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# Φ-cohomology Theory

By

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**Abstract**

In this paper, we introduce Φ-cohomology theory based on an associator Φ. This cohomology theory takes a value in the category, whose objects are triple of two **Q**-vector spaces with a comparison map. Using this cohomology theory, the periods can be expressed by coefficients of the given associators. We prove that the coefficients of associator can be expressed as iterated integrals.

## § 1. Introduction

**1.0.1. Introduction** There are many algebraic relations over **Q** among the multiple zeta values. Associator relation is one of the most interesting relation. It is a series of relations and to describe this relation, it is convenient to consider the generating function of multiple zeta values, which is called the Drinfeld associator. The Drinfeld associator satisfies, 2-cycle, 3-cycle and 5-cycle relations. In general a non-commutative formal power series satisfying these relations is called an associator. It is conjectured that all the relations among the multiple zeta values are satisfied by the coefficients of any associator Φ.

Many relations on multiple zeta values are proved by considering integral expression or hypergeometric functions. They are interpreted as period integrals which are obtained by comparison of de Rham and Betti cohomologies. It is natural to ask that these methods of period integrals can be applicable to any associator. In this paper, we establish a new cohomology theory, called Φ-cohomology theory whose "fake" period is expressed by the coefficients of an associator Φ. For example, using Φ-cohomology

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theory, we can prove Brown-Zagier identities for associators. This will be proved in a subsequent paper (arXiv:1301.7474). In this paper, most of the part is spent to rewrite well known facts on usual cohomology theory in the context of algebroids.

**1.0.2. Notations** A *locally compact* vector space $V$ is an inductive limit of projective limits of finite dimensional vector spaces equipped with a linear topology after Lefschetz. (See [L], [D].) In the literature [K], these spaces are called Tate vector spaces. From now on a vector space is assumed to be a locally compact vector space and a homomorphism is assume to be continuous with respect to linear topology. Tensor product means completed tensor product. If $L$ is discrete, $M, N$ are compact, then $Hom(M, L)$ is discrete and $M \otimes N$ is compact. Moreover, we have $Hom(M, Hom(N, L)) = Hom(M \otimes N, L)$. A compact discrete vector space is called an *artinian vector space*. An artinian vector space is finite dimensional.

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## § 2. Cohomology of algebroids

### § 2.1. Algebroids

Let $\mathbf{k}$ be a field and $S$ a non-empty finite set. A $\mathbf{k}$-*algebroid* $A = (A, S)$ over $S$ consists of a data of a set $\{A_{pq}\}_{p,q \in S}$ of compact $\mathbf{k}$-vector spaces equipped with a structure of multiplications

$$\mu_{pqr} : A_{pq} \otimes A_{qr} \to A_{pr}.$$

These data should satisfy the following axioms.

1. There exists a specified element $1_p$ in $A_{pp}$ such that

$$A_{pq} \xrightarrow{1_p \otimes id} A_{pp} \otimes A_{pq} \xrightarrow{\mu_{ppq}} A_{pq},$$
$$A_{pq} \xrightarrow{id \otimes 1_q} A_{pq} \otimes A_{qq} \xrightarrow{\mu_{pqq}} A_{pq}$$

   are identities.

2. Associativity for multiplication holds. That is, the following diagram is commutative

$$\begin{array}{ccc} A_{pq} \otimes A_{qr} \otimes A_{rs} & \xrightarrow{\mu_{pqr} \otimes id} & A_{pr} \otimes A_{rs} \\ id \otimes \mu_{qrs} \downarrow & & \downarrow \mu_{prs} \\ A_{pq} \otimes A_{qs} & \xrightarrow{\mu_{pqs}} & A_{ps}. \end{array}$$

3. For any $p, q \in S$, there exist elements $x \in A_{pq}$ and $y \in A_{qp}$ such that $xy = 1_p$ and $yx = 1_q$.

The above condition 3. is called the *condition for connectedness.* Using above axioms, we can prove that $A_p = A_{pp}$ is an algebra and $A_{pq}$ is a free left $A_p$-module (right $A_q$-module) of rank one.

An algebroid $\mathcal{A} = (\mathcal{A}, S)$ equipped with the counit homomorphism $\epsilon_{pq} : \mathcal{A}_{pq} \to \mathbf{Q}$, coproduct homomorphisms $\Delta_{pq} : \mathcal{A}_{pq} \to \mathcal{A}_{pq} \otimes \mathcal{A}_{pq}$ and the antipodal homomorphisms $S_{ab} : \mathcal{A}_{pq} \to \mathcal{A}_{qp}$ is called a Hopf algebroid if they satisfies the following properties.

1. The coproduct homomorphisms are coassociative, i.e. $\Delta_{pq} \circ (\Delta_{pq} \otimes id) = \Delta_{pq} \circ (id \otimes \Delta_{pq})$, and counitary, i.e. $(id \otimes \epsilon_{pq}) \circ \Delta_{pq} = (\epsilon_{pq} \otimes id) \circ \Delta_{pq} = id$.

2. Antipodal is the inverse of multiplication, i.e. $1_b \circ \epsilon_{ab} = \mu_{bab} \circ (S_{ab} \otimes id) \circ \Delta_{ab}$ and $1_a \circ \epsilon_{ab} = \mu_{aba} \circ (id \otimes S_{ab}) \circ \Delta_{ab}$, and anti-multiplicative, i.e. $\mu_{bca} \circ sw \circ (S_{ab} \otimes S_{bc}) = S_{ac} \circ \mu_{abc}$. Here $sw : \mathcal{A}_{ba} \otimes A_{cb} \to \mathcal{A}_{cb} \otimes A_{ab}$ is the map obtained by switching the first and the second tensor forctors.

*Remark* 1. One can define an *algebroid object* in any $\mathbf{k}$-linear abelian tensor categories with an identity object $\mathbf{1}$. In this case, we do not assume property 3 of the axiom of algebroid. In the example in §3.1, we do not assume the existence of a base in $\mathcal{A}_{ab}$. We assume the existence of basis for each $\mathcal{A}_{dR,ab}$ and $\mathcal{A}_{B,ab}$. We can not take basis which correspond to each other via the comparison isomorphism in general.

Let $\mathcal{A} = (\mathcal{A}, S)$ and $\mathcal{B} = (B, T)$ be two $\mathbf{k}$-algebroids. We define an *external tensor product* $\mathcal{A} \boxtimes \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ by the algebroid over $S \times T$ as follows. For three points $x_1 = (s_1, t_1), x_2 = (s_2, t_2)$ and $x_3 = (s_3, t_3)$ in $S \times T$, we set

$$(\mathcal{A} \boxtimes \mathcal{B})_{x_1 x_2} = \mathcal{A}_{s_1 s_2} \otimes B_{t_1 t_2}.$$

and define the mutliplication

$$(\mathcal{A} \boxtimes \mathcal{B})_{x_1 x_2} \otimes (\mathcal{A} \boxtimes \mathcal{B})_{x_2 x_3} \to (\mathcal{A} \boxtimes \mathcal{B})_{x_1 x_3}.$$

by $\mu_{s_1 s_2 s_3} \otimes \mu_{t_1 t_2 t_3}$.

**Example 2.1.** Let $X$ be a connected topological space, $S$ be a finite set of points in $X$. Let $\pi_1(X)$ be the fundamental groupoid over $S$. That is, for two points $p, q$ in $S$, $\pi_1(X, p, q)$ is the set of homotopy equivalent classes of paths in $X$ beginning from $p$ and ending with $q$. Then the unipotent completion $A_{p,q}$ of $\mathbf{Q}[\pi_1(X, p, q)]$ forms an Hopf algebroid by the standard multiplication of the groupiods $\pi_1(X)$.

## § 2.2. $A$-Modules and their extensions

**Definition 2.2.** Let $A = (A, S)$ be an algebroid over $S$. Let $M = \{M_p\}_{p\in S}$ be a set of locally compact vector spaces and $\mu_M = \{\mu_{M,pq}\}$ be a set of linear maps

$$\mu_{M,pq} : A_{pq} \otimes M_q \to M_p.$$

The data $M = \{M_p, \mu_{M,pq}\}$ is called a *left $A$-module(=left $(A, S)$-module)* if it satisfies the following axioms

1. The element $1_p$ acts as an identity.
2. The action $\mu_M$ and the action of multiplications of $A$ are associative.

We define a *right $A$-module* similarly.

Let $\mathcal{A}$ be a Hopf algebroid, and $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{A}$ modules. We introduct an $\mathcal{A}$-module structure on $\mathcal{M} \otimes \mathcal{N} = \{\mathcal{M}_a \otimes \mathcal{N}_a\}_a$ by

$$\mathcal{A}_{ab} \otimes (\mathcal{M}_b \otimes N_b) \xrightarrow{\Delta_{ab}\otimes id} (\mathcal{A}_{ab} \otimes \mathcal{A}_{ab}) \otimes (\mathcal{M}_b \otimes N_b) \xrightarrow{\mu_M\otimes\mu_N} \mathcal{M}_a \otimes \mathcal{N}_a$$

One can define a left $\mathcal{A}$-module structure on a space of $\mathbf{k}$-linear map $M^* = Hom_{\mathbf{k}}(M, \mathbf{Q})$ by

$$\begin{aligned} &\mathcal{A}_{ab} \otimes Hom_{\mathbf{k}}(M_b, \mathbf{Q}) \to Hom_{\mathbf{k}}(M_a, \mathbf{Q}) \\ &(x \otimes \varphi)(m) = \varphi(\mu_M(S_{ab}(x) \otimes m)) \text{ for } \varphi \in Hom_{\mathbf{k}}(M_b, \mathbf{Q}), m \in M_a, x \in \mathcal{A}_{ab}. \end{aligned}$$

It is called the dula of $M$.

