RIMS Kôkyûroku Bessatsu
B68 (2017), 27–66

Desingularization of multiple zeta-functions of generalized Hurwitz-Lerch type and evaluation of p-adic multiple L-functions at arbitrary integers

By

Hidekazu Furusho [*], Yasushi Komori [**], Kohji Matsumoto [***] and Hirofumi Tsumura [†]

Abstract

We study analytic properties of multiple zeta-functions of generalized Hurwitz-Lerch type. First, as a special type of them, we consider multiple zeta-functions of generalized Euler-Zagier-Lerch type and investigate their analytic properties which were already announced in our previous paper. Next we give 'desingularization' of multiple zeta-functions of generalized Hurwitz-Lerch type, which include those of generalized Euler-Zagier-Lerch type, the Mordell-Tornheim type, and so on. As a result, the desingularized multiple zeta-function turns out to be an entire function and can be expressed as a finite sum of ordinary multiple zeta-functions of the same type. As applications, we explicitly compute special values of desingularized double zeta-functions of Euler-Zagier type. We also extend our previous results concerning a relationship between p-adic multiple L-functions and p-adic multiple star polylogarithms to more general indices with arbitrary (not necessarily all positive) integers.

Contents

Received April 10, 2014. Revised October 23, 2015, August 26, 2016
2010 Mathematics Subject Classification(s): 11M32, 11S40, 11G55
Key Words: *multiple zeta-function, multiple polylogarithm, desingularization, p-adic multiple L-function, p-adic multiple polylogarithm.*
Research of the authors is supported by Grants-in-Aid for Science Research
(no. 24684001 for HF, no. 25400026 for YK, no. 25287002 for KM, no. 15K04788 for HT, respectively), JSPS.

[*]Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan.
e-mail: furusho@math.nagoya-u.ac.jp
[**]Department of Mathematics, Rikkyo University, Nishi-Ikebukuro, Toshima-ku, Tokyo 171-8501, Japan.
e-mail: komori@rikkyo.ac.jp
[***]Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan.
e-mail: kohjimat@math.nagoya-u.ac.jp
[†]Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1, Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan.
e-mail: tsumura@tmu.ac.jp

§ 0. Introduction

In the present paper we continue our study developed in our previous papers [6, 7], with supplying some proofs of results in [6] which were stated with no proof. In [6], we studied multiple zeta-functions of generalized Euler-Zagier-Lerch type (see below) and considered their analytic properties. Based on those considerations, we introduced the method of *desingularization* of multiple zeta-functions, which is to resolve all singularities of them. By this method we constructed the desingularized multiple zeta-function which is entire and can be expressed as a finite sum of ordinary multiple zeta-functions.

The first main purpose of the present paper is to extend our theory of desingularization to the following more general situation.

Let $\xi_k, \gamma_{jk}, \beta_j$ $(1 \leqslant j \leqslant d, 1 \leqslant k \leqslant r)$ be complex parameters with $|\xi_k| \leqslant 1$, real parts $\Re\gamma_{jk} \geqslant 0$, $\Re\beta_j > 0$, and let s_j $(1 \leqslant j \leqslant d)$ be complex variables. We assume that for each k $(1 \leqslant k \leqslant r)$, at least one of $\Re\gamma_{jk} > 0$. We define the **multiple zeta-functions of generalized Hurwitz-Lerch type** by

$$
\begin{aligned}
(0.1)\quad & \zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j)) \\
&= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{\xi_1^{m_1}\cdots\xi_r^{m_r}}{(\beta_1+\gamma_{11}m_1+\cdots+\gamma_{1r}m_r)^{s_1}\cdots(\beta_d+\gamma_{d1}m_1+\cdots+\gamma_{dr}m_r)^{s_d}} \\
&= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{\prod_{k=1}^{r}\xi_k^{m_k}}{\prod_{j=1}^{d}(\beta_j+\sum_{k=1}^{r}\gamma_{jk}m_k)^{s_j}}.
\end{aligned}
$$

Obviously this is convergent absolutely when $\Re s_j > r$ for $1 \leqslant j \leqslant d$, and it is known that this can be continued meromorphically to the whole space $\mathbb{C}^d$ (see [12]).

In the present paper we will construct desingularized multiple zeta-functions, which will be expressed as a finite sum of $\zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j))$.

The **multiple zeta-function of generalized Euler-Zagier-Lerch type** defined by

$$
(0.2)\qquad \zeta_r((s_j);(\xi_j);(\gamma_j)) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{j=1}^{r}\xi_j^{m_j}(m_1\gamma_1+\cdots+m_j\gamma_j)^{-s_j}
$$

for parameters $\xi_j, \gamma_j \in \mathbb{C}$ $(1 \leqslant j \leqslant r)$ with $|\xi_j| = 1$ and $\Re\gamma_j > 0$, is a special case of (0.1). In fact, putting $d = r$, $\gamma_{jk} = \gamma_k$ $(j \geqslant k)$, $\gamma_{jk} = 0$ $(j < k)$, and $\beta_j = \gamma_1 + \cdots + \gamma_j$, (0.1) reduces to (0.2). This (0.2) was the main actor of the previous paper [6].

When $\xi_j = \gamma_j = 1$ for all j, (0.2) is the famous Euler-Zagier multiple sum (Hoffman [10], Zagier [20]):

$$\zeta_r(s_1, \ldots, s_r) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{j=1}^{r} (m_1 + \cdots + m_j)^{-s_j}. \tag{0.3}$$

Singularities of (0.3) have been determined explicitly (see Akiyama, Egami and Tanigawa [1]).

On the other hand, when $r = 1$ and $\gamma_1 = 1$, then the above series coincides with the Lerch zeta-function

$$\phi(s_1, \xi_1) = \sum_{m_1=1}^{\infty} \xi_1^{m_1} m_1^{-s_1}. \tag{0.4}$$

It is known that $\phi(s_1, \xi_1)$ is entire if $\xi_1 \neq 1$, while if $\xi_1 = 1$ then $\phi(s_1, 1)$ is nothing but the Riemann zeta-function $\zeta(s_1)$ and has a simple pole at $s_1 = 1$.

The plan of the present paper is as follows.

In Section 1 we prove that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be continued meromorphically to the whole space $\mathbb{C}^r$, and its singularities can be explicitly given (Theorems 1.1 and 1.4). This result was announced in [6, Section 2] without proof. The assertion of the meromorphic continuation is, as mentioned above, already given in [12]. However in Section 1 we give an alternative argument, based on Mellin-Barnes integrals, which is probably more suitable to obtain explicit information on singularities.

In Section 2, we give desingularization of the multiple zeta-functions of generalized Hurwitz-Lerch type (see (0.1)), which include those of generalized Euler-Zagier-Lerch type, the Mordell-Tornheim type, and so on. In fact, we will show that these desingularized multiple zeta-functions are entire (see Theorem 2.2), which was already announced in [6, Remark 4.5]. Actually this includes our previous result shown in [6, Theorem 3.4]. We further show that these desingularized multiple zeta-functions can be expressed as finite sums of ordinary multiple zeta-functions (see Theorem 2.7).

In Section 3, we give some examples of desingularization of various multiple zeta-functions. The main technique is a certain generalization of ours used in the proof of [6, Theorem 3.8]. In particular, we give desingularization of multiple zeta-functions of root systems introduced by the second, the third and the fourth authors (see, for example, [13]).

In Section 4, we study special values of desingularized double zeta-functions of Euler-Zagier type. More generally, we give some functional relations for desingularized double zeta-functions and ordinary double zeta-functions of Euler-Zagier type (see

Propositions 4.3, 4.5 and 4.7). By marvelous cancellations among singularities of ordinary double zeta-functions, we can explicitly compute special values of desingularized double zeta-functions of Euler-Zagier type at any integer points (see Examples 4.4, 4.6, 4.8 and Proposition 4.9).

An important aspect of [7] is the construction of the theory of p-adic multiple L-functions. The second main purpose of the present paper is to give a certain extension of our result on special values of p-adic multiple L-functions.

In [14], the second, the third and the fourth authors introduced p-adic double L-functions, as the double analogue of the classical Kubota-Leopoldt p-adic L-functions. In [7], we generalized the argument in [14] to define p-adic multiple L-functions. On the other hand, the first author [4] [5] developed the theory of p-adic multiple polylogarithms under a very different motivation. A remarkable discovery in [7] is that there is a connection between these two multiple notions. In fact, we proved that the values of p-adic multiple L-functions at positive integer points can be described in terms of p-adic multiple star polylogarithms ([7, Theorem 3.41]).

In Section 5 of the present paper, we extend this result to obtain the description of the values of p-adic multiple L-functions at arbitrary (not necessarily all positive) integer points in terms of p-adic multiple star polylogarithms (Theorem 5.8).

§ 1. The meromorphic continuation and the location of singularities

The purpose of this section is to prove the following result which was announced in [6, Theorem 2.3].

Theorem 1.1. *The function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be continued meromorphically to the whole space $\mathbb{C}^r$. Moreover,*

(i) *If $\xi_j \neq 1$ for all j $(1 \leqslant j \leqslant r)$, then $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is entire.*

(ii) *If $\xi_j \neq 1$ for all j $(1 \leqslant j \leqslant r-1)$ and $\xi_r = 1$, then $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has a unique simple singular hyperplane $s_r = 1$.*

(iii) *If $\xi_j = 1$ for some j $(1 \leqslant j \leqslant r-1)$, then $\zeta_r((s_j);(\xi_j);(\gamma_j))$ has infinitely many simple singular hyperplanes.*

Actually the location of the singular hyperplanes will be more explicitly described in Theorem 1.4.

Remark 1. The multiple polylogarithm is defined by

$$Li_{n_1,\ldots,n_r}(z_1,\ldots,z_r) = \sum_{0<k_1<\cdots<k_r} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}} = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\prod_{j=1}^{r}(z_j\cdots z_r)^{m_j}(m_1+\cdots+m_j)^{-n_j}, \tag{1.1}$$

where $(n_j) \in \mathbb{N}^r$ and $(z_j) \in \mathbb{C}^r$ with $|z_j| = 1$ $(1 \leqslant j \leqslant r)$ (see Goncharov [9]). Inspired by this definition, we generally define

$$Li_{s_1,\ldots,s_r}(z_1,\ldots,z_r) = \zeta_r((s_j);(\prod_{\nu=j}^{r} z_\nu);(1)) \tag{1.2}$$

for $(s_j) \in \mathbb{C}^r$ and $(z_j) \in \mathbb{C}^r$ with $|z_j| = 1$ $(1 \leqslant j \leqslant r)$ (see (0.2)). In fact, it follows from Theorem 1.1 that the right-hand side of (1.2) can be meromorphically continued to $(s_j) \in \mathbb{C}^r$. Moreover, when $\prod_{\nu=j}^{r} z_\nu \neq 1$ for all j, the right-hand side is entire. In particular, setting $\xi_j = \prod_{\nu=j}^{r} z_\nu$ $(1 \leqslant j \leqslant r)$ and $\xi_{r+1} = 1$, we obtain

$$\zeta_r((n_j);(\xi_j);(1)) = Li_{n_1,\ldots,n_r}\left(\frac{\xi_1}{\xi_2}, \frac{\xi_2}{\xi_3}, \ldots, \frac{\xi_r}{\xi_{r+1}}\right) \tag{1.3}$$

for all $(n_j) \in \mathbb{Z}^r$ when $\xi_j \neq 1$ $(1 \leqslant j \leqslant r)$. In Section 5, we will show a p-adic version of (1.3) (see Theorem 5.8 and Remark 7).

Now we start the proof of Theorem 1.1. Let $C(j,r)$ be the number of h $(j \leqslant h \leqslant r)$ for which $\xi_h = 1$ holds. We first prove the following lemma.

Lemma 1.2. *The function $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be continued meromorphically to the whole space $\mathbb{C}^r$, and its possible singularities can be listed as follows, where $\ell \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.*

- *If $\xi_j = 1$, then $s_j + s_{j+1} + \cdots + s_r = C(j,r) - \ell$ $\quad (1 \leqslant j \leqslant r-1)$,*
- *If $\xi_r = 1$, then $s_r = 1$,*
- *If $\xi_j \neq 1$ for all j $(1 \leqslant j \leqslant r)$, then $\zeta_r((s_j);(\xi_j);(\gamma_j))$ is entire.*

Proof. We prove the theorem by induction on r. In the case $r = 1$, our zeta-function is essentially the Lerch zeta-function (0.4), so the assertion of the lemma is classical.

Now let $r \geqslant 2$, and assume that the assertion of the lemma is true for $r-1$. The proof is based on the Mellin-Barnes integral formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz, \tag{1.4}$$

where $s, \lambda \in \mathbb{C}$, $\Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, $-\Re s < c < 0$ and the path of integration is the vertical line $\Re z = c$. This formula has been frequently used to show the meromorphic continuation of various multiple zeta-functions (e.g. [15], [16], [17]). In particular, the following argument is quite similar to that in [17]. In what follows, ε denotes an arbitrarily small positive number, not necessarily the same at each occurrence.

First of all, using [15, Theorem 3], we see that series (0.2) is absolutely convergent in the region

$$\{(s_1,\ldots,s_r)\mid \sigma_{r-j+1}+\cdots+\sigma_r>j\ (1\leqslant j\leqslant r)\}, \tag{1.5}$$

where $\sigma_j=\Re s_j$ $(1\leqslant j\leqslant r)$. At first we assume that $(s_1,\ldots,s_r)$ is in this region. Divide

$$\begin{aligned}&(m_1\gamma_1+\cdots+m_r\gamma_r)^{-s_r}\\&=(m_1\gamma_1+\cdots+m_{r-1}\gamma_{r-1})^{-s_r}\left(1+\frac{m_r\gamma_r}{m_1\gamma_1+\cdots+m_{r-1}\gamma_{r-1}}\right)^{-s_r},\end{aligned}$$

and apply (1.4) to the second factor on the right-hand side with $\lambda=m_r\gamma_r/(m_1\gamma_1+\cdots+m_{r-1}\gamma_{r-1})$ to obtain

$$\begin{aligned}&\zeta_r((s_j);(\xi_j);(\gamma_j))\\&=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{\xi_1^{m_1}}{(m_1\gamma_1)^{s_1}}\times\cdots\times\frac{\xi_{r-1}^{m_{r-1}}}{(m_1\gamma_1+\cdots+m_{r-1}\gamma_{r-1})^{s_{r-1}}}\\&\quad\times\frac{\xi_r^{m_r}}{(m_1\gamma_1+\cdots+m_{r-1}\gamma_{r-1})^{s_r}}\left(\frac{m_r\gamma_r}{m_1\gamma_1+\cdots+m_{r-1}\gamma_{r-1}}\right)^{z}dz\\&=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);(\xi_1,\ldots,\xi_{r-1});(\gamma_1,\ldots,\gamma_{r-1}))\\&\quad\times\gamma_r^z\phi(-z,\xi_r)dz,\end{aligned}\tag{1.6}$$

where $-\sigma_r<c<-1$. (To apply (1.4) it is enough to assume $c<0$, but to ensure the convergence of the above multiple series it is necessary to assume $c<-1$.)

