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Microlocal analysis on PDE : Some contributions by Yoshinori Morimoto around kinetic equations

By

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Abstract

In this short paper, we give a brief presentation of some contribution to microlocal analysis on the partial differential equations by Yoshinori Morimoto. The first part concerns the study of the degenerate elliptic equations, by using the microlocal analysis based on the theory of pseudo-differential operators. The second part is about the analysis of non-cutoff Boltzmann equations where the microlocal analysis contribute lots of progress, in particular, using Fefferman-Phong's uncertainty principle to prove the smoothing effect of solutions and Littlewood-Paley theory to study the existence of classical solutions for non-cutoff Boltzmann equations. So that we focus only on the contribution to the study of kinetic equations by Yoshinori Morimoto.

§ 1. Hypoellipticity of infinite degenerate elliptic opertators

Microlocal analysis (Wikipedia Encyclopedia)

In mathematical analysis, microlocal analysis comprises techniques developed from the 1950s onwards based on Fourier transforms related to the study of linear PDE. This includes pseudo-differential operators and Fourier integral operators. The term microlocal implies localisation not only with respect to location in the space, but also with respect to cotangent space directions at a given point. The techniques of microlocal

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analysis were developed in the 1960s and 1970s.

Hypoellipticity : A partial differential operator P is called hypoelliptic, if $u \in \mathcal{D}', Pu \in C^\infty$ imply that $u \in C^\infty$.

- Elliptic operator with C^∞ coefficients is hypoelliptic.
- The following Hörmander's operators is hypoelliptic,

$$H = \sum_{j=1}^{m} X_j^2 + X_0,$$

where $X = \{X_0, X_1, \cdots, X_m\}$ is a system of vector fields which satisfy the Hörmander's condition: $X_0, X_1, \cdots, X_m$ and their repeated commutators of at most k times span the tangent space .

A typic exemple of Hörmander's operators is

$$P_1 = \partial_t^2 + \partial_x^2 + x^{2k}\partial_y^2.$$

It is possible to take a system of pseudo-differential operators X.

The work of Y. Morimoto was motivated by the previous result of S. Kusuoka and D. Stroock by means of the probabilistic method (Malliavin calculus). Namely, they gave a remarkable example:

$$P = D_t^2 + D_x^2 + g(x)D_y^2, \;\; 0 \le g(x) \in C^\infty(\mathbb{R}), \;\; g(x) > 0 \text{ if } x \neq 0$$

is hypoelliptic in $\mathbb{R}^3$, if and only if

$$\lim_{x\to\infty} |x| \log g(x) = 0.$$

If $g(x) = \exp -|x|^\sigma, \sigma > 0$ then the condition is interpreted to $0 < \sigma < 1$. The term D_t^2 is significant, in fact, $A = D_x^2 + g(x)D_y^2$ is hypoelliptic in $\mathbb{R}^2$ without the degenerate rate condition. This phenomena is quite similar to non-analytic hypoellipticity for Baouendi-Goulaouic operator

$$P_0 = D_t^2 + D_x^2 + x^{2m}D_y^2, \;\; , m = 1, 2, \cdots,$$

that is, the Gevrey singularity of order G^{m+1} propagates along the t direction. The propagation of the singularity of Gevrey class can be roughly understood by the uncertainty principle (mathematically formalized by Fefferman-Phong) as follows: If we denote the symbol of $D_x^2 + x^{2m}D_y^2$ by $\xi^2 + x^{2m}\eta^2$, then

$$0 \le \xi^2 + x^{2m}\eta^2 \le 2|\eta|^{2/(m+1)} \text{ on the box } \{|x| \le |\eta|^{-1/(m+1)}, |\xi| \le |\eta|^{1/(m+1)}\},$$

and P_0 behaves like, on a tube $\{(t,x,y,\xi,\eta); t\in(-\infty,\infty), |x|\le|\eta|^{-1/(m+1)}, |\xi|\le |\eta|^{1/(m+1)}\}$,

$$P_0 \sim D_t^2 + 2|D_y|^{2/(m+1)}.$$

As for C^∞ singularity, the log order is critical. Based on this idea, the characterization of partial differential operators with the logarithmic regularity estimate has been studied by means of the Fefferman-Phong uncertainty principle.

Infinite degenerate elliptic operators The following exemple is an infinite degenerate elliptic operators

$$P_2 = \partial_t^2 + \partial_x^2 + e^{-2|x|^{-\delta}}\partial_y^2,$$

with $0<\delta<1$. This operator is also hypoelliptic. More generally, for the operators

$$P_3 = D_t^2 + \mathcal{A}(x, D_x), \quad \mathbb{R}_t \times \mathbb{R}_x^n$$

where $\mathcal{A}(x,D_x)$ is a formally self-adjoint second order PDO such that

$$(\mathcal{A}(x,D_x)u,u) \ge c_0\|u\|_{L^2}^2, \quad \forall u\in C_0^\infty.$$

There are a criterion for hypoellipticity.