Let $M$ and $N$ be left $A$ modules. A family of linear maps $\varphi_p : M_p \to N_p$ is called an $A$-homomorphism if it is compatible with the left action of $A$, i.e.

$$\begin{array}{ccc} A_{pq} \otimes M_q & \to & M_p \\ \downarrow & & \downarrow \\ A_{pq} \otimes N_q & \to & N_p. \end{array}$$

The set of continuous $A$-homomorphism from $M$ to $N$ is denoted by $Hom_A(M, N)$. Then the space $Hom_A(M, N)$ is isomorphic to the kernel of the map

$$\begin{array}{ccc} \prod_p Hom_{\mathbf{k}}(M_p, N_p) & \to & \prod_{pq} Hom_{\mathbf{k}}(A_{pq} \otimes M_q, N_p) \\ (\varphi_p)_p & \mapsto & (\mu_{pq}(1 \otimes \varphi_q) - \varphi_p\mu_{pq})_{pq} \end{array}$$

We define the *extension group* $Ext^i_A(M, N)$ by the cohomology of the complex $\mathbf{R}Hom_A(M, N)$ defined by the following complex $C^\bullet(M, N) = C^\bullet_{(A,S)}(M, N)$:

$$C^\bullet(M, N) : 0 \to C^0(M, N) \to C^1(M, N) \to C^2(M, N) \to \cdots , \tag{2.1}$$

where

$$C^i(M,N) = \prod_{p_0,\dots,p_i} Hom_{\mathbf{k}}(A_{p_0p_1}\otimes A_{p_1p_2}\otimes\cdots\otimes A_{p_{i-1}p_i}\otimes M_{p_i}, N_{p_0})$$

and the differential $\delta : C^i(M,N)\to C^{i+1}(M,N)$ is defined by

$$\begin{aligned}&\delta(\varphi)_{p_0\dots p_{i+1}}(a_1\otimes\cdots\otimes a_{i+1}\otimes m)\\ =&a_1\varphi_*(a_2\otimes\cdots\otimes a_{i+1}\otimes m) - \varphi_*(a_1a_2\otimes\cdots\otimes a_{i+1}\otimes m)\\ &+\varphi_*(a_1\otimes a_2a_3\otimes\cdots\otimes a_{i+1}\otimes m) - \cdots\\ &+(-1)^{i+1}\varphi_*(a_1\otimes a_2\otimes\cdots\otimes a_{i+1}m).\end{aligned}$$

Here we write the obvious index of $\varphi$ as $*$. For example,

$$\varphi_*(a_2\otimes\cdots\otimes a_{i+1}\otimes m) = \varphi_{p_1\cdots p_{i+1}}(a_2\otimes\cdots\otimes a_{i+1}\otimes m)\in N_{p_1}.$$

*Remark* 2.

1. Since all the vector spaces are locally compact in linear topology, and tensor products are completed tensor products, the extension group commute with the scalar extension for an extension of fields $\mathbf{k}\to\mathbf{k}'$.
2. If the set of base points $S$ is one point set $\{s\}$, then a data for algebroid over $S$ is equal to that of algebra and the extension group defined as above is equal to that for modules over the algebra.

### § 2.3. Homomorphism of algebroids

Let $S,T$ be non-empty finite sets and $(A,S)$ and $(B,T)$ be algebroids over $S$ and $T$, respectively. The *homomorphism* $\varphi : (A,S)\to(B,T)$ *of algebroids* is a pair of map $\psi : S\to T$ and a set of linear maps

$$\varphi_{pq} : A_{pq}\to B_{\psi(p)\psi(q)} \tag{2.2}$$

such that

1. the map $\varphi$ preserves the identities: i.e. $\varphi_{pp}(1_p) = 1_{\varphi(p)}$ for all $p\in S$, and
2. the set of maps $\{\varphi_{pq}\}$ is compatible with the multiplications.

$$\begin{array}{ccc} A_{pq}\otimes A_{qr} & \xrightarrow{\mu_{pqr}} & A_{pr}\\ \downarrow & & \downarrow\\ B_{\psi(p)\psi(q)}\otimes B_{\psi(q)\psi(r)} & \xrightarrow{\mu_{\psi(p)\psi(q)\psi(r)}} & B_{\psi(p)\psi(r)}.\end{array}$$

When $S = T$ and $\psi$ is the identity map, a homomorphims of algebroids $\varphi : (A, S) \to (B, S)$ is called a homomorphism of algebroids over $S$.

**Example 2.3.** Let $(A, S)$ be an algebroid and $s \in S$. Then we have a natural morphism of algebroids: $i_s : (A, \{s\}) \to (A, S)$.

Let $\varphi : A = (A, S) \to B = (B, T)$ be a homomorphism of algebroids and $M$ be a $B$-module. The pull back $\varphi^* M$ of $M$ is an $A$-module defined by $(\varphi^* M)_s = M_{\varphi(s)}$ and the action of $A_{pq}$ on $(\varphi^* M)_s$ is defined via the map (2.2).

Now we consider the functoriality of extensions. Let $\varphi : A \to B$ be a homomorphism of algebroids, and $N$, $M$ be $B$-modules. We define a map $\mathbf{R}Hom_B(M, N) \to \mathbf{R}Hom_A(\varphi^* M, \varphi^* N)$ by

$$\prod_{q_0,\dots,q_i \in T} Hom_{\mathbf{k}}(B_{q_0 q_1} \otimes B_{q_1 q_2} \otimes \cdots \otimes B_{q_{i-1} q_i} \otimes M_{q_i}, N_{q_0}) \ni \alpha$$
$$\mapsto \beta \in \prod_{p_0,\dots,p_i \in S} Hom_{\mathbf{k}}(A_{p_0 p_1} \otimes A_{p_1 p_2} \otimes \cdots \otimes A_{p_{i-1} p_i} \otimes M_{\varphi(p_i)}, N_{\varphi(p_0)}),$$

where $\beta_{p_0 \cdots p_i}$ is defined by the composite of

$$A_{p_0 p_1} \otimes \cdots \otimes A_{p_{i-1} p_i} \otimes M_{\varphi(p_i)}$$
$$\to B_{\varphi(p_0)\varphi(p_1)} \otimes \cdots \otimes B_{\varphi(p_{i-1})\varphi(p_i)} \otimes M_{\varphi(p_i)} \xrightarrow{\alpha} N_{\varphi(p_0)}.$$

It defines a homomorphism of complexes.

**Proposition 2.4.** *Let $A$ be an algebroid and $M$ be a compact $A$-module and $N$ be a discrete $A$-module. Then the natural homomorphism*

$$\mathbf{R}Hom_{(A,S)}(M, N) \to \mathbf{R}Hom_{(A,\{s\})}(M_s, N_s)$$

*induced by the restriction map $\{s\} \to S$ is a quasi-isomorphism.*

**Corollary 2.5.** *Under the same assumption as the above proposition, there is a natural isomorphism of cohomologies*

$$Ext^i_{(A,\{s\})}(M_s, N_s) \simeq Ext^i_{(A,\{t\})}(M_t, N_t)$$

*for elements $s, t$ in $S$.*

*Proof.* Let $p \in S$. We define a left $A$-module $\mathcal{A}_{*p}$ by $(\mathcal{A}_{*p})_q = A_{qp}$. For a fixed $N$, the functor

$$F^i : M \mapsto Ext^i_A(M, N) \quad (i = 0, 1, \dots)$$

is a $\delta$-functor for compact $\mathcal{A}$-modules. We prove the following two statements.

1. $F^0(\mathcal{A}_{*p}) = N_p$ and

2. $F^i(\mathcal{A}_{*p}) = 0$ for $i \geq 1$.

By the above statements for $S' = \{s\}$, both the functor $F$ for $S$ and $F'$ for $S'$ are universal and identical in degree zero. Therefore by the naturality of universal delta functor, we have the statement of the proposition. To prove the above statements, it is enough to show that

$$0 \to N_p \to C^0(\mathcal{A}_{*p}, N) \to C^1(\mathcal{A}_{*p}, N) \to C^2(\mathcal{A}_{*p}, N) \to \dots \tag{2.3}$$

is exact. We define a map $\alpha : C^{i+1}(\mathcal{A}_{*p}, N) \to C^i(\mathcal{A}_{*p}, N)$. Let $u = (u_{p_0 \cdots p_{i+1}})$ be an element in $C^{i+1}(\mathcal{A}_{*p}, N)$. We define an element $\alpha(u) \in C^i(\mathcal{A}_{*p}, N)$ by

$$\alpha(u)_{q_0 \cdots q_i}(a_1 \otimes \cdots a_i \otimes m) = u_{q_0 \cdots q_i p}(a_1 \otimes \cdots a_i \otimes m \otimes 1).$$

Then, for $u \in C^i(\mathcal{A}_{*p}, N)$ and

$$a_1 \otimes \cdots \otimes a_i \otimes m \in A_{p_0 p_1} \otimes A_{p_1 p_2} \otimes \cdots \otimes A_{p_{i-1} p_i} \otimes N_{p_i},$$

we have

$$\delta\alpha(u) - \alpha\delta(u) = (-1)^i u.$$

Therefore the sequence (2.3) is exact. Thus we have the statement of the proposition. □

## § 2.4. Higher direct images and Hochschild-Serre-Leray quasi-isomorphism

**2.4.1.** In this section, we define relative cohomologies and study their properties.

Let $f : (A, S) \to (B, T)$ be a homomorphism of algebroids, $M$ be a discrete $A$-module and $t \in T$. Let $B_{*t}$ be the left $A$-module $(B_{f(s),t})_s$. We define a complex $(\mathbf{R}f_*M)_t$ to be $\mathbf{R}Hom_A(B_{*t}, M)$. The right $B$ action on $A \otimes \cdots \otimes A \otimes B_{*t}$ induces a left $B$-module structure on $\mathbf{R}f_*M$. As a consequence, we have a left $B$-module structure on $\mathbf{R}^i f_* M = H^i(\mathbf{R}f_*M)$.

