ satisfies the growth condition $|\psi(v)| \lesssim \langle v\rangle^\alpha$, then*

$$\lim_{n\to\infty} \int \psi(v) dF_n(v) = \int \psi(v) dF(v). \tag{2.4}$$

The following proposition shows that zero mean vector condition, $\int v_j dF(v) = 0$, in $P_\alpha(\mathbb{R}^d)$ with $\alpha \ge 1$ is not restrictive.

Proposition 2.2. *Let $\alpha \geq 1$. Let $\{F_n\}$ be a sequence of probability measures in $P_\alpha(\mathbb{R}^d)$, and let $F \in P_0(\mathbb{R}^d)$. If $W_\alpha(F_n, F) \to 0$ ($n \to \infty$) then $F_n(\cdot + a_n) \rightharpoonup F(\cdot + a)$ in $P_0(\mathbb{R}^d)$, and $\{F_n(\cdot + a_n)\}$ satisfies the* **tightness** *condition:*

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{|v|\geq R\}} |v|^\alpha dF_n(v + a_n) = 0\,, \tag{2.5}$$

where $a_n = \int v dF_n(v), a = \int v dF(v)$.

Proof. By means of Theorem 2.1, (2.3) in (iii) implies $F \in P_\alpha$ with $\alpha \geq 1$, and hence $a = \int v dF(v)$ is well-defined. (iv) assures the convergence of $a_n \to a$, which leads to

$$e^{ia_n\cdot\xi}\varphi_n(\xi) \to e^{ia\cdot\xi}\varphi(\xi), \text{ pointwise in } \mathbb{R}^d,$$

where $\varphi_n = \mathcal{F}(F_n), \varphi = \mathcal{F}(F)$. Therefore, we obtain that $F_n(\cdot + a_n) \rightharpoonup F(\cdot + a)$. The tightness condition (2.5) is a direct consequence of the original tightness condition (2.2). □

Since $\int v dF_n(v + a_n) = \int v dF(v + a) = 0$, zero mean vector condition can be supposed in the convergence for a sequence in $P_\alpha(\mathbb{R}^d)$ if $\alpha \geq 1$. Conversely, if $\{G_n\} \subset P_\alpha(\mathbb{R}^d)$ with $\alpha \geq 1$ satisfies zero mean vector condition and

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{|v|\geq R\}} |v|^\alpha dG_n(v) = 0\,,$$

then for any sequence $\{a_n\} \subset \mathbb{R}^d$ satisfying $a_n \to a$ we have

$$W_\alpha(G_n(\cdot - a_n), G(\cdot - a)) \to 0.$$

In what follows we assume zero mean vector condition for $P_\alpha(\mathbb{R}^d)$ when $\alpha \geq 1$. Under this condition, the inclusion $\mathcal{F}(P_\alpha(\mathbb{R}^d)) \subset \mathcal{K}^\alpha$ holds (see [5, proof of Lemma 3.15]), while we have

Proposition 2.3. *Let $1 < \alpha \leq 2$. If $\varphi(\xi)$ belongs to $\mathcal{K}^\alpha$ then $F = \mathcal{F}^{-1}(\varphi)$ satisfies*

$$\int v_j dF(v) = 0, \;\; j = 1, \cdots, d. \tag{2.6}$$

Proof. Suppose that there exists a $\mathcal{F}(F) = \varphi \in \mathcal{K}^\alpha$ such that $\int v dF(v) = a \neq 0$. Since $|1 - \varphi(\xi)| \leq \|1 - \varphi\|_\alpha |\xi|^\alpha$, for any $1 < \beta < \alpha$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1 - \operatorname{Re}\varphi(\xi)}{|\xi|^{d+\beta}} d\xi \leq \int_{\mathbb{R}^d} \frac{|1 - \varphi(\xi)|}{|\xi|^{d+\beta}} d\xi &\leq \int_{|\xi|>R} \frac{2d\xi}{|\xi|^{d+\beta}} + \|1 - \varphi\|_\alpha \int_{|\xi|\leq R} \frac{d\xi}{|\xi|^{d+\beta-\alpha}} \\ &\lesssim \|1 - \varphi\|_\alpha^{\beta/\alpha}.\end{aligned}$$