Theorem 1.1 (Y. Morimoto, Osaka J. Math. 1987). P_3 is hypoelliptic in $\mathbb{R}_t\times\mathbb{R}_x^n$ if and only if for any $x_0\in\mathbb{R}_x^n$ there exists a neighborhood ω of x_0 such that for any $\epsilon>0$ the estimate

$$\|\log\langle D_t, D_x\rangle\phi\|_{L^2}^2 \le \epsilon \mathrm{Re}(P_3\phi,\phi) + C_\epsilon\|\phi\|_{L^2}^2, \quad \forall\phi\in C_0^\infty(\mathbb{R}_t\times\omega)$$

holds with a constant C_ϵ.

The above results can be extended to more general cases.

Another necessary and sufficient condition for the hypoellipticity Let

$$P_4 = D_x^2 + f(x)D_t^2 + g(x)D_y^2, \quad \mathbb{R}^3_{x,t,y}, \quad f,\, g\ge 0.$$

Let f_I denote the average of f in $I\subset I_0$ an open fixed interval in $\mathbb{R}_x$, $f_I, g_I>0$. We say that f,g satisfy (M,f,g) condition in I_0 if

$$(M,f,g) \qquad \inf_{\delta>0}\big(\sup\{f_I^{1/2}\,|I|\,|\log g_{3I}|; I \,\mathrm{with}\, 3I\subset I_0, g_{3I}<\delta\}\big) = 0.$$

Theorem 1.2 (Y. Morimoto-T. Morioka, Bull. Sci. Math. 1997). P_4 is hypoelliptic in $I_0\times\mathbb{R}_t\times\mathbb{R}_y$ if and only if f,g satisfy (M,f,g) and (M,g,f) in I_0.

The above results is also true for more general cases. An example :

$$f = x^{2m}, \quad g = e^{-|x|^{-s}}, \quad s < m+1, (by T. Hoshiro)$$

The uncertainty principle Consider a symbol of the form, for $0 < \lambda \le 1$,

$$a(x,\xi) = |\xi|^{2\lambda} + V(x), \quad (x,\xi) \in \mathbb{R}^{2n}$$

where $V(x)$ is a non-negative function and depends on a large parameter $M > 0$. Let $\mathcal{C}$ denote the set of boxes

$$B = \{(x,\xi) \in \mathbb{R}^{2n};\ |x_j - x_j^0| \le \frac{\delta}{2},\ |\xi_j - \xi_j^0| \le \frac{\delta^{-1}}{2}\}.$$

Theorem 1.3 (Y. Morimoto, Publ. RIMS. Kyoto Univ. 1987).

$$\inf_{B\in\mathcal{C}} m\big(\{(x,\xi) \in B; a(x,\xi) \ge R\}\big) \ge c > 0,$$

imply

$$\|\,|D_x|^\lambda u\|_{L^2}^2 + (V(x)u,\, u) \ge c' R \|u\|_{L^2}^2, \quad u \in C_0^\infty(\mathbb{R}^n),$$

where $c' > 0$ depends only on c, n, λ.

As an application, we consider the operators

$$P_5 = a(x,y,D_x) + g(x')b(x,y,D_y), \quad \text{in } \mathbb{R}^n \times \mathbb{R}^m,$$

where $x = (x', x'') \in \mathbb{R}^n$ and

$$\mathrm{Re}\, a(x,y,\xi) \ge c_1|\xi|^2, \quad \text{for large } |\xi|,$$
$$\mathrm{Re}\, b(x,y,\xi) \ge c_2|\eta|^2, \quad \text{for large } |\eta|,$$

$g(x') > 0$ for all $x' \neq 0$ and vanish infinite order at 0. Then we have

$$\lim_{|x'|\to 0} |x'|\,|\log g(x')| = 0 \implies P_5 \text{ is hypoelliptic.}$$

Let $a(x,\xi) = |\xi|^2 + c\, g(x')|\eta|^2$, then we can apply the uncertainty principle with $R = (k \log|\eta|)^2$ which deduce the logarithmic regularity estimates.

