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Outflow/inflow problems to symmetric hyperbolic–parabolic systems

By

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§ 1. Introduction

The present paper surveys the results on authors' papers [1, 9, 10, 11], which study a time global existence of a solution and its asymptotic behavior to hyperbolic–parabolic conservation laws over one-dimensional half space $\mathbb{R}_+ := (0, \infty)$,

$$U_t + f(U)_x = (G(U)U_x)_x, \quad x \in \mathbb{R}_+,\ t > 0. \tag{1.1}$$

Here $U = U(t, x)$ is an unknown m-vector valued function taking values in an open convex set $\mathcal{O}_U \subset \mathbb{R}^m$; $f(U)$ is a smooth m-vector valued function defined on $\mathcal{O}_U$; $G(U)$ is a smooth $m \times m$ matrix valued function defined on $\mathcal{O}_U$. The researches in [1, 9] show the existence and the asymptotic stability of the stationary solution to the system (1.1) and the research in [11] obtains a convergence rate of a time global solution towards the stationary solution.

To study the system (1.1), we rewrite it to a normal form of a symmetric hyperbolic-parabolic system under assumptions [**A1**] and [**N**].

[**A1**] There exist real valued smooth functions $\eta = \eta(U)$ and $q = q(U)$ on $\mathcal{O}_U$, which are called an entropy function and an entropy flux for the system (1.1), having the following properties (i)–(iii):

(i) $\eta(U)$ is a smooth strictly convex scalar function, that is, the Hessian matrix $D_U^2\eta(U)$ is positive definite for $U \in \mathcal{O}_U$;

(ii) $D_U q(U) = D_U\eta(U) D_U f(U)$ for $U \in \mathcal{O}_U$;

(iii) the matrix $G(U)(D_U^2\eta(U))^{-1}$ is real symmetric and non-negative definite for $U \in \mathcal{O}_U$.

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The assumption [**A1**] allows us to rewrite the system (1.1) to that in a symmetric form. Furthermore we transform the symmetric system to the normal form, which is a coupled system of hyperbolic and parabolic equations, by assuming a null condition:

[**N**] the null space of the viscosity matrix $G(U)$ is independent of the dependent variable U.

Using the assumptions [**A1**] and [**N**], we see that there exists a diffeomorphism $U \mapsto u$ from $\mathcal{O}_U$ onto $\mathcal{O}_u \subset \mathbb{R}^m$, which allows us to rewrite the system (1.1) to that for a new dependent variable u as

$$A^0(u)u_t + A(u)u_x = B(u)u_{xx} + g(u, u_x). \tag{1.2}$$

Here $A^0(u)$, $A(u)$ and $B(u)$ are real symmetric matrices

$$A^0(u) = \begin{pmatrix} A_1^0(u) & 0 \\ 0 & A_2^0(u) \end{pmatrix}, \ A(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix}, \ B(u) = \begin{pmatrix} 0 & 0 \\ 0 & B_2(u) \end{pmatrix}.$$

Precisely, $A^0(u)$ is real symmetric and positive definite, that is, $A_1^0(u)$ and $A_2^0(u)$ are real symmetric and positive definite; $A(u)$ is real symmetric, that is, $A_{11}(u)$ and $A_{22}(u)$ are symmetric and ${}^\top A_{12}(u) = A_{21}(u)$; $B_2(u)$ is real symmetric and positive definite; $g(u, u_x)$ is a nonlinear term in a form

$$g(u, u_x) = \begin{pmatrix} 0 \\ g_2(u, u_x) \end{pmatrix}.$$

The null condition [**N**] is one of sufficient conditions to enable the above transformation. Actually some model equations are transformed to the normal symmetric form (1.2) unless the null condition [**N**] holds. A typical example is viscous gas without heat-conductivity. The conditions [**A1**] and [**N**] are introduced in [2, 3, 6] for analysis on the asymptotic stability of a constant state.