**2.4.2.** Let $(A, \{s\}) \xrightarrow{g} (B, \{t\}) \xrightarrow{f} (C, \{u\})$ be morphisms of algebroids and $M$ a discrete $(A, \{s\})$ module. We have the following analog of Hochschild-Serre-Leray quasi-isomorphism.

**Proposition 2.6.**

1. *For a compact vector space $N$, the sequence*

$$K(N): 0 \to Hom(N \otimes C, M) \xrightarrow{\partial} Hom(N \otimes B \otimes C, M) \xrightarrow{\partial} Hom(N \otimes B \otimes B \otimes C, M) \xrightarrow{\partial} Hom(N \otimes B \otimes B \otimes B \otimes C, M)$$

*is exact. Here the differential $\partial$ is given by the following formula.*

$$\begin{aligned}(\partial\varphi)(n \otimes b_1 \otimes \cdots \otimes b_{q+1} \otimes c) =& \varphi(n \otimes b_1 b_2 \otimes \cdots \otimes b_{q+1} \otimes c) \\ &- \varphi(n \otimes b_1 \otimes b_2 b_3 \otimes \cdots \otimes b_{q+1} \otimes c) + \cdots \\ &+ (-1)^q \varphi(n \otimes b_1 \otimes \cdots \otimes f(b_{q+1})c)\end{aligned}$$

*for $\varphi \in Hom(N \otimes B^{\otimes q} \otimes C, M)$ and $n \in N$, $b_i \in B$, $c \in C$.*

2. *Let $M$ be a discrete vector space. The homomorphism*

$$Hom(A^{\otimes k} \otimes C, M) \xrightarrow{\psi} \oplus_{p+q=k} Hom(B^{\otimes p} \otimes C, Hom(A^{\otimes q} \otimes B, M)) \tag{2.4}$$

*is a quasi-isomorphism, here $\psi$ is defeind by*

$$\begin{aligned}Hom(A^{\otimes k} \otimes C, M) \to Hom(C, Hom(A^{\otimes k} \otimes B, M)) = Hom(A^{\otimes k} \otimes B \otimes C, M) \\ \varphi \mapsto \Big(a \otimes b \otimes c \mapsto \varphi(a \otimes f(b)c)\Big) \text{ for } a \in A^{\otimes k}, b \in B, c \in c.\end{aligned}$$

3. *Let $M$ be a discrete $A$-module. There is a canonical quasi-isomorphism*

$$\mathbf{R}(fg)_* M \xrightarrow{\sim} \mathbf{R}f_*(\mathbf{R}g_* M). \tag{2.5}$$

*Proof.* 1. The exactness is reduce to the case where $N = \mathbf{k}$. We use the adjointness for tensor product for compact modules. The null homotopy $\theta$ is given by the formula:

$$(\theta\varphi)(b_1 \otimes \cdots \otimes b_q \otimes c) = \varphi(1 \otimes b_1 \otimes \cdots \otimes b_q \otimes c)$$

2. We apply the statement 1. for $N = \mathbf{k}, A, A^{\otimes 2}, \ldots$ and get the following long exact sequences $K^p$ for $p = 0, 1, \ldots$:

$$\begin{aligned}K^p : & Hom(A^{\otimes p} \otimes C, M) \to Hom(A^{\otimes p} \otimes B \otimes C, M) \\ & \to Hom(A^{\otimes p} \otimes B \otimes B \otimes C, M) \to \cdots\end{aligned}$$

One can check that the cone of (2.4) is quasi-isomorphic to the simple complex associated to the double complex $K^0 \to K^1 \to \cdots$. Therefore it is exact. □

Using Proposition 2.4, we have the following corollary.

**Corollary 2.7.** *For homomorphisms of algebroids* $(A,S) \xrightarrow{g} (B,T) \xrightarrow{f} (C,U)$, *we have the similar quasi-isomorphism (2.5).*

**Corollary 2.8.** *We have a spectral sequence*

$$E_2^{p,q} = \mathbf{R}^p f_*(\mathbf{R}^q g_* M) \Rightarrow \ E_\infty^{p+q} = \mathbf{R}^{p+q}(fg)_*(M).$$

### § 2.5. Semi-simplicial algebroids and semi-simplicial modules

**2.5.1.** Let $A_\bullet = ((A_I, S_I))_I$ be a semi-simplicial algebroid indexed by $I = (i_0 < i_1 < \cdots < i_p)$ of the following type:

$$\cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{i_0<i_1<i_2} A_{i_0 i_1 i_2} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{i_0<i_1} A_{i_0 i_1} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{i_0} A_{i_0}.$$

An $A_\bullet$ *semi-simplicial module* $M_\bullet = (M_I)_I$ is defined by a collection of $A_I$-module $M_I$ endowed with a collection of $A_I$-homomorphism

$$\partial_{M,I,i} : f_{I,\partial_i I}^* M_{\partial_i I} \to M_I,$$

which is functorial for face maps. Here $f_{I,\partial_i I} : A_I \to A_{\partial_i I}$ is induced by the face map.

**2.5.2.** Let $A_\bullet$ be a simplicial algebroid, $B$ be an algebroid and $\psi_i : A_i \to B$ be an augmentation of $A_\bullet$ to $B$. Let $M_\bullet$ be a discrete $A_\bullet$-module and and $N$ be a compact $B$-module. By composing face maps, we get a homomorphism of algebroids $\psi_{i_0\cdots i_p} : A_{i_0\cdots i_p} \to B$. Then we have the complex $\mathbf{R}Hom_{A_\bullet}(\psi^* N, M_\bullet)$ defined as follows

$$\begin{aligned} 0 \to \prod_{i_0} \mathbf{R}Hom_{A_{i_0}}(\psi_{i_0}^* N, M_{i_0}) &\to \prod_{i_0<i_1} \mathbf{R}Hom_{A_{i_0 i_1}}(\psi_{i_0 i_1}^* N, M_{i_0 i_1}) \\ &\to \prod_{i_0<i_1<i_2} \mathbf{R}Hom_{A_{i_0 i_1 i_2}}(\psi_{i_0 i_1 i_2}^* N, M_{i_0 i_1 i_2}) \to \cdots . \end{aligned}$$

## § 3. Category $\mathcal{C}$ and associator

### § 3.1. Category $\mathcal{C}$

**Definition 3.1.** (Category $\mathcal{C}$) We define the abelian category $\mathcal{C}$ as follows. An object $V$ of $\mathcal{C}$ is a triple $(V_{dR}, V_B, c_V)$ consisting of

1. $\mathbf{Q}$-vector space $V_{dR}$,
2. $\mathbf{Q}$-vector space $V_B$, and
3. a $\mathbf{C}$-linear isomorphism $c_V : V_B \otimes \mathbf{C} \simeq V_{dR} \otimes \mathbf{C}$.

The vector space $V_{dR}$ and $V_B$ are called de Rham part and Betti part of $V$, respectively. ¿From now on, we consider locally compact version. Here, $-\otimes \mathbf{C}$ means the completed tensor product. Morphism $f : V \to W$ is a pair of $\mathbf{Q}$-linear maps $f_{dR} : V_{dR} \to W_{dR}$ and $f_B : V_B \to W_B$ compatible with the comparison maps. The category $\mathcal{C}$ becomes a $\mathbf{Q}$-linear tensor category by tensoring each of de Rham and Betti components.

Let $T$ be a set. We define a notion of Hopf algebroid object $\mathcal{A}$ over $T$ in $\mathcal{C}$ as in §2.1. As for the axiom 1., we assume that the identity $1_p$ maps to identity $1_p$ under the comparison map. For the axiom 3, we assume the existence of $x$ and $y$ for each components. We define the notion of $\mathcal{A}$-module as follows.

**Definition 3.2.** Let $M = (M_a)_{a\in T} = (M_{dR,a}, M_{B,a}, c_{M,a})_{a\in T}$ be an object in $\mathcal{C}$ indexed by $a \in T$. The triple $M$ is called an *$\mathcal{A}$-module* if it is equipped with an action

$$\mathcal{A}_{ab} \otimes M_b \to M_a.$$

of $\mathcal{A}$ in $\mathcal{C}$, which is associative and unitary. Here the action of algebroid is given by a morphism in $\mathcal{C}$.