§ 2. The non-cutoff Boltzmann equations

From 2005, Yoshinori Morimoto and his collaborators (Alexandre-Morimoto-Ukai-Xu-Yang (AMUXY)) start to work on the Kinetic equations such as non-cutoff Boltzmann equations and Landau equation with more than 30 publications.

$$f_t + v\cdot\nabla_x f = Q(f,f)\,, \quad f|_{t=0} = f_0, \tag{2.1}$$

where the Boltzmann collisional operator is

$$Q(g,f)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left(v-v_*,\sigma\right)\left\{g'_*f'-g_*f\right\}d\sigma dv_*\,,$$

and the collision kernel is

$$B(v-v_*,\sigma)=|v-v_*|^\gamma b(\cos\theta),\ \ 0\leq\theta\leq\frac{\pi}{2},$$

with angular non-cutoff factor,

$$b(\cos\theta)\ \approx\ \theta^{-2-2s},\ \ 0\leq\theta\leq\frac{\pi}{2}. \tag{2.2}$$

Smoothing effect of non-cutoff Boltzmann equation

A first microlocal analysis results of Boltzmann equation is the following smoothing effect work.

Theorem 2.1 (AMUXY, ARMA, 2010). Assume (2.2) holds true, with $0<s<1$, $\gamma>\max\{-3,-3/2-2s\}$, $0<T\leq+\infty$. $\Omega\subset\mathbb{R}^3_x, f\geq 0, f\not\equiv 0, f\in L^\infty([0,T];H^5_\ell(\Omega\times\mathbb{R}^3))$ for any $\ell\in\mathbb{N}$, be a solution of Cauchy problem (2.1). Then we have

$$f\in C^\infty(]0,T[\times\Omega,\,\mathcal{S}(\mathbb{R}^3))\,.$$

The main ingredient for the proof of the above Theorem are :

- The coercivity and upper bound estimates of Boltzmann operators.
- Uncertainty principle and regularity of the following transport equation:

$$f_t+v\cdot\nabla_x f=g\in\mathcal{D}'(\mathbb{R}^7). \tag{2.3}$$

The coercivity and upper bound estimates We have the following coercive estimate :

Let $g\geq 0$, $g\not\equiv 0$, $g\in L^1_2\bigcap L\log L(\mathbb{R}^3_v)$. Then for any function $f\in H^s_{\frac{\gamma}{2}+s}(\mathbb{R}^3_v)$, we have

$$C_g^{-1}\|f\|^2_{H^s_{\frac{\gamma}{2}}(\mathbb{R}^3_v)}\leq(-Q(g,f),f)_{L^2(\mathbb{R}^3_v)}+C\|g\|_{L^1_{\gamma+2s}(\mathbb{R}^3_v)}\|f\|^2_{L^2_{\frac{\gamma}{2}+s}(\mathbb{R}^3_v)}.$$

We have also the upper bound estimate

$$\|Q(f,\,g)\|_{H^m_\alpha(\mathbb{R}^3_v)}\leq C\|f\|_{L^1_{\alpha^++(\gamma+2s)^+}(\mathbb{R}^3_v)}\|g\|_{H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)}\,.$$

The results are due to : Alexendre-Desvillettes-Villani-Wennberg and also Alexendre-Morimoto-Ukai-Xu-Yang (AMUXY).

The Littlewood-Paley theory

- For the coercivity and upper bound estimate of non-cutoff Boltzmann operators, in the Maxwellian case ($\gamma = 0$), the proof is just by using the Fourier transformation (Bobylov formula).

- But for general non Maxwellian case ($\gamma > -3$), we need to use the following Littlewood-Paley decomposition :

$$\phi_0(v) + \sum_{k=1}^{\infty} \phi(2^{-k}v) = 1, \quad v \in \mathbb{R}^3,$$

with $\phi_0, \phi \in C_0^\infty(\mathbb{R}^3)$ and

$$\text{supp } \phi_0 \subset \{|v| < 2\}, \quad \text{supp } \phi \subset \{1 < |v| < 3\}.$$

Then we can study the Boltzmann operators with kinetic factors as

$$B_k(|v - v_*|, \sigma) = \phi(2^{-k}(v - v_*))|v - v_*|^\gamma b(\cos\theta).$$

The Fefferman-Phong's uncertainty principle and kinetic equations

The smoothing property of Boltzmann equation is like the following Hörmander's type equation:

$$f_t + v \cdot \nabla_x f + (-\triangle_v)^s f = g,$$

with $X_0 = \partial_t + v \cdot \nabla_x$, $X_1 = (-\triangle_v)^{\frac{s}{2}}$. And it is in the form of transport equation as (2.3).