Next we shift the path of integration from $\Re z=c$ to $\Re z=M-\varepsilon$, where M is a large positive integer, and ε is a small positive number. This is possible because, by virtue of Stirling's formula, we see that the integrand is of rapid decay when $\Im z\to\infty$. Relevant poles are $z=0,1,2,\ldots$ (coming from $\Gamma(-z)$) and $z=-1$ if $\xi_r=1$ (coming from $\phi(-z,\xi_r)$). Counting the residues of those poles, we obtain

$$\begin{aligned}&\zeta_r((s_j);(\xi_j);(\gamma_j))\\&=\delta(r)\frac{\gamma_r^{-1}}{s_r-1}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r-1);(\xi_1,\ldots,\xi_{r-1});(\gamma_1,\ldots,\gamma_{r-1}))\\&+\sum_{k=0}^{M-1}\binom{-s_r}{k}\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+k);(\xi_1,\ldots,\xi_{r-1});(\gamma_1,\ldots,\gamma_{r-1}))\\&\quad\times\gamma_r^k\phi(-k,\xi_r)\end{aligned}\tag{1.7}$$

$$+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_r+z)\Gamma(-z)}{\Gamma(s_r)}$$
$$\times\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);(\xi_1,\ldots,\xi_{r-1});(\gamma_1,\ldots,\gamma_{r-1}))$$
$$\times\gamma_r^z\phi(-z,\xi_r)dz$$
$$=X+\sum_{k=0}^{M-1}Y(k)+Z,$$

say, where

$$\delta(r)=\begin{cases}1 & (\xi_r=1),\\ 0 & (\xi_r\neq 1).\end{cases} \tag{1.8}$$

From (1.5) we see that

$$\zeta_{r-1}((s_1,\ldots,s_{r-2},s_{r-1}+s_r+z);(\xi_1,\ldots,\xi_{r-1});(\gamma_1,\ldots,\gamma_{r-1}))$$

is absolutely convergent if

$$\sigma_{r-j}+\cdots+\sigma_r+\Re z>j\qquad(1\leqslant j\leqslant r-1),$$

so the integral Z is convergent (and hence holomorphic) in the region

$$\{(s_1,\ldots,s_r)\mid\sigma_{r-j}+\cdots+\sigma_r>j-M+\varepsilon\quad(0\leqslant j\leqslant r-1)\}. \tag{1.9}$$

(Here, the condition corresponding to $j=0$ is necessary to assure that the factor $\Gamma(s_r+z)$ in the integrand does not encounter the poles.) Therefore by (1.7) and the assumption of induction we can continue $\zeta_r((s_j);(\xi_j);(\gamma_j))$ meromorphically to region (1.9). Since M is arbitrary, we can now conclude that $\zeta_r((s_j);(\xi_j);(\gamma_j))$ can be continued meromorphically to the whole space $\mathbb{C}^r$.

Next we examine the possible singularities on the right-hand side of (1.7). By the assumption of induction, we see that the possible singularities of $Y(k)$ are

$$s_j+\cdots+s_{r-2}+s_{r-1}+s_r+k=C(j,r-1)-\ell\quad\text{if}\quad\xi_j=1\quad(1\leqslant j\leqslant r-2) \tag{1.10}$$

and

$$s_{r-1}+s_r+k=1\quad\text{if}\quad\xi_{r-1}=1. \tag{1.11}$$

If $\xi_j\neq 1$ for all j $(1\leqslant j\leqslant r-1)$, then $Y(k)$ is entire. The term X appears only in case $\xi_r=1$, and in this case, $s_r=1$ is a possible singularity. Moreover, by the assumption of induction we find the following possible singularities of X:

(1.12)
$$s_j+\cdots+s_{r-2}+s_{r-1}+s_r-1=C(j,r-1)-\ell\quad\text{if}\quad\xi_j=1\quad(1\leqslant j\leqslant r-2,\ j=r)$$

and

$$s_{r-1}+s_r-1=1 \quad \text{if} \quad \xi_{r-1}=1 \text{ and } \xi_r=1. \tag{1.13}$$

If $\xi_j \neq 1$ for all j $(1 \leqslant j \leqslant r-1)$, then X is entire. Since k also runs over $\mathbb{N}_0$, renaming $k+\ell$ in (1.10) and k in (1.11) as ℓ, we find that the above list of possible singularities can be rewritten as follows (where $\ell \in \mathbb{N}_0$).

- $s_j+\cdots+s_r=(C(j,r-1)+1)-\ell$ and $s_r=1$ if $\xi_j=1$ $(1 \leqslant j \leqslant r-2)$ and $\xi_r=1$,
- $s_{r-1}+s_r=2-\ell$ and $s_r=1$ if $\xi_{r-1}=1$ and $\xi_r=1$ (given by (1.11) and (1.13)),
- $s_j+\cdots+s_r=C(j,r-1)-\ell$ if $\xi_j=1$ $(1 \leqslant j \leqslant r-2)$ and $\xi_r \neq 1$,
- $s_{r-1}+s_r=1-\ell$ if $\xi_{r-1}=1$ and $\xi_r \neq 1$.

Since $C(j,r)=C(j,r-1)+1$ when $\xi_r=1$ and $C(j,r)=C(j,r-1)$ when $\xi_r \neq 1$, the factors $C(j,r-1)+1$ and $C(j,r-1)$ in the above list are all equal to $C(j,r)$. This completes the proof of the lemma, because we also notice that $C(r-1,r)=2$ if $\xi_{r-1}=\xi_r=1$ and $C(r-1,r)=1$ if $\xi_{r-1}=1$ and $\xi_r \neq 1$. □

Next we discuss whether the possible singularities listed in Lemma 1.2 are indeed singularities, or not. For this purpose, we first prepare the following

Lemma 1.3. *Let $\xi \in \mathbb{C}$ with $|\xi|=1$. If $\xi \neq \pm 1$, then $\phi(-k,\xi) \neq 0$ for all $k \in \mathbb{N}_0$. If $\xi=\pm 1$, then $\phi(-k,\xi) \neq 0$ for all odd $k \in \mathbb{N}$ and $k=0$, and $\phi(-k,\xi)=0$ for all even $k \in \mathbb{N}$.*

Proof. If $\xi=\pm 1$, then we have

$$\phi(-k,1)=\zeta(-k),$$
$$\phi(-k,-1)=(2^{1+k}-1)\zeta(-k),$$

which reduces to the well-known cases. In the following we assume that $\xi \neq \pm 1$. Put $\xi=e^{2\pi i\theta}$ with $0<\theta<1$ and $\theta \neq 1/2$. It is known that

$$\frac{1}{1-\xi e^t}=\sum_{k=0}^{\infty}\phi(-k,\xi)\frac{t^k}{k!} \tag{1.14}$$

(cf. [6, Section 1]). If $k=0$, then we have

$$\phi(0,\xi)=\frac{1}{1-\xi} \neq 0.$$

Assume $k \geqslant 1$. For any sufficiently small $\varepsilon>0$, we have

$$\begin{aligned}\frac{\phi(-k,\xi)}{k!} &= \frac{1}{2\pi i}\int_{|t|=\varepsilon}\frac{t^{-k-1}dt}{1-\xi e^t} \\ &= \frac{1}{2\pi i}\int_{|t|=\varepsilon}\frac{t^{-k-1}dt}{1-e^{t+2\pi i\theta}} = -\sum_{n\in\mathbb{Z}}\frac{1}{(2\pi i(n-\theta))^{k+1}},\end{aligned}$$

where the last equality follows by counting residues at the poles $t = 2\pi i(n-\theta)$. Therefore it is sufficient to show that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n-\theta)^{k+1}} \neq 0. \tag{1.15}$$

If k is odd, then the left-hand side is clearly positive. If k is even, then

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n-\theta)^{k+1}} = \sum_{n=0}^{\infty} \left(\frac{1}{(n+1-\theta)^{k+1}} - \frac{1}{(n+\theta)^{k+1}} \right) \neq 0,$$

because for all $n \geqslant 0$,

$$\frac{1}{(n+1-\theta)^{k+1}} - \frac{1}{(n+\theta)^{k+1}} \begin{cases} < 0 & (0 < \theta < 1/2) \\ > 0 & (1/2 < \theta < 1). \end{cases}$$

The lemma is proved. □

Now our aim is to prove the following theorem, from which Theorem 1.1 immediately follows.

Theorem 1.4. *Among the list of possible singularities of $\zeta_r((s_j);(\xi_j);(\gamma_j))$ given in Lemma 1.2, the "true" singularities are listed up as follows, where $\ell \in \mathbb{N}_0$.*

(I) *If $\xi_j = 1$, then $s_j + \cdots + s_r = C(j,r) - \ell$ $\quad (1 \leqslant j \leqslant r-2)$,*

(II) *If $\xi_{r-1} = 1$ and $\xi_r = 1$, then $s_{r-1} + s_r = 2, 1, -2\ell$,*

(III) *If $\xi_{r-1} = 1$ and $\xi_r = -1$, then $s_{r-1} + s_r = 1, -2\ell$,*

(IV) *If $\xi_{r-1} = 1$ and $\xi_r \neq \pm 1$, then $s_{r-1} + s_r = 1 - \ell$,*

(V) *If $\xi_r = 1$, then $s_r = 1$.*

Remark 2. When $\xi_j = \gamma_j = 1$ $(1 \leqslant j \leqslant r)$, this theorem recovers [1, Theorem 1].

Proof. The proof is by induction on r. The case $r = 1$ is obvious, so we assume $r \geqslant 2$ and the theorem is true for $r-1$.

First we put $s_{r-1} + s_r = u$, and regard (1.7) as a formula in variables $s_1, \ldots, s_{r-2}, u, s_r$. This idea of "changing variables" is originally due to Akiyama, Egami and Tanigawa [1]. We have

$$X = \delta(r) \frac{\gamma_r^{-1}}{s_r - 1} \zeta_{r-1}((s_1, \ldots, s_{r-2}, u-1); (\xi_1, \ldots, \xi_{r-1}); (\gamma_1, \ldots, \gamma_{r-1})),$$

$$Y(k) = \binom{-s_r}{k} \zeta_{r-1}((s_1, \ldots, s_{r-2}, u+k); (\xi_1, \ldots, \xi_{r-1}); (\gamma_1, \ldots, \gamma_{r-1})) \gamma_r^k \phi(-k, \xi_r).$$

Consider $Y(k)$. The singularities (1.10) and (1.11) are coming from the ζ_{r-1} factor. These singularities do not be canceled by the factor $\binom{-s_r}{k}$, because the ζ_{r-1} factor (after the above "changing variables") does not include the variable s_r. Also, if $k' \neq k$, then the singularities of $Y(k')$ and of $Y(k)$ do not cancel with each other, because $Y(k')$ and $Y(k)$ is of different order with respect to s_r.

When $\xi_r = 1$, the term X appears. The possible singularities coming from X are (1.12), (1.13), and $s_r = 1$. These singularities do not cancel with each other. Also, these singularities do not cancel the singularities coming from $Y(k)$, which can be seen again by observing the order with respect to s_r.

Therefore now we can say:

(i) The possible singularities of $Y(k)$ are "true" if they are "true" singularities of ζ_{r-1} and $\phi(-k, \xi_r) \neq 0$,

(ii) When $\xi_r = 1$, the hyperplane $s_r = 1$ is a "true" singularity, while the other possible singularities of X are "true" if they are "true" singularities of ζ_{r-1}.

Consider (i). By the assumption of induction, the "true" singularities of

$$\zeta_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + k); (\xi_1, \ldots, \xi_{r-1}); (\gamma_1, \ldots, \gamma_{r-1}))$$

are

(i-1) $s_j + \cdots + s_r + k = C(j, r-1) - \ell$ if $\xi_j = 1$ $(1 \leqslant j \leqslant r-3)$,

(i-2) $s_{r-2} + s_{r-1} + s_r + k = 2, 1, -2\ell$ if $\xi_{r-2} = 1$, $\xi_{r-1} = 1$,

(i-3) $s_{r-2} + s_{r-1} + s_r + k = 1, -2\ell$ if $\xi_{r-2} = 1$, $\xi_{r-1} = -1$,

(i-4) $s_{r-2} + s_{r-1} + s_r + k = 1 - \ell$ if $\xi_{r-2} = 1$, $\xi_{r-1} \neq \pm 1$,

(i-5) $s_{r-1} + s_r + k = 1$ if $\xi_{r-1} = 1$.

Here, by Lemma 1.3 we see that $k \in \mathbb{N}_0$ if $\xi_r \neq \pm 1$, while k is 0 or an odd positive integer if $\xi_r = \pm 1$. Renaming $k + \ell$ in (i-1) as ℓ, we can rewrite (i-1) as

(i-1') $s_j + \cdots + s_r = C(j, r-1) - \ell$ if $\xi_j = 1$ $(1 \leqslant j \leqslant r-3)$.

Next, the equality in (i-2) is $s_{r-2} + s_{r-1} + s_r = 2 - k, 1 - k, -2\ell - k$, and the right-hand side exhausts all integers $\leqslant 2$ even in the case when k is 0 or an odd positive integer. Therefore (i-2) can be rewritten as

(i-2') $s_{r-2} + s_{r-1} + s_r = 2 - \ell$ if $\xi_{r-2} = 1$, $\xi_{r-1} = 1$.

Similarly we rewrite (i-3) and (i-4) as

(i-3') $s_{r-2} + s_{r-1} + s_r = 1 - \ell$ if $\xi_{r-2} = 1$, $\xi_{r-1} = -1$,

(i-4') $s_{r-2} + s_{r-1} + s_r = 1 - \ell$ if $\xi_{r-2} = 1$, $\xi_{r-1} \neq \pm 1$.

These (i-1')–(i-4') and (i-5) give the list of "true" singularities coming from the case (i).

Next consider (ii). By the assumption of induction, the "true" singularities of

$$\delta(r) \frac{1}{s_r - 1} \zeta_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r - 1); (\xi_1, \ldots, \xi_{r-1}); (\gamma_1, \ldots, \gamma_{r-1}))$$

are

(ii-1) $s_j + \cdots + s_r - 1 = C(j, r-1) - \ell$ if $\xi_j = 1$ $(1 \leqslant j \leqslant r-3)$, $\xi_r = 1$,

(ii-2) $s_{r-2} + s_{r-1} + s_r - 1 = 2, 1, -2\ell$ if $\xi_{r-2} = 1$, $\xi_{r-1} = 1$, $\xi_r = 1$,

(ii-3) $s_{r-2}+s_{r-1}+s_r-1=1,-2\ell$ if $\xi_{r-2}=1$, $\xi_{r-1}=-1$, $\xi_r=1$,
(ii-4) $s_{r-2}+s_{r-1}+s_r-1=1-\ell$ if $\xi_{r-2}=1$, $\xi_{r-1}\neq\pm1$, $\xi_r=1$,
(ii-5) $s_{r-1}+s_r-1=1$ if $\xi_{r-1}=1$, $\xi_r=1$,

and

(ii-6) $s_r=1$ if $\xi_r=1$.

The last (ii-6) is singularity (V) in the statement of Theorem 1.4.

From (i-1'), (ii-1) and the definition of $C(j,r)$ we obtain $s_j+\cdots+s_r=C(j,r)-\ell$ if $\xi_j=1$ $(1\leqslant j\leqslant r-3)$. This gives singularity (I) for $1\leqslant j\leqslant r-3$.

Consider the case $j=r-2$. From (i-2') and (ii-2) we find that $s_{r-2}+s_{r-1}+s_r=3-\ell$ are singularities if $\xi_{r-2}=1$, $\xi_{r-1}=1$, $\xi_r=1$. From (i-3'), (i-4'), (ii-3) and (ii-4) we find that $s_{r-2}+s_{r-1}+s_r=2-\ell$ are singularities if $\xi_{r-2}=1$, $\xi_{r-1}\neq1$, $\xi_r=1$. These observations and (i-2')–(i-4') imply that $s_{r-2}+s_{r-1}+s_r=C(r-2,r)-\ell$ are singularities if $\xi_{r-2}=1$. This is singularity (I) for $j=r-2$.

Finally, from (i-5) we obtain the singularities $s_{r-1}+s_r=1-\ell$ if $\xi_{r-1}=1$, $\xi_r\neq\pm1$, and $s_{r-1}+s_r=1,-2\ell$ if $\xi_{r-1}=1$, $\xi_r=\pm1$. The former case gives singularity (IV). The latter case, combined with (ii-5), gives singularities (II) and (III). This completes the proof of the theorem.

□

Remark 3. In the above proof, an important fact is that there are infinitely many $k\in\mathbb{N}$ with $\phi(-k,\xi)\neq0$. Actually, Lemma 1.3 ensures this fact. We can give another approach to ensure this fact. The number defined by

$$H_k(\xi^{-1}):=(1-\xi)\phi(-k,\xi)\qquad(k\in\mathbb{N}_0)\tag{1.16}$$

is called the kth Frobenius-Euler number studied by Frobenius in [8]. He showed that, if ξ is the primitive cth root of unity with $c>1$ and p is an odd prime number with $p\nmid c$, then

$$H_k(\xi^{-1})\equiv\frac{1}{\xi^{-1}-1}\qquad(\mathrm{mod}\ p)$$

for any $k\in\mathbb{N}_0$ with $k\equiv1$ (mod $p-1$). Thus there are infinitely many $k\in\mathbb{N}$ with

$$\phi(-k,\xi)=\frac{1}{1-\xi}H_k(\xi^{-1})\neq0.$$

Remark 4. It is desirable to generalize the results proved in this section to more general multiple zeta-functions defined by (0.1), but it seems not easy, because the argument based on Mellin-Barnes integrals will become more complicated (see [18]).

