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Functional equations for double series of Euler-Hurwitz-Barnes type with coefficients

By

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Abstract

We first survey the known results on functional equations for the double zeta-function of Euler type and its various generalizations. Then we prove two new functional equations for double series of Euler-Hurwitz-Barnes type with complex coefficients. The first one is of general nature, while the second one is valid when the coefficients are Fourier coefficients of a cusp form.

§ 1. Introduction

Functional equations play a very important role in the theory of zeta and L-functions. In the case of the most classical Riemann zeta-function $\zeta(s)$, Riemann proved the beautiful symmetric functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{1.1}$$

This symmetry, however, does not remain when we consider the Hurwitz zeta-function $\zeta(s,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s}$, where $0<\alpha\leq 1$. The functional equation for $\zeta(s,\alpha)$ is of the form

$$\zeta(s,\alpha)=\frac{\Gamma(1-s)}{i(2\pi)^{1-s}}\left\{e^{\pi is/2}\phi(1-s,\alpha)-e^{-\pi is/2}\phi(1-s,-\alpha)\right\}, \tag{1.2}$$

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where $i = \sqrt{-1}$ and $\phi(s,\alpha) = \sum_{n=1}^{\infty} e^{2\pi i n\alpha} n^{-s}$ is the Lerch zeta-function (see [28, (2.17.3)]). When $\alpha = 1$, (1.2) reduces to (1.1), but the form (1.2) is no longer symmetric.

In recent decades, the theory of various multiple zeta-functions has been developed extensively. Therefore it is natural to search for functional equations for those multiple zeta-functions. For example, the Barnes multiple zeta-function

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} (\omega_1 n_1 + \cdots + \omega_r n_r + \alpha)^{-s} \tag{1.3}$$

(Barnes [1]) with complex parameters $\omega_1, \ldots, \omega_r, \alpha$ is a direct generalization of Hurwitz zeta-functions, so a kind of functional equation similar to (1.2) is expected to hold. In fact, such an equation has been proved under certain condition of parameters (see Hardy and Littlewood [8] in the case $r = 2$, and [15] [27] in general).

Another important class of multiple zeta-functions is the multi-variable sum

$$\zeta_r(s_1, \ldots, s_r) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} n_1^{-s_1} (n_1 + n_2)^{-s_2} \cdots (n_1 + \cdots + n_r)^{-s_r}, \tag{1.4}$$

where $s_j = \sigma_j + it_j$ $(1 \leq j \leq r)$ are complex variables. This sum was first considered by Euler in the case $r = 2$, and then introduced by Hoffman [9] and Zagier [29] independently of each other for general r. To find some kind of functional equations for the sum (1.4) or its variants/generalizations seems a rather complicated problem. Let us consider the simplest case $r = 2$:

$$\zeta_2(s_1, s_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-s_1} (m+n)^{-s_2}. \tag{1.5}$$

In the following sections we will report our attempt to search for functional equations for (1.5) and its various variants.

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§ 2. Functional equations for double zeta-functions

Let $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\omega > 0$, and define

$$\zeta_2(s_1, s_2; \alpha, \beta, \omega) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i n\beta}}{(\alpha + m)^{s_1} (\alpha + m + n\omega)^{s_2}}. \tag{2.1}$$

This is a generalization of (1.5) of the Hurwitz-Lerch type. The functional equation for $\zeta_2(s_1, s_2; \alpha, \beta, \omega)$ can be stated in terms of the confluent hypergeometric function

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{e^{i\rho}\infty} e^{-xy} y^{a-1} (1+y)^{c-a-1} dy, \tag{2.2}$$

where $a, c, x \in \mathbb{C}$, $|\arg x| < \pi$, $\Re a > 0$, $-\pi < \rho < \pi$, $|\rho + \arg x| < \pi/2$. Define

$$F_{\pm}(s_1, s_2; \alpha, \beta, \omega) = \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k; \alpha, \beta) \Psi(s_2, s_1 + s_2; \pm 2\pi i k\omega), \tag{2.3}$$

where

$$\sigma_c(k; \alpha, \beta) = \sum_{0<d|k} e^{2\pi i d\alpha} e^{2\pi i (k/d)\beta} d^c.$$