Let $a(r) = r^{-2s}$, Q is a cube in $\mathbb{R}^n$, $\ell(Q)$ is the side length. Q^* is any cube such that $Q \subset Q^*$ with $\ell(Q^*) = 2\ell(Q)$. For $a^+, a^- \geq 0$,

$$E(Q, Q^*) = \left\{v \in Q^*;\ a^+(v) \geq ||a^-||_{L^\infty(Q)} - a(\ell(Q))\right\}.$$

The main assumption on the functions a^+, a^- is

$$(H_0) \qquad \inf_{Q,Q^*} \left\{ \frac{|E(Q,Q^*)|}{|Q^*|};\quad Q \subset Q^* \text{ with } 2\ell(Q) = \ell(Q^*) \right\} \geq \kappa > 0.$$

Regularity for transport Equations

Theorem 2.2 (Uncertainty Principle, JFA, AMUXY, 2008). If the condition (H_0) holds, then

$$\int_{\mathbb{R}^n} a^-(v)|f(v)|^2 dv \leq C \int_{\mathbb{R}^n} \left(||D_v|^s f(v)|^2 + a^+(v)|f(v)|^2\right) dv,$$

for any $f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 2.3 (JFA, AMUXY, 2008). Assume that $g \in H^{-s'}(\mathbb{R}^7), 0 \le s' < 1$, $f \in L^2(\mathbb{R}^7)$ is a weak solution (2.3) such that $|D_v|^s f \in L^2(\mathbb{R}^7)$ for some $0 < s \le 1$. Then

$$\frac{\Lambda_x^{s(1-s')/(s+1)} f}{(1+|v|^2)^{s\,s'/2(s+1)}}, \quad \frac{\Lambda_t^{s(1-s')/(s+1)} f}{(1+|v|^2)^{s/2(s+1)}} \in L^2(\mathbb{R}^7).$$

The main idea in the proof of this regularity theorem is to define the corresponding functions $a_\pm(v)$ and show that they satisfy the condition (H_0) with the Fourier variables τ and η as parameters.

$$a^-(v,\tau,\eta) = c_0 \frac{|\tau|^{2s(1-s')/(s+1)}}{(1+|v|^2)^{s/(s+1)}} + c_1 \frac{|\eta|^{2s(1-s')/(s+1)}}{(1+|v|^2)^{s\,s'/(s+1)}},$$

and

$$a^+(v,\tau,\eta) = 1 + \frac{|\tau + v\cdot\eta|^2}{(1+\tau^2+|\eta|^2)^{s'}}.$$

Existence of the classical solutions

Consider the perturbation $f = \mu + \sqrt{\mu} g$ around a normalized Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Since $Q(\mu,\mu) = 0$, we have

$$Q(f,\, f) = Q(\mu,\, \sqrt{\mu}\, g) + Q(\sqrt{\mu}\, g,\, \mu) + Q(\sqrt{\mu}\, g,\, \sqrt{\mu}\, g).$$

Denote

$$\Gamma(g,\, h) = \mu^{-1/2} Q(\sqrt{\mu}\, g,\, \sqrt{\mu}\, h).$$

Then the linearized Boltzmann operator takes the form

$$\mathcal{L} g = -\Gamma(\sqrt{\mu}\,,\, g) - \Gamma(g,\, \sqrt{\mu}\,).$$

We consider the Cauchy problem for the perturbation g

$$\begin{cases} g_t + v\cdot\nabla_x g + \mathcal{L} g = \Gamma(g,\, g), \;\; t > 0\,, \\ g|_{t=0} = g_0. \end{cases} \tag{2.4}$$

We have the following existence of classical solutions for non-cutoff Boltzmann equation.

Theorem 2.4 (JFA, AMUXY, 2012). Assume $\gamma + 2s \le 0$, $0 < s < 1$ and $\gamma > \max\{-3, -\frac{3}{2} - 2s\}$. Let $N \ge 4, \ell \ge N$. There exists $\varepsilon_0 > 0$, such that if $||g_0||_{\mathcal{H}^N_\ell(\mathbb{R}^6)} \le \varepsilon_0$, then the Cauchy problem (2.4) admits a global solution

$$g \in L^\infty([0, +\infty[;\, \mathcal{H}^N_\ell(\mathbb{R}^6)).$$

Here

$$\|f\|^2_{\mathcal{H}^N_\ell(\mathbb{R}^6)} = \sum_{|\alpha|+|\beta|\le N} \|\langle v\rangle^{|2s+\gamma|(\ell-|\beta|)}\partial^\alpha_\beta f\|^2_{L^2(\mathbb{R}^6)}.$$

Theorem 2.5 (AA, AMUXY, 2011). Assume $0 < s < 1$ and $\gamma + 2s > 0$. Let $g_0 \in H^k_\ell(\mathbb{R}^6)$ for some $k \ge 6$, $\ell > 3/2 + 2s + \gamma$. There exists $\varepsilon_0 > 0$, such that if $\|g_0\|_{H^k_\ell(\mathbb{R}^6)} \le \varepsilon_0$, the Cauchy problem (2.4) admits a global solution

$$g \in L^\infty([0, +\infty[\,;\ H^k_\ell(\mathbb{R}^6)).$$

By apply Theorem 2.1, then we have the regularity of solutions of Theorem 2.4-2.5,

$$f = \mu + \sqrt{\mu}\, g \in C^\infty(]0, T[\times\mathbb{R}^3,\, \mathcal{S}(\mathbb{R}^3))\,.$$

§ 3. The smoothing effect of the non-cutoff Boltzmann operators

Gelfand-Shilov regularizing effect

Another important work of Y. Morimoto on the microlocal analysis of kinetic equation is the regularizing effect of Cauchy problem in Gelfand-Shilov class.