§ 2. Desingularization of multiple zeta-functions

In this section, we define desingularization of multiple zeta-functions of generalized Hurwitz-Lerch type (0.1), which includes those of generalized Euler-Zagier-Lerch type

(0.2).

Combining the integral representation of gamma function

$$\Gamma(s) = a^s \int_0^\infty e^{-ax} x^{s-1} dx \tag{2.1}$$

for $a \in \mathbb{C}$ with $\Re a > 0$, and

$$\frac{1}{e^y - \xi} = \sum_{n=0}^\infty \xi^n e^{-(n+1)y} \tag{2.2}$$

for $|\xi| \leqslant 1$ and $y > 0$, the multiple zeta-function of generalized Hurwitz-Lerch type defined by (0.1) is rewritten in the integral form as

$$\begin{aligned}\zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j)) &= \frac{1}{\Gamma(s_1)\cdots\Gamma(s_d)} \\ &\times \int_{[0,\infty)^d} \frac{e^{(\gamma_{11}+\cdots+\gamma_{1r}-\beta_1)x_1}\cdots e^{(\gamma_{d1}+\cdots+\gamma_{dr}-\beta_d)x_d} x_1^{s_1-1}\cdots x_d^{s_d-1}}{(e^{x_1\gamma_{11}+\cdots+x_d\gamma_{d1}}-\xi_1)\cdots(e^{x_1\gamma_{1r}+\cdots+x_d\gamma_{dr}}-\xi_r)} dx_1\cdots dx_d \\ &= \frac{1}{\prod_{j=1}^d \Gamma(s_j)} \int_{[0,\infty)^d} \prod_{j=1}^d x_j^{s_j-1} \exp\Big(\big(\sum_{k=1}^r \gamma_{jk}-\beta_j\big)x_j\Big) dx_j \prod_{k=1}^r \frac{1}{\exp\Big(\sum_{j=1}^d \gamma_{jk}x_j\Big)-\xi_k}.\end{aligned} \tag{2.3}$$

If $\xi_k \neq 1$ for all k, then, as was shown in [12], it can be analytically continued to the whole space in (s_j) as an entire function via the integral representation:

$$\begin{aligned}\zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j)) &= \frac{1}{\prod_{j=1}^d (e^{2\pi i s_j}-1)\Gamma(s_j)} \\ &\times \int_{\mathcal{C}^d} \prod_{j=1}^d x_j^{s_j-1} \exp\Big(\big(\sum_{k=1}^r \gamma_{jk}-\beta_j\big)x_j\Big) dx_j \prod_{k=1}^r \frac{1}{\exp\Big(\sum_{j=1}^d \gamma_{jk}x_j\Big)-\xi_k},\end{aligned} \tag{2.4}$$

where $\mathcal{C}$ is the Hankel contour, that is, the path consisting of the positive real axis (top side), a circle around the origin of radius ε (sufficiently small), and the positive real axis (bottom side). The replacement of $[0,\infty)^d$ by the contour $\mathcal{C}^d$ can be checked directly (for the details, see [12], where, more generally, the cases $\xi_j = 1$ for some j are treated).

Motivated as in [6], we introduce the notion of desingularization.

Definition 2.1. Let $\xi_k, \gamma_{jk}, \beta_j \in \mathbb{C}$ with $|\xi_k| \leqslant 1$, $\Re\gamma_{jk} \geqslant 0$, $\Re\beta_j > 0$, and for each j, at least one of $\Re\gamma_{jk} > 0$. Define the **desingularized multiple zeta-function**, which we also call the **desingularization of** $\zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j))$, by

$$(2.5)\quad \begin{aligned}&\zeta_r^{\rm des}((s_j);(\xi_k);(\gamma_{jk});(\beta_j))\\ &:=\lim_{c\to 1}\frac{1}{\prod_{k=1}^r(1-\delta(k)c)}\\ &\times\frac{1}{\prod_{j=1}^d(e^{2\pi i s_j}-1)\Gamma(s_j)}\int_{\mathcal{C}^d}\prod_{j=1}^d x_j^{s_j-1}\exp\Big((\sum_{k=1}^r\gamma_{jk}-\beta_j)x_j\Big)dx_j\\ &\times\prod_{k=1}^r\left(\frac{1}{\exp\Big(\sum_{j=1}^d\gamma_{jk}x_j\Big)-\xi_k}-\delta(k)\frac{c}{\exp\Big(c\sum_{j=1}^d\gamma_{jk}x_j\Big)-1}\right)\end{aligned}$$

for $(s_j)\in\mathbb{C}^r$, where the limit is taken for $c\in\mathbb{R}$ and $\delta(k)$ is as in (1.8).

Remark 5. If $\xi_k\neq 1$ for all k, then $\zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j))$ is already entire as we mentioned above, so there is no need of desingularization. In fact, since in this case $\delta(k)=0$ for all k, (2.5) coincides with (2.4).

For $c\in\mathbb{R}$, $y,\xi\in\mathbb{C}$, $\delta\in\{0,1\}$ with $\delta=1$ if $\xi=1$, and $\delta=0$ otherwise, let

$$F_{c,\delta}(y,\xi)=\begin{cases}\dfrac{1}{1-\delta c}\Big(\dfrac{1}{e^y-\xi}-\delta\dfrac{c}{(e^{cy}-1)}\Big) & (c\neq 1),\\ \dfrac{1}{e^y-\xi}-\delta\dfrac{ye^y}{(e^y-1)^2} & (c=1),\end{cases}$$

and further we write $F_\delta(y,\xi)=F_{1,\delta}(y,\xi)$.

Theorem 2.2. *For $\xi_k,\gamma_{jk},\beta_j\in\mathbb{C}$ as in Definition 2.1, we have*

$$(2.6)\quad \begin{aligned}&\zeta_r^{\rm des}((s_j);(\xi_k);(\gamma_{jk});(\beta_j))\\ &=\frac{1}{\prod_{j=1}^d(e^{2\pi i s_j}-1)\Gamma(s_j)}\int_{\mathcal{C}^d}\prod_{j=1}^d x_j^{s_j-1}\exp\Big((\sum_{k=1}^r\gamma_{jk}-\beta_j)x_j\Big)dx_j\\ &\times\prod_{k=1}^r F_{\delta(k)}\left(\sum_{j=1}^d\gamma_{jk}x_j,\xi_k\right),\end{aligned}$$

and is analytically continued to $\mathbb{C}^r$ as an entire function in (s_j).

Theorem 2.2 can be shown in almost the same way as in [6, Theorem 3.4]. We first use Lemma 2.4 below in place of [6, Lemma 3.6] to find that the limit and the multiple integrals on the right-hand side of (2.5) can be interchanged. Then we use the following Lemma 2.3 to obtain the assertion of Theorem 2.2.

Lemma 2.3.

$$F_{1,\delta}(y,\xi)=\lim_{c\to 1}F_{c,\delta}(y,\xi).$$

Proof. If $\xi=\delta=1$, then

$$\lim_{c\to 1}\frac{1}{1-\delta c}\Big(\frac{1}{e^y-\xi}-\delta\frac{c}{e^{cy}-1}\Big)=\lim_{c\to 1}\frac{1}{1-c}\Big(\frac{1}{e^y-1}-\frac{c}{e^{cy}-1}\Big)$$
$$(2.7)\qquad =-\frac{1-e^y+ye^y}{(e^y-1)^2}$$
$$=\frac{1}{e^y-1}-\frac{ye^y}{(e^y-1)^2},$$

while if $\delta=0$ and $\xi\neq 1$, the assertion is obvious. □

Let $\mathcal{N}(\varepsilon)=\{z\in\mathbb{C}\mid |z|\leqslant\varepsilon\}$ and $\mathcal{S}(\theta)=\{z\in\mathbb{C}\mid |\arg z|\leqslant\theta\}$.

Lemma 2.4. *Let $0<\theta<\pi/2$. Assume $|\xi|\leqslant 1$. Then there exist $A>0$ and sufficiently small $\varepsilon>0$ such that for all $c\in\mathbb{R}$ with sufficiently small $|1-c|$,*

$$(2.8)\qquad |F_{c,\delta}(y,\xi)|<Ae^{-\Re y/2}$$

for any $y\in\mathcal{N}(\varepsilon)\cup\mathcal{S}(\theta)$.

Proof. (1) Assume $\delta=0$ and $\xi\neq 1$. Then there exist $\varepsilon, C>0$ such that for all $y\in\mathcal{N}(\varepsilon)$,

$$|F_{c,\delta}(y,\xi)|=\Big|\frac{1}{e^y-\xi}\Big|<C.$$

Further for $y\in\mathcal{S}(\theta)\setminus\mathcal{N}(\varepsilon)$, we have

$$|F_{c,\delta}(y,\xi)|\leqslant\frac{1}{|e^y|-1}=\frac{e^{-\Re y}}{1-e^{-\Re y}}\leqslant C'e^{-\Re y}.$$

(2) Assume $\delta=\xi=1$. Then this case reduces to [6, Lemma 3.6]. □

It is to be noted that the following continuity properties hold.

Theorem 2.5. *The desingularization $\zeta_r^{\rm des}((s_j);(\xi_k);(\gamma_{jk});(\beta_j))$ is continuous in both (s_j) and (ξ_k). In particular, if $\xi_k\neq 1$ for all k, then $\zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j))$ is continuous in both (s_j) and (ξ_k).*

Proof. The first statement follows easily from Lemma 2.4 by using the dominated convergence theorem. The second statement is just a special case of the first statement in view of Remark 5. □

Next we give a generating function of special values of $\zeta_r((s_j);(\xi_k);(\gamma_{jk});(\beta_j))$ at non-positive integers. Write the Taylor expansion of $F_\delta(y,\xi)$ with respect to y as

$$(2.9)\qquad F_\delta(y,\xi)=\frac{1}{e^y-\xi}-\delta\frac{ye^y}{(e^y-1)^2}=\sum_{n=0}^{\infty}F_\delta^n(\xi)\frac{y^n}{n!}.$$

Then

$$(2.10)\qquad F_\delta^n(\xi)=\begin{cases} B_{n+1} & (\xi=1,\delta=1),\\ \dfrac{H_n(\xi)}{1-\xi} & (\xi\neq 1,\delta=0),\end{cases}$$

where B_{n+1} denotes the $(n+1)$-th Bernoulli number. The first formula of (2.10) can be shown by differentiating the definition of Bernoulli numbers

$$(2.11)\qquad \frac{y}{e^y-1}=\sum_{n=0}^{\infty}B_n\frac{y^n}{n!},$$

while the second formula follows from (1.14) and (1.16).

Theorem 2.6. *Let $\lambda_1,\ldots,\lambda_d\in\mathbb{N}_0$. Then we have*

$$(2.12)\quad \zeta_r^{\mathrm{des}}((-\lambda_j);(\xi_k);(\gamma_{jk});(\beta_j))= \prod_{j=1}^{d}(-1)^{\lambda_j}\lambda_j! \sum_{\substack{m_j+\nu_{j1}+\cdots+\nu_{jr}=\lambda_j\\(1\leqslant j\leqslant d)}}\Big(\prod_{k=1}^{r}F_{\delta(k)}^{\nu_{1k}+\cdots+\nu_{dk}}(\xi_k)\Big) \times\Big(\prod_{j=1}^{d}\frac{(\sum_{k=1}^{r}\gamma_{jk}-\beta_j)^{m_j}}{m_j!}\Big)\Big(\prod_{j=1}^{d}\prod_{k=1}^{r}\frac{\gamma_{jk}^{\nu_{jk}}}{\nu_{jk}!}\Big).$$

Proof. Let $D_j=\sum_{k=1}^{r}\gamma_{jk}-\beta_j$. It is sufficient to calculate the Taylor expansion with respect to x_j's of the integrand on the right-hand side of (2.6). Using (2.9) we have

$$(2.13)\quad \prod_{j=1}^{d}\exp\Big((\sum_{k=1}^{r}\gamma_{jk}-\beta_j)x_j\Big)\prod_{k=1}^{r}F_{\delta(k)}\Big(\sum_{j=1}^{d}\gamma_{jk}x_j,\xi_k\Big)$$

$$= \sum_{m_1,\ldots,m_d=0}^{\infty} \left(\prod_{j=1}^{d} \frac{D_j^{m_j}}{m_j!} x_j^{m_j} \right) \sum_{n_1,\ldots,n_r=0}^{\infty} \prod_{k=1}^{r} \frac{F_{\delta(k)}^{n_k}(\xi_k)}{n_k!} \Big(\sum_{j=1}^{d} \gamma_{jk} x_j \Big)^{n_k}$$

$$= \sum_{m_1,\ldots,m_d=0}^{\infty} \left(\prod_{j=1}^{d} \frac{D_j^{m_j}}{m_j!} x_j^{m_j} \right) \sum_{n_1,\ldots,n_r=0}^{\infty} \prod_{k=1}^{r} \frac{F_{\delta(k)}^{n_k}(\xi_k)}{n_k!}$$

$$\times \sum_{\nu_{1k}+\cdots+\nu_{dk}=n_k} \binom{n_k}{\nu_{1k},\ldots,\nu_{dk}} \prod_{j=1}^{d} \gamma_{jk}^{\nu_{jk}} x_j^{\nu_{jk}}$$

$$= \sum_{m_1,\ldots,m_d=0}^{\infty} \left(\prod_{j=1}^{d} \frac{D_j^{m_j}}{m_j!} x_j^{m_j} \right) \sum_{n_1,\ldots,n_r=0}^{\infty} \prod_{k=1}^{r} \sum_{\nu_{1k}+\cdots+\nu_{dk}=n_k} \frac{F_{\delta(k)}^{n_k}(\xi_k)}{\nu_{1k}!\cdots\nu_{dk}!} \prod_{j=1}^{d} \gamma_{jk}^{\nu_{jk}} x_j^{\nu_{jk}}$$

$$= \sum_{m_1,\ldots,m_d=0}^{\infty} \sum_{n_1,\ldots,n_r=0}^{\infty} \sum_{\substack{\nu_{1k}+\cdots+\nu_{dk}=n_k \\ (1\leqslant k\leqslant r)}} \Big(\prod_{k=1}^{r} F_{\delta(k)}^{n_k}(\xi_k) \Big) \Big(\prod_{j=1}^{d} \frac{D_j^{m_j}}{m_j!} \Big) \Big(\prod_{j=1}^{d} \prod_{k=1}^{r} \frac{\gamma_{jk}^{\nu_{jk}}}{\nu_{jk}!} \Big)$$

$$\times \prod_{j=1}^{d} x_j^{m_j+\nu_{j1}+\cdots+\nu_{jr}},$$

which gives the formula (2.12).

□

Remark 6. Since $D_j = 0$ for all $j = 1,\ldots,d$ in the case of multiple zeta-functions of generalized Euler-Zagier-Lerch type, only terms with $m_j = 0$ with $j = 1,\ldots,d$ contributes to the sum in the formula (2.12), which recovers [6, Theorem 3.7].

Lastly, we give a formula which expresses the desingularized zeta-function as a linear combination of ordinary zeta-functions of the same type, which is a generalization of [6, Theorem 3.8]. To this end, we prepare some notation and assume the following condition: There exists a set of constants c_{mj} $(1 \leqslant k, m \leqslant r)$ such that

$$\sum_{j=1}^{d} c_{mj}\gamma_{jk} = \delta_{mk} \tag{2.14}$$

for all k, m, where δ_{mk} is the Kronecker delta. Under this assumption, for indeterminates $\boldsymbol{u} = (u_j), \boldsymbol{v} = (v_j)$ $(j = 1,\ldots,d)$, we define the generating function

$$G(\boldsymbol{u},\boldsymbol{v}) = \prod_{k=1}^{r} \Big\{ 1 - \delta(k) \Big(1 + \sum_{j=1}^{d} c_{kj}(v_j^{-1} - \beta_j) \Big) \Big(\sum_{j=1}^{d} \gamma_{jk} u_j v_j \Big) \Big\}, \tag{2.15}$$

and also define constants $\alpha_{\boldsymbol{l},\boldsymbol{m}}$ as the coefficients of the expansion

$$G(\boldsymbol{u},\boldsymbol{v}) = \sum_{\boldsymbol{l},\boldsymbol{m}} \alpha_{\boldsymbol{l},\boldsymbol{m}} \prod_{j=1}^{d} u_j^{l_j} v_j^{m_j} \quad \text{with} \quad \boldsymbol{l} = (l_1,\ldots,l_d), \quad \boldsymbol{m} = (m_1,\ldots,m_d). \tag{2.16}$$

We define the Pochhammer symbol $(s)_k = s(s+1)\cdots(s+k-1)$ as usual. Then we have the following theorem, which is a generalization of [6, Theorem 3.8].