Noting

$$\int_{\mathbb{R}^d} \frac{1-\operatorname{Re}\varphi(\xi)}{|\xi|^{d+\beta}} d\xi = \int_{\mathbb{R}^d} |v|^{\beta} \Big(\int_{\mathbb{R}^d} \frac{2\sin^2 \frac{(v/|v|)\cdot\zeta}{2}}{|\zeta|^{d+\beta}} d\zeta \Big) dF(v)$$

we have $\int \langle v\rangle^{\beta} dF(v+a) < \infty$. Since $\int v dF(v+a) = 0$, we see $\varphi_a(\xi) = e^{ia\cdot\xi}\varphi(\xi)$ belongs to $\mathcal{K}^{\beta}$, by means of [5, Lemma 3.15]. Therefore, we have

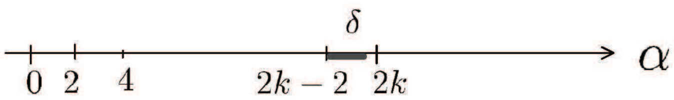
$$\begin{aligned} \sup_{\xi} \frac{|1-e^{-i\xi\cdot a}|}{|\xi|^{\beta}} &\le \sup_{\xi} \frac{|1-\varphi(\xi)|}{|\xi|^{\beta}} + \sup_{\xi} \frac{|\varphi(\xi)-e^{-i\xi\cdot a}|}{|\xi|^{\beta}} \\ &= \sup_{\xi} \frac{|1-\varphi(\xi)|}{|\xi|^{\beta}} + \sup_{\xi} \frac{|\varphi_a(\xi)-1|}{|\xi|^{\beta}} < \infty. \end{aligned}$$

This gives a contradiction to the fact that if $\beta > 1, a \neq 0$, then $e^{-i\xi\cdot a} \notin \mathcal{K}^{\beta}$.

□

The rest of this note is devoted to characterizing the Fourier images of spaces $P_{2k-2+\delta}(\mathbb{R}^d)$, writing, for $\alpha > 0$;

$$\alpha = 2k-2+\delta, \;\; k \in \mathbb{N}^+, \delta \in [0,2), k+\delta > 1.$$



In [8], the Fourier images of probability measure having finite moment, without zero mean vector condition was characterized, in terms of the forward difference operator and its iterates. As a modification of this result, we introduce a new classification of characteristic functions defined in terms of the symmetric central difference operator and its iterates as follows: Let Δ denote the symmetric central difference operator which acts on each characteristic function φ on $\mathbb{R}^d$, $\varphi = \widehat{F}$ with $F \in P_0(\mathbb{R}^d)$, by the rule

$$\begin{aligned} \Delta\varphi(\xi) &= \frac{2\varphi(0)-\varphi(\xi)-\varphi(-\xi)}{4} \\ &= \frac{1-\operatorname{Re}\,\varphi(\xi)}{2} \\ &= \int_{\mathbb{R}^d} \sin^2\left(\frac{v\cdot\xi}{2}\right) dF(v). \end{aligned} \tag{2.7}$$

To define the iteration of Δ, we introduce a more general notation

$$\begin{aligned}\Delta_\xi\varphi(x) &= \frac{2\varphi(x)-\varphi(x+\xi)-\varphi(x-\xi)}{4}\\ &= \int_{\mathbb{R}^d} e^{-iv\cdot x}\sin^2\left(\frac{v\cdot\xi}{2}\right)dF(v).\end{aligned}$$