Letting $u = {}^\top(v, w)$ where $v = v(t, x) \in \mathbb{R}^{m_1}$ and $w = w(t, x) \in \mathbb{R}^{m_2}$, the system (1.2) is written as

$$A_1^0(u)v_t + A_{11}(u)v_x + A_{12}(u)w_x = 0, \tag{1.3a}$$

$$A_2^0(u)w_t + A_{21}(u)v_x + A_{22}(u)w_x = B_2(u)w_{xx} + g_2(u, u_x). \tag{1.3b}$$

The initial data for (1.3) is prescribed as

$$u(0, x) = u_0(x) = {}^\top(v_0, w_0)(x), \quad \text{i.e.,} \quad (v, w)(0, x) = (v_0, w_0)(x), \tag{1.4}$$

where a spatial asymptotic state of the initial data is supposed to be a constant,

$$\lim_{x\to\infty} u_0(x) = u_+ = {}^\top(v_+, w_+), \quad \text{i.e.,} \quad \lim_{x\to\infty} (v_0, w_0)(x) = (v_+, w_+). \tag{1.5}$$

The spatial asymptotic state $u_+ = {}^\top(v_+, w_+)$ is chosen to satisfy a condition below.

[A2] The matrix $A_{11}(u_+)$ is nonsingular.

The assumption **[A2]** is introduced in [1] and allows, without loss of generality, to assume that the matrix $A_{11}(u)$ is given in a form

$$A_{11} = \begin{pmatrix} a_{11} \; a_{12} \\ a_{21} \; a_{22} \end{pmatrix}, \quad \#^- \left(A_{11}(u_+)\right) = \#^- \left(a_{11}(u_+)\right), \tag{1.6}$$

where $\#^-(M)$ denote the number of negative eigenvalues of a matrix M. Let v^- and v^+ correspond to negative and positive characteristics, respectively,

$$v(t,x) = \begin{pmatrix} v^-(t,x) \\ v^+(t,x) \end{pmatrix}, \quad v^-(t,x) \in \mathbb{R}^{m_1^-}, \; v^+(t,x) \in \mathbb{R}^{m_1^+},$$

where $m_1^- := \#^-(A_{11}(u_+))$ and $m_1^+ := \#^+(A_{11}(u_+))$. To construct the solution to (1.3a) in a neighborhood of the asymptotic state $u_+ = {}^\top(v_+, w_+)$, we have to prescribe the boundary condition on v^+ for the hyperbolic equations with the positive characteristics and on w for the parabolic equations. Hence

$$v^+(t,0) = v_b^+, \; w(t,0) = w_b, \tag{1.7}$$

where $v_b^+ \in \mathbb{R}^{m_1^+}$ and $w_b \in \mathbb{R}^{m_2}$ are constants.

The existence of the stationary solution to (1.1) is first showed in [1] under the assumption **[A2]**. It is also proved in [9] under an assumption

[A2-1] The matrix $A_{11}(u_+)$ is negative definite.

This condition, introduced in [9], stronger than **[A2]** and means all of the characteristics of the hyperbolic equations (1.3a) are negative.

The asymptotic stability of the stationary solution is proved in [1] under the assumption **[A2]** and [9] under **[A2-1]**, respectively. Author's research in [10] also shows the asymptotic stability of a constant state u_+ for a system linearized from (1.3) under **[A2-1]**. In these researches, the stability condition **[SK]** is assumed.

[SK] If $\lambda A^0(u_+)\phi = A(u_+)\phi$ and $B(u_+)\phi = 0$ for $\lambda \in \mathbb{R}$ and $\phi \in \mathbb{R}^m$, then $\phi = 0$.

The stability condition **[SK]** is known as a condition to ensure L^2-energy dissipation. However, in some problems, the asymptotic stability be able to be shown without **[SK]**. Especially, the paper [11] introduce an assumption,

[A2-2] the matrix $A(u_+)$ is negative definite and shows the asymptotic stability by a weighted energy method. This method also gives the convergence rate towards the stationary solution. Apparently, the condition **[A2-2]** is stronger than **[A2-1]** as the matrix $A(u)$ is symmetric.