The Gelfand-Shilov spaces $S^\mu_\nu(\mathbb{R}^d)$, with $\mu, \nu > 0$, $\mu + \nu \ge 1$, are defined as the spaces of smooth functions $f \in C^\infty(\mathbb{R}^d)$ satisfying the estimates

$$\exists A, C > 0, \quad |\partial^\alpha_v f(v)| \le CA^{|\alpha|}(\alpha!)^\mu e^{-\frac{1}{A}|v|^{1/\nu}}, \quad v \in \mathbb{R}^d,\ \alpha \in \mathbb{N}^d,$$

or, equivalently

$$\exists A, C > 0, \quad \sup_{v\in\mathbb{R}^d} |v^\beta \partial^\alpha_v f(v)| \le CA^{|\alpha|+|\beta|}(\alpha!)^\mu(\beta!)^\nu, \quad \alpha, \beta \in \mathbb{N}^d.$$

Also equivalently, $f \in \mathcal{S}(\mathbb{R}^d)$ satisfying to the estimates

$$\exists C > 0, \epsilon > 0, \quad |f(v)| \le Ce^{-\epsilon|v|^{1/\nu}}, \qquad |\widehat{f}(\xi)| \le Ce^{-\epsilon|\xi|^{1/\mu}}.$$

Remark that Gelfand-Shilov space is much strong than the Gevrey space(for which $\nu = +\infty$) and the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

The Cauchy problem defined by the evolution equation associated to the harmonic oscillator

$$\begin{cases} \partial_t f + \mathcal{H} f = 0, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^d), \end{cases}$$

enjoys nice regularizing properties in $S^{\frac{1}{2}}_{\frac{1}{2}}(\mathbb{R}^d)$ for any positive time, where

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}$$

is the d-dimensional harmonic oscillator.

Moreover, the smoothing effect for the solutions to the Cauchy problem defined by the evolution equation associated to the fractional harmonic oscillator, $0 < s < 1$,

$$\begin{cases} \partial_t f + \mathcal{H}^s f = 0, \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^d), \end{cases}$$

occurs for any positive time in the symmetric Gelfand-Shilov space $S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}^d)$.

The first result about kinetic equation is the regularizing effect of Cauchy problem to the following Kolmogorov equation :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \triangle_v f = 0, \qquad (x, v) \in \mathbb{R}^{2d} \\ f|_{t=0} = f_0 \in L^2(\mathbb{R}^{2d})\,. \end{cases}$$

The solution of above equation satisfying $f(t) \in G^{\frac{1}{2}}(\mathbb{R}^{2d})$, for all $t > 0$, in fact, we have

$$e^{-c(t\triangle_v + t^3\triangle_x)} f(t, \cdot, \cdot) \in L^2(\mathbb{R}^{2d})\,,$$

for some $c > 0$ (Morimoto-Xu [26]).

For the solution of the spatial homogeneous radial symmetric non-cutoff Boltzmann equation (Maxwellian molecule), we consider also a perturbation $f = \mu + \sqrt{\mu} g$, then we have

Theorem 3.1 (Lerner-Morimoto-Pravda-Starov-Xu, JDE, 2013). Let $g_0 \in L^1(\mathbb{R}^3_v) \cap L^2(\mathbb{R}^3_v)$ radially symmetric and $\|g_0\|_{L^2} << 1$. Assume that $g(t)$ is the classical solution of the Cauchy problem

$$\partial_t g + \mathcal{L} g = \Gamma(g\,, g), \qquad g|_{t=0} = g_0\,.$$

Then, $\exists c_0 > 0, C > 0, \forall t \geq 0$,

$$\|e^{c_0 t \mathcal{H}^s} g(t)\|_{L^2} \leq e^{-Ct} \|g_0\|_{L^2},$$

this implies that

$$\forall t > 0, \quad g(t) \in S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}^3).$$

It is also true for the case $s = 1$, i. e. the homogeneous Landau equation.