The series (2.3) is absolutely convergent in the region $\sigma_1 < 0$, $\sigma_2 > 1$. In fact, choosing $\rho = \mp(\pi/2)$ and putting $y = \mp i\eta$ in (2.2), we have

$$|\Psi(s_2, s_1 + s_2; \pm 2\pi i k\omega)| \ll \int_0^{\infty} e^{-2\pi k\omega\eta} \eta^{\sigma_2 - 1} |1 \mp i\eta|^{\sigma_1 - 1} d\eta. \tag{2.4}$$

Since $|1 \mp i\eta| \geq 1$, we see that $|1 \mp i\eta|^{\sigma_1 - 1} = O(1)$ if $\sigma_1 < 1$. Therefore the right-hand side of (2.4) is

$$\ll \int_0^{\infty} e^{-2\pi k\omega\eta} \eta^{\sigma_2 - 1} d\eta \ll k^{-\sigma_2} \Gamma(\sigma_2).$$

Using this estimate, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} |\sigma_{s_1+s_2-1}(k; \alpha, \beta) \Psi(s_2, s_1 + s_2; \pm 2\pi i k\omega)| \\ &\ll \sum_{k=1}^{\infty} \sum_{d|k} d^{\sigma_1+\sigma_2-1} k^{-\sigma_2} \ll \sum_{d=1}^{\infty} d^{\sigma_1 - 1} \sum_{l=1}^{\infty} l^{-\sigma_2}, \end{aligned} \tag{2.5}$$

which is convergent when $\sigma_1 < 0$ and $\sigma_2 > 1$.

It is known that $\zeta_2(s_1, s_2; \alpha, \beta, \omega)$ satisfies the following functional equation.

Theorem 2.1. *The functions $F_{\pm}(s_1, s_2; \alpha, \beta, \omega)$ can be continued meromorphically to the whole space $\mathbb{C}^2$, and the functional equation*

$$\begin{aligned} \zeta_2(s_1, s_2; \alpha, \beta, \omega) &= \frac{\Gamma(1-s_1)}{\Gamma(s_2)} \Gamma(s_1+s_2-1) \phi(s_1+s_2-1, \beta) \omega^{1-s_1-s_2} \\ &+ \Gamma(1-s_1) \omega^{1-s_1-s_2} \\ &\times \{F_+(1-s_2, 1-s_1; \beta, \alpha, \omega) + F_-(1-s_2, 1-s_1; \beta, -\alpha, \omega)\} \end{aligned} \tag{2.6}$$

holds.

Here, the first term on the right-hand side of (2.6) is an "additional" term, and the main body of the right-hand side is the second term involving $F_\pm$. This term, compared with the left-hand side, expresses a duality between the values at (s_1, s_2) and at $(1-s_2, 1-s_1)$ (and also a duality between (α, β) and (β, α)), and hence formula (2.6) can be regarded as a double analogue of (1.2). The functions $F_\pm$ are not Dirichlet series in the usual sense, but the confluent hypergeometric function satisfies the asymptotic expansion

(2.7)
$$\Psi(a, c; x) = x^{-a} - a(a-c+1)x^{-a-1} + \frac{a(a+1)(a-c+1)(a-c+2)}{2}x^{-a-2} + \cdots,$$

so we may say that $\Psi(a, c; x)$ can be approximated by x^{-a}. From this viewpoint it is possible to say that $F_\pm$ can be approximated by the Dirichlet series

$$\sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k; \alpha, \beta)(\pm 2\pi i k\omega)^{-s_2}.$$

In this sense $F_\pm$ may be regarded as generalized Dirichlet series.