The hyperbolic-parabolic system (1.1) is a generalization of the concrete models arising in physics, especially in fluid dynamics. The assumption **[A2-1]** corresponds to the outflow problem for the model system of compressible viscous gases. This problem is studied in [4, 5, 12] of which results are stated below. For the heat-conductive model of compressible viscous gases in $\mathbb{R}^3$, Matsumura and Nishida in [7] show the asymptotic stability of a constant state (or a stationary solution corresponding to an external potential force) and establish a technical energy method. For the system (1.1), Umeda, Kawashima and Shizuta in [14] consider a sufficient condition which guarantees a dissipative structure of the system (1.1) and show the asymptotic stability of the constant state in the full space $\mathbb{R}^n$.

The half space problem to the hyperbolic-parabolic coupled systems is studied by Kawashima, Nishibata and Zhu in [5], where they consider outflow problems for a barotropic model of compressible and viscous gases. They show the existence and the asymptotic stability of stationary solutions. For this problem, convergence rate towards the stationary solution is obtained in Nakamura, Nishibata, Yuge [12]. For the heat-conductive model, Kawashima, Nakamura, Nishibata and Zhu [4] prove the existence and the asymptotic stability of stationary solutions for the outflow problem, too.

Notations. For vectors $u, v \in \mathbb{R}^m$, $|u|$ and $\langle u, v\rangle$ denote the standard Euclidean norm and inner product, respectively. For a matrix A, ${}^\top A$ denotes a transport matrix of A. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}_+)$ denotes a standard Lebesgue space over $\mathbb{R}_+$ equipped with a norm $\|\cdot\|_{L^p}$. For a non-negative integer s, $H^s(\mathbb{R}_+)$ denotes an s-th order the Sobolev space over $\mathbb{R}_+$ in the L^2 sense with a norm $\|\cdot\|_{H^s}$. Notice that $H^0(\mathbb{R}_+) = L^2(\mathbb{R}_+)$ and $\|\cdot\|_{H^0} = \|\cdot\|_{L^2}$. For a function $f = f(u)$, $D_u f(u)$ denotes a Fréchet derivative of f with respect to u. Especially, in the case of $u = {}^\top(u_1, \ldots, u_n) \in \mathbb{R}^n$ and $f(u) = {}^\top(f_1, \ldots, f_m)(u) \in \mathbb{R}^m$, the Fréchet derivative $D_u f = (\frac{\partial f_i}{\partial u_j})_{ij}$ is an $m \times n$ matrix. For a function $f = f(v, w)$, we sometimes abbreviate partial Fréchet derivatives $D_v f(v, w)$ and $D_w f(v, w)$ to $f_v(v, w)$ and $f_w(v, w)$, respectively. For a square matrix M, $[M]$ means a symmetric part, that is, $[M] := (M + {}^\top M)/2$. Notations $\#^-(M)$ and $\#^+(M)$ denote the numbers of negative eigenvalues and positive eigenvalues of a matrix M, respectively.

§ 2. Existence of stationary solution

The stationary wave $\tilde{U}(x)$ is defined as a smooth stationary solution to (1.1) which converges to a constant state $U_+ = U(u_+)$ as $x \to \infty$. Thus $\tilde{U}$ satisfies a system of ordinary differential equations equations

$$f(\tilde{U})_x = (G(\tilde{U})\tilde{U}_x)_x, \quad x \in \mathbb{R}_+. \tag{2.1}$$

Let $\tilde{u} = {}^\top(\tilde{v}, \tilde{w})$ be a stationary solution for (1.3). By a diffeomorphism $U \mapsto u$, we have a relation $\tilde{u} = u(\tilde{U})$ and $\tilde{U} = U(\tilde{u})$. Here we prescribe same boundary and spatial asymptotic conditions (1.7) and (1.5) as $\tilde{u}$. Namely

$$v^+(0) = v_b^+, \ \tilde{w}(0) = w_b, \tag{2.2a}$$

$$\lim_{x\to\infty} \tilde{u}(x) = u_+, \ \text{ i.e., } \ \lim_{x\to\infty} (\tilde{v}, \tilde{w})(x) = (v_+, w_+). \tag{2.2b}$$

The existence of the stationary solution for the boundary value problem (2.1) and (2.2) is summarized in the following theorem of which detailed proof is stated in the papers [1, 9]. The property of the stationary solution depends on the matrix $D_U f(U_+))$. Here and hereafter the stationary solution corresponding to a zero eigenvalue is called a degenerate stationary solution and the others are called non-degenerate. The non-degenerate stationary solution exists if the number of negative characteristics of the system (1.1) with $G(U) \equiv 0$ is greater than that of hyperbolic equations (1.3a). The existence of the degenerate stationary solution is showed under the assumptions **[A2-1]** and that the matrix $D_U f(U_+)$ has a simple zero-eigenvalue as well as the corresponding characteristic field is genuinely nonlinear. For details, the readers are referred to [1, 9].