Then we have $\Delta\varphi(\xi)=\Delta_\xi\varphi(0)$, and we can define the kth iterate $\Delta^k\varphi(\xi)$ by $\Delta_\xi^k\varphi(0)$, where $\Delta_\xi^k\varphi(0)$ is defined inductively by $\Delta_\xi^k\varphi(x)=\Delta_\xi(\Delta_\xi^{k-1}\varphi)(x)$ for $k=2,3,\cdots$. As for the second iterate $\Delta^2\varphi(\xi)$ we note

$$\begin{aligned}\Delta_\xi^2\varphi(x) &= \frac{2(\Delta_\xi\varphi)(x)-(\Delta_\xi\varphi)(x+\xi)-(\Delta_\xi\varphi)(x-\xi)}{4}\\ &= \int_{\mathbb{R}^d}\frac{2e^{-iv\cdot x}-e^{-iv\cdot(x+\xi)}-e^{-iv\cdot(x-\xi)}}{4}\sin^2\left(\frac{v\cdot\xi}{2}\right)dF(v)\\ &= \int_{\mathbb{R}^d} e^{-iv\cdot x}\sin^4\left(\frac{v\cdot\xi}{2}\right)dF(v).\end{aligned}$$

Therefore

$$\begin{aligned}\Delta^2\varphi(\xi)=\Delta_\xi^2\varphi(0) &= \int_{\mathbb{R}^d}\sin^4\left(\frac{v\cdot\xi}{2}\right)dF(v)\\ &= \frac{3-4\,\mathrm{Re}\ \varphi(\xi)+\mathrm{Re}\ \varphi(2\xi)}{8}\\ &= \frac{6\varphi(0)-4\varphi(\xi)-4\varphi(-\xi)+\varphi(2\xi)+\varphi(-2\xi)}{16}.\end{aligned}$$

Furthermore, the kth iterate of Δ can be written as

$$\begin{aligned}\Delta^k\varphi(\xi)=\Delta_\xi^k\varphi(0) &= \int_{\mathbb{R}^d}\sin^{2k}\left(\frac{v\cdot\xi}{2}\right)dF(v)\\ &= \sum_{j=0}^k c_{k,j}\ \mathrm{Re}\ \varphi(j\xi) = \frac{1}{2}\sum_{j=0}^k c_{k,j}\ \left[\varphi(j\xi)+\varphi(-j\xi)\right].\end{aligned}\tag{2.8}$$

It follows from the definition of Fourier transform that the $c_{k,j}$ coincide with the coefficients of the trigonometric identity

$$\sin^{2k}\left(\frac{x}{2}\right)=\sum_{j=0}^k c_{k,j}\ \cos(jx)\qquad(x\in\mathbb{R})\tag{2.9}$$

and an inductive calculation gives

$$c_{k,j}=\begin{cases}2^{-2k}\dbinom{2k}{k} & \text{for}\quad j=0,\\ (-1)^j 2^{-2k+1}\dbinom{2k}{k+j} & \text{for}\quad j=1,\cdots,k.\end{cases}\tag{2.10}$$

We are now in a position to introduce the spaces of characteristic functions in question. For each positive integer k and $\delta \in [0, 2)$ with $k + \delta > 1$, we put $\alpha^* = \delta$ if $k = 1$, $\alpha^* = 2$ if $k \geq 2$ and define

$$\mathcal{M}_k^\delta(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{K}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\delta}} \, d\xi + \|\varphi - 1\|_{\alpha^*} < \infty \right\}. \tag{2.11}$$

For each $0 < \beta \leq \alpha^*$, we also introduce the metric

$$\begin{aligned} dis_{k,\delta,\beta}(\varphi, \tilde{\varphi}) &= \|\varphi - \tilde{\varphi}\|_{\mathcal{M}_k^\delta} + \|\varphi - \tilde{\varphi}\|_\beta \quad \text{where} \\ \|\varphi - \tilde{\varphi}\|_{\mathcal{M}_k^\delta} &= \int_{\mathbb{R}^d} \frac{|\Delta^k \varphi(\xi) - \Delta^k \tilde{\varphi}(\xi)|}{|\xi|^{d+2k-2+\delta}} \, d\xi \,. \end{aligned} \tag{2.12}$$

Concerning the image spaces of the $P_{2k-2+\delta}(\mathbb{R}^d)$ under the Fourier transform and complete metric structures, we have the following result.