The meromorphic continuation of $F_\pm$ was shown in [22, Proposition 2], where their functional equation

$$F_\pm(1-s_2, 1-s_1; \beta, \alpha, \omega) = (\pm 2\pi i\omega)^{s_1+s_2-1} F_\pm(s_1, s_2; \alpha, \beta, \omega) \tag{2.8}$$

was also proved. The transformation formula

$$\Psi(a, c; x) = x^{1-c}\Psi(a-c+1, 2-c; x) \tag{2.9}$$

of the confluent hypergeometric function is used in the proof of (2.8). Applying (2.8) to [22, Proposition 1], we can immediately obtain formula (2.6). Therefore the above Theorem 2.1 is essentially included in [22], though it is first explicitly stated in [23].

The main statement of [22] is as follows. Let

$$\begin{aligned} g(s_1, s_2; \alpha, \beta, \omega) &= \zeta_2(s_1, s_2; \alpha, \beta, \omega) \\ &\quad - \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\phi(s_1+s_2-1, \beta)\omega^{1-s_1-s_2}. \end{aligned} \tag{2.10}$$

Then

Theorem 2.2 ([22, Theorem 2]). *We have*

$$\begin{aligned} \frac{g(s_1, s_2; \alpha, \beta, \omega)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} &= \frac{g(1-s_2, 1-s_1; 1-\beta, 1-\alpha, \omega)}{(i\omega)^{s_1+s_2-1}\Gamma(s_2)} \\ &+ e^{\pi i(s_1+s_2-1)/2}F_+(s_1, s_2; \alpha, \beta, \omega) - e^{\pi i(1-s_1-s_2)/2}F_+(s_1, s_2; 1-\alpha, 1-\beta, \omega), \end{aligned} \tag{2.11}$$

and especially, when $\alpha = \beta = 1$, *we have*

$$\frac{g(s_1, s_2; 1, 1, \omega)}{(2\pi)^{s_1+s_2-1}\Gamma(1-s_1)} = \frac{g(1-s_2, 1-s_1; 1, 1, \omega)}{(i\omega)^{s_1+s_2-1}\Gamma(s_2)} + 2i \sin\left(\frac{\pi}{2}(s_1+s_2-1)\right) F_+(s_1, s_2; 1, 1, \omega). \tag{2.12}$$

This theorem can also be easily deduced from Proposition 1 and Proposition 2 of [22].

Remark 1. (Historical note) The idea of the proof of Theorem 2.1 goes back to [12], where certain mean values of Dirichlet L-functions were studied. The application of the confluent hypergeometric function in this context was first done by Katsurada [11]. In order to study Barnes' double zeta-functions (the case $r = 2$ of (1.3)), the second-named author [19] introduced the two-variable double series

$$\zeta_2(s_1, s_2; \alpha, \omega) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} (\alpha+m)^{-s_1}(\alpha+m+n\omega)^{-s_2} \tag{2.13}$$

(the case $\beta = 0$ of (2.1)), and studied its properties, invoking the methods in [12], [11]. In particular, the special case $\beta = 0$ of [22, Proposition 1] was already given in [19, (5.5)].

§ 3. The symmetric form

In this and the next section we present the contents of two joint papers of Komori, Tsumura and the second-named author. Theorem 2.1 is a non-symmetric functional equation, similarly to (1.2) for Hurwitz zeta-functions. Is there any *symmetric* functional equation for double zeta-functions? One of the main point of [13] is that such equations do exist on certain special hyperplanes.

In [13], the following generalization of (1.5) was introduced:

$$\zeta_2(s_1, s_2; \omega_1, \omega_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m\omega_1)^{-s_1}(m\omega_1+n\omega_2)^{-s_2}, \tag{3.1}$$

where $\omega_1, \omega_2 \in \mathbb{C}$ with $\Re\omega_1 > 0$, $\Re\omega_2 > 0$. The functional equation similar to Theorem 2.1 holds for $\zeta_2(s_1, s_2; \omega_1, \omega_2)$, which is

$$\begin{aligned}\zeta_2(s_1, s_2; \omega_1, \omega_2) &= \frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1)\omega_1^{-1}\omega_2^{1-s_1-s_2} \\ &+ \Gamma(1-s_1)\omega_1^{-1}\omega_2^{1-s_1-s_2} \\ &\times \{F_+(1-s_2, 1-s_1; 1, 1, \omega_2/\omega_1) + F_-(1-s_2, 1-s_1; 1, 1, \omega_2/\omega_1)\}\end{aligned} \tag{3.2}$$