Main ingredient of proofs of above results is the spectral decomposition of harmonic oscillator on the Hermite basis, where the Hermite function is

$$\Psi_\alpha = \frac{1}{\sqrt{\alpha_1! ... \alpha_d!}} A_1^{\alpha_1} ... A_d^{\alpha_d} \Psi_0, \quad \alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d,$$

with

$$\Psi_0(v) = (2\pi)^{-\frac{d}{4}} e^{-\frac{|v|^2}{4}}, \qquad A_j = \frac{v_j}{2} - \frac{\partial}{\partial v_j}\,.$$

The family $(\Psi_\alpha)_{\alpha\in\mathbb{N}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ composed by the eigenfunctions of the d-dimensional harmonic oscillator

$$\mathcal{H} = -\Delta_v + \frac{|v|^2}{4} = \sum_{k\geq 0}\Big(\frac{d}{2}+k\Big)\mathbb{P}_k, \quad \mathrm{Id} = \sum_{k\geq 0}\mathbb{P}_k,$$

where $\mathbb{P}_k$ is the orthogonal projection onto $\mathcal{E}_k$ =Span$\{\Psi_\alpha; |\alpha| = k\}$.

For the linearized radially symmetric Boltzmann operators, we have the following spectral decomposition

$$\mathcal{L} = \sum_{k\geq 1}\lambda_k\mathbb{P}_k, \quad \lambda_k \ \approx\ k^s\,,\ (k\to+\infty).$$

Then, we have

$$\Big(\mathcal{L}g,\, g\Big)_{L^2} \ \approx\ \|\mathcal{H}^{s/2}g\|_{L^2},$$

and the upper bounded estimate of Boltzmann operators,

$$\Big(\Gamma(f,g),\ e^{t\mathcal{L}}\mathbf{S}_n h\Big)_{L^2} \leq C\|e^{\frac{t}{2}\mathcal{L}}\mathbf{S}_{n-2}f\|_{L^2}\|e^{\frac{t}{2}\mathcal{L}}\mathbf{S}_{n-2}\mathcal{H}^{s/2}g\|_{L^2}\|e^{\frac{t}{2}\mathcal{L}}\mathbf{S}_n\mathcal{H}^{s/2}h\|_{L^2}$$

with $\mathbf{S}_n = \sum_{k=0}^n \mathbb{P}_k$. This spectral decomposition permits to study also the smoothing effect of the spatially inhomogeneous non-cutoff Kac equation,

$$\begin{cases}\partial_t f + v\partial_x f = K(f,f),\\ f|_{t=0} = f_0,\end{cases}$$

the Kac collision operator is defined as

$$K(g,f) = \int_{|\theta|\leq\frac{\pi}{4}}\beta(\theta)\left(\int_{\mathbb{R}}(g'_*f' - g_*f)dv_*\right)d\theta,$$

where

$$\beta(\theta)\ \approx\ |\theta|^{-1-2s}, \quad 0<s<1.$$

The Kac equation for the fluctuation around the Maxwellian distribution reads

$$\begin{cases}\partial_t g + v\partial_x g + \mathcal{K}g = \Gamma(g,g),\\ g|_{t=0} = g_0,\end{cases} \tag{3.1}$$

Theorem 3.2. There exist some positive constants $\epsilon_0>0$, $c_0>1$ such that for all $g_0\in H^1(\mathbb{R}_x; L^2(\mathbb{R}_v))$ satisfying

$$\|g_0\|_{H^1(\mathbb{R}_x;L^2(\mathbb{R}_v))}\leq\epsilon_0,$$

the Cauchy problem (3.1) admits a unique weak solution $g \in L^\infty([0,T]; H^1(\mathbb{R}_x; L^2(\mathbb{R}_v)))$. Furthermore, this solution is smooth for all positive time $0 < t \leq T$, and satisfies the Gelfand-Shilov-Gevrey type estimates:

$$\forall \delta > 0, \exists C > 1, \forall 0 < t \leq T, \forall k, \ell, p \mathbb{R},$$

$$\|v^k \partial_v^\ell \partial_x^p g(t)\|_{L^\infty(\mathbb{R}^2_{x,v})} \leq \frac{C^{k+l+p+1}}{t^{\frac{2s+1}{2s}(k+l+p+3)+\delta}} (k!)^{\frac{2s+1}{2s}} (l!)^{\frac{2s+1}{2s}} (p!)^{\frac{2s+1}{2s}} \|g_0\|_{H^1(\mathbb{R}_x; L^2(\mathbb{R}_v))}.$$

This result establishes a Gelfand-Shilov smoothing effect in the velocity variable and a Gevrey smoothing effect in the position variable :

$$\forall t > 0, \quad g(t, x, \cdot) \in S^{1+\frac{1}{2s}}_{1+\frac{1}{2s}}(\mathbb{R}_v)$$

and

$$\forall t > 0, g(t, \cdot, v) \in G^{1+\frac{1}{2s}}(\mathbb{R}_x).$$

So that there are similar regularity properties as Kolomogrov equation with some loss.