Theorem 2.7. *Under the assumption* (2.14), *we have*

$$\zeta_r^{\mathrm{des}}((s_j);(\xi_k);(\gamma_{jk});(\beta_j)) = \sum_{\boldsymbol{l},\boldsymbol{m}} \alpha_{\boldsymbol{l},\boldsymbol{m}} \Big(\prod_{j=1}^{d} (s_j)_{l_j}\Big) \zeta_r((s_j+m_j);(\xi_k);(\gamma_{jk});(\beta_j)). \tag{2.17}$$

Proof. First note that it is sufficient to show the statement with sufficiently large $\Re s_j$ due to the analytic continuation. Then we can write

$$\zeta_r^{\mathrm{des}}((s_j);(\xi_k);(\gamma_{jk});(\beta_j)) = \lim_{c\to 1} \frac{I_c((s_j);(\xi_k);(\gamma_{jk});(\beta_j))}{\prod_{k=1}^{r}(1-\delta(k)c)}, \tag{2.18}$$

where

$$I_c((s_j);(\xi_k);(\gamma_{jk});(\beta_j)) := \frac{1}{\prod_{j=1}^{d}\Gamma(s_j)} \int_{[0,\infty)^d} \prod_{j=1}^{d} x_j^{s_j-1} \exp\Big((\sum_{k=1}^{r}\gamma_{jk}-\beta_j)x_j\Big)dx_j \times \prod_{k=1}^{r}\Bigg(\frac{1}{\exp\Big(\sum_{j=1}^{d}\gamma_{jk}x_j\Big)-\xi_k} - \delta(k)\frac{c}{\exp\Big(c\sum_{j=1}^{d}\gamma_{jk}x_j\Big)-1}\Bigg). \tag{2.19}$$

We obtain

$$\begin{aligned} &\lim_{c\to 1} \frac{I_c((s_j);(\xi_k);(\gamma_{jk});(\beta_j))}{\prod_{k=1}^{r}(1-\delta(k)c)} \\ &= \lim_{c\to 1} \frac{1}{\prod_{j=1}^{d}\Gamma(s_j)} \int_{[0,\infty)^d} \prod_{j=1}^{d} x_j^{s_j-1} \exp\Big((\sum_{k=1}^{r}\gamma_{jk}-\beta_j)x_j\Big)dx_j \\ &\qquad \times \prod_{k=1}^{r} F_{c,\delta(k)}\Big(\sum_{j=1}^{d}\gamma_{jk}x_j,\xi_k\Big) \\ &= \frac{1}{\prod_{j=1}^{d}\Gamma(s_j)} \int_{[0,\infty)^d} \prod_{j=1}^{d} x_j^{s_j-1} \exp\Big((\sum_{k=1}^{r}\gamma_{jk}-\beta_j)x_j\Big)dx_j \\ &\qquad \times \prod_{k=1}^{r} F_{\delta(k)}\Big(\sum_{j=1}^{d}\gamma_{jk}x_j,\xi_k\Big). \end{aligned} \tag{2.20}$$

For $|\xi| \leqslant 1$ and $y>0$, equation (2.2) and

$$\frac{e^y}{(e^y-1)^2} = \sum_{n=0}^{\infty}(n+1)e^{-(n+1)y} \tag{2.21}$$

holds. Using these formulas, for any $K \subset \{1, \ldots, r\}$ we have

$$\begin{aligned}(2.22)\quad & \int_{[0,\infty)^d} \prod_{j=1}^{d} x_j^{s_j-1} \exp\Big((\sum_{k=1}^{r} \gamma_{jk} - \beta_j) x_j \Big) dx_j \\ & \times \prod_{k \notin K} \frac{1}{\exp\Big(\sum_{j=1}^{d} \gamma_{jk} x_j\Big) - \xi_k} \prod_{k \in K} \delta(k) \frac{\Big(\sum_{j=1}^{d} \gamma_{jk} x_j\Big) \exp\Big(\sum_{j=1}^{d} \gamma_{jk} x_j\Big)}{\Big(\exp\Big(\sum_{j=1}^{d} \gamma_{jk} x_j\Big) - 1\Big)^2} \\ & = \int_{[0,\infty)^d} \prod_{j=1}^{d} x_j^{s_j-1} \exp\Big((\sum_{k=1}^{r} \gamma_{jk} - \beta_j) x_j \Big) dx_j \prod_{k \in K} \delta(k) \Big(\sum_{j=1}^{d} \gamma_{jk} x_j \Big) \\ & \quad \times \prod_{k \notin K} \Bigg(\sum_{h_k=0}^{\infty} \xi_k^{h_k} \exp\Big(-(h_k+1) \sum_{j=1}^{d} \gamma_{jk} x_j \Big) \Bigg) \\ & \quad \times \prod_{k \in K} \Bigg(\sum_{h_k=0}^{\infty} (h_k+1) \exp\Big(-(h_k+1) \sum_{j=1}^{d} \gamma_{jk} x_j \Big) \Bigg) \\ & = \sum_{\substack{h_k \geqslant 0 \\ 1 \leqslant k \leqslant r}} \Big(\prod_{k \in K} (h_k+1) \Big) \int_{[0,\infty)^d} \prod_{j=1}^{d} x_j^{s_j-1} \exp\Big(-(\sum_{k=1}^{r} \gamma_{jk} h_k + \beta_j) x_j \Big) dx_j \\ & \quad \times \prod_{k \notin K} \xi_k^{h_k} \prod_{k \in K} \delta(k) \Big(\sum_{j=1}^{d} \gamma_{jk} x_j \Big).\end{aligned}$$

(When $K = \emptyset$, the empty product is to be regarded as 1.) Since $\delta(k) = \delta(k)\xi_k^{h_k}$, we have

$$\prod_{k \in K} \delta(k) \prod_{k \notin K} \xi_k^{h_k} = \prod_{k \in K} \delta(k) \prod_{k=1}^{r} \xi_k^{h_k}. \tag{2.23}$$

Also, since we assume (2.14), we can write

$$\prod_{k \in K} (h_k+1) = \prod_{l \in K} \Big(\sum_{j=1}^{d} c_{lj} (\beta_j + \sum_{k=1}^{r} \gamma_{jk} h_k - \beta_j) + 1 \Big). \tag{2.24}$$

Therefore, introducing constants $B_{K,\boldsymbol{l}}$ with $\boldsymbol{l} = (l_1, \ldots, l_d) \in \mathbb{N}_0^d$ as the coefficients of the expansion

$$\prod_{k \in K} \delta(k) \Big(\sum_{j=1}^{d} \gamma_{jk} x_j \Big) = \sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}} \prod_{j=1}^{d} x_j^{l_j}, \tag{2.25}$$

we find that (2.22) is equal to

$$\sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}} \sum_{\substack{h_k \geqslant 0 \\ 1 \leqslant k \leqslant r}} \Big(\prod_{m \in K} \Big(\sum_{j=1}^{d} c_{mj} (\beta_j + \sum_{k=1}^{r} \gamma_{jk} h_k - \beta_j) + 1 \Big) \Big) \prod_{k=1}^{r} \xi_k^{h_k} \tag{2.26}$$

$$\times \int_{[0,\infty)^d} \prod_{j=1}^{d} x_j^{s_j+l_j-1} \exp\Big(-(\sum_{k=1}^{r}\gamma_{jk}h_k+\beta_j)x_j\Big)dx_j$$

$$= \sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}} \sum_{\substack{h_k\geqslant 0\\ 1\leqslant k\leqslant r}} \Big(\prod_{m\in K}\Big(\sum_{j=1}^{d} c_{mj}(\beta_j+\sum_{k=1}^{r}\gamma_{jk}h_k)+c_{m0}\Big)\Big)\prod_{k=1}^{r}\xi_k^{h_k}$$

$$\times \prod_{j=1}^{d}\Gamma(s_j+l_j)\frac{1}{\Big(\beta_j+\sum_{k=1}^{r}\gamma_{jk}h_k\Big)^{s_j+l_j}},$$

where $c_{m0} := 1-\sum_{j=1}^{d} c_{mj}\beta_j$. Consider the factor

$$\prod_{m\in K}\Big(\sum_{j=1}^{d} c_{mj}(\beta_j+\sum_{k=1}^{r}\gamma_{jk}h_k)+c_{m0}\Big) \quad (=: Q,\ \text{say})$$

on the right-hand side of (2.26). Putting

$$\alpha_j := \begin{cases} \beta_j+\sum_{k=1}^{r}\gamma_{jk}h_k & (1\leqslant j\leqslant d),\\ 1 & (j=0),\end{cases} \tag{2.27}$$

we find that

$$Q = \prod_{m\in K}\sum_{j=0}^{d} c_{mj}\alpha_j = \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}}\Big(\prod_{m\in K} c_{mj_m}\Big)\Big(\prod_{m\in K}\alpha_{j_m}\Big).$$

For each $\{j_m \mid m\in K\}$, define

$$J(j) = J(j;\{j_m\}) := |\{m\in K \mid j_m=j\}| = \sum_{m\in K}\delta_{j,j_m} \qquad (1\leqslant j\leqslant d).$$

Then we see that

$$\prod_{m\in K}\alpha_{j_m} = \prod_{j=1}^{d}\alpha_j^{J(j)}.$$

Therefore we obtain

$$Q = \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}}\Big(\prod_{m\in K} c_{mj_m}\Big)\prod_{j=1}^{d}(\beta_j+\sum_{k=1}^{r}\gamma_{jk}h_k)^{J(j)}. \tag{2.28}$$

Using (2.28) we find that (2.26) can be rewritten as

$$\sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}}\Big(\prod_{j=1}^{d}\Gamma(s_j+l_j)\Big)$$

$$\times \sum_{\substack{h_k\geqslant 0\\ 1\leqslant k\leqslant r}} \left(\sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}} \Big(\prod_{m\in K} c_{mj_m}\Big) \frac{\prod_{k=1}^r \xi_k^{h_k}}{\prod_{j=1}^d \Big(\beta_j + \sum_{k=1}^r \gamma_{jk}h_k\Big)^{s_j+l_j-J(j)}} \right)$$

$$= \sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}} \Big(\prod_{j=1}^d \Gamma(s_j+l_j)\Big) \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}} \Big(\prod_{m\in K} c_{mj_m}\Big) \zeta_r\Big((s_j+l_j-J(j));(\xi_k);(\gamma_{jk});(\beta_j)\Big).$$

Therefore from (2.18) we obtain

$$\begin{aligned}(2.29)\quad & \zeta_r^{\rm des}((s_j);(\xi_k);(\gamma_{jk});(\beta_j)) \\ &= \sum_{K\subset\{1,\ldots,r\}} (-1)^{|K|} \sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}} \Big(\prod_{j=1}^d (s_j)_{l_j}\Big) \\ &\qquad \times \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}} \Big(\prod_{m\in K} c_{mj_m}\Big) \zeta_r\Big((s_j+l_j-J(j));(\xi_k);(\gamma_{jk});(\beta_j)\Big).\end{aligned}$$

Put

$$(2.30)\qquad H(\boldsymbol{u},\boldsymbol{v}) := \sum_{K\subset\{1,\ldots,r\}} (-1)^{|K|} \sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}} \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}} \Big(\prod_{m\in K} c_{mj_m}\Big) \prod_{j=1}^d u_j^{l_j} v_j^{l_j-J(j)}.$$

Our last task is to show that

$$(2.31)\qquad G(\boldsymbol{u},\boldsymbol{v}) = H(\boldsymbol{u},\boldsymbol{v}).$$

From (2.25), we have

(2.32)

$$\begin{aligned} H(\boldsymbol{u},\boldsymbol{v}) &= \sum_{K\subset\{1,\ldots,r\}} (-1)^{|K|} \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}} \Big(\prod_{m\in K} c_{mj_m}\Big) \Big(\prod_{j=1}^d v_j^{-J(j)}\Big) \sum_{\boldsymbol{l}} B_{K,\boldsymbol{l}} \prod_{j=1}^d u_j^{l_j} v_j^{l_j} \\ &= \sum_{K\subset\{1,\ldots,r\}} (-1)^{|K|} \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}} \Big(\prod_{m\in K} c_{mj_m}\Big) \Big(\prod_{m\in K}\prod_{j=1}^d v_j^{-\delta_{j,j_m}}\Big) \prod_{k\in K} \delta(k) \Big(\sum_{j=1}^d \gamma_{jk} u_j v_j\Big). \end{aligned}$$

Since we see that

$$\prod_{j=1}^d v_j^{-\delta_{j,j_m}} = \begin{cases} v_{j_m}^{-1} & (j_m \geqslant 1), \\ 1 & (j_m = 0), \end{cases}$$

under the convention $v_0 = 1$, we find that the right-hand side of (2.32) is equal to

$$\sum_{K\subset\{1,\ldots,r\}} (-1)^{|K|} \sum_{\substack{0\leqslant j_m\leqslant d\\ m\in K}} \Big(\prod_{m\in K} c_{mj_m} v_{j_m}^{-1}\Big) \prod_{k\in K} \delta(k) \Big(\sum_{j=1}^d \gamma_{jk} u_j v_j\Big)$$

$$= \sum_{K\subset\{1,\ldots,r\}} (-1)^{|K|}\Big(\prod_{m\in K}\Big(\sum_{j=1}^{d} c_{mj}v_j^{-1} + 1 - \sum_{j=1}^{d} c_{mj}\beta_j\Big)\Big)$$

$$\times \prod_{k\in K}\delta(k)\Big(\sum_{j=1}^{d}\gamma_{jk}u_jv_j\Big)$$

$$= \sum_{K\subset\{1,\ldots,r\}} (-1)^{|K|}\Big\{\prod_{k\in K}\delta(k)\Big(1+\sum_{j=1}^{d} c_{kj}(v_j^{-1}-\beta_j)\Big)\Big(\sum_{j=1}^{d}\gamma_{jk}u_jv_j\Big)\Big\}$$

$$= \prod_{k=1}^{r}\Big\{1-\delta(k)\Big(1+\sum_{j=1}^{d} c_{kj}(v_j^{-1}-\beta_j)\Big)\Big(\sum_{j=1}^{d}\gamma_{jk}u_jv_j\Big)\Big\} = G(\boldsymbol{u},\boldsymbol{v}),$$

hence (2.31). Therefore, regarding $(s_j)_{l_j}$ and $\zeta_r\Big((s_j+l_j-J(j));(\xi_k);(\gamma_{jk});(\beta_j)\Big)$ as indeterminates $u_j^{l_j}$ and $v_j^{l_j-J(j)}$, respectively, we arrive at the assertion of the theorem. □

§ 3. Examples of desingularization

Our Theorem 2.7 in the preceding section requires the assumption (2.14). In this section we see how this assumption is satisfied in examples.

Example 3.1. In the case of the triple zeta-function of generalized Euler-Zagier-Lerch type ($d=r=3$), we have

$$(\xi_k) = \begin{pmatrix}1\,1\,1\end{pmatrix}, \qquad (\beta_j) = \begin{pmatrix}\gamma_1 & \gamma_1+\gamma_2 & \gamma_1+\gamma_2+\gamma_3\end{pmatrix}, \tag{3.1}$$

$$(c_{mj}) = \begin{pmatrix}\gamma_1^{-1} & 0 & 0\\ -\gamma_2^{-1} & \gamma_2^{-1} & 0\\ 0 & -\gamma_3^{-1} & \gamma_3^{-1}\end{pmatrix}, \qquad (\gamma_{jk}) = \begin{pmatrix}\gamma_1 & 0 & 0\\ \gamma_1 & \gamma_2 & 0\\ \gamma_1 & \gamma_2 & \gamma_3\end{pmatrix}. \tag{3.2}$$

The generating function constructed by using these data coincides with $G(\boldsymbol{u},\boldsymbol{v})$ in [6, Example 4.4].