([13, Theorem 2.1]). This formula itself can be proved just similarly to Theorem 2.1. However, from this formula, it is possible to deduce the following symmetric functional equation. Let

$$\begin{aligned}(3.3)\ \xi(s_1,s_2;\omega_1,\omega_2) = &\left(\frac{2\pi i}{\omega_1\omega_2}\right)^{(1-s_1-s_2)/2}\Gamma(s_2)\\ &\times\left\{\zeta_2(s_1,s_2;\omega_1,\omega_2)-\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\Gamma(s_1+s_2-1)\zeta(s_1+s_2-1)\omega_1^{-1}\omega_2^{1-s_1-s_2}\right\}.\end{aligned}$$

Then we have

Theorem 3.1 ([13, Theorem 2.2]). *The hyperplane*

$$\Omega_{2k+1} = \{(s_1,s_2)\in\mathbb{C}^2 \mid s_1+s_2=2k+1\} \tag{3.4}$$

($k\in\mathbb{Z}\setminus\{0\}$) *is not a singular locus of* $\zeta_2(s_1,s_2;\omega_1,\omega_2)$*, and when* $(s_1,s_2)\in\Omega_{2k+1}$*, the functional equation*

$$\xi(s_1,s_2;\omega_1,\omega_2)=\xi(1-s_2,1-s_1;\omega_1,\omega_2) \tag{3.5}$$

holds.

When $\omega_1=1$ and $\omega_2=\omega$, this theorem is actually almost immediately obtained from (2.12). (But the second-named author did not notice this point when he wrote [22]).

From Theorem 3.1, we can evaluate certain values of $\zeta_2(s_1,s_2;\omega_1,\omega_2)$ when s_1,s_2 are non-positive integers ([13, Corollary 2.4]).

§ 4. Functional equations for double L-functions

In the statement of aforementioned functional equations ((2.6), (3.2)), an "additional" term involving $\phi(s_1+s_2-1,\beta)$ or $\zeta(s_1+s_2-1)$ appears as the first term of the right-hand side, which also appears in the definitions (2.10) and (3.3). The reason of the appearance of such a term can be clarified when we consider a little more general situation.

Let $f\in\mathbb{Z}_{\geq 2}$, and let $a_j:\mathbb{Z}\to\mathbb{C}$ ($j=1,2$) be periodic functions with period f. Assume a_j is an even (or odd) function, and define the "sign" λ of a_j by $\lambda(a_j)=1$ (resp. -1) if a_j is even (resp. odd). Define the associated L-function by $L(s,a_j)=\sum_{m=1}^{\infty}a_j(m)m^{-s}$. In [14] the double series

$$L_2(s_1,s_2;a_1,a_2;\omega_1,\omega_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_1(m)a_2(n)}{(m\omega_1)^{s_1}(m\omega_1+n\omega_2)^{s_2}} \tag{4.1}$$

was introduced. Denote the finite Fourier expansion of a_j by

$$a_j(m) = \sum_{\nu=1}^{f} \widehat{a}_j(\nu) e^{2\pi i \nu m/f}.$$

The series $L_2(s_1, s_2; a_1, a_2; \omega_1, \omega_2)$ can be continued meromorphically to the whole space $\mathbb{C}^2$. As an analogue of Theorem 3.1, the following symmetric functional equation holds.