The homogeneous Boltzmann equation with measure-valued initial datum

Recently, Y. Morimoto work for the homogeneous Boltzmann equation with measure-valued initial datum.

For $\alpha \in [0,2]$, $\mathcal{P}^\alpha(\mathbb{R}^3)$ denotes the probability density function f on $\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} |v|^\alpha f(v) dv < \infty,$$

and moreover when $\alpha \geq 1$, it requires that

$$\int_{\mathbb{R}^3} v_j f(v) dv = 0, \quad j = 1, 2, 3.$$

A characteristic function $\varphi(t, \xi)$ is the Fourier transform of $f(t, v) \in \mathcal{P}^0(\mathbb{R}^3)$:

$$\varphi(t, \xi) = \hat{f}(t, \xi) = \mathcal{F}(f)(t, \xi) = \int_{\mathbb{R}^3} e^{-iv\cdot\xi} f(t, v) dv.$$

For each $\alpha \in [0,2]$, set $\widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3) = \mathcal{F}^{-1}\left(\mathcal{K}^\alpha(\mathbb{R}^3)\right)$ with $\mathcal{K}(\mathbb{R}^3) = \mathcal{F}(\mathcal{P}^0(\mathbb{R}^3))$ and

$$\mathcal{K}^\alpha(\mathbb{R}^3) = \Big\{\varphi \in \mathcal{K}(\mathbb{R}^3) : \|\varphi - 1\|_{\mathcal{D}^\alpha} < \infty\Big\}.$$

Here the distance $\mathcal{D}^\alpha$ between $\varphi(\xi)$ and $\tilde{\varphi}(\xi)$ with $\alpha > 0$ is defined by

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{D}^\alpha} \equiv \sup_{0 \neq \xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}.$$

It is easy to see that

- The set $\mathcal{K}^\alpha(\mathbb{R}^3)$ endowed with the distance $\mathcal{D}^\alpha$ is a complete metric space;
- $\mathcal{K}^\alpha(\mathbb{R}^3) = \{1\}$ for all $\alpha > 2$;
- the embeddings $\{1\} \subset \mathcal{K}^\alpha(\mathbb{R}^3) \subset \mathcal{K}^\beta(\mathbb{R}^3) \subset \mathcal{K}(\mathbb{R}^3)$ hold for $2 \geq \alpha \geq \beta \geq 0$.

By taking the Fourier transform of the Boltzmann equation(Maxwellian case), we have the Bobylev formula:

$$\partial_t \varphi(t,\xi) = \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\varphi\left(t,\xi^+\right)\varphi\left(t,\xi^-\right) - \varphi(t,\xi)\right) d\sigma, \tag{3.2}$$

where we have used

$$\varphi(t,0) = \int_{\mathbb{R}^3} f(t,v)dv = 1,$$

here,

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}$$

satisfying

$$\xi^+ + \xi^- = \xi, \quad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2.$$

From now on, we consider the Cauchy problem for (3.2) with initial condition

$$\varphi(0,\xi) = \varphi_0(\xi). \tag{3.3}$$

Results available for the Cauchy problem (3.2)-(3.3):

- Global solvability: For $\alpha \in]2s,2]$, this Cauchy problem admits a unique global solution $\varphi(t,\xi) \in C\left([0,\infty), \mathcal{K}^\alpha(\mathbb{R}^3)\right)$ for every $\varphi_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$
- Regularity: If $\mathcal{F}^{-1}(\varphi_0)(v)$ is not a single Dirac mass, $f(t,\cdot) \in L^1(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for any $t > 0$ (see [5]).

When $2s < \alpha < 2$, the initial energy is infinite so that the solution will no longer tend to an equilibrium, but to a self-similar solution: For any given constant $K > 0$

$$f_{\alpha,K}(t,v) = e^{-3\mu_\alpha t}\Psi_{\alpha,K}\left(ve^{-\mu_\alpha t}\right).$$

$$\mu_\alpha = \frac{\lambda_\alpha}{\alpha}, \quad \lambda_\alpha \equiv \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma,$$

$$\int_{\mathbb{R}^3} \Psi_{\alpha,K}(v)dv = 1, \quad \hat{\Psi}_{\alpha,K}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3), \lim_{|\eta|\to 0} \frac{1-\hat{\Psi}_{\alpha,K}(\eta)}{|\eta|^\alpha} = K.$$

- Convergence to $f_{\alpha,K}(t,v)$: not well understood, even the convergence in distribution sense is a problem.