Theorem 1. (i) (Non-degenerate flow) *Suppose* **[A1]**, **[A2]** *and* **[N]** *hold. If*

$$\#^-(D_U f(U_+)) > m_1^-, \tag{2.3}$$

then there exists a local stable manifold $\mathcal{M}^{\mathrm{s}} \subset \mathbb{R}^{m_2}$ around the equilibrium w_+ such that if $w_b \in \mathcal{M}^{\mathrm{s}}$ and $\delta := |w_+ - w_b|$ is sufficiently small, then there exists a unique smooth solution $\tilde{u}(x)$ to (2.1) *and* (2.2) *satisfying an exponential decay estimate*

$$|\partial_x^k(\tilde{u}(x) - u_+)| \le C\delta e^{-cx} \ \ \textit{for} \ \ k = 0, 1, \dots.$$

(ii) (Degenerate flow) *Assume that* **[A1]**, **[A2-1]** *and* **[N]** *hold. Moreover we assume that $D_U f(U_+)$ has a simple zero-eigenvalue $\mu(U_+) = 0$ and that the characteristic field corresponding to $\mu(U_+) = 0$ is genuinely nonlinear. Namely,*

$$D_U \mu(U_+) R(U_+) \neq 0,$$

where $\mu(U)$ is an eigenvalue of the matrix $D_U f(U)$ satisfying $\mu(U_+) = 0$ and $R(U)$ be a right eigenvector of $D_U f(U)$ corresponding to $\mu(U)$. Then there exists a local center manifold $\mathcal{M} \subset \mathbb{R}^{m_2}$ such that if $w_b \in \mathcal{M}$ and δ is sufficiently small, then there exists a unique smooth solution $\tilde{u}(x)$ to (2.1) *and* (2.2) *satisfying an algebraic decay estimate*

$$|\partial_x^k(\tilde{u}(x) - u_+)| \le C\frac{\delta^{k+1}}{(1+\delta x)^{k+1}} + C\delta e^{-cx} \ \ \textit{for} \ \ k = 0, 1, \dots.$$

The asymptotic stability of the stationary solution constructed in the above theorem is proved in the papers [1, 9]. It is discussed in section 3. In section 4, we give the survey on the paper [11], which studies the convergence rate towards the stationary solution. Here we study the convergence rate only for the non-degenerate flow for simplicity. For the detailed discussion on the degenerate flow, readers are referred to [11].

§ 3. Asymptotic stability of stationary solution

We study the asymptotic stability, of the stationary solution of which existence is shown in Theorem 1, under the assumption **[SK]**. It ensures a dissipative structure of the system (1.1). In the computations below, we use an equivalent condition.

> **[K]** There exists an $m \times m$ real matrix K such that $KA^0(u_+)$ is skew-symmetric and $[KA(u_+)] + B(u_+)$ is symmetric and positive definite.

The condition **[K]** is proposed by Kawashima in [2] to show the asymptotic stability of a constant state in full space. Shizuta and Kawashima in [13] prove the equivalence of the conditions **[K]** and **[SK]**. The main purpose of author's researches in [1, 9, 10, 11] is to generalize their methods to the half space problem for the asymptotic analysis. Precisely, we prove the asymptotic stability of the non-degenerate and the degenerate stationary solutions. However the present survey only shows the asymptotic stability of the non-degenerate stationary solution in Theorem 1-(i) for simplicity. For the degenerate stationary solution, see [9, 11].

We are at the position to state the results obtained under the assumption **[A2-1]**, which implies that $m_1^- = m_1$ and $m_1^+ = 0$. Hence the boundary condition (1.7) is prescribed only for the parabolic equations (1.3b). Namely,

$$w(0,t) = w_b. \tag{3.1}$$

The following theorem is proved in [9].