Theorem 2.4. *For each positive integer k and $\delta \in [0, 2)$ with $k + \delta > 1$,*

$$\mathcal{M}_k^\delta(\mathbb{R}^d) = \mathcal{F}\left(P_{2k-2+\delta}(\mathbb{R}^d)\right). \tag{2.13}$$

Moreover, the space $\mathcal{M}_k^\delta(\mathbb{R}^d)$ is complete with respect to each metric $dis_{k,\delta,\beta}$ and the condition

$$\lim_{n \to \infty} dis_{k,\delta,\beta}(\varphi_n, \varphi) = 0 \ \textit{ with } \ \varphi_n, \varphi \in \mathcal{M}_k^\delta(\mathbb{R}^d) \tag{2.14}$$

is equivalent to

$$W_{2k-2+\delta}(F_n, F) \to 0, \tag{2.15}$$

where F_n, $F \in P_{2k-2+\delta}(\mathbb{R}^d)$ defined by $\widehat{F_n} = \varphi_n$, $\widehat{F} = \varphi$.

Remark.

(1) In the case $k = 1, 0 < \delta < 2$, the newly defined space $\mathcal{M}_k^\delta$ coincides with $\widetilde{\mathcal{M}}^\delta$.

(2) For a probability measure F on the real line, it has been shown in [14] that, if $\int |x|^{2k-2+\delta} dF(x) < \infty$, then there exists $C_{k,\delta} > 0$, which depends only on k, δ, such that

$$\int_0^\infty \frac{1}{t^{1+\delta}} \Big\{ 1 - \text{Re } \varphi(t) + \sum_{j=1}^{k-1} \frac{t^{2j} \varphi^{(2j)}(0)}{(2k)!} \Big\} dt = C_{k,\delta} \int_{-\infty}^\infty |x|^{2k-2+\delta} dF(x),$$

where $\varphi = \mathcal{F}(F)$. However, this characterization is different from the one given in (2.11). We also remark that the family of metric (2.12) with $k = 1$ in one dimensional case was introduced earlier by [1] in their probabilistic work on random convex combinations.

(3) It was proved in [24, Theorem 1] that

$$W_2(F_n, F) \to 0, \text{ Wasserstein metric} \Leftrightarrow \ \|\varphi_n - \varphi\|_2 \to 0, \text{ Toscani metric},$$

therefore, we have moreover, for any $\beta \in (0,2)$

$$\Leftrightarrow \ dis_{2,0,\beta}(\varphi_n, \varphi) = \int_{\mathbb{R}^d} \frac{|\Delta^2 \varphi_n(\xi) - \Delta^2 \varphi(\xi)|}{|\xi|^{d+2}} d\xi + \sup_{\xi \neq 0} \frac{|\varphi_n(\xi) - \varphi(\xi)|}{|\xi|^\beta} \to 0.$$

For the proof of Theorem 2.4 we prepare

Proposition 2.5. *Let $k \in \mathbb{N}, \delta \in [0,2), k + \delta > 1$ and let $\mathcal{M}_k^\delta$ be a subspace of $\mathcal{K} = \mathcal{F}(P_0(\mathbb{R}^d))$ defined by (2.11). Then we have the formula (2.13). Furthermore, for $M \in [1, \infty]$, if we put*

$$c_{d,k,\delta,M} = \int_{\{|\zeta| \le M\}} \frac{\sin^{2k}(\mathbf{e}_1 \cdot \zeta/2)}{|\zeta|^{d+2k-2+\delta}} d\zeta > 0, \tag{2.16}$$

and if $F = \mathcal{F}^{-1}(\varphi)$ for $\varphi \in \mathcal{M}_k^\delta$, then for any $R > 0$ we have

$$\int_{\{|v| \ge R\}} |v|^{2k-2+\delta} dF(v) \le \frac{1}{c_{d,k,\delta,1}} \int_{\{|\xi| \le 1/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\delta}} d\xi. \tag{2.17}$$