Let $\mathcal{A}, \mathcal{B}$ be algebroid objects (or simply algebroids) in $\mathcal{C}$, $f : \mathcal{A} \to \mathcal{B}$ a homomorphism of algebroids in $\mathcal{C}$, and $M = (M_{dR}, M_B, c_M)$ a discrete $\mathcal{A}$-module. Then $\mathbf{R}^i f_* M = H^i(\mathbf{R} f_* M)$ becomes an object in $\mathcal{C}$ and we have the natural isomorphisms:

$$(\mathbf{R}^i f_* M)_{dR} = \mathbf{R}^i f_{dR*}(M_{dR}), \quad (\mathbf{R}^i f_* M)_B = \mathbf{R}^i f_{B*}(M_B).$$

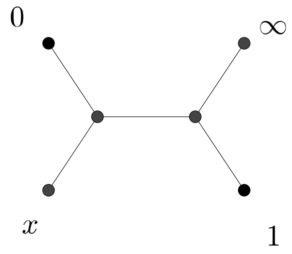
## § 3.2. Fundamental algebroid of moduli spaces

In this subsection, we recall the fact on the structure of Hopf algebroids $\mathcal{A}_{n,dR}, \mathcal{A}_{n,B}$ of the moduli space $\mathcal{M}_n = \mathcal{M}_{0,n}$ of $n$-punctured genus zero curves. As for the fundamental groups and the de Rham fundamental groups, see [OT].

**Definition 3.3.** We define the *set of tangential points* $T_n$ of $n$ points in genus zero curve by the set of planer trivalent tree with $n$ terminals up to mirror. A graph is called trivalent if every vertex is terminal or adjacent to three edges. A graph with a cyclic ordering for edges adjacent to each non-terminal vertex. By giving a numbering of four points as $0, 1, \infty, x$, we have

$$T_4 = \{\overline{01}, \overline{10}, \overline{0\infty}, \overline{\infty 0}, \overline{1\infty}, \overline{\infty 1}\}$$

For example, the tangential point $\overline{01}$ is tangent vector at 0 and its direction is going to 1. It is expressed as the following tree. In this tree 0 and $x$ is very close and 1 and $\infty$ is also very close. The point $x$ lies between 0 and 1.




Thus $\#T_4 = 3 \times 2$, $\#T_5 = 15 \times 4$, etc.

Then we can define the pro-nilpotent $\mathbf{Q}$-algebroids $\mathcal{A}_{n,dR}, \mathcal{A}_{n,B}$ over the set $T_n$ as follows.

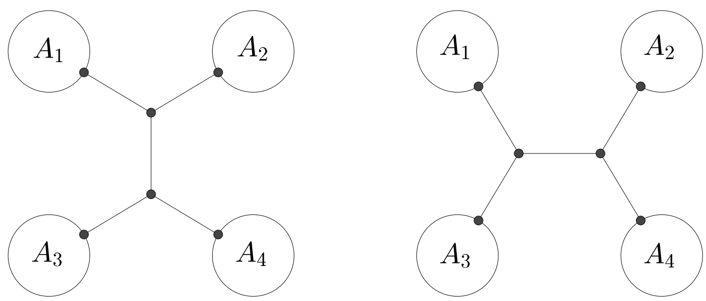
**Definition 3.4.** To define an algebroid $\mathcal{A}_{n,dR}$, we consider a complete associative algebra defined by the following generators and relations. Each component $\mathcal{A}_{n,dR,ab}$ of the algebroid $\mathcal{A}_{n,dR}$ is defined as this algebra. In other words, an algebroid $\mathcal{A}_{n,dR}$ is constant on $a, b \in T_n$.

1. (Generators) $t_{ij}$ with $1 \leq i < j \leq n$. We use the notation $t_{ji} = t_{ij}$ for $i < j$.
2. (Relations)
   (a) $[t_{ij}, t_{kl}] = 0$ for $\#\{i,j,k,l\} = 4$.
   (b) $[t_{ij}, t_{ik} + t_{kj}] = 0$ for $\#\{i,j,k\} = 3$.
   (c) $\sum_{j \neq i} t_{ij} = 0$ for all $i$.

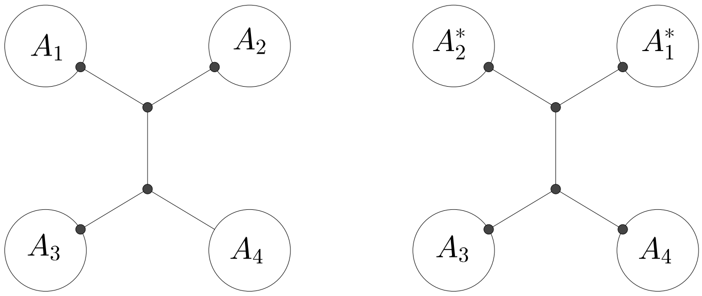
The multiplications are defined by the product structure of the algebra $\mathcal{A}_{n,dR,ab}$. Then $\mathcal{A}_{n,dR}$ is the completed de Rham fundamental group algebra of $\mathcal{M}_n$ and has a standard coproduct $\Delta(t_{ij}) = t_{ij} \otimes 1 + 1 \otimes t_{ij}$.

**Definition 3.5.**

1. Two tangential base points $a, b \in T_n$ are *adjacent* if it can be transformed by an elementary $IH$ change. See the following figure. The shape of the tree in the left hand side looks "I" and that in the right hand side looks "H".



2. Two tangential base points $a, b \in T_n$ are *neighbors* if $a$ is obtained by a half twisting of $b$ with respect to a edge.




Here $A_i^*$ is the reflection of $A_i$.

3. $\mathcal{A}_{n,B} = \{\mathcal{A}_{n,B,ab}\}_{ab}$ is a pro-nilpotent algebroid generated by two type of generators:
   (a) path $p_{ab}$ connecting two adjacent tangential base points.
   (b) small half circle $c_{ab}$ connecting two neighbors.

   Relations on $\mathcal{A}_{n,B}$ are generated by the following relations.

$$p_{\alpha\beta}p_{\beta\alpha} = e \quad (a, b \text{ are adjacent to each other, 2-cycle relations})$$

$$p_{\alpha'\beta}c_{\beta\beta'}p_{\beta'\gamma}c_{\gamma\gamma'}p_{\gamma'\alpha}c_{\alpha\alpha'} = e$$
$$(\alpha'\beta, \beta'\gamma, \gamma'\alpha \text{ are dejacent,}$$
$$\alpha\alpha', \beta\beta', \gamma\gamma' \text{ are neibors, 3-cycle relations})$$

$$p_{\alpha_1\alpha_2}p_{\alpha_2\alpha_3}p_{\alpha_3\alpha_4}p_{\alpha_4\alpha_5}p_{\alpha_5\alpha_1} = e$$
$$(\alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_4, \alpha_4\alpha_5, \alpha_5\alpha_1 \text{ are adjacent, 5-cycle relations})$$

   Then the $\mathcal{A}_{n,B}$ is the completed groupoid algebra of $\mathcal{M}_n$.

**Definition 3.6.** We define the category $M^{inf}$ as follows. The objects of $M^{inf}$ are products $\mathcal{M}_n$ for various $n$ ($n \geq 4$), disc $\Delta$ and punctured disc $\Delta^*$, and the morphisms are generated by infinitesimal inclusions and projections. As for the infinitesimal inclusions, see [De].

**Proposition 3.7.** *By attaching $\mathcal{A}_{n,dR}$ (resp. $\mathcal{A}_{n,B}$) to $\mathcal{M}_n$ and attaching $\mathcal{A}_{\Delta^*,dR} = \mathbf{Q}[[\frac{dx}{x}]]$ (resp. $\mathcal{A}_{\Delta^*,B} = \mathbf{Q}[[\log c]]$) to $\Delta^*$, we have a functor $\mathcal{A}_{dR}$ (resp. $\mathcal{A}_B$) from the category $M^{inf}$ to the category of Hopf algbroids.*

## § 3.3. Functorial characterization of associators

In this subsection, we recall the definition of associator after Drinfeld and its functorial characterization.

**Definition 3.8.** An element $\Phi(e_0, e_1)$ of $\mathbf{Q}\langle\langle e_0, e_1\rangle\rangle$ is called an *associator* if it satisfies the following conditions.

1. $\Phi(e_0+f_0, e_1+f_1) = \Phi(e_0, e_1)\Phi(f_0, f_1)$ in $\mathbf{Q}\langle\langle e_0, e_1\rangle\rangle\langle\langle f_0, f_1\rangle\rangle$. Here $\mathbf{Q}\langle\langle e_0, e_1\rangle\rangle\langle\langle f_0, f_1\rangle\rangle$ is a complete associative algebra generated by $e_0, e_1, f_0, f_1$ with the commutativity relations $[e_i, f_j] = 0$ for $i, j = 0, 1$.
2. $\Phi(e_0, e_1) \equiv 1 \mod (e_0, e_1)^2$.
3. $\Phi(e_0, e_1)\Phi(e_1, e_0) = 1$.
4. $\Phi(e_0, e_1)\mathbf{e}(\frac{e_1}{2})\Phi(e_1, e_\infty)\mathbf{e}(\frac{e_\infty}{2})\Phi(e_\infty, e_0)\mathbf{e}(\frac{e_0}{2}) = 1$.
5. The equality $\Phi(t_{34}, t_{45})\Phi(t_{12}, t_{23})\Phi(t_{45}, t_{51})\Phi(t_{23}, t_{34})\Phi(t_{51}, t_{12}) = 1$ holds in $\mathcal{A}_5$.

The next proposition is a reformulation of the result of [Dr].