Theorem 4.1 ([14, Theorem 2.1]). *The functional equation*

$$\begin{aligned}(4.2)\qquad &\left(\frac{2\pi i}{f\omega_1\omega_2}\right)^{(1-s_1-s_2)/2} \Bigg\{\Gamma(s_2)L_2(s_1, s_2; a_1, a_2; \omega_1, \omega_2) \\ &\quad - \frac{\omega_2^{1-s_1-s_2}}{f\omega_1}\Gamma(1-s_1)\Gamma(s_1+s_2-1)L(s_1+s_2-1, a_2)\sum_{\nu=1}^{f} a_1(\nu)\Bigg\} \\ &= \left(\frac{2\pi i}{f\omega_1\omega_2}\right)^{(s_1+s_2-1)/2} \Bigg\{\Gamma(1-s_1)L_2(1-s_2, 1-s_1; \widehat{a}_2, \widehat{a}_1; \omega_1, \omega_2) \\ &\quad - \frac{\omega_2^{s_1+s_2-1}}{f\omega_1}\Gamma(s_2)\Gamma(1-s_1-s_2)L(1-s_1-s_2, \widehat{a}_1)\sum_{\nu=1}^{f} \widehat{a}_2(\nu)\Bigg\}\end{aligned}$$

holds on the hyperplane $s_1+s_2 = 2k+1$ $(k \in \mathbb{Z})$ *if* $\lambda(a_1)\lambda(a_2) = 1$, *and on the hyperplane* $s_1 + s_2 = 2k$ $(k \in \mathbb{Z})$ *if* $\lambda(a_1)\lambda(a_2) = -1$.

Therefore if

$$(4.3)\qquad \sum_{\nu=1}^{f} a_1(\nu) = \sum_{\nu=1}^{f} \widehat{a}_2(\nu) = 0$$

then the "additional" terms do not appear. In particular, if a_1, a_2 are primitive Dirichlet characters χ_1, χ_2 of conductor f, then (4.3) holds, and so the functional equation becomes the following very simple form:

$$\begin{aligned}(4.4)\qquad &\left(\frac{2\pi i}{f\omega_1\omega_2}\right)^{(1-s_1-s_2)/2} \Gamma(s_2)L_2(s_1, s_2; \chi_1, \chi_2; \omega_1, \omega_2) \\ &= \left(\frac{2\pi i}{f\omega_1\omega_2}\right)^{(s_1+s_2-1)/2} \Gamma(1-s_1)L_2(1-s_2, 1-s_1; \widehat{\chi}_2, \widehat{\chi}_1; \omega_1, \omega_2)\end{aligned}$$

on the hyperplanes indicated in the statement of Theorem 4.1.

Remark 2. Using the relation $\widehat{\chi}_j(m) = \overline{\chi}_j(m)/\tau(\overline{\chi}_j)$ (where $\tau(\cdot)$ denotes the Gauss sum), we can restate formula (4.4) in which $\widehat{\chi}_j$ is replaced by $\overline{\chi}_j$. Such a formula is stated as [14, Corollary 2.3], but the factor $\chi_1(-1)f$ is lacking on the left-hand side of the statement there.

From this result we can evaluate certain values of double L-functions at non-positive integer points ([14, Section 3]). Those results further motivate the construction of the theory of p-adic multiple L-functions. A double analogue of the Kubota-Leopoldt p-adic L-function was introduced in [14, Section 4], and then, a more general theory of p-adic multiple L-functions has been developed in [7].

To prove Theorem 4.1, the following double zeta-function of Hurwitz-Lerch type was introduced:

$$\zeta_2(s_1, s_2; \alpha, \beta; \omega_1, \omega_2) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i n\beta}}{((\alpha+m)\omega_1)^{s_1}((\alpha+m)\omega_1 + n\omega_2)^{s_2}}, \tag{4.5}$$

where $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\omega_1, \omega_2 \in \mathbb{C}$ with $\Re\omega_1 > 0$, $\Re\omega_2 > 0$. This is a generalization of both (2.1) and (3.1). (Frankly speaking, this is almost equivalent to consider (2.1) with $\omega \in \mathbb{C}$, $|\arg\omega| < \pi$.) Since $L_2(s_1, s_2; a_1, a_2; \omega_1, \omega_2)$ can be written as a linear combination of functions of the form (4.5), it is sufficient to show a functional equation for (4.5). This can be done in a way similar to the proofs of Theorem 2.1 and Theorem 3.1 (see [14, Sections 5, 6]).

§ 5. Functional equations for double series with complex coefficients

Let $