Moreover,

$$\int_{\mathbb{R}^d} |v|^{2k-2+\delta} dF(v) \le \frac{1}{c_{d,k,\delta,\infty}} \int_{\mathbb{R}^d} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\delta}} d\xi\,. \tag{2.18}$$

Proof. Note

$$\int_{\{|\xi| \le M/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\delta}} d\xi = \int_{\mathbb{R}^d_v} \Big(\int_{\{|\xi| \le M/R\}} \frac{\sin^{2k}(v \cdot \xi/2)}{|\xi|^{d+2k-2+\delta}} d\xi \Big) dF(v).$$

By the change of variable $|v|\xi = \zeta$ and by using the invariance of the rotation, we have

$$\begin{aligned}\int_{\{|\xi| \le M/R\}} \frac{\sin^{2k}(v \cdot \xi/2)}{|\xi|^{d+2k-2+\delta}} d\xi &= |v|^{2k-2+\delta} \int_{\{|\zeta| \le M|v|/R\}} \frac{\sin^{2k}(\mathbf{e}_1 \cdot \zeta/2)}{|\zeta|^{d+2k-2+\delta}} d\zeta \\ &\ge |v|^{2k-2+\delta} \mathbf{1}_{\{|v| \ge R\}} c_{d,k,\delta,M}\,,\end{aligned}$$

which yields (2.17), with the choice of $M = 1$. By letting $M \to \infty$ and $R \to 0$, we obtain (2.18). The formula (2.13) is now obvious since

$$\lim_{M \to \infty} \int_{\{|\xi| \le M\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\delta}} d\xi \le c_{d,k,\delta,\infty} \int |v|^{2k-2+\delta} dF(v). \tag{2.19}$$

□

Proof of Theorem 2.4. Suppose that $\{\varphi_n\}_{n=1}^\infty \subset \mathcal{M}_k^\delta$ satisfies

$$dis_{k,\delta,\beta}(\varphi_n, \varphi_m) \to 0 \ \ (n, m \to \infty) \ .$$

Since it follows from [5, Proposition 3.10] that $\mathcal{K}^\beta$ is a complete metric space, we have the limit (pointwise convergence)

$$\varphi(\xi) = \lim_{n\to\infty} \varphi_n(\xi) \in \mathcal{K}^\beta \subset \mathcal{K}.$$

For any fixed $R > 1$ we have

$$\int_{\{R^{-1}\le|\xi|\le R\}} \frac{|\Delta^k \varphi_n(\xi)|}{|\xi|^{d+2k-2+\delta}} d\xi \le \sup_n \|\varphi_n\|_{\mathcal{M}_k^\delta} < \infty.$$

Taking the limit with respect to n and letting $R \to \infty$, we have $\varphi \in \mathcal{M}_k^\delta$. Now it is easy to see that $dis_{k,\delta,\beta}(\varphi_n, \varphi) \to 0$, so that $\mathcal{M}_k^\delta$ is a complete metric space.

We show (2.14) implies (2.15). Suppose that, for $F_n, F \in P_\alpha(\mathbb{R}^d)$, we have

$$\varphi_n = \mathcal{F}(F_n), \varphi = \mathcal{F}(F) \in \mathcal{M}_k^\delta, \ \text{ and } \ \lim_{n\to\infty} dis_{k,\delta,\beta}(\varphi_n, \varphi) = 0\,.$$

The weak convergence $F_n \rightharpoonup F$ in $P_0(\mathbb{R}^d)$ follows from the fact that $\|\varphi_n - \varphi\|_\beta \to 0$, because $\varphi_n \to \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$ and so $F_n \to F$ in $\mathcal{S}'(\mathbb{R}^d)$. Note that for $R > 1$

$$\int_{\{|\xi|\le 1/R\}} \frac{\Delta^k \varphi_n(\xi)}{|\xi|^{d+\alpha}} d\xi \le \int_{\{|\xi|\le 1/R\}} \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+\alpha}} d\xi + \|\varphi_n - \varphi\|_{\mathcal{M}_k^\delta}\,.$$