**Proposition 3.9.** *The set of functorial isomorphisms $\varphi$ from $\mathcal{A}_B\otimes\mathbf{C}$ to $\mathcal{A}_{dR}\otimes\mathbf{C}$ such that*

1. *the isomorphism $\varphi$ sends the logarithm of the small half circle $\log(c_{ij})$ to $\pi\mathbf{i}t_{ij}$, and*
2. *the abelianization of $\varphi$ sends $[0,1]^{ab}$ to $1$*

*is identified with the set of associators. The one-to-one correspondence is given by*

$$\mathcal{A}_{4,B,\overline{01},\overline{10}} \ni [0,1] \mapsto \Phi \in \mathcal{A}_{dR,4} = \mathbf{C}\langle\langle e_0, e_1\rangle\rangle.$$

*Here $e_0$ and $e_1$ are the dual basis of $\omega_0 = \dfrac{dx}{x}$ and $\omega_1 = \dfrac{dx}{x-1}$, respectively.*

By the proposition above, for a given associator $\Phi$, we have an isomorphism of Hopf algebroids

$$c_{\Phi,n} : \mathcal{A}_{n,B} \otimes \mathbf{C} \xrightarrow{\simeq} \mathcal{A}_{n,dR} \otimes \mathbf{C}.$$

This isomorphism gives a Hopf algebroid object $\mathcal{A}_n = (\mathcal{A}_{n,dR}, \mathcal{A}_{n,B}, c_{\Phi,n})$ in $\mathcal{C}$. The isomorphism $c_{\Phi,n}$ is called the $\Phi$-*comparison map.*

## § 4. Examples of $\mathcal{A}$-modules

### § 4.1. Choice of coordinates and homomorphism of algebroids

Let $C$ be a genus zero curve and $P = (C, p_1, \dots, p_n)$ $(p_i \in C)$ an element in $\mathcal{M}_n$. We choose a coordinate $t$ of $C$ such that $t(p_{n-2}) = 0, t(p_{n-1}) = 1, t(p_n) = \infty$. Using the coordinate $t$, $\mathcal{M}_n$ is identified with an open set of $\mathbf{A}^{n-3}$ defined by

$$\{(x_1, \dots, x_{n-3}) \mid \ x_i \neq x_j \text{ for } i \neq j, x_i \neq 0, 1 \text{ for all } i\}$$

by setting $x_k = t(p_k)$. This coordinate is called the *distinguished coordinate*. By taking the distinguished coordinate of $\mathcal{M}_4$, it is identified with $\mathbf{P}^1 - \{0, 1, \infty\}$.

**Definition 4.1** (admissible function, admissible differential form)**.**

1. Let $S = (i, j, k, l)$ be an ordered subset of distinct elements in $[1, n]$. Let $P = (C, p_1, \ldots, p_n)$ be an element of $\mathcal{M}_n$. There is a unique coordinate $t$ of $C$ such that $t(p_i) = 0, t(p_j) = 1, t(p_k) = \infty$. The value $t(p_l)$ at $p_l$ gives rise to an algebraic function on $\mathcal{M}_n$, which is denoted by $\varphi_S$. A function on $\mathcal{M}_n$ of this form is called an *admissible function* on $\mathcal{M}_n$. The set of admissible functions is denoted by $Ad(\mathcal{M}_n)$.
2. Let $x_1, \ldots, x_{n-3}$ be the distinguished coordinate. An element in the linear span of $\frac{dx_i}{x_i}, \frac{dx_i}{x_i - 1}, \frac{d(x_i - x_j)}{x_i - x_j}$ is called an *admissible differential form.*

*Remark* 3.

1. $\varphi \in Ad(\mathcal{M}_n)$ defines a morphism $\mathcal{M}_n \to \mathcal{M}_4$ and a homomorphism of algebroids $\mathcal{A}_n \to \mathcal{A}_4$.
2. If $S \cap \{n-2, n-1, n\} = \emptyset$, using the distinguished coordinates of $\mathcal{M}_n$, we have
$$\varphi_S(P) = \frac{(x_l - x_i)(x_j - x_k)}{(x_l - x_k)(x_j - x_i)}.$$
Therefore $\varphi_S$ is invariant under the substitutions $i \leftrightarrow l, j \leftrightarrow k$ and $i \leftrightarrow j, k \leftrightarrow l$.
3. The following functions are admissible functions.
$$\frac{x_i}{x_j} = \frac{(x_i - 0)(x_j - \infty)}{(x_j - 0)(x_i - \infty)},\ 1 - \frac{x_i}{x_j} = \frac{(x_j - x_i)(\infty - 0)}{(x_j - 0)(\infty - x_j)},\ 1 - x_i = \frac{(1 - x_i)(\infty - 0)}{(1 - 0)(\infty - x_i)}$$

Since the fundmental groups and de Rham fundamental groups are functorial on morphisms in $M^{\mathrm{inf}}$ and products, we have the following theorem.

**Proposition 4.2.** *Let $\Phi$ be an associator.*

1. *Let $3 \leq m < n$ be integers and $f$ morphism defined by*
$$f : \mathcal{M}_n \to \mathcal{M}_m : (x_1, \ldots, x_{n-3}) \to (x_1, \ldots, x_{m-3}).$$
*Then for $\star = dR, B$, the induced maps of algebroids $\mathcal{A}_{n,\star} \to \mathcal{A}_{m,\star}$ are compatible with the $\Phi$-comparison maps.*
2. *Let $3 \leq m, n$ be integers. Then a morphism*
$$f : \mathcal{M}_{n+m-3} \to \mathcal{M}_n \times \mathcal{M}_m$$
$$(x_1, \ldots, x_{n-3}, y_1, \ldots, y_{m-3}) \mapsto (x_1, \ldots, x_{n-3}) \times (y_1, \ldots, y_{m-3})$$
*induces a homomorphism of algebroids $f : \mathcal{A}_{n+m-3} \to \mathcal{A}_n \boxtimes \mathcal{A}_m$ in $\mathcal{C}$.*

*3. Let $3 \leq m < n_1, n_2$ be integers. Then the natural morphism*

$$f : \mathcal{M}_{n_1} \times_{\mathcal{M}_m} \mathcal{M}_{n_2} \to \mathcal{M}_{n_1} \times \mathcal{M}_{n_2}$$

*induces a homomorphism of algebroids in $\mathcal{C}$.*

## § 4.2. Fundamental examples of $\mathcal{A}$-modules

**4.2.1.** Let $4 \leq m < n$ and $f : \mathcal{M}_n \to \mathcal{M}_m$ be the map defined by $(x_1, \ldots, x_{n-3}) \mapsto (x_1, \cdots, x_{m-3})$. Then we have an algebroid homomorphism $f : \mathcal{A}_n \to \mathcal{A}_m$.

**4.2.2.** By taking an abelianization $\mathcal{A}_n^{ab}$ of $\mathcal{A}_n$, we have a homomorphism of Hopf algebroids $\mathcal{A}_n \to \mathcal{A}_n^{ab}$. By choosing a base point $p \in T_n$, we have have an $\mathcal{A}_n$-module $\mathcal{A}_{n,p*}^{ab}$. In particular, by using the distinguished coordinate $x$, $\mathcal{A}_4$-module $x^\alpha \mathbf{Q}[[a]]$ is defined by taking the base point as $\overline{01}$,

**4.2.3.** Let $\varphi$ be an admissible function on $\mathcal{M}_n$ and $\alpha$ formal parameter. The morphism $\mathcal{M}_n \to \mathcal{M}_4$ induced by $\varphi$ is also denoted by $\varphi$ and $x$ be the distinguished coordinate of $\mathcal{A}_4$. We define the $\mathcal{A}_n[[\alpha]]$-module $\varphi^\alpha \mathbf{Q}[[\alpha]]$ as the pull back $\varphi^*(x^\alpha \mathbf{Q}[[\alpha]])$ of $x^\alpha \mathbf{Q}[[\alpha]]$. It is compact and $x^\alpha \big(\mathbf{Q}[[\alpha]]/\alpha^k \mathbf{Q}[[\alpha]]\big)$ is discrete. We define

$$\left(\prod_{i=1}^{m} \varphi^{\alpha_i}\right)\mathbf{Q}[[\alpha_1, \ldots, \alpha_m]] = \varphi_1^{\alpha_1}\mathbf{Q}[[\alpha_1]] \otimes \cdots \otimes \varphi_m^{\alpha_m}\mathbf{Q}[[\alpha_m]]$$

**Proposition 4.3.** *Let $\varphi_i$, $(i = 1, \ldots, m)$, $\psi_j$, $(j = 1, \ldots, l)$ be admissible functions on $\mathcal{M}_n$ and $a_{ij} \in \mathbf{Z}$. We assume that $\psi_j = \prod_{i=1}^m \varphi_i^{a_{ij}}$, and set $L_i = \sum_i^m a_{ij}\alpha_j$ for $i = 1, \ldots, m$. Then we have an isomorphism*

$$\left(\prod_{i=1}^{m} \varphi_i^{L_i}\right)\mathbf{Q}[[\alpha_i]] = \left(\prod_{j=1}^{l} \psi_j^{\alpha_j}\right)\mathbf{Q}[[\alpha_i]].$$

*as $\mathcal{A}_n$-modules.*

*Remark* 4. In the example $\varphi^\alpha \mathbf{Q}[[\alpha]]$ in 4.2.3, the action of $\mathbf{Q}[[\alpha]]$ by multiplication commutes with the action of $\mathcal{A}_4$.

## § 4.3. Φ-cohomology, semi-simplicial sheaf and compact support cohomology

**4.3.1.** Let $M$ be a discrete $\mathcal{A}_n$-module and $f$ the map $\mathcal{M}_n \to \mathcal{M}_3 = pt$, the higher direct image $\mathbf{R}f_*M$ is denoted by $\mathbf{R}\Gamma(\mathcal{M}_n, M)$ and $\mathbf{R}^i f_* M$ is denoted by $H^i_\Phi(\mathcal{M}_n, M)$. It is called the Φ-*cohomology with coefficients in $M$*.