For $f_0(v), g_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$, set

$$\widetilde{P}(t,\xi) = e^{-At}\frac{1}{2}\sum_{j,l=1}^{3}\xi_j\xi_l P_{jl}(0)X(\xi),$$
$$P_{jl}(0) = \int_{\mathbb{R}^3}\left(v_jv_l - \frac{\delta_{jl}}{3}|v|^2\right)(f_0(v)-g_0(v))dv,$$

where

$$A = \frac{3}{4}\int_{\mathbb{S}^2}\mathcal{B}\left(\frac{\sigma\cdot\xi}{|\xi|}\right)\left(1-\left(\frac{\sigma\cdot\xi}{|\xi|}\right)^2\right)d\sigma,$$

$X(\xi) \equiv X(|\xi|)$ is a smooth radially symmetric function satisfying $0 \le X(\xi) \le 1$ and $X(\xi) = 1$ for $|\xi| \le 1$ and $X(\xi) = 0$ for $|\xi| \ge 2$.

The first result on the $\mathcal{D}^{2+\delta}-$time asymptotic stability of the solutions is the following theorem.

Theorem 3.3 (Morimoto-Yang-Zhao, *to appear JEMS*). Suppose $f_0(v), g_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ for $\alpha \in]\max\{2s,1\},2]$. Let $\hat{f}(t,\xi)$ and $\hat{g}(t,\xi)$ be the corresponding two global solutions of the Cauchy problem (3.2) with initial data $\hat{f}_0(\xi)$ and $\hat{g}_0(\xi)$ respectively. Assume for some $\delta \in]0,\alpha] \cap \left]0, \frac{A}{\mu_\alpha}\right]$, the initial data satisfy

$$\int_{\mathbb{R}^3}|v|^2(f_0(v)-g_0(v))dv = 0, \tag{3.4}$$

$$\begin{cases}\displaystyle\int_{\mathbb{R}^3}|v|^2|f_0(v)-g_0(v)|dv < +\infty,\\ \left\|\hat{f}_0(\cdot)-\hat{g}_0(\cdot)-\widetilde{P}(0,\cdot)\right\|_{\mathcal{D}^{2+\delta}} < +\infty.\end{cases} \tag{3.5}$$

Then there exists some positive constant $C_1 > 0$ independent of t and ξ such that

$$\left\|\hat{f}(t,\cdot)-\hat{g}(t,\cdot)-\widetilde{P}(t,\cdot)\right\|_{\mathcal{D}^{2+\delta}} \le C_1e^{-\eta_0 t}, \quad t \ge 0.$$

Here, $\eta_0 = \min\{A-\delta\mu_\alpha, B\}$ and

$$B = \int_{\mathbb{S}^2}\mathcal{B}\left(\frac{\sigma\cdot\xi}{|\xi|}\right)\left(1-\left|\cos\frac{\theta}{2}\right|^{2+\delta}-\left|\sin\frac{\theta}{2}\right|^{2+\delta}\right)d\sigma, \quad \cos\theta = \frac{\sigma\cdot\xi}{|\xi|}.$$

Note that for the $\mathcal{D}^{2+\delta}-$convergence to the self-similar solution, one can simply take $g_0 = \Psi_{\alpha,K}(v)$. Based on this, in order to obtain a convergence in strong topology, such as in the Sobolev norms, we will give a uniform in time estimate on the solution in H^N-norm that is given in

Theorem 3.4 (Morimoto-Yang-Zhao, *to appear JEMS*). For $\max\{1,2s\} < \alpha < 2$, assume that $f_0(v) \in \widetilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ satisfies (3.4)-(3.5) and is not a single Dirac mass, $g_0(v) = \Psi_{\alpha,K}(v)$. Then for any given positive constant $t_1 > 0$ and any $N \in \mathbb{N}$, there exists a positive constant $C_2(t_1, N)$ such that

$$\sup_{t\in[t_1,+\infty)} \Big\{ \|f(t,\cdot)\|_{H^N} \Big\} \le C_2(t_1, N).$$

Consequently, there exists a positive constant $C_3(t_1, N)$ such that

$$\left\|f(t,\cdot) - f_{\alpha,K}(t,\cdot)\right\|_{H^N} = \left\|f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}\left(ve^{-\mu_\alpha t}\right)\right\|_{H^N} \le C_3(t_1,N)e^{-\frac{\eta_0 t}{2}} \tag{3.6}$$

holds for any $t \ge t_1$.

the convergence rate given in (3.6) is faster than the decay rate of the self-similar solution itself. Hence in this case, the infinite energy solution $f(t,v)$ converges to the self-similar solution $f_{\alpha,K}(t,v)$ exponentially in time.

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