It follows from (2.17) that for any $\varepsilon > 0$ there exist $R > 1$ and $N \in \mathbb{N}$ such that

$$\int_{\{|v|\ge R\}} |v|^{2k-2+\delta} dF_n(v) + \int_{\{|v|\ge R\}} |v|^{2k-2+\delta} dF(v) < \varepsilon \ \text{ if } n \ge N,$$

which shows the tightness condition (2.2) of Theorem 2.1. The equivalence in Theorem 2.1 gives (2.15).

Conversely, we first show (2.15) implies

$$\|\varphi_n - \varphi\|_{\mathcal{M}_k^\delta} \to 0\,. \tag{2.20}$$

Indeed, it follows from (2.2) that

$$\begin{gathered} \forall\varepsilon > 0,\ \exists K_\varepsilon > 1 \text{ independent of } n; \\ \int_{|v|\ge K_\varepsilon} |v|^{2k-2+\delta} (dF_n(v) + dF(v)) < \frac{\varepsilon}{2c_{d,k,\delta,\infty}}. \end{gathered} \tag{2.21}$$

For any $R > 1$ we have, with $\alpha = 2k - 2 + \delta$,

$$\begin{aligned}
J_n(R) &:= \int_{|\xi|\le 1/R} \frac{|\Delta^k\varphi_n(\xi) - \Delta^k\varphi(\xi)|}{|\xi|^{d+\alpha}} d\xi \\
&\le \int_{\mathbb{R}^d_v} \Big(\int_{|\xi|\le 1/R} \frac{\sin^{2k} \frac{v\cdot\xi}{2}}{|\xi|^{d+\alpha}} d\xi \Big)(dF_n(v) + dF(v)) = \int_{|v|\ge K_\varepsilon} + \int_{|v|<K_\varepsilon} \\
&< \frac{\varepsilon}{2} + \int_{|v|<K_\varepsilon} |v|^\alpha \Big(\int_{|\zeta|\le K_\varepsilon/R} \frac{\sin^{2k} \frac{\mathbf{e}_1\cdot\zeta}{2}}{|\zeta|^{d+2k-2+\delta}} d\zeta \Big)(dF_n(v) + dF(v)).
\end{aligned}$$

Hence we can take a large $R_\varepsilon > 1$ such that $J_n(R_\varepsilon) < \varepsilon$. Since $|\varphi_n|, |\varphi| \le 1$, by taking another large R_ε, if necessary, we may assume

$$\int_{|\xi|\ge R_\varepsilon} \frac{|\Delta^k\varphi_n(\xi) - \Delta^k\varphi(\xi)|}{|\xi|^{d+2k-2+\delta}} d\xi < \varepsilon.$$

The weak convergence $F_n \rightharpoonup F$ implies $\varphi_n \to \varphi$ uniformly on any compact set in $\mathbb{R}^d$ because $\{\varphi_n\}$ are equi-continuous (see, for example, [15, (19)]). Therefore we have

$$\int_{1/R_\varepsilon \le|\xi|\le R_\varepsilon} \frac{|\Delta^k\varphi_n(\xi) - \Delta^k\varphi(\xi)|}{|\xi|^{d+2k-2+\delta}} d\xi \to 0,$$

which concludes the proof of (2.20).