Example 3.2. Consider the case of the Mordell-Tornheim double zeta-function, which is defined by the double series

$$\zeta_{MT,2}(s_1,s_2,s_3) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{1}{m_1^{s_1}m_2^{s_2}(m_1+m_2)^{s_3}} \tag{3.3}$$

(cf. [15] [18]), corresponding to $d=3$ and $r=2$. In this case, constants c_{mj} are not uniquely determined. For any $a,b\in\mathbb{C}$, we have

$$(\xi_k) = \begin{pmatrix}1\,1\end{pmatrix}, \qquad (\beta_j) = \begin{pmatrix}1\,1\,2\end{pmatrix}, \tag{3.4}$$

$$(c_{mj}) = \begin{pmatrix} a+1 & a & -a \\ b & b+1 & -b \end{pmatrix}, \qquad (\gamma_{jk}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.5}$$

Therefore we have

$$\begin{aligned}
G(\boldsymbol{u},\boldsymbol{v}) &= (1 - v_1^{-1}(u_1v_1 + u_3v_3))(1 - v_2^{-1}(u_2v_2 + u_3v_3)) \\
&\quad - (1 - v_2^{-1}(u_2v_2 + u_3v_3))(v_1^{-1} + v_2^{-1} - v_3^{-1})(u_1v_1 + u_3v_3)a \\
&\quad - (1 - v_1^{-1}(u_1v_1 + u_3v_3))(v_1^{-1} + v_2^{-1} - v_3^{-1})(u_2v_2 + u_3v_3)b \\
&\quad + (v_1^{-1} + v_2^{-1} - v_3^{-1})^2(u_1v_1 + u_3v_3)(u_2v_2 + u_3v_3)ab \\
&= (u_1 - 1)(u_2 - 1) + u_3(u_1 - 1)v_2^{-1}v_3 + u_3(u_2 - 1)v_1^{-1}v_3 + u_3^2v_1^{-1}v_2^{-1}v_3^2 \\
&\quad + \Big\{(u_2 - 1)(u_1 - u_3) - u_3(1 - u_1 - u_2 + u_3)v_2^{-1}v_3 + u_3^2v_2^{-2}v_3^2 \\
&\qquad + u_1(u_2 - u_3 - 1)v_1v_2^{-1} - u_1(u_2 - 1)v_1v_3^{-1} + u_1u_3v_1v_2^{-2}v_3 \\
&\qquad + (u_2 - 1)u_3v_1^{-1}v_3 + u_3^2v_1^{-1}v_2^{-1}v_3^2\Big\}a \\
&\quad + \Big\{(u_1 - 1)(u_2 - u_3) - u_3(1 - u_1 - u_2 + u_3)v_1^{-1}v_3 + u_3^2v_1^{-2}v_3^2 \\
&\qquad + u_2(u_1 - u_3 - 1)v_1^{-1}v_2 - u_2(u_1 - 1)v_2v_3^{-1} + u_2u_3v_1^{-2}v_2v_3 \\
&\qquad + (u_1 - 1)u_3v_2^{-1}v_3 + u_3^2v_1^{-1}v_2^{-1}v_3^2\Big\}b \\
&\quad + \Big\{u_3^2 - 2u_1u_3 - 2u_2u_3 + 2u_1u_2 \\
&\qquad + u_3(u_1 + 2u_2 - 2u_3)v_1^{-1}v_3 + u_3(2u_1 + u_2 - 2u_3)v_2^{-1}v_3 \\
&\qquad + u_1u_3v_1v_2^{-2}v_3 + u_2u_3v_1^{-2}v_2v_3 \\
&\qquad + u_1(u_2 - 2u_3)v_1v_2^{-1} + u_2(u_1 - 2u_3)v_1^{-1}v_2 \\
&\qquad - u_1(2u_2 - u_3)v_1v_3^{-1} - u_2(2u_1 - u_3)v_2v_3^{-1} \\
&\qquad + u_3^2v_1^{-2}v_3^2 + u_3^2v_2^{-2}v_3^2 \\
&\qquad + u_1u_2v_1v_2v_3^{-2} + 2u_3^2v_1^{-1}v_2^{-1}v_3^2\Big\}ab.
\end{aligned}$$

From the constant part of this expression with respect to a and b, we obtain the following identity, which is an example of Theorem 2.7.

(3.6)

$$\begin{aligned}
\zeta_{MT,2}^{\rm des}(s_1,s_2,s_3) &= (s_1-1)(s_2-1)\zeta_{MT,2}(s_1,s_2,s_3) + s_3(s_1-1)\zeta_{MT,2}(s_1,s_2-1,s_3+1) \\
&\quad + s_3(s_2-1)\zeta_{MT,2}(s_1-1,s_2,s_3+1) + s_3(s_3+1)\zeta_{MT,2}(s_1-1,s_2-1,s_3+2).
\end{aligned}$$

On the other hand, coefficients of a, b, and ab give rise to the following identities[1]:

(3.7)
$$\begin{aligned}(s_2-1)(s_1-s_3)\zeta_{MT,2}(s_1,s_2,s_3)-s_3(2-s_1-s_2+s_3)\zeta_{MT,2}(s_1,s_2-1,s_3+1)\\ +s_3(s_3+1)\zeta_{MT,2}(s_1,s_2-2,s_3+2)+s_1(s_2-s_3-1)\zeta_{MT,2}(s_1+1,s_2-1,s_3)\\ -s_1(s_2-1)\zeta_{MT,2}(s_1+1,s_2,s_3-1)+s_1s_3\zeta_{MT,2}(s_1+1,s_2-2,s_3+1)\\ +(s_2-1)s_3\zeta_{MT,2}(s_1-1,s_2,s_3+1)+s_3(s_3+1)\zeta_{MT,2}(s_1-1,s_2-1,s_3+2)=0,\end{aligned}$$

(3.8)
$$\begin{aligned}(s_3(s_3+1)-2s_1s_3-2s_2s_3+2s_1s_2)\zeta_{MT,2}(s_1,s_2,s_3)\\ +s_3(s_1+2s_2-2s_3-2)\zeta_{MT,2}(s_1-1,s_2,s_3+1)\\ +s_3(2s_1+s_2-2s_3-2)\zeta_{MT,2}(s_1,s_2-1,s_3+1)\\ +s_1s_3\zeta_{MT,2}(s_1+1,s_2-2,s_3+1)+s_2s_3\zeta_{MT,2}(s_1-2,s_2+1,s_3+1)\\ +s_1(s_2-2s_3)\zeta_{MT,2}(s_1+1,s_2-1,s_3)+s_2(s_1-2s_3)\zeta_{MT,2}(s_1-1,s_2+1,s_3)\\ -s_1(2s_2-s_3)\zeta_{MT,2}(s_1+1,s_2,s_3-1)-s_2(2s_1-s_3)\zeta_{MT,2}(s_1,s_2+1,s_3-1)\\ +s_3(s_3+1)\zeta_{MT,2}(s_1-2,s_2,s_3+2)+s_3(s_3+1)\zeta_{MT,2}(s_1,s_2-2,s_3+2)\\ +s_1s_2\zeta_{MT,2}(s_1+1,s_2+1,s_3-2)+2s_3(s_3+1)\zeta_{MT,2}(s_1-1,s_2-1,s_3+2)=0.\end{aligned}$$

The coefficients of a and of b give the same identity (3.7) (because of the symmetry of s_1 and s_2 in (3.3)), while (3.8) follows from the coefficient of ab.

However it should be noted that each coefficient of s_j in (3.7) and (3.8) can be shown to be equal to 0 by partial fractional decompositions. Hence these equations do not yield new relations. Similarly in general cases, it may be expected that only the constant term will give a non-trivial result.

The following example can be regarded as a root-theoretic generalization of Example 3.2, because $\zeta_{MT,2}(s_1,s_2,s_3)$ is the zeta-function of the root system of type A_2.

Example 3.3. In the case of zeta-functions of root systems (cf. [13]), we have

$$(\xi_k)=(\xi_k)_{1\leqslant k\leqslant r},\qquad (\beta_\alpha)=(\langle\alpha^\vee,\rho\rangle)_{\alpha\in\Delta_+}, \tag{3.9}$$

$$(c_{m\alpha})=\begin{pmatrix}I_r & 0\end{pmatrix},\qquad (\gamma_{\alpha k})=(\langle\alpha^\vee,\lambda_k\rangle)_{\alpha\in\Delta_+,1\leqslant k\leqslant r}, \tag{3.10}$$

where I_r is the $r\times r$ identity matrix, $\Delta_+=\{\alpha_1,\ldots,\alpha_r,\ldots\}$ is the set of all positive roots in a given root system, whose first r elements $\alpha_1,\ldots,\alpha_r$ are fundamental roots,

[1]Here, the second term of (3.7) is not $s_3(1-s_1-s_2+s_3)$, but $s_3(2-s_1-s_2+s_3)$, because the factor corresponding to $u_3(1+u_3)=u_3+u_3^2$ is not $s_3(1+s_3)$, but $s_3+s_3(s_3+1)=s_3(2+s_3)$.

$d = |\Delta_+|$, ρ is the Weyl vector, and $\lambda_1, \cdots, \lambda_r$ are fundamental weights. Thus

$$\begin{aligned} G(\boldsymbol{u}, \boldsymbol{v}) &= \prod_{k=1}^{r} \Bigl(1 - \delta(k)\Bigl(1 + \sum_{\alpha\in\Delta_+} c_{k\alpha}(v_\alpha^{-1} - \beta_\alpha)\Bigr)\Bigl(\sum_{\alpha\in\Delta_+} \gamma_{\alpha k} u_\alpha v_\alpha\Bigr)\Bigr) \\ &= \prod_{k=1}^{r} \Bigl(1 - \delta(k)\Bigl(1 + v_{\alpha_k}^{-1} - \langle \alpha_k^\vee, \rho\rangle\Bigr)\Bigl(\sum_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda_k\rangle u_\alpha v_\alpha\Bigr)\Bigr) \\ &= \prod_{k=1}^{r} \Bigl(1 - \delta(k) \sum_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda_k\rangle u_\alpha v_\alpha v_{\alpha_k}^{-1}\Bigr). \end{aligned} \tag{3.11}$$

In particular, if $\xi_k = 1$ $(1 \leqslant k \leqslant r)$, then

$$G(\boldsymbol{u}, \boldsymbol{v}) = \prod_{k=1}^{r} \Bigl(1 - \sum_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda_k\rangle u_\alpha v_\alpha v_{\alpha_k}^{-1}\Bigr). \tag{3.12}$$

§ 4. Special values of $\zeta_2^{\rm des}$ at any integer points

The multiple zeta-function of Euler-Zagier type defined by (0.3) can be meromorphically continued to the whole complex space with many singularities (see [1]). In the case $r = 2$, the singularities of $\zeta_2(s_1, s_2)$ are located on

$$s_2 = 1,\ \ s_1 + s_2 = 2, 1, 0, -2, -4, -6, \ldots$$

([1, Theorem 1]), which implies that its special values of many integer points cannot be determined.

Here we consider the desingularized double zeta-function of Euler-Zagier type defined by

$$\zeta_2^{\rm des}(s_1, s_2) = \zeta_2^{\rm des}(s_1, s_2; 1, 1; 1, 0, 1, 1; 1, 1)$$

in (2.1) with $(r, d) = (2, 2)$. We showed in [6, (4.3)] that

$$\begin{aligned} \zeta_2^{\rm des}(s_1, s_2) &= (s_1 - 1)(s_2 - 1)\zeta_2(s_1, s_2) \\ &\quad + s_2(s_2 + 1 - s_1)\zeta_2(s_1 - 1, s_2 + 1) - s_2(s_2 + 1)\zeta_2(s_1 - 2, s_2 + 2), \end{aligned} \tag{4.1}$$

which is entire. Therefore its special values of all integer points can be determined, though each term on the right-hand side has singularities. We give their explicit expressions as follows. Note that a part of the examples mentioned below were already introduced in [6, Examples 4.7 and 4.9] with no proof.

First we consider the case $s_2 \in \mathbb{Z}_{\leqslant 0}$. We prepare the following lemma.

Lemma 4.1. *For $N \in \mathbb{N}_0$,*

$$\zeta_2(s,-N) = -\frac{1}{N+1}\zeta(s-N-1) + \sum_{k=0}^{N}\binom{N}{k}\zeta(s-N+k)\zeta(-k), \tag{4.2}$$

$$\begin{aligned}\zeta_2(-N,s) = \frac{1}{N+1}\zeta(s-N-1) - \sum_{k=0}^{N}\binom{N}{k}\zeta(s-N+k)\zeta(-k) \\ + \zeta(s)\zeta(-N) - \zeta(s-N)\end{aligned} \tag{4.3}$$

hold for $s \in \mathbb{C}$ except for singularities.

Proof. It follows from [15, (4.4)] that

$$\begin{aligned}\zeta_2(s_1,s_2) = \frac{1}{s_2-1}\zeta(s_1+s_2-1) + \sum_{k=0}^{M-1}\binom{-s_2}{k}\zeta(s_1+s_2+k)\zeta(-k) \\ + \frac{1}{2\pi i\Gamma(s_2)}\int_{(M-\varepsilon)}\Gamma(s_2+z)\Gamma(-z)\zeta(s_1+s_2+z)\zeta(-z)dz\end{aligned} \tag{4.4}$$

for $M \in \mathbb{N}$ and $(s_1,s_2) \in \mathbb{C}^2$ with $\Re s_2 > -M+\varepsilon$, $\Re(s_1+s_2) > 1-M+\varepsilon$ for any small $\varepsilon > 0$. Setting $(s_1,s_2) = (s,-N)$ and $M = N+1$ in (4.4), we see that (4.2) holds for any $s \in \mathbb{C}$ except for singularities because the both sides of (4.2) can be continued meromorphically to $\mathbb{C}$. Next, using the well-known relation

$$\zeta_2(s_1,s_2) + \zeta_2(s_2,s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1+s_2),$$

we can immediately obtain (4.3). □

Example 4.2. From (4.2) and (4.3), we have

$$\zeta_2(s,0) = -\zeta(s-1) - \frac{1}{2}\zeta(s), \tag{4.5}$$

$$\zeta_2(s,-1) = -\frac{1}{2}\zeta(s-2) - \frac{1}{2}\zeta(s-1) - \frac{1}{12}\zeta(s), \tag{4.6}$$

$$\zeta_2(0,s) = \zeta(s-1) - \zeta(s), \tag{4.7}$$

$$\zeta_2(-1,s) = \frac{1}{2}\left\{\zeta(s-2) - \zeta(s-1)\right\}. \tag{4.8}$$

Proposition 4.3. *For $s \in \mathbb{C}$ and $N \in \mathbb{N}_0$,*

$$\zeta_2^{\mathrm{des}}(s,-N) = -\sum_{k=0}^{N}\binom{N}{k}(k+1)(s-N+k-1)\zeta(s-N+k)\zeta(-k). \tag{4.9}$$

Proof. From (4.1) we have

$$\zeta_2^{\mathrm{des}}(s,-N) = (s-1)(-N-1)\zeta_2(s,-N) - N(-N+1-s)\zeta_2(s-1,-N+1)$$

$$+ N(-N+1)\zeta_2(s-2,-N+2).$$

Substituting (4.2) with $(s,-N)$, $(s-1,-N+1)$ and $(s-2,-N+2)$ into the right-hand side of the above equation, we have

$$\zeta_2^{\mathrm{des}}(s,-N) = \sum_{k=0}^{N}\left\{(s-1)(-N-1)\binom{N}{k} - N(-N+1-s)\binom{N-1}{k} + N(-N+1)\binom{N-2}{k}\right\}\zeta(s-N+k)\zeta(-k),$$

and the right-hand side of the above formula can be transformed to the right-hand side of (4.9). $\square$

Example 4.4. Setting $N=3$ in (4.9), we obtain

$$\zeta_2^{\mathrm{des}}(s,-3) = \frac{s-4}{2}\zeta(s-3) + \frac{s-3}{2}\zeta(s-2) - \frac{s-1}{30}\zeta(s). \tag{4.10}$$