Theorem 2. *Assume that the same assumptions as in Theorem* 1-(i) *hold. Namely the conditions* **[A1]**, **[A2]**, **[N]**, (2.3) *with* $m_1^- = m_1$ *and* $w_b \in \mathcal{M}^{\mathrm{s}}$ *are assumed to hold. (Hence the stationary solution* $\tilde{u}$ *to* (2.1), (3.1) *and* (2.2b) *exists if* δ *is sufficiently small.) Then there exists a positive constant* ε *such that if*

$$\|u_0 - \tilde{u}\|_{H^2} + \delta \leq \varepsilon,$$

and the initial boundary value problem (1.3), (1.4) *and* (1.7) *has a unique solution* $u(t,x)$ *globally in time satisfying*

$$u - \tilde{u} \in C([0,\infty), H^2(\mathbb{R}_+)).$$

Moreover the solution converges to the stationary solution $\tilde{u}$:

$$\lim_{t\to\infty}\|u(t)-\tilde{u}\|_{L^\infty}=0.$$

The crucial point of proof of Theorem 2 is to obtain a uniform a priori estimate of a perturbation from the stationary solution

$$(\varphi,\psi)(t,x):=(v,w)(t,x)-(\tilde{v},\tilde{w})(x).$$

We have from (1.3) the equation for (φ,ψ)as

$$A_1^0(u)\varphi_t+A_{11}(u)\varphi_x+A_{12}(u)\psi_x=h_1, \tag{3.2a}$$

$$A_2^0(u)\psi_t+A_{21}(u)\varphi_x+A_{22}(u)\psi_x=B_2(u)\psi_{xx}+h_2. \tag{3.2b}$$

h_1 and h_2 are remainder terms given by

$$\begin{aligned}
h_1 &:= g_1-\tilde{g}_1-(A_{11}-\tilde{A}_{11})\tilde{v}_x-(A_{12}-\tilde{A}_{12})\tilde{w}_x,\\
h_2 &:= g_2-\tilde{g}_2-(A_{21}-\tilde{A}_{21})\tilde{v}_x-(A_{22}-\tilde{A}_{22})\tilde{w}_x+(B_2-\tilde{B}_2)\tilde{w}_{xx},
\end{aligned}$$

where $\tilde{g}_1:=g_1(\tilde{u},\tilde{w}_x)$, $\tilde{g}_2:=g_2(\tilde{u},\tilde{u}_x)$, $\tilde{A}_{ij}:=A_{ij}(u)$ for $i,j=1,2$ and $\tilde{B}_2:=B_2(\tilde{u})$. The initial and the boundary conditions for the equations (3.2) are derived from (1.4), (1.7) and (2.2). Namely,

$$(\varphi,\psi)(0,x)=(\varphi_0,\psi_0)(x):=(v_0,w_0)(x)-(\tilde{v},\tilde{w})(x), \tag{3.3}$$

$$\psi(t,0)=0. \tag{3.4}$$

To state the a priori estimate, we define an energy norm

$$N(t):=\sup_{0\le\tau\le t}\|(\varphi,\psi)(\tau)\|_{H^2}.$$

Proposition 3. *Let $(\varphi,\psi)\in C([0,T];H^2(\mathbb{R}_+))$ be a solution to (3.2)–(3.4) for a certain $T>0$. Then there exists a positive constant ε such that if $N(T)+\delta\le\varepsilon$, the solution satisfies*

$$\|(\varphi,\psi)(t)\|_{H^2}^2+\int_0^t\|\varphi_x(\tau)\|_{H^1}^2+\|\psi_x(\tau)\|_{H^2}^2\,d\tau\le C\|(\varphi_0,\psi_0)\|_{H^2}^2$$

for $t\in[0,T]$.