**4.3.2.** Let $X_\bullet = (X_I)_I$ be a semi-simplicial object in $M^{inf}$ indexed by $I = (i_0 < i_1 < \cdots < i_p)$ of the following type:

$$\cdots \begin{smallmatrix}\to\\\to\\\to\\\to\end{smallmatrix} \coprod_{i_0<i_1<i_2} X_{i_0 i_1 i_2} \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} \coprod_{i_0<i_1} X_{i_0 i_1} \rightrightarrows \coprod_{i_0} X_{i_0}$$

Let $Y$ be an object in $M^{inf}$ and $\varphi_i : X_i \to Y$ be an augmentation of $X_\bullet$ to $Y$. Let $M_\bullet$ be a discrete $\mathcal{A}(X_\bullet)$-module and $N$ a compact $\mathcal{A}(Y)$-module. Then we have the complex $\mathbf{R}Hom_{\mathcal{A}(X_\bullet)}(\psi^* N, M_\bullet)$ defined in 2.5.2. If $Y = pt$ and $N = \mathbf{Q}$, the complex $\mathbf{R}Hom(\psi^* N, M)$ is denoted by $\mathbf{R}\Gamma(X_\bullet, M_\bullet)$. The $i$-th cohomologies of $\mathbf{R}Hom_{\mathcal{A}(X_\bullet)}(\psi^* N, M_\bullet)$ and $\mathbf{R}\Gamma(X_\bullet, M_\bullet)$ are denoted by $Ext^i_{\mathcal{A}(X_\bullet)}(\psi^* N, M_\bullet)$ and $H^i_\Phi(X_\bullet, M_\bullet)$.

**4.3.3.** Using the following diagram of sheaves and algebroids, we define the cohomology with partial compact support on $\mathcal{M}_4$ for a discrete $\mathcal{A}_4$-module $\mathcal{F}$.

$$(\mathcal{A}_{\Delta^*}, j_0^*\mathcal{F}) \xrightarrow{j_0} (\mathcal{A}_4, \mathcal{F}) \xleftarrow{j_1} (\mathcal{A}_{\Delta^*}, j_1^*\mathcal{F}),$$

Let $j : \mathcal{M}_4 \to \overline{\mathcal{M}_4} - \{\infty\}$ be the open immersion. We define $H^*_\Phi(\overline{\mathcal{M}_4} - \{\infty\}, j_! \mathcal{F})$ by the cohomology of the cone of the following homomorphism of complexes:

$$\mathbf{R}\Gamma(\mathcal{M}_4, \mathcal{F}) \to \mathbf{R}\Gamma(\Delta_0^*, j_0^*\mathcal{F}) \oplus \mathbf{R}\Gamma(\Delta_1^*, j_1^*\mathcal{F})$$

## § 5. Comparison to classical de Rham theory and Betti theory

### § 5.1. Φ-integral

**5.1.1.** Let $M$ be an artinian $\mathcal{A}_n$-module. We have an isomorphism

$$H^i_\Phi(\mathcal{M}_n, M)_{dR} \simeq H^i_{dR}(\mathcal{M}_n, M_{dR})$$

and

$$H^i_\Phi(\mathcal{M}_n, M)_B \simeq H^i_B(\mathcal{M}_n, M_B).$$

We set $M^* = Hom(M, \mathbf{Q})$. Then the space $M^*$ is equipped with an $\mathcal{A}_n$-module structure via the antipodal. Let $C_*(\mathcal{M}_n, M^*)$ be the topological dual complex of $\mathbf{R}\Gamma(\mathcal{M}_n, M)$. Then it is isomorphic to

$$\cdots \to \bigoplus_{x,y,z} \mathcal{A}_{n,xy} \otimes \mathcal{A}_{n,yz} \otimes M^*_{k,z} \to \bigoplus_{x,y} \mathcal{A}_{n,xy} \otimes M^*_{k,y} \to \bigoplus_{x} M^*_{k,x} \to 0. \tag{5.1}$$

Its homology is denoted by $H^B_i(\mathcal{M}_n, M^*)$ and is identified with the homology group of chain complex with the coefficient in $M^*$: An element $\sigma$ of the chain complex is a linear combination of $[\gamma, f]$ where $\gamma$ is an $i$-chain in $\mathcal{M}_n$ and $f$ is a section of $M^*$ on $\gamma$.

**5.1.2. Homology and cohomology over a formal power series ring** We set $R_k = \mathbf{Q}[[\alpha]]/(\alpha)^{k+1}$. Let $M_k$ be an $\mathcal{A}_n$-module equipped with $R_k$-module structure, such that each fibers are free $R_k$-module of finite rank. We set $M_k^* = Hom_{R_k}(M_k, \mathbf{Q})$. Then the complex $\mathbf{R}\Gamma(\mathcal{M}_n, M_k)$ is a complex of discrete $R_k$-modules. Let $C_*(\mathcal{M}_n, M_k^*)$ be the $R_k$-dual complex of $\mathbf{R}\Gamma(\mathcal{M}_n, M_k)$. Then $C(\mathcal{M}_n, M_k^*)$ is a complex of compact $R_k$-modules. This complex is expressed as (5.1), where the action of $\mathcal{A}_n$ on $M_k^*$ is obtained by using antipodal. The $i$-th homology of $C(\mathcal{M}_n, M_k^*)$ is denoted by $H_i(\mathcal{M}_n, M_k^*)$. Then we have the natural pairing

$$H_i(\mathcal{M}_n, M_k^*) \otimes H^i(\mathcal{M}_n, M_k) \to \mathbf{C}[[\alpha]]/(\alpha)^{k+1}.$$

If $M_k$ is obtained by the quotient $M/\alpha^{k+1}M$ of a free $\mathbf{Q}[[\alpha]]$-module $M$, we can take the projective limit, and get a homomorphism

$$H_i(\mathcal{M}_n, M^*) \otimes H^i(\mathcal{M}_n, M) \to \mathbf{C}[[\alpha]].$$

Using the comparison map, we have the following pairing.

$$H_i^B(\mathcal{M}_n, M_B^*) \otimes H_{dR}^i(\mathcal{M}_n, M_{dR}) \to \mathbf{C}[[\alpha]]. \tag{5.2}$$

**Definition 5.1** (Φ-integral, twisted chain)**.** Let $\Phi$ be an associator.

1. Let $\sigma = [\gamma, f] \in H_i^B(\mathcal{M}_n, M_B^*)$ and $\omega \in H_{dR}^i(\mathcal{M}_n, M_{dR})$. We define a $\mathbf{Q}[[\alpha]]$-*valued* Φ-*integral* by the pairing
$$\int_\gamma^\Phi f\omega = (\sigma, \omega) \in \mathbf{C}[[\alpha]],$$
where the pairing is defined in (5.2).

2. Let $\varphi_i$ $(i = 1, \dots, l)$ be admissible functions on $\mathcal{M}_n$, $D$ a domain in $\mathcal{M}_n(\mathbf{R})$ defined by $0 \le x_1 \le x_2 \le \cdots \le x_{n-3} \le 1$ for some distinguished coordinates $x_1, \dots, x_{n-3}$. Assume that the values of $\varphi_i$ are positive and real on $D$. The twisted chain on $D$ with the product of positive real branches of $\varphi_i^{\alpha_i}$ is denoted by $\prod_{i=1}^l \varphi_D^{\alpha_i}$.

## § 5.2. Cohomology of nearby fibers and higher direct image

**5.2.1. Higher direct images and nearby cohomology** Let $\mathcal{M}_n \to \mathcal{M}_m$ be a morphism appeared in §4.2.1. We give a method to compute the higher direct image of $f : \mathcal{M}_n \to \mathcal{M}_m$ for $dR$ and $B$. Let $f : T_n \to T_m$ be the corresponding map for infinitesimal points, and $y$ an element of $T_m$. We set $T_{n,m}(y) = f^{-1}(y)$.

**Definition 5.2.** Let $\mathcal{A}_{n,m,B,y}$ (resp. $\mathcal{A}_{n,m,dR,y}$) be the subalgebroid of $\mathcal{A}_{n,B}$, (resp. $\mathcal{A}_{n,dR}$) generated by the images of $\mathcal{A}_{4,B}$ (resp. $\mathcal{A}_{4,dR}$) induced by infinitesimal inclusions of $\mathcal{M}_4 \to \mathcal{M}_n$ contained in the fiber of $y$. Then the image of $\mathcal{A}_{n,m,B,y} \otimes \mathbf{C}$ is equal to $\mathcal{A}_{n,m,dR,y} \otimes \mathbf{C}$. Therefore $\mathcal{A}_{n,m,B,y}$ and $\mathcal{A}_{n,m,dR,y}$ defines a Hopf algebroid object in $\mathcal{C}$ on $T_{n,m}(y)$, which is denoted by $\mathcal{A}_{n,m,y}$ For $x \in T_{n,m}(y)$, $\mathcal{A}_{n,m,y,xx}$ is denoted by $\mathcal{A}_{n,m,x}$.

*Remark* 5. The $B$-part $\mathcal{A}_{n,m,B}$ can be interpreted as follows. Let $N_{n,m}$ be the kernel of $\pi_1^B(\mathcal{M}_n) \to \pi_1^B(\mathcal{M}_m)$. Then $N_{n,m}$ becomes a fibered groupoid over the map $T_n \to T_m$. We can easily see that $\mathcal{A}_{n,m}$ is the nilpotent completion of $N_{n,m}$.