For example,

$$\zeta_2^{\mathrm{des}}(1,-3) = \frac{1}{20}, \qquad \zeta_2^{\mathrm{des}}(2,-3) = \frac{1}{3} - \frac{1}{30}\zeta(2),$$
$$\zeta_2^{\mathrm{des}}(3,-3) = \frac{3}{4} - \frac{1}{15}\zeta(3), \qquad \zeta_2^{\mathrm{des}}(4,-3) = \frac{1}{2} + \frac{1}{2}\zeta(2) - \frac{1}{10}\zeta(4).$$

Also we have

$$\zeta_2^{\mathrm{des}}(0,0) = \frac{1}{4}, \quad \zeta_2^{\mathrm{des}}(-1,-1) = \frac{1}{36}, \quad \zeta_2^{\mathrm{des}}(0,-2) = \frac{1}{18}.$$

Proposition 4.5. *For $s \in \mathbb{C}$ and $N \in \mathbb{N}_0$,*

$$\begin{aligned} \zeta_2^{\mathrm{des}}(-N,s) &= \frac{(s-N-3)(s-N-2)}{(N+3)(N+2)}\zeta(s-N-1) \\ &+ \sum_{k=0}^{N+1}\frac{(ks+N-k+2)(s-N+k-1)}{N+2}\binom{N+2}{k}\zeta(s-N+k)\zeta(-k) \\ &- (N+1)(s-1)\zeta(s)\zeta(-N) + s(s+1+N)\zeta(s+1)\zeta(-N-1) \\ &+ (s-N-1)\zeta(s-N). \end{aligned} \tag{4.11}$$

Proof. From (4.1), we have

$$\zeta_2^{\mathrm{des}}(-N,s) = (-N-1)(s-1)\zeta_2(-N,s) + s(s+1+N)\zeta_2(-N-1,s+1)$$

$$- s(s+1)\zeta_2(-N-2, s+2).$$

Similar to the proof of Proposition 4.3, substituting (4.3) with $(-N, s)$, $(-N-1, s+1)$ and $(-N-2, s+2)$ into the right-hand side of the above equation, we can obtain (4.11). Note that, in this case, we apply (4.3) with the sum on the right-hand side from 0 to $N+2$, but the term corresponding to $k = N+2$ is canceled and does not appear in the final statement. ☐

Example 4.6. Setting $N = 1$ in (4.11), we have

$$\zeta_2^{\mathrm{des}}(-1, s) = \frac{(s-4)(s-3)}{12}\zeta(s-2) + \frac{s-2}{2}\zeta(s-1) - \frac{s(s-1)}{12}\zeta(s). \tag{4.12}$$

For example,

$$\zeta_2^{\mathrm{des}}(-1, 1) = \frac{1}{8}, \qquad \zeta_2^{\mathrm{des}}(-1, 2) = \frac{5}{12} - \frac{1}{6}\zeta(2),$$
$$\zeta_2^{\mathrm{des}}(-1, 3) = -\frac{1}{12} + \frac{1}{2}\zeta(2) - \frac{1}{2}\zeta(3).$$

Next we consider $\zeta_2^{\mathrm{des}}(N, 1)$ $(N \in \mathbb{N})$. From (4.1) with $s_1 = N \in \mathbb{Z}_{>1}$ and $s_2 \to 1$, we have

$$\zeta_2^{\mathrm{des}}(N, 1) = (N-1) \lim_{s_2 \to 1}(s_2 - 1)\zeta_2(N, s_2) + (2-N)\zeta_2(N-1, 2) - 2\zeta_2(N-2, 3).$$

We know from Arakawa and Kaneko [2, Proposition 4] that

$$\zeta_2(N, s) = \frac{\zeta(N)}{s-1} + O(1) \quad (N \in \mathbb{Z}_{>1}). \tag{4.13}$$

Thus we obtain the following.

Proposition 4.7. *For* $N \in \mathbb{N}_{>1}$,

$$\zeta_2^{\mathrm{des}}(N, 1) = (N-1)\zeta(N) + (2-N)\zeta_2(N-1, 2) - 2\zeta_2(N-2, 3). \tag{4.14}$$

Example 4.8. Using well-known results for double zeta-values, we obtain

$$\zeta_2^{\mathrm{des}}(2, 1) = \zeta(2) - 2\zeta_2(0, 3) = 2\zeta(3) - \zeta(2),$$
$$\zeta_2^{\mathrm{des}}(3, 1) = 2\zeta(3) - \zeta_2(2, 2) - 2\zeta_2(1, 3) = 2\zeta(3) - \frac{5}{4}\zeta(4),$$
$$\zeta_2^{\mathrm{des}}(4, 1) = 3\zeta(4) - 2\zeta_2(3, 2) - 2\zeta_2(2, 3) = 3\zeta(4) + 2\zeta(5) - 2\zeta(2)\zeta(3),$$

where we note $\zeta_2(0, 3) = \zeta(2) - \zeta(3)$.

The case $N = 1$ should be treated separately.

Proposition 4.9.

$$\zeta_2^{\mathrm{des}}(1,1)=\frac{1}{2}.$$

Proof. Denote the first, the second and the third term on the right-hand side of (4.1) by I_1, I_2 and I_3, respectively. Setting $M=1$ in (4.4), we have

$$\lim_{s_2\to 1}\lim_{s_1\to 1} I_1=\lim_{s_2\to 1}(s_2-1)\lim_{s_1\to 1}(s_1-1)\zeta_2(s_1,s_2)=0. \tag{4.15}$$

Using (4.7) and (4.8), we obtain

$$\begin{aligned}\lim_{s_2\to 1}\lim_{s_1\to 1}(I_2+I_3)&=\lim_{s_2\to 1}\left\{s_2^2\zeta_2(0,s_2+1)-s_2(s_2+1)\zeta_2(-1,s_2+2)\right\}\\&=\lim_{s_2\to 1}\left(s_2^2-\frac{s_2(s_2+1)}{2}\right)\{\zeta(s_2)-\zeta(s_2+1)\}\\&=\lim_{s_2\to 1}\frac{s_2}{2}(s_2-1)\{\zeta(s_2)-\zeta(s_2+1)\}=\frac{1}{2}.\end{aligned} \tag{4.16}$$

From (4.15) and (4.16), we obtain the assertion. Note that, since $\zeta_2^{\mathrm{des}}(s_1,s_2)$ is entire, the final result does not depend on the choice how to take the limit. □

§ 5. p-adic multiple star polylogarithm for indices with arbitrary integers

Now we proceed to our second main topic of the present paper. Our aim is to extend the result of [7, Theorem 3.41] to the case of indices with arbitrary (not necessarily all positive) integers (Theorem 5.8), which is a p-adic analogue of the equation (1.3).

First we prepare ordinary notation. For a prime number p, let $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$ be the set of p-adic integers, p-adic numbers, the algebraic closure of $\mathbb{Q}_p$ and the p-adic completion of $\overline{\mathbb{Q}}_p$ respectively. For a in $\mathbf{P}^1(\mathbb{C}_p)$ $(=\mathbb{C}_p\cup\{\overline{\infty}\})$, $\bar{a}$ means the image $\mathrm{red}(a)$ by the reduction map $\mathrm{red}:\mathbf{P}^1(\mathbb{C}_p)\to\mathbf{P}^1(\overline{\mathbb{F}}_p)\left(=\overline{\mathbb{F}}_p\cup\{\overline{\infty}\}\right)$, where $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$. For a finite subset $S\subset\mathbf{P}^1(\overline{\mathbb{F}}_p)$, we define $]S[:=\mathrm{red}^{-1}(S)\subset\mathbf{P}^1(\mathbb{C}_p)$. Denote by $|\cdot|_p$ the p-adic absolute value, and by μ_c the group of cth roots of unity in $\mathbb{C}_p$ for $c\in\mathbb{N}$. We put $q=p$ if $p\neq 2$ and $q=4$ if $p=2$. We denote by $\omega:\mathbb{Z}_p^\times\to\mathbb{Z}_p^\times$ the Teichmüller character and define $\langle x\rangle:=x/\omega(x)$ for $x\in\mathbb{Z}_p^\times$.

We recall that, for $r\in\mathbb{N}$, $k_1,\ldots,k_r\in\mathbb{Z}$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$, the **p-adic multiple L-function** of depth r, a $\mathbb{C}_p$-valued function on

$$(s_j)\in\mathfrak{X}_r\left(q^{-1}\right):=\left\{(s_1,\ldots,s_r)\in\mathbb{C}_p^r\,\middle|\,|s_j|_p<qp^{-1/(p-1)}\ (1\leqslant j\leqslant r)\right\},$$

is defined in [7] by

$$L_{p,r}(s_1,\ldots,s_r;\omega^{k_1},\ldots,\omega^{k_r};c)$$

$$:= \int_{(\mathbb{Z}_p^r)'} \langle x_1\rangle^{-s_1}\langle x_1+x_2\rangle^{-s_2}\cdots\langle\sum_{j=1}^r x_j\rangle^{-s_r}\omega^{k_1}(x_1)\cdots\omega^{k_r}(\sum_{j=1}^r x_j)\prod_{j=1}^r d\widetilde{\mathfrak{m}}_c(x_j),$$

where $\left(\mathbb{Z}_p^r\right)' := \left\{(x_j)\in\mathbb{Z}_p^r \,\middle|\, p\nmid x_1,\ p\nmid(x_1+x_2),\dots,\ p\nmid\sum_{j=1}^r x_j\right\}$, and $\widetilde{\mathfrak{m}}_c$ is the p-adic measure given in [7, §1]. The function is equal to $L_{p,r}(s_1,\dots,s_r;\omega_1^k,\dots,\omega_r^k;1,\dots,1;c)$ in [7, Definition 1.16]. When $r=1$, we have

$$L_{p,1}(s;\omega^{k-1};c) = (\langle c\rangle^{1-s}\omega^k(c)-1)L_p(s;\omega^k), \tag{5.1}$$

where $L_p(s;\omega^k)$ is the Kubota-Leopoldt p-adic L-function (see [7, Example 1.19]).

The p-adic rigid TMSPL can be defined for indices with arbitrary integers in the same way as [7, Definition 3.4]: Let $n_1,\dots,n_r\in\mathbb{Z}$ and $\xi_1,\dots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p\leqslant 1$ $(1\leqslant j\leqslant r)$. The **$p$-adic rigid TMSPL** [2] is defined by the following p-adic power series:

$$\ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r,z) := \sum_{\substack{0<k_1\leqslant\cdots\leqslant k_r\\ (k_1,p)=\cdots=(k_r,p)=1}} \frac{\xi_1^{k_1}\cdots\xi_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}} z^{k_r} \tag{5.2}$$

which converges for $z\in]\bar{0}[=\{x\in\mathbb{C}_p \mid |x|_p<1\}$ by $|\xi_j|_p\leqslant 1$ for $1\leqslant j\leqslant r$.

When $|\xi_j|_p=1$ for all $1\leqslant j\leqslant r$, by the completely same way as the arguments in [7, §3], we can show that it can be extended to a rigid analytic function (consult [7, §3.1]) on $\mathbf{P}^1(\mathbb{C}_p)-]S[$ with

$$S := \{\overline{\xi_r^{-1}}, \overline{(\xi_{r-1}\xi_r)^{-1}},\dots,\overline{(\xi_1\cdots\xi_r)^{-1}}\}. \tag{5.3}$$

Namely,

$$\ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)\in A^{\mathrm{rig}}(\mathbf{P}^1\setminus S).$$

We also note that

$$\ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;0) = \ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;\infty)=0, \tag{5.4}$$

and the following equality:

Proposition 5.1. *For $n_1,\dots,n_r\in\mathbb{Z}$ and $c\in\mathbb{N}_{>1}$ with $(c,p)=1$,*

$$L_{p,r}(n_1,\dots,n_r;\omega^{-n_1},\dots,\omega^{-n_r};c) = \sum_{\substack{\xi_1^c=\cdots=\xi_r^c=1\\ \xi_1\cdots\xi_r\neq1,\ \dots,\ \xi_{r-1}\xi_r\neq1,\ \xi_r\neq1}} \ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;1).$$

[2] TMSPL stands for the twisted multiple star polylogarithm. Here 'star' means that we add equalities in the running indices of the summation.

The p-adic partial TMSPL can also be defined for indices with arbitrary integers in the same way as [7, Definition 3.4]: Let $n_1,\dots,n_r\in\mathbb{Z}$ and $\xi_1,\dots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p\leqslant 1$ $(1\leqslant j\leqslant r)$. Let $\alpha_1,\dots,\alpha_r\in\mathbb{N}$ with $0<\alpha_j<p$ $(1\leqslant j\leqslant r)$. The ***p*-adic partial TMSPL** $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots\xi_r;z)$ is defined by the following p-adic power series:

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z):=\sum_{\substack{0<k_1\leqslant\cdots\leqslant k_r\\ k_1\equiv\alpha_1,\dots,k_r\equiv\alpha_r \bmod p}}\frac{\xi_1^{k_1}\cdots\xi_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}z^{k_r} \tag{5.5}$$

which converges for $z\in]\bar{0}[$.

When $|\xi_j|_p=1$ for all $1\leqslant j\leqslant r$, by the completely same way as the arguments in [7, §3.2], we can show that it is a rigid analytic function on $\mathbf{P}^1(\mathbb{C}_p)-]S[$. Namely,

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)\in A^{\mathrm{rig}}(\mathbf{P}^1\setminus S). \tag{5.6}$$

We have

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;0)=\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;\infty)=0 \tag{5.7}$$

by the equality

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)=\frac{1}{p^r}\sum_{\rho_1^p=\cdots=\rho_r^p=1}\rho_1^{-\alpha_1}\cdots\rho_r^{-\alpha_r}\ell^{(p),\star}_{n_1,\dots,n_r}(\rho_1\xi_1,\dots,\rho_r\xi_r;z).$$

We also note

$$\ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)=\sum_{0<\alpha_1,\dots,\alpha_r<p}\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z). \tag{5.8}$$

The following formulas are extensions of [7, Lemma 3.19] to the case of indices with arbitrary integers.

Lemma 5.2. *Let $n_1,\dots,n_r\in\mathbb{Z}$, $\xi_1,\dots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p\leqslant 1$ $(1\leqslant j\leqslant r)$ and $\alpha_1,\dots,\alpha_r\in\mathbb{N}$ with $0<\alpha_j<p$ $(1\leqslant j\leqslant r)$.*

(i) *For any index $(n_1,\dots,n_r)$,*

$$\frac{d}{dz}\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)=\frac{1}{z}\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r-1}(\xi_1,\dots,\xi_r;z).$$

(ii) *For $n_r=1$ and $r\neq 1$,*

$$\frac{d}{dz}\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$$
$$=\begin{cases}\frac{\xi_r(\xi_rz)^{\alpha_r-\alpha_{r-1}-1}}{1-(\xi_rz)^p}\ell^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p),\star}_{n_1,\dots,n_{r-1}}(\xi_1,\dots,\xi_{r-2},\xi_{r-1};\xi_rz) & \textit{if}\quad \alpha_r\geqslant\alpha_{r-1},\\ \frac{\xi_r(\xi_rz)^{\alpha_r-\alpha_{r-1}+p-1}}{1-(\xi_rz)^p}\ell^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p),\star}_{n_1,\dots,n_{r-1}}(\xi_1,\dots,\xi_{r-2},\xi_{r-1};\xi_rz) & \textit{if}\quad \alpha_r<\alpha_{r-1}.\end{cases}$$

(iii) *For $n_r = 1$ and $r = 1$ with $\xi_1 = \xi$ and $\alpha_1 = \alpha$,*

$$\frac{d}{dz}\ell_1^{\equiv\alpha,(p),\star}(\xi;z) = \frac{\xi(\xi z)^{\alpha-1}}{1-(\xi z)^p}.$$

Proof. They can be proved by direct computations. □

The following result is an extension of [7, Theorem 3.21] to the case of indices with arbitrary integers.