This theorem is proved by deriving an a-priori estimate in H^2-Sobolev space. To this end, we define an energy form

$$\mathcal{E}:=\eta(U)-\eta(\tilde{U})-D_U\eta(\tilde{U})(U-\tilde{U}).$$

The straightforward computation yields the equation for $\mathcal{E}$

$$
\begin{aligned}
(3.5) \qquad & \mathcal{E}_t + \mathcal{F}_x + \langle B_2(u)\psi_x, \psi_x\rangle + \mathcal{G}_1 = \mathcal{B}_x + \mathcal{R}_1 + \mathcal{R}_2, \\
& \mathcal{F} := q(U) - q(\tilde{U}) - D_U\eta(\tilde{U})(f(U) - f(\tilde{U})), \\
& \mathcal{B} := \big(D_U\eta(U) - D_U\eta(\tilde{U})\big)(G(U)U_x - G(\tilde{U})\tilde{U}_x), \\
& \mathcal{G}_1 := D_U\eta(\tilde{U})_x(f(U) - f(\tilde{U})) - \big(D_U\eta(U) - D_U\eta(\tilde{U})\big)f(\tilde{U})_x, \\
& \mathcal{R}_1 := -\langle D_U^2\eta(U)^\top U_u(u)\xi_x, (G(U)U_u(u) - G(\tilde{U})U_u(\tilde{u}))\tilde{u}_x\rangle \\
& \qquad - \langle G(U)U_u(u)\xi_x, \big(D_U^2\eta(U)U_u(u) - D_U^2\eta(\tilde{U})U_u(\tilde{u})\big)\tilde{u}_x\rangle, \\
& \mathcal{R}_2 := -\langle \big(D_U^2\eta(U)U_u(u) - D_U^2\eta(\tilde{U})U_u(\tilde{u})\big)\tilde{u}_x, (G(U)U_u(u) - G(\tilde{U})U_u(\tilde{u}))\tilde{u}_x\rangle.
\end{aligned}
$$

If $N(t)$ is sufficiently small, the energy form $\mathcal{E}$ is equivalent to $|(\varphi, \psi)|^2$ as the Hessian matrix $D_U^2\eta$ is positive. Hence the integration of (3.5) with respect to x and t yields the basic L^2 estimate. To derive the estimates for the higher order derivatives, we make use of the symmetric system (3.2). In these computations, we use the assumption **[A2-1]** and the dissipative estimate for the hyperbolic part under the stability condition **[K]**. For the detailed proofs, see [10].

To study the general case under **[A2]**, we have to impose additional conditions on initial condition data, comparing to Theorem 2. To state the result, we define a function space:

$$
\mathfrak{X}(0,T) = \left\{ (\varphi, \psi) \;\middle|\; \begin{array}{l} \varphi \in \bigcap_{k=0}^{2} C^k([0,T]; H^{2-k}(\mathbb{R}_+)), \;\; \psi \in C([0,T]; H^3(\mathbb{R}_+)), \\ \phi_t \in \bigcap_{k=0}^{1} C^k([0,T]; H^{2-2k}(\mathbb{R}_+)), \;\; \phi_t \in L^2(0,T; H^1(\mathbb{R}_+)). \end{array} \right\}.
$$

The following result is proved in [1].

Theorem 4. *Assume that the conditions* **[A1]**, **[A2]**, **[N]** *and* **[SK]** *hold. Then there exists a positive constant* ε *such that if*

$$
\|v_0 - \tilde{v}\|_{H^2} + \|w_0 - \tilde{w}\|_{H^3} + \|w_{tt}(0)\|_{L^2} + \delta \le \varepsilon,
$$

the problem (1.3), (1.4) *and* (1.5) *has a unique solution* u *in the function space* $\mathfrak{X}(0,\infty)$*. Moreover the solution* u *converges to the stationary solution* $\tilde{u}$ *in* L^∞ *norm as time variable* t *goes to infinity.*

§ 4. Convergence rate

Provided that the assumption **[A2-2]** holds, it is possible to derive the convergence rate of the solution towards the stationary solution. This result hold even though the stability condition **[SK]** does not hold. The first result is summarized in