It remains to show $\|\varphi_n - \varphi\|_\beta \to 0$ for $\beta = \min\{\alpha, 2\}$. The case $\beta = 2$ means that (i) of Theorem 2.1 holds with $\alpha = 2k - 2 + \delta \ge 2$. Then it follows from (2.2) that there exists a strictly increasing function $\phi(t)$ in $(0,\infty)$ independent of n such that $\phi(t)/t$ is decreasing in $(0,\infty)$ and

$$M := \sup_n \int_{\mathbb{R}^d} |v|^2 \phi(|v|) \Big(dF_n + dF(v)\Big) < \infty, \ \ \text{(see [24, Lemma 2])}.$$

The proof of $\|\varphi_n - \varphi\|_2 \to 0$ is quite the same as in [24, p. 624-625]. For the sake of convenience of readers, we give it here. As in the proof of [12, Lemma 3.1], we have

$$\begin{aligned}
|\partial_j\partial_k\varphi_n(\xi) - \partial_j\partial_k\varphi_n(\eta)| &\le \int_{\mathbb{R}^d} 2\left|\sin\frac{|\xi-\eta||v|}{2}\right| |v|^2 dF_n(v) \\
&\le 2\Big(\int_{\mathbb{R}^d} |v|^2\phi(|v|)dF_n(v)\Big) \sup_{|v|} \frac{|\sin(|\xi-\eta||v|/2)|}{\phi(|v|)} \\
&\le 2M \frac{1}{\phi(2/|\xi-\eta|)},
\end{aligned}$$

because $\min\{xy,1\}/\phi(x) \leq 1/\phi(1/y)$. Since

$$\begin{aligned}\frac{\varphi_n(\xi)-\varphi(\xi)}{|\xi|^2} &= \int_0^1 \Big(\nabla\otimes\nabla\varphi_n(t\xi)-\nabla\otimes\nabla\varphi_n(0)\Big)\Big(\frac{\xi}{|\xi|}\cdot\frac{\xi}{|\xi|}\Big)(1-t)dt\\ &+\frac{1}{2}\Big(\nabla\otimes\nabla\varphi_n(0)-\nabla\otimes\nabla\varphi(0)\Big)\Big(\frac{\xi}{|\xi|}\cdot\frac{\xi}{|\xi|}\Big)\\ &-\int_0^1 \Big(\nabla\otimes\nabla\varphi(t\xi)-\nabla\otimes\nabla\varphi(0)\Big)\Big(\frac{\xi}{|\xi|}\cdot\frac{\xi}{|\xi|}\Big)(1-t)dt,\end{aligned}$$

the equi-continuity of $\{\partial_j\partial_k\varphi_n(\xi)\}$ and the condition (iv) of Theorem 2.1 equivalent to (i) show that $\forall\varepsilon>0$, $\exists\delta>0$;

$$\lim_{n\to\infty}\sup_{|\xi|\leq\delta}\frac{|\varphi_n(\xi)-\varphi(\xi)|}{|\xi|^2}<\varepsilon\,,$$

because of $|\partial_j\partial_k\varphi_n(0)-\partial_j\partial_k\varphi(0)|=|\int v_jv_kdF_n(v)-\int v_jv_kdF(v)|$. On the other hand, it is obvious that for a large $R_\varepsilon>1$ we have

$$\sup_{|\xi|\geq R_\varepsilon}\frac{|\varphi_n(\xi)-\varphi(\xi)|}{|\xi|^2}<\varepsilon.$$

Since the pointwise convergence of $\{\varphi_n\}$ follows from the weak convergence of $\{F_n\}$ and $\{\varphi_n\}\subset\mathcal{K}^2$ is equi-continuous on any compact set in $\mathbb{R}^d$, we see that

$$\lim_{n\to\infty}\sup_{\delta\leq|\xi|\leq R_\varepsilon}\frac{|\varphi_n(\xi)-\varphi(\xi)|}{|\xi|^2}=0\,.$$

Therefore we have showed that (i) with $\alpha=2k-2+\delta\geq 2$, that is, (2.15) implies $\|\varphi_n-\varphi\|_2\to 0$.