Proposition 5.3. *Let $n_1,\dots,n_r \in \mathbb{Z}$, $\xi_1,\dots,\xi_r \in \mathbb{C}_p$ with $|\xi_j|_p = 1$ $(1 \leqslant j \leqslant r)$ and $\alpha_1,\dots,\alpha_r \in \mathbb{N}$ with $0 < \alpha_j < p$ $(1 \leqslant j \leqslant r)$. Set S as in (5.3). The function $\ell_{n_1,\dots,n_r}^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}(\xi_1,\dots,\xi_r;z)$ is an overconvergent function on $\mathbf{P}^1 \setminus S$. Namely,*

$$\ell_{n_1,\dots,n_r}^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}(\xi_1,\dots,\xi_r;z) \in A^\dagger(\mathbf{P}^1 \setminus S).$$

Here $A^\dagger(\mathbf{P}^1 \setminus S)$ means the space of overconvergent functions on $\mathbf{P}^1 \setminus S$ (consult [7, Notation 3.13]).

Proof. The proof of [7, Theorem 3.21] was done by the induction on the weight but here it is achieved by the induction on the depth r.

(i) Assume that $r = 1$. By [7, Theorem 3.21], we know $\ell_{n_1}^{\equiv\alpha_1,(p),\star}(\xi_1;z) \in A^\dagger(\mathbf{P}^1 \setminus S)$ when $n_1 > 0$. When $n_1 \leqslant 0$, by Lemma 5.2 (i) and (iii) we know that the function is a rational function and the degree of whose numerator is less than that of whose denominator which is a power of $1-(\xi_1 z)^p$, which implies that the poles of the function are of the form ζ_p/ξ_1 $(\zeta_p \in \mu_p)$.

(ii) Assume that $r > 1$ and $n_r = 1$. We put

$$S_\infty = S \cup \{\overline{\infty}\} \qquad \text{and} \qquad S_{\infty,0} = S \cup \{\overline{\infty}\} \cup \{\bar{0}\}$$

and take a lift $\{\widehat{s_0},\widehat{s_1},\dots,\widehat{s_d}\}$ of $S_{\infty,0}$ with $\widehat{s_0} = \infty$ and $\widehat{s_1} = 0$. Put

$$\beta(z) := \begin{cases} \frac{\xi_r(\xi_r z)^{\alpha_r-\alpha_{r-1}-1}}{1-(\xi_r z)^p} & \text{if} \quad \alpha_r \geqslant \alpha_{r-1}, \\ \frac{\xi_r(\xi_r z)^{\alpha_r-\alpha_{r-1}+p-1}}{1-(\xi_r z)^p} & \text{if} \quad \alpha_r < \alpha_{r-1}. \end{cases}$$

By our assumption

$$\ell_{n_1,\dots,n_{r-1}}^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p),\star}(\xi_1,\dots,\xi_{r-2},\xi_{r-1};\xi_r z) \in A^\dagger(\mathbf{P}^1 \setminus \{\overline{\xi_r^{-1}},\dots,\overline{(\xi_1\cdots\xi_r)^{-1}}\})$$

and by the fact $\beta(z)dz \in \Omega^{\dagger,1}(\mathbf{P}^1 \setminus \{\bar{0},\overline{\infty},\overline{\xi_r^{-1}}\})$, we have

$$\ell_{n_1,\dots,n_{r-1}}^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p),\star}(\xi_1,\dots,\xi_{r-2},\xi_{r-1};\xi_r z)\cdot\beta(z)dz \in \Omega^{\dagger,1}(\mathbf{P}^1 \setminus S_{\infty,0}).$$

For the symbol $\Omega^{\dagger,1}$, consult [7, §3.2]. Put

$$f(z) := \ell_{n_1,\dots,n_{r-1}}^{\equiv(\alpha_1,\dots,\alpha_{r-1}),(p),\star}(\xi_1,\dots,\xi_{r-2},\xi_{r-1};\xi_r z)\cdot\beta(z) \in A^\dagger(\mathbf{P}^1 \setminus S_{\infty,0}). \tag{5.9}$$

Since $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$ belongs to $A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)\Big(\subset A^{\mathrm{rig}}(\mathbf{P}^1\setminus S_{\infty,0})\Big)$ by (5.6) and it satisfies the differential equation in Lemma 5.2 (ii), i.e. its differential is equal to $f(z)$, we have particularly, in the expression of [7, Lemma 3.14],

$$a_m(\widehat{s_1};f)=0 \qquad (m>0) \tag{5.10}$$

(recall $\widehat{s_1}=0$) and

$$a_1(\widehat{s_l};f)=0 \qquad (2\leqslant l\leqslant d). \tag{5.11}$$

By (5.9) and (5.10),

$$f(z)\in A^{\dagger}(\mathbf{P}^1\setminus S_\infty).$$

By (5.11) and [7, Lemma 3.15], there exists a unique function $F(z)$ in $A^{\dagger}(\mathbf{P}^1\setminus S_\infty)$, i.e. a function $F(z)$ which is rigid analytic on an affinoid V containing

$$\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[\,=\mathbf{P}^1(\mathbb{C}_p)-]\overline{\infty},S[$$

such that

$$F(0)=0 \quad \text{and} \quad dF(z)=f(z)dz. \tag{5.12}$$

Since $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$ is also a unique function in $A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)$ satisfying (5.12), the restrictions of both $F(z)$ and $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$ to the subspace $\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[$ must coincide, i.e.

$$F(z)\Big|_{\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[}\equiv \ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)\Big|_{\mathbf{P}^1(\mathbb{C}_p)-]S_\infty[}.$$

Hence by the coincidence principle of rigid analytic functions ([7, Proposition 3.3]), there is a rigid analytic function $G(z)$ on the union of V and $\mathbf{P}^1(\mathbb{C}_p)-]S[$ whose restriction to V is equal to $F(z)$ and whose restriction to $\mathbf{P}^1(\mathbb{C}_p)-]S[$ is equal to $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$. So we can say that

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)\in A^{\mathrm{rig}}(\mathbf{P}^1\setminus S)$$

can be rigid analytically extended to a bigger rigid analytic space by $G(z)$. Namely,

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)\in A^{\dagger}(\mathbf{P}^1\setminus S).$$

(iii) Assume that $r>1$ and $n_r<1$. In our (ii) above, we showed that

$$\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_{r-1},1}(\xi_1,\dots,\xi_r;z)\in A^{\dagger}(\mathbf{P}^1\setminus S). \tag{5.13}$$

Now showing that $\ell^{\equiv(\alpha_1,\dots,\alpha_r),(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)\in A^{\dagger}(\mathbf{P}^1\setminus S)$ is immediate, which follows from the differential equation in Lemma 5.2 (i) and (5.7).

(iv) Assume that $r>1$ and $n_r>1$. The proof in this case can be achieved by the induction on n_r. Recall that we have (5.13) by our (ii) above. By our assumption

$$\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p),\star}_{n_1,\ldots,n_{r-1},n_r-1}(\xi_1,\ldots,\xi_{r-1},\xi_r;z)\in A^\dagger(\mathbf{P}^1\setminus S)$$

and by the fact $\frac{dz}{z}\in\Omega^{\dagger,1}(\mathbf{P}^1\setminus\{\overline{\infty},\overline{0}\})$, we have

$$\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p),\star}_{n_1,\ldots,n_{r-1},n_r-1}(\xi_1,\ldots,\xi_{r-1},\xi_r;z)\frac{dz}{z}\in\Omega^{\dagger,1}(\mathbf{P}^1\setminus S_{\infty,0}).$$

Put

$$f(z):=\frac{1}{z}\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p),\star}_{n_1,\ldots,n_{r-1},n_r-1}(\xi_1,\ldots,\xi_{r-1},\xi_r;z)\in A^\dagger(\mathbf{P}^1\setminus S_{\infty,0}).$$

Then it follows that

$$\ell^{\equiv(\alpha_1,\ldots,\alpha_r),(p),\star}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z)\in A^\dagger(\mathbf{P}^1\setminus S)$$

by the same arguments as those given in (ii) above. □

By (5.8) and Proposition 5.3, we have

Corollary 5.4. *Let $n_1,\ldots,n_r\in\mathbb{Z}$, $\xi_1,\ldots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p=1$ $(1\leqslant j\leqslant r)$. Set S as in (5.3). The function $\ell^{(p),\star}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z)$ is an overconvergent function on $\mathbf{P}^1\setminus S$. Namely, $\ell^{(p),\star}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z)\in A^\dagger(\mathbf{P}^1\setminus S)$.*

The p-adic TMSPL can also be defined for indices with arbitrary integers in the same way as [7, Definition 3.29]: Let $n_1,\ldots,n_r\in\mathbb{Z}$ and $\xi_1,\ldots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p\leqslant 1$ $(1\leqslant j\leqslant r)$. The **$p$-adic TMSPL** $Li^{(p),\star}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z)$ is defined by the following p-adic power series:

$$Li^{(p),\star}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z):=\sum_{0<k_1\leqslant\cdots\leqslant k_r}\frac{\xi_1^{k_1}\cdots\xi_r^{k_r}z^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}} \tag{5.14}$$

which converges for $z\in]\bar{0}[$ by $|\xi_j|_p\leqslant 1$ for $1\leqslant j\leqslant r$. By direct computations one obtains the following differential equations which are extensions of [7, Lemma 3.31] to the case of indices with arbitrary integers.

Lemma 5.5. *Let $n_1,\ldots,n_r\in\mathbb{Z}$, $\xi_1,\ldots,\xi_r\in\mathbb{C}_p$ with $|\xi_j|_p\leqslant 1$ $(1\leqslant j\leqslant r)$.*

(i) *For any index $(n_1,\ldots,n_r)$,*

$$\frac{d}{dz}Li^{(p),\star}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z)=\frac{1}{z}Li^{(p),\star}_{n_1,\ldots,n_{r-1},n_r-1}(\xi_1,\ldots,\xi_r;z).$$

(ii) *For $n_r=1$ and $r\neq 1$,*

$$\frac{d}{dz}Li^{(p),\star}_{n_1,\ldots,n_r}(\xi_1,\ldots,\xi_r;z)=\left\{\frac{\xi_r}{1-\xi_r z}+\frac{1}{z}\right\}Li^{(p),\star}_{n_1,\ldots,n_{r-1}}(\xi_1,\ldots,\xi_{r-2},\xi_{r-1};\xi_r z).$$

(iii) *For* $n_r = 1$ *and* $r = 1$ *with* $\xi_1 = \xi$,

$$\frac{d}{dz} Li_1^{(p),\star}(\xi; z) = \frac{\xi}{1 - \xi z}.$$

The following result is an extension of [7, Theorem-Definition 3.32] to the case of indices with arbitrary integers.

Proposition 5.6. *Fix a branch of the* p*-adic logarithm by* $\varpi \in \mathbb{C}_p$. *Let* $n_1, \ldots, n_r \in \mathbb{Z}$, $\xi_1, \ldots, \xi_r \in \mathbb{C}_p$ *with* $|\xi_j|_p \leqslant 1$ $(1 \leqslant j \leqslant r)$. *Put*

$$S_r := \{\bar{0}, \overline{\infty}, \overline{(\xi_r)^{-1}}, \overline{(\xi_{r-1}\xi_r)^{-1}}, \ldots, \overline{(\xi_1 \cdots \xi_r)^{-1}}\} \subset \mathbf{P}^1(\overline{\mathbb{F}}_p).$$

Then the function $Li_{n_1,\ldots,n_r}^{(p),\star}(\xi_1, \ldots, \xi_r; z)$ *can be analytically continued as a Coleman function attached to* $\varpi \in \mathbb{C}_p$, *that is,*

$$Li_{n_1,\ldots,n_r}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_r; z) \in A_{\mathrm{Col}}^{\varpi}(\mathbf{P}^1 \setminus S_r)$$

whose restriction to $]\bar{0}[$ *is given by* $Li_{n_1,\ldots,n_r}^{(p),\star}(\xi_1, \ldots, \xi_r; z)$ *and which is constructed by the following iterated integrals:*

$$Li_1^{(p),\star,\varpi}(\xi_1; z) = -\log^{\varpi}(1 - \xi_1 z) = \int_0^z \frac{\xi_1}{1 - \xi_1 t} dt, \tag{5.15}$$

$$\begin{aligned} & Li_{n_1,\ldots,n_r}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_r; z) \\ & = \begin{cases} \int_0^z Li_{n_1,\ldots,n_{r-1},n_r-1}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_r; t) \frac{dt}{t} & \text{if} \quad n_r \neq 1, \\ \int_0^z Li_{n_1,\ldots,n_{r-1}}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_{r-2}, \xi_{r-1}\xi_r; t) \{\frac{\xi_r}{1-\xi_r t} + \frac{1}{t}\} dt & \text{if} \quad n_r = 1. \end{cases} \end{aligned} \tag{5.16}$$

Here $A_{\mathrm{Col}}^{\varpi}(\mathbf{P}^1 \setminus S_r)$ means the space of Coleman functions of $\mathbf{P}^1 \setminus S_r$ (consult [7, Notation 3.25]).

Proof. The proof of [7, Theorem-Definition 3.32] was done by the induction on the weight but here it is achieved by the induction on the depth r.

(i) Assume that $r = 1$. By [7, Theorem-Definition 3.32], we know $Li_{n_1}^{(p),\star,\varpi}(\xi_1; z) \in A_{\mathrm{Col}}^{\varpi}(\mathbf{P}^1 \setminus S_1)$ when $n_1 > 0$. When $n_1 \leqslant 0$, it is immediate to see the assertion by the differential equation in Lemma 5.5 (iii) because differentials of Coleman functions are again Coleman functions.

(ii) Assume that $r > 1$ and $n_r = 1$. Then by our induction assumption on r, $Li_{n_1,\ldots,n_{r-1}}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_{r-1}; z) \in A_{\mathrm{Col}}^{\varpi}(\mathbf{P}^1 \setminus S_{r-1})$ and also $Li_{n_1,\ldots,n_{r-1}}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_{r-1}; 0) = 0$. Hence $Li_{n_1,\ldots,n_{r-1}}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_{r-1}; t)$ has a zero at $t = 0$. Therefore the integrand on the right-hand side of (5.16) has no pole at $t = 0$. So the integration (5.16) starting from 0 makes sense and whence we have

$$Li_{n_1,\ldots,n_{r-1},1}^{(p),\star,\varpi}(\xi_1, \ldots, \xi_{r-1}, \xi_r; z) \in A_{\mathrm{Col}}^{\varpi}(\mathbf{P}^1 \setminus S_r). \tag{5.17}$$

(iii) Assume that $r > 1$ and $n_r < 1$. It is immediate to prove

$$Li^{(p),\star,\varpi}_{n_1,\dots,n_{r-1},n_r}(\xi_1,\dots,\xi_{r-1},\xi_r;z) \in A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S_r)$$

by (5.17) and the differential equation in Lemma 5.5 (i).

(iv) Assume that $r > 1$ and $n_r > 1$. The proof can be achieved by the induction on n_r. By our induction assumption, $Li^{(p),\star,\varpi}_{n_1,\dots,n_r-1}(\xi_1,\dots,\xi_r;z) \in A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S_r)$ and also $Li^{(p),\star,\varpi}_{n_1,\dots,n_r-1}(\xi_1,\dots,\xi_r;0) = 0$. Hence $Li^{(p),\star,\varpi}_{n_1,\dots,n_r-1}(\xi_1,\dots,\xi_r;t)$ has a zero at $t = 0$. Therefore the integrand on the right-hand side of (5.16) has no pole at $t = 0$. The integration (5.16) starting from 0 makes sense and thus we have $Li^{(p),\star,\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z) \in A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S_r)$. □

It should be noted that the restriction of the p-adic TMSPL $Li^{(p),\star,\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$ to $\mathbf{P}^1(\mathbb{C}_p)-\,]S_r \setminus \{\bar{0}\}[$ does not depend on any choice of the branch $\varpi \in \mathbb{C}_p$, which can be proved in the same way as [7, Proposition 3.34].

In particular, we remind that it is shown in [7, Theorem-Definiton 3.38] that, for $\rho_1,\dots,\rho_r \in \mu_p$ and $\xi_1,\dots,\xi_r \in \mu_c$ with $(c,p) = 1$ and

$$\xi_1\cdots\xi_r \neq 1, \quad \xi_2\cdots\xi_r \neq 1, \quad \dots, \quad \xi_{r-1}\xi_r \neq 1, \quad \xi_r \neq 1,$$

the special value of $Li^{(p),\star,\varpi}_{n_1,\dots,n_r}(\rho_1\xi_1,\dots,\rho_r\xi_r;z)$ at $z = 1$ is independent of the choice of ϖ. This value, denoted by $Li^{(p),\star}_{n_1,\dots,n_r}(\rho_1\xi_1,\dots,\rho_r\xi_r)$ for short, is called the **p-adic twisted multiple L-star value**.