Theorem 5. *Assume the same assumptions as in Theorem 2 and* **[A2-2]** *hold.*
(i) (Exponential decay.) *Let* $u_0 - \tilde{u} \in H^2(\mathbb{R}_+)$ *and* $e^{\alpha x/2}(u_0 - \tilde{u}) \in L^2(\mathbb{R}_+)$ *for a positive constant* α. *Then there exists a positive constant* ε *such that if*

$$\|u_0 - \tilde{u}\|_{H^2} + \|e^{\alpha x/2}(u_0 - \tilde{u})\|_{L^2} + \delta \leq \varepsilon,$$

then the initial boundary value problem (1.3), (1.4) *and* (1.7) *has a unique solution globally in time as*

$$u - \tilde{u} \in C([0,\infty); H^2(\mathbb{R}_+)).$$

Moreover the solution u *verifies the decay estimate*

$$\|u(t) - \tilde{u}\|_{H^2} + \|e^{\alpha x/2}(u(t) - \tilde{u})\|_{L^2} \leq C(\|u_0 - \tilde{u}\|_{H^2} + \|e^{\alpha x/2}(u_0 - \tilde{u})\|_{L^2})e^{-\nu t/2}$$

for a certain positive constant ν.
(ii) (Algebraic decay.) *Let* $u_0 - \tilde{u} \in H^2(\mathbb{R}_+)$ *and* $(1+x)^{\alpha/2}(u_0 - \tilde{u}) \in L^2(\mathbb{R}_+)$ *for a positive constant* α. *Then there exists a positive constant* ε *such that if*

$$\|u_0 - \tilde{u}\|_{H^2} + \|(1+x)^{\alpha/2}(u_0 - \tilde{u})\|_{L^2} + \delta \leq \varepsilon,$$

then the initial boundary value problem (1.3), (1.4) *and* (1.7) *has a unique solution globally in time satisfying*

$$u - \tilde{u} \in C([0,\infty); H^2(\mathbb{R}_+)).$$

It verifies the decay estimate

$$\|u(t) - \tilde{u}\|_{H^2} \leq C(\|u_0 - \tilde{u}\|_{H^2} + \|(1+x)^{\alpha/2}(u_0 - \tilde{u})\|_{L^2})(1+t)^{-\alpha/2}.$$

This theorem is proved by the weighted energy method. The detailed proof is given in [11]. The second result is summarized in the theorem below.

Theorem 6. *Let the condition* **[A2-2]** *and the same assumptions as in Theorem 2 except* **[SK]** *hold.*
(i) (Exponential decay.) *Let* $e^{\alpha x/2}(u_0 - \tilde{u}) \in H^2(\mathbb{R}_+)$ *hold for a positive constant* α. *Then there exists a positive constant* ε, *for a constant* $\beta \in (0, \alpha]$, *such that if*

$$(\|e^{\beta x/2}(u_0 - \tilde{u})\|_{H^2} + \delta)\beta^{-1} \leq \varepsilon,$$

then the initial boundary value problem (1.3), (1.4) *and* (1.7) *has a unique solution globally in time as*

$$e^{\beta x/2}(u - \tilde{u}) \in C([0,\infty); H^2(\mathbb{R}_+)).$$

It verifies the decay estimate

$$\|e^{\beta x/2}(u(t) - \tilde{u})\|_{H^2} \leq C\, \|e^{\beta x/2}(u_0 - \tilde{u})\|_{H^2}\, e^{-\nu t/2}$$

for a certain positive constant ν*.*
(ii) (Algebraic decay.) *Let* $(1+\gamma x)^{\alpha/2}(u_0-\tilde{u}) \in H^2(\mathbb{R}_+)$ *hold for a positive constant* γ *and a constant* $\alpha \geq 2$*. Then there exists a positive constant* ε *such that if*

$$(\|(1+\gamma x)^{\alpha/2}(u_0-\tilde{u})\|_{H^2}+\delta)\gamma^{-1}+\gamma \leq \varepsilon,$$

then the initial boundary value problem (1.3)*,* (1.4) *and* (1.7) *has a unique solution globally in time as*

$$(1+\gamma x)^{\alpha/2}(u-\tilde{u}) \in C([0,\infty); H^2(\mathbb{R}_+)).$$

It verifies the decay estimate

$$\|u(t)-\tilde{u}\|_{H^2} \leq C\,\|(1+\gamma x)^{\alpha/2}(u_0-\tilde{u})\|_{H^2}\,(1+t)^{-(\alpha-\theta)/2}$$

for an arbitrarily small positive constant θ*.*

This theorem is also proved by the weighted energy method. See [11] for the details.

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