We show that the condition (i) with $0<\alpha<2$, (i.e. $\alpha=\delta, k=1$), implies $\|\varphi_n-\varphi\|_\alpha\to 0$. Noting that

$$\frac{\varphi_n(\xi)-\varphi(\xi)}{|\xi|^\alpha}=\int_{\mathbb{R}^d}\frac{e^{-i\xi\cdot v}-1}{(|\xi||v|)^\alpha}|v|^\alpha\Big(dF_n(v)-dF(v)\Big),\qquad \text{if } \alpha\in(0,1),$$

$$\frac{\varphi_n(\xi)-\varphi(\xi)}{|\xi|^\alpha}=\int_{\mathbb{R}^d}\frac{e^{-i\xi\cdot v}-1+i\xi\cdot v}{(|\xi||v|)^\alpha}|v|^\alpha\Big(dF_n(v)-dF(v)\Big),\qquad \text{if } \alpha\in[1,2),$$

and $\dfrac{|e^{-iz}-1|}{|z|^\alpha}\leq\min\{|z|^{1-|\alpha|},2\}$ $(\alpha<1)$, $\dfrac{|e^{-iz}-1+iz|}{|z|^\alpha}\leq\min\{|z|^{2-|\alpha|},3\}$ $(\alpha\geq 1)$, we have, for any $1<\tilde{R}<R$,

$$\begin{aligned}\sup_{|\xi|<R^{-1}}\frac{|\varphi_n(\xi)-\varphi(\xi)|}{|\xi|^\alpha} &\leq \sup_{|\xi|<R^{-1}}\int_{\{|v|<\tilde{R}\}}\Big(|v||\xi|\Big)^{1+[\alpha]-\alpha}|v|^\alpha\Big(dF_n(v)+dF(v)\Big)\\ &\quad+3\int_{\{|v|\geq\tilde{R}\}}|v|^\alpha\Big(dF_n(v)+dF(v)\Big)\\ &=I_1(R,\tilde{R})+I_2(\tilde{R})\,.\end{aligned}$$

It follows from (2.2) that for any $\varepsilon > 0$ there exists a $\tilde{R}_\varepsilon > 1$ such that $I_2(\tilde{R}_\varepsilon) < \varepsilon$. If $\tilde{R}_\varepsilon > 1$ is fixed as above, then we have

$$I_1(R, \tilde{R}_\varepsilon) \le 2\frac{\tilde{R}_\varepsilon^{1+[\alpha]}}{R^{1+[\alpha]-\alpha}} \to 0\,, \quad R \to \infty\,.$$

Consequently, for any $\varepsilon > 0$ there exists $R > 1$ such that

$$\sup_{|\xi|<R^{-1}} \frac{|\varphi_n(\xi) - \varphi(\xi)|}{|\xi|^\alpha} \le 2\varepsilon,$$

which leads us to $\|\varphi_n - \varphi\|_\alpha \to 0$. This completes the proof of Theorem 2.4 $\Box$

As stated in Introduction and Remark (3) of Theorem 2.4, the equivalence between the metric $W_2(\cdot,\cdot)$ and Toscani metric with $\alpha = 2$ was proved in [24, Theorem 1] under a fixed energy condition ($\int |v|^2 dF(v) = Ed$) and zero mean vector ($\int v_j dF(v) = 0$), however the former condition is unnecessary since one has an equivalent condition (iii) in Theorem 2.1. The condition (iii) is also useful to show that the measure valued solution F_t in Theorem 1.1 belongs to $C([0,\infty), P_\alpha(\mathbb{R}^3))$, directly (, not through the Bobylev formula,) after proving $F_t \in L^\infty_{loc}([0,\infty); P_\alpha(\mathbb{R}^3))$ (see [16, Proposition 3.5] and [17, Corollary 1.7]). Though the Toscani metric was applicable in [24], to show the uniqueness of solutions in Maxwellian case, as the same as $W_2(\cdot,\cdot)$ in [23] (cf., [22, Theorem 4.1]), it is an open problem whether Toscani metric or its generalization is applicable for non Maxwellian case, like $W_\alpha(\cdot,\cdot)$ with $\alpha = 1, 2$, respectively, working in [11, 10], respectively.

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