The following result is an extension of [7, Theorem 3.36] to the case of indices with arbitrary integers, where we give a relationship between our p-adic rigid TMSPL $\ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$ and our p-adic TMSPL $Li^{(p),\star,\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z)$.

Proposition 5.7. *Fix a branch $\varpi \in \mathbb{C}_p$. Let $n_1,\dots,n_r \in \mathbb{Z}$, $\xi_1,\dots,\xi_r \in \mathbb{C}_p$ with $|\xi_j|_p = 1$ $(1 \leqslant j \leqslant r)$. The equality*

$$(5.18)\quad \ell^{(p),\star}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z) = Li^{(p),\star,\varpi}_{n_1,\dots,n_r}(\xi_1,\dots,\xi_r;z) + \sum_{d=1}^{r}\left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\ \sum_{\rho^p_{i_1}=1}\cdots\sum_{\rho^p_{i_d}=1} Li^{(p),\star,\varpi}_{n_1,\dots,n_r}\Big(((\prod_{l=1}^{d}\rho_{i_l}^{\delta_{i_l j}})\xi_j);z\Big)$$

holds for $z \in \mathbf{P}^1(\mathbb{C}_p)-]S_r \setminus \{\bar{0}\}[$, where δ_{ij} is the Kronecker delta.

Proof. By using the power series expansions (5.5) and (5.14), direct calculations show that the equality holds on $]\bar{0}[$. By Corollary 5.4, the left-hand side belongs to $A^{\dagger}(\mathbf{P}^1 \setminus S_r)$ $(\subset A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S_r))$, while by Proposition 5.6, the right-hand side belongs to $A^{\varpi}_{\mathrm{Col}}(\mathbf{P}^1 \setminus S_r)$. Therefore by the coincidence principle (consult [7, Proposition 3.27]), the equality holds on the whole space of $\mathbf{P}^1(\mathbb{C}_p)- \,]S_r \setminus \{\bar{0}\}[$, in fact, on an affinoid bigger than the space. □

Our main theorem in this section is the following, which could be regarded as an extension of [7, Theorem 3.41] to the case of indices with arbitrary integers and might be also regarded as an extension of [7, Theorem 2.1] to the case of indices with arbitrary integers in the special case of $\gamma_1 = \cdots = \gamma_r = 1$.

Theorem 5.8. *For $n_1, \ldots, n_r \in \mathbb{Z}$ and $c \in \mathbb{N}_{>1}$ with $(c,p) = 1$,*

(5.19)

$$\begin{aligned} &L_{p,r}(n_1, \ldots, n_r; \omega^{-n_1}, \ldots, \omega^{-n_r}; c) \\ &= \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} Li^{(p),\star}_{n_1,\ldots,n_r}\left(\frac{\xi_1}{\xi_2}, \frac{\xi_2}{\xi_3}, \ldots, \frac{\xi_r}{\xi_{r+1}}\right) \\ &+ \sum_{d=1}^{r} \left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} \sum_{\rho_{i_1}^p=1} \cdots \sum_{\rho_{i_d}^p=1} \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} Li^{(p),\star}_{n_1,\ldots,n_r}\Big(\Big(\frac{\prod_{l=1}^d \rho_{i_l}^{\delta_{i_l j}} \xi_j}{\xi_{j+1}}\Big)\Big), \end{aligned}$$

where we put $\xi_{r+1} = 1$.

Proof. It follows from Proposition 5.1 and Proposition 5.7. □

Remark 7. From (1.3), we have

$$\sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \zeta_r((n_j);(\xi_j);(1)) = \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} Li_{n_1,\ldots,n_r}\left(\frac{\xi_1}{\xi_2}, \frac{\xi_2}{\xi_3}, \ldots, \frac{\xi_r}{\xi_{r+1}}\right), \tag{5.20}$$

where $\xi_{r+1} = 1$. Similarly, we obtain

$$\sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} \zeta_r^\star((n_j);(\xi_j);(1)) = \sum_{\substack{\xi_1^c=1 \\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1 \\ \xi_r\neq 1}} Li^\star_{n_1,\ldots,n_r}\left(\frac{\xi_1}{\xi_2}, \frac{\xi_2}{\xi_3}, \ldots, \frac{\xi_r}{\xi_{r+1}}\right), \tag{5.21}$$

where

$$\zeta_r^\star((n_j);(\xi_j);(1)) = \sum_{0<k_1\leqslant\cdots\leqslant k_r} \frac{(\xi_1/\xi_2)^{k_1}\cdots(\xi_r/\xi_{r+1})^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}, \tag{5.22}$$

with $\xi_{r+1} = 1$ and

$$Li^\star_{n_1,\ldots,n_r}(z_1, \ldots, z_r) = \sum_{0<k_1\leqslant\cdots\leqslant k_r} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}} \tag{5.23}$$

for $(n_j) \in \mathbb{N}^r$ and $(z_j) \in \mathbb{C}^r$ with $|z_j| = 1$, which are star-versions of (0.2) and (1.1), respectively. Also (5.23) should be compared with (5.14). Note that Theorem 5.8 can be regarded as a p-adic analogue of (5.21). Therefore $L_{p,r}((s_j);(\omega^{k_j});c)$ might be called the *p-adic multiple L-star function.*

Corollary 5.9. *For $n_1,\ldots,n_r \in \mathbb{N}_0$ and $c \in \mathbb{N}_{>1}$ with $(c,p)=1$,*

$$\begin{aligned}(5.24)\quad & L_{p,r}(-n_1,\ldots,-n_r;\omega^{n_1},\ldots,\omega^{n_r};c) \\ &= \sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1\\ \xi_r\neq 1}} \widetilde{\mathfrak{B}}((n_j);(\xi_j)) \\ &+ \sum_{d=1}^{r}\left(-\frac{1}{p}\right)^d \sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\ \sum_{\rho_{i_1}^p=1}\cdots\sum_{\rho_{i_d}^p=1}\ \sum_{\substack{\xi_1^c=1\\ \xi_1\neq 1}} \cdots \sum_{\substack{\xi_r^c=1\\ \xi_r\neq 1}} \widetilde{\mathfrak{B}}((n_j);((\prod_{j\leqslant i_l}\rho_{i_l})\xi_j)),\end{aligned}$$

where $\{\widetilde{\mathfrak{B}}((n_j);(\xi_j))\}$ are certain twisted multiple Bernoulli numbers defined by

$$\begin{aligned}(5.25)\quad & \frac{\xi_1 \exp\left(\sum_{\nu=1}^r t_\nu\right)}{1-\xi_1\exp\left(\sum_{\nu=1}^r t_\nu\right)} \prod_{j=2}^{r} \frac{1}{1-\xi_j \exp\left(\sum_{\nu=j}^r t_\nu\right)} \\ &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} \widetilde{\mathfrak{B}}((n_j);(\xi_j))\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}.\end{aligned}$$

Proof. We first show the following result which can be proved by the same method as in the proof of [11, Lemma 5.9]. For $z \in]\bar{0}[$, we obtain from the definition (5.14) that

$$\begin{aligned}(5.26)\quad & \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty} Li^{(p),\star}_{-n_1,\ldots,-n_r}\left(\frac{\xi_1}{\xi_2},\frac{\xi_2}{\xi_3},\ldots,\frac{\xi_r}{\xi_{r+1}};z\right)\frac{t_1^{n_1}\cdots t_r^{n_r}}{n_1!\cdots n_r!} \\ &= \frac{\xi_1 z e^{\sum_{\nu=1}^r t_\nu}}{1-\xi_1 z e^{\sum_{\nu=1}^r t_\nu}} \prod_{j=2}^{r}\frac{1}{1-\xi_j z e^{\sum_{\nu=j}^r t_\nu}}\end{aligned}$$

(cf. [11, (5.16)]). Since $Li^{(p),\star}_{-n_1,\ldots,-n_r}(\xi_1/\xi_2,\xi_2/\xi_3,\ldots,\xi_r/\xi_{r+1};z)$ is a rational function in z, we can let $z \to 1$ on the both sides of (5.26). Hence it follows from (5.25) that

$$(5.27)\quad Li^{(p),\star}_{-n_1,\ldots,-n_r}\left(\frac{\xi_1}{\xi_2},\frac{\xi_2}{\xi_3},\ldots,\frac{\xi_r}{\xi_{r+1}}\right) = \widetilde{\mathfrak{B}}((n_j);(\xi_j)) \quad ((n_j)\in\mathbb{N}_0^r).$$

Therefore we can see that the right-hand side of (5.19) coincides with the right-hand side of (5.24). This completes the proof. □

Remark 8. It should be emphasized that (5.24) with replacing $\widetilde{\mathfrak{B}}((n_j);(\xi_j))$ by $\mathfrak{B}((n_j);(\xi_j))$ defined by

$$\prod_{j=1}^{r}\frac{1}{1-\xi_j\exp\left(\sum_{\nu=j}^r t_\nu\right)} = \sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\mathfrak{B}((n_j);(\xi_j))\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$

(see [7, Definition 1.4]) is also valid; in fact, it is [7, Theorem 2.1].

Finally, we consider the case $r = 1$. Since

$$\sum_{\substack{\xi^c=1\\ \xi\neq 1}} \frac{\xi e^t}{1-\xi e^t} = \frac{e^t}{e^t-1} - \frac{ce^{ct}}{e^{ct}-1} = \sum_{n=0}^{\infty}(1-c^{n+1})B_{n+1}\frac{t^n}{n!} + (1-c),$$

$$\sum_{\rho^p=1}\sum_{\substack{\xi^c=1\\ \xi\neq 1}} \frac{\rho\xi e^t}{1-\rho\xi e^t} = \sum_{\rho^p=1}\left\{\frac{\rho e^t}{\rho e^t-1} - \frac{c\rho^c e^{ct}}{\rho^c e^{ct}-1}\right\} = \frac{pe^{pt}}{e^{pt}-1} - \frac{cpe^{cpt}}{e^{cpt}-1}$$

$$= \sum_{n=0}^{\infty}(1-c^{n+1})p^{n+1}B_{n+1}\frac{t^n}{(n+1)!} + (1-c)p,$$

we have

$$\sum_{\substack{\xi^c=1\\ \xi\neq 1}} \widetilde{\mathfrak{B}}(n;\xi) = \begin{cases}(1-c^{n+1})\frac{B_{n+1}}{n+1} & (n>0),\\ \frac{1-c}{2} & (n=0),\end{cases}$$

$$\sum_{\substack{\xi^c=1\\ \xi\neq 1}}\sum_{\rho^p=1} \widetilde{\mathfrak{B}}(n;\rho\xi) = \begin{cases}(1-c^{n+1})p^{n+1}\frac{B_{n+1}}{n+1} & (n>0),\\ \frac{(1-c)p}{2} & (n=0).\end{cases}$$

Hence (5.24) implies that

$$L_{p,1}(-n;\omega^n;c) = \sum_{\substack{\xi^c=1\\ \xi\neq 1}} \widetilde{\mathfrak{B}}(n;\xi) - \frac{1}{p}\sum_{\rho^p=1}\sum_{\substack{\xi^c=1\\ \xi\neq 1}} \widetilde{\mathfrak{B}}(n;\rho\xi)$$

$$= \begin{cases}(1-c^{n+1})(1-p^n)\frac{B_{n+1}}{n+1} & (n>0),\\ 0 & (n=0).\end{cases}$$

By (5.1), this can be rewritten as the Kubota-Leopoldt formula ([19, Theorem 5.11]):

$$L_p(1-n;\omega^n) = -(1-p^{n-1})\frac{B_n}{n} \quad (n\in\mathbb{N}). \tag{5.28}$$

On the other hand, combining (5.19) in the case $r = 1$ and (5.1), we obtain the Coleman formula ([3]):

$$L_p(n;\omega^{1-n}) = \left(1-\frac{1}{p^n}\right) Li_n^{(p),\star}(1) \quad (n\in\mathbb{N}) \tag{5.29}$$

(see [7, Example 3.42]). Therefore Theorem 5.8 can be regarded as a generalization of both (5.28) and (5.29).

Acknowledgements. The authors express their sincere gratitude to the referee for important suggestions.

References

[1] Akiyama, S., Egami, S. and Tanigawa, Y., Analytic continuation of multiple zeta-functions and their values at non-positive integers, *Acta Arith.*, **98** (2001), 107–116.

[2] Arakawa, T and Kaneko, M., Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, *Nagoya Math. J.*, **153** (1999), 189–209.

[3] Coleman, R., Dilogarithms, regulators and p-adic L-functions, *Invent. Math.* **69** (1982), 171–208.

[4] Furusho, H., p-adic multiple zeta values I, p-adic multiple polylogarithms and the p-adic KZ equation, *Invent. Math.*, **155** (2004), 253–286.

[5] Furusho, H., p-adic multiple zeta values II, Tannakian interpretations, *Amer. J. Math.*, **129** (2007), 1105-1144.

[6] Furusho, H., Komori, Y., Matsumoto, K. and Tsumura, H., Desingularization of complex multiple zeta-functions, preprint, to appear in *Amer. J. Math.*, `arXiv:1508.06920`.[3]

[7] Furusho, H., Komori, Y., Matsumoto, K. and Tsumura, H., Fundamentals of p-adic multiple L-functions, and evaluation of their special values, preprint, to appear in *Selecta Math.*, `arXiv:1508.07185`.[3]

[8] Frobenius, G., Über die Bernoullischen Zahlen und die Eulerschen Polynome, *Preuss. Akad. Wiss. Sitzungsber.*, (1910), no. 2, 809–847.

[9] Goncharov, A. B., Multiple polylogarithms, cyclotomy, and modular complexes, *Math. Res. Lett.*, **5** (1998), 497–516.

[10] Hoffman, M. E., Multiple harmonic series, *Pacific J. Math.*, **152** (1992), 275–290.

[11] Kaneko, M. and Tsumura, H., Multi-poly-Bernoulli numbers and related zeta functions, preprint, `arXiv:1503.02156`.

[12] Komori, Y., An integral representation of multiple Hurwitz–Lerch zeta functions and generalized multiple Bernoulli numbers, *Quart. J. Math.*, **61** (2010), 437–496.

[13] Komori, Y., Matsumoto, K. and Tsumura, H., On Witten multiple zeta-functions associated with semisimple Lie algebras II, *J. Math. Soc. Japan*, **62** (2010), 355–394.

[14] Komori, Y., Matsumoto, K. and Tsumura, H., Functional equations for double L-functions and values at non-positive integers, *Intern. J. Number Theory*, **7** (2011), 1441–1461.

[15] Matsumoto, K., On the analytic continuation of various multiple zeta-functions, *Number Theory for the Millennium II*, Proc. Millennial Conf. on Number Theory, Bennett, M. A. et al. (eds.), A K Peters, 2002, pp. 417–440.

[16] Matsumoto, K., Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, *Nagoya Math. J.*, **172** (2003), 59–102.

[17] Matsumoto, K., The analytic continuation and the asymptotic behaviour of certain multiple zeta-functions I, *J. Number Theory*, **101** (2003), 223–243.

[18] Matsumoto, K., On Mordell-Tornheim and other multiple zeta-functions, *Proc. Session in Analytic Number Theory and Diophantine Equations*, Heath-Brown, D. R. and Moroz, B. Z. (eds.), *Bonner Math. Schriften* **360** (2003), n. 25, 17pp.

[19] Washington, L. C., *Introduction to Cyclotomic Fields*, Second edition, Graduate Texts in Mathematics, **83**, Springer-Verlag, New York, 1997.

[3]Note that [6] and [7] were at first combined and written as the following single paper: Furusho, H., Komori, Y., Matsumoto, K. and Tsumura, H., "Desingularization of complex multiple zeta-functions, fundamentals of p-adic multiple L-functions, and evaluation of their special values", preprint (`arXiv:1309.3982v2`).

[20] Zagier, D., Values of zeta functions and their applications, *First European Congress of Mathematics*, Vol.II, Joseph, A. et al. (eds.), *Progr. Math.*, **120**, Birkhäuser, 1994, pp. 497–512.