**Proposition 5.3.** *We choose $x \in T_n, y \in T_m$ such that $f(x) = y$. We have the following exact sequence:*

$$0 \leftarrow \mathcal{A}_{m,y} \leftarrow \mathcal{A}_{n,x} \xleftarrow{d_0} \mathcal{A}_{n,x} \otimes \mathcal{A}_{n,m,x} \xleftarrow{d_1} \mathcal{A}_{n,x} \otimes \mathcal{A}_{n,m,x} \otimes \mathcal{A}_{n,m,x} \leftarrow \cdots$$

*Here $d_0(x \otimes y) = xy - x\epsilon(y)$, $d_1(x \otimes y \otimes z) = xy \otimes z - x \otimes yz + x \otimes y\epsilon(z)$, $\dots$, where $\epsilon : \mathcal{A}_{n,m,x} \to \mathbf{Q}$ is the augmentation. This becomes a free $\mathcal{A}_{n,x}$-resolution of $\mathcal{A}_{m,y}$.*

*Proof.* We reduce the proposition to the $B$-part. Let $f : G \to H$ be a surjective homomorphism of group and $N$ be the kernel of $f$. We prove that the sequence

$$0 \leftarrow \mathbf{Q}[H] \leftarrow \mathbf{Q}[G] \xleftarrow{d_0} \mathbf{Q}[G \times N] \xleftarrow{d_1} \mathbf{Q}[G \times N^2] \leftarrow \cdots \tag{5.3}$$

is exact. We choose a set theoretic section $s : H \to G$. Then

$$\begin{aligned}
&\theta_0 : \mathbf{Q}[H] \to \mathbf{Q}[G] : h \to s(h) \\
&\theta_1 : \mathbf{Q}[G] \to \mathbf{Q}[G \times N] : g \to g \otimes g^{-1}s(g) \\
&\theta_2 : \mathbf{Q}[G \times N] \to \mathbf{Q}[G \times N^2] : g \otimes n \to g \otimes n \otimes n^{-1}g^{-1}s(ng) \\
&\cdots
\end{aligned}$$

gives a null homotopy. Therefore the sequence (5.3) is an exact sequence. By taking a nilpotent completion, we have the proposition for the $B$-part. □

**Corollary 5.4.** *The complex $\mathbf{R}f_*M_y$ is quasi-isomorphic to the complex*

$$Hom(\mathbf{Q}, M_x) \xrightarrow{d_0} Hom(\mathcal{A}_{n,m,x}, M_x) \xrightarrow{d_1} Hom(\mathcal{A}_{n,m,x} \otimes \mathcal{A}_{n,m,x} M_x) \to \cdots$$

*where $f(x) = y$. For example $d_0(\varphi)(a) = a\varphi(1) - \varphi(\epsilon(a))$. The action of $\mathcal{A}_{m,B}$ on $\mathbf{R}f_*M_B$ is given by the monodromy action.*

## § 5.3. De Rham cohomologies and Gauss-Manin connection

**5.3.1. Comparison to de Rham complexes** We show that the $dR$-part is equal to the Gauss-Manin connection with coefficients in $M_{dR}$. The action of $t_{ij}$ yields a nilpotent endomorphism $E_{ij}$ on $M_{dR}$. Then the connection $(M_{dR}, \nabla_M)$ on $\mathcal{M}_n$ defined by

$$\nabla_M(m) = \sum_{\substack{1 \le i < j \le n-1 \\ (i,j) \ne (n-2,n-1)}} \frac{(dx_i - dx_j)E_{ij}}{x_i - x_j} m$$

is integrable.

**Proposition 5.5.** *Let $M_{dR}$ be a discrete $\mathcal{A}_{n,dR}$-module and $f : \mathcal{M}_n \to \mathcal{M}_m$ be the map given in Proposition 4.2, 1.*

1. *As a vector space $\mathbf{R}f_* M_{dR}$ is quasi-isomorphic to the following relative de Rham complexes:*

$$\mathbf{R}f'_* M_{dR} : M_{dR} \xrightarrow{\nabla} M_{dR} \otimes \Omega^1_{n/m} \xrightarrow{\nabla} M_{dR} \otimes \Omega^2_{n/m} \to \dots$$

   *Here $\Omega^\bullet_{n/m}$ is a subcomplex of the relative de Rham complex $\Omega^\bullet_{\mathcal{M}_{0,n}/\mathcal{M}_{0,m}}$ generated by $\dfrac{d(x_i - x_j)}{x_i - x_j}$ with $i, j > m - 3$.*

2. *if $M$ is finite dimensional, then $\mathbf{R}^i f_* M_{dR}$ is also finite dimensional, and*
3. *$\mathbf{R}^i f_* M_{dR} = 0$ if $i > n - m$.*

*Proof.* Since the action of $\langle E_{ij} \rangle$ are nilpotent, we can show that $\mathbf{R}f'_* M_{dR}$ is quasi-isomorphism to $\mathbf{R}f_* M_{dR}$ by the induction of the length of nilpotent filtrations. □

To give an explicit quasi-isomorphism, it is convenient to introduce the bar complex. Let $\overline{B_n}$ be the reduced bar complex of the DGA of logarithmic differentials $\Omega^\bullet_n$. (For the definition of simplicial bar complex, see [T] §5.) Then the topological dual of $\mathcal{A}_n$ is isomorphic $B_n = H^0(\overline{B_n}) \subset \overline{B_n}$ and $H^i(B_n) = 0$ for $i \ne 0$. Then $B_n$ becomes a Hopf algebra and the $\mathcal{A}_n$-action on $M_{dR}$ yields a right $B_n$-comodule structure on $M_{dR}$. By the definition (2.1), and Proposition 2.4, $\mathbf{R}f_* M_{dR}$ is quasi-isomorphic to

$$0 \to M_{dR} \otimes B_m \xrightarrow{d_0} M_{dR} \otimes B_n \otimes B_m \xrightarrow{d_1} M_{dR} \otimes B_n \otimes B_n \otimes B_m \to \dots.$$

For example $d_0, d_1$ is given by the formula:

$$\begin{aligned} d_0(m \otimes a) &= \Delta_M(m) \otimes a - m \otimes \Delta_m(a) \\ d_1(m \otimes b \otimes a) &= \Delta_M(m) \otimes b \otimes a - m \otimes \Delta(b) \otimes a + m \otimes b \otimes \Delta_m(a). \end{aligned}$$

Here $\Delta_m : B_m \to B_n \otimes B_m$, $\Delta : B_n \to B_n \otimes B_n$ and $\Delta_M : M_{dR} \to M_{dR} \otimes B_n$ are the coproducts.

**Proposition 5.6.**

1. *Let* $\psi^k : M_{dR} \otimes B_n^{\otimes k} \otimes B_m \to M_{dR} \otimes \Omega^k_{n/m}$ *be a map defined by*

$$m \otimes a_1 \otimes \cdots \otimes a_k \otimes b \mapsto m \otimes \pi(a_1) \cdots \pi(a_k)\epsilon(b),$$

*where* $\epsilon : B_m \to \mathbf{Q}$ *is the augmentation, and* $\pi : B \to \Omega^1_{n/m}$ *is the composite of the projection to* $\Omega^\bullet_{n/m}$ *and the projection to* $\Omega^1_{n/m}$.

*Then*

$$\sum_k \psi^k : M_{dR} \otimes B_n^{\otimes \bullet} \otimes B_m \to M_{dR} \otimes \Omega^\bullet_{n/m}$$

*is a homomorphism of complex and quasi-isomorphism.*

2. *The coaction* $\Delta_M$ *of* $B_{m,dR}$ *on* $\mathbf{R}^i f_* M_{dR}$ *is equal to the exponential of the Gauss-Manin connection defined by*

$$\nabla^{(1)} = \nabla, \quad \nabla^{(i)} = (\nabla \otimes id) \circ \nabla^{(i-1)} \in M \otimes \Omega^{\otimes i}_{n/m}, \quad \Delta_M(m) = \sum_{i=0}^{\infty} \nabla^{(i)}(m).$$

*Proof.* 1. We can show the map $\Psi$ is a map of complex using the fact $B_m$ is graded by the bar length. □

## § 5.4. Framing and $\mathcal{A}$-actions

Let $M$ be a discrete $\mathcal{A}_4$-module and $c_M : M_B \to M_{dR}$ be the comparison map of $M$.

**Definition 5.7.**

1. Let $y \in T_4$. A *framing* of $M$ at $y$ is a pair of homomorphisms $\alpha : \mathbf{Q} \to M_{B,y}$ and $\beta : M_{dR} \to \mathbf{Q}$.
2. Let $f = (\alpha, \beta)$ be a framing of $M$ at $y$, and $\gamma$ be an element in $\mathcal{A}_{4,B,yz}$. The *value* $f(\gamma)$ *of* $f$ *at* $\gamma$ is defined by $\beta \circ c_M \circ \gamma \circ \alpha \in \mathbf{C}$

Let $f = (\alpha, \beta)$ be a framing of $M$ at $\overline{01}$. The $dR$-part $M_{dR}$ of $M$ is a $\mathcal{A}_{4,dR} \simeq \mathbf{C}\langle\langle e_0, e_1\rangle\rangle$-module. Let $E_0, E_1$ be actions of $e_0$ and $e_1$ on $M_{dR}$. The action of $\varphi = \varphi(e_0, e_1) \in \mathbf{C}\langle\langle e_0, e_1\rangle\rangle$ on $M_{dR}$ is denoted by $\varphi(E_0, E_1)$. Since the actions of $\mathcal{A}_{4,B}$ and $\mathcal{A}_{4,dR}$ on $M_B$ and $M_{dR}$ are compatible via the comparison map, using the associator $\Phi$, we have

$$f([0,1]) = \beta c_M [0,1] \alpha = \beta c_{\mathcal{A}_4}([0,1]) c_M \alpha = \beta \Phi(E_0, E_1) c_M \alpha \in \mathbf{Q}[[\alpha_i]].$$

of $M_{dR}$.

## § 6. Iterated Φ-integral and coefficients of the associator

### § 6.1. Iterated Φ-integral

Let $\Phi$ be an associator and $\omega_0 = \frac{dx}{x}, \omega_1 = \frac{dx}{x-1}$ be one forms on $\mathcal{M}_4$. In this subsection, we define the iterated Φ-integral $\int_{[0,1]}^{\Phi} \omega_{i_1} \dots \omega_{i_k}$ for $i_1, \dots, i_k \in \{0, 1\}$. We can check the following lemma. We set $X = \mathcal{M}_4$.

**Lemma 6.1.** *Let $0 \le m < n$ be integers. Let $f : [1, n] \to [1, m]$ be a weakly increasing surjective map. Then $f$ defines a partition of $[1, n]$ into numbered $m$-subset. According to this partition, we have a map $\Delta_f : X^m \to X^n$. Then $X^m$ $(m \le n)$ can be expressed as a simplicial objects in $M^{inf}$ such that the morphisms $\Delta_f$ are compatible with these simplicial structures.*

We set $X_0^{m+2} = \{(p, x_1, \dots, x_m, q) \in X^{m+2} \mid p \ne q\}$. Using the lemma above, we have the following co-semi-simplicial variety:

$$\coprod_{I_n} X_0^{n+2} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \coprod_{I_{n-1}} X_0^{n+1} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \coprod_{I_1} X_0^{3} \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \coprod_{I_0} X_0^{2} \tag{6.1}$$

where $I_m$ is the set of weakly increasing surjective map from $[1, n+2]$ to $[1, m+2]$. We define a map $\pi_m : X_0^{m+2} \to X_0^2 = \mathcal{M}_5$ by $(p, x_1, \dots, x_m, q) \mapsto (p, q)$. Then the above co-semi-simplicial variety is compatible with the map $\pi_m$. Therefore we have the following associated double complex:

$$0 \to \mathbf{R}\pi_{n*}\mathbf{Q} \to \bigoplus_{I_{n-1}} \mathbf{R}\pi_{n-1*}\mathbf{Q} \to \cdots \to \bigoplus_{I_1} \mathbf{R}\pi_{1*}\mathbf{Q} \to \bigoplus_{I_0} \mathbf{R}\pi_{0*}\mathbf{Q} \to 0 \tag{6.2}$$

We set $U = X_0^{n+2} - \cup_{i=0}^{n}\{x_i = x_{i+1}\}$, where $p = x_0, \dots, x_{n+1} = q$ is the coordinate of $X_0^{n+2}$ and $j : U \to X_0^{n+2}$ is the natural inclusion.

**Definition 6.2.** The $k$-th cohomology of the total complex of (6.2) is denoted by $\mathbf{R}^k\pi_* j_! \mathbf{Q}$. It is an $\mathcal{A}_5$-module.

For $(i_1, \dots, i_m) \in \{0, 1\}^m$, we set

$$V_m = \bigoplus_{(i_1, \dots, i_m) \in \{0,1\}^m} \omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_m} \mathbf{Q}.$$

Then we have the following proposition. (See also [DG].)

**Proposition 6.3.** *Under the above notations, we have*

$$(\mathbf{R}^k\pi_* j_! \mathbf{Q})_{dR,x} \simeq \begin{cases} \bigoplus_{m=0}^{n} V_m & (k = n) \\ 0 & (k \ne n) \end{cases}$$

*for any* $x \in T_2$. *Moreover, the Gauss-Manin connection*

$$\nabla : V_m \to \langle\langle \frac{dp}{p}, \frac{dp}{p-1} \rangle\rangle V_{m-1} \oplus V_{m-1} \langle\langle \frac{dq}{q}, \frac{dq}{q-1} \rangle\rangle$$

*on this space is given by*

$$\nabla(\omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_m}) = \omega_{i_1}(\omega_{i_2} \boxtimes \cdots \boxtimes \omega_{i_m}) - (\omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_{m-1}})\omega_{i_m}.$$

As for Betti-part, we have the similar proposition.

**Proposition 6.4.** *Let* $(x, y) \in T_4 \times T_4$. *The fiber of the Betti-part of* $\mathbf{R}^n \pi_* j_! \mathbf{Q}$ *at* $(x, y)$ *is canonically isomorphic to the dual vector space of* $\mathbf{Q}[\pi_1(\mathcal{M}_4)_{xy}]/I^{n+1}$, *where* $I$ *is the augmentation ideal. Under this isomorphism, the path* $[0,1]$ *in* $\pi_1(\mathcal{M}_4)_{\vec{01},\vec{10}}$ *corresponds to the element* $\Delta_n = \{0 \le x_n \le \cdots \le x_1 \le 1\}$ *in the dual vector space of* $(\mathbf{R}^n \pi_* j_! \mathbf{Q})_{B,(\vec{01},\vec{10})}$.

We define the iterated integral as follows.

**Definition 6.5.** We define $\Phi$-*iterated integral* $\int_{[0,1]}^{\Phi} \omega_{i_1} \cdots \omega_{i_n}$ by the pairing

$$([0,1], \omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_n})$$

obtained by the comparison map in $\mathbf{R}^n \pi_* j_! \mathbf{Q}$.

We can state the relation between iterated integral and the coefficient of the associator.

**Theorem 6.1.** *The coefficient* $c_{\Phi,I}$ *of* $e_{i_1} e_{i_2} \dots e_{i_k}$ *in* $\Phi(e_0, e_1) = c_{\Phi,4}([0,1])$ *is equal to* $\int_{[0,1]}^{\Phi} \omega_{i_1} \dots \omega_{i_k}$.

*Proof.* Let $\gamma : [0,1] \to \mathcal{M}_4$ be a path such that $\gamma(0) = x, \gamma(1) = y$. We set $U \cap \pi_n^{-1}(x,y) = U_{xy}$ Then $\gamma$ corresponds to the relative cycle $\widetilde{\gamma}$ in $H_n(X^n; X^n - U_{xy})$ defined by $\widetilde{\gamma} = (\gamma(x_1), \dots, \gamma(x_n)) : \Delta_n \to X^n$. For $x = y$, the constant path $c_x$ corresponds to $\widetilde{c_x}$. By the functoriality of this construction, the constant path is in the image of $\pi_1(pt)_{xx} \to \pi_1(X)_{xx}$. Therefore, we have

$$(\widetilde{c_x}, \omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_m}) = \begin{cases} 1 & (m = 0) \\ 0 & (m \ne 0) \end{cases} \tag{6.3}$$

We consider the infinitesimal inclusion

$$i : \mathcal{M}_4 \to \mathcal{M}_5 : y \mapsto (\vec{01}, y).$$

The dual $\mathcal{A}_4$-module of the restriction of $\mathbf{R}^n\pi_*j_!\mathbf{Q}$ is denoted by

$$\Pi^{(n)}_{\vec{01},*} := (i^*\mathbf{R}^n\pi_*j_!\mathbf{Q})^*.$$

Then we have the Betti framing $\alpha : \mathbf{Q} \to (\Pi^{(n)}_{\vec{01},\vec{01}})_B$ at $\vec{01}$ corresponding to the constant path $c_{\vec{01}}$. On the other hand, we have

$$(\Pi^{(n)}_{\vec{01},\vec{01}})_{dR} \simeq \mathbf{Q}\langle\langle e_0, e_1\rangle\rangle/(e_0, e_1)^{n+1}$$

as $\mathcal{A}_{4,dR}$-module, where $e_{i_1}\cdots e_{i_n}$ is the dual basis of $\omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_n}$. Therefore the element $\omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_n}$ defines a de Rham framing $\beta : (\Pi^{(n)}_{\vec{01},\vec{01}})_{dR} \to \mathbf{Q}$. By the equality (6.3), by the comparison map, we have $\alpha(1) = 1 \in \mathbf{Q}\langle\langle e_0, e_1\rangle\rangle/(e_0, e_1)^{n+1}$ via the comparison map. Therefore, we have

$$\begin{aligned}([0,1], \omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_n}) =&([0,1]c_{\vec{01}}, \omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_n}) \\ =&(\Phi(e_0, e_1), \omega_{i_1} \boxtimes \cdots \boxtimes \omega_{i_n}).\end{aligned}$$

Therefore we have the statement of the theorem. □

We define the Φ-*multiple zeta value* by

$$\zeta_\Phi(m_1, \ldots, m_k) = \int_{[0,1]}^{\Phi} \omega_0^{m_k-1}\omega_1 \ldots \omega_0^{m_1-1}\omega_1.$$

By the above theorem, it is a coefficient of the associator Φ.

## § 6.2. Remarks on the action of the Grothendieck-Teichmüller group

In this paper, we introduced algebroids of $\mathcal{M}_n$ and simplicial objects consisting of these algebroids. We also introduced standard $\mathcal{A}_n$-modules arising from abelianization of $\mathcal{A}_4$. Periods of Φ-cohomologies with coefficients in such sheaves are expressed by coefficients of associators. Similar stories can be applied to the action of functorial automorphism group of fiber functors of Betti or de Rham realization. These groups of functorial automorphisms are called Grothendieck-Teichmüller groups. We can consider similar cohomology theory on which Grothendieck-Teichmüller group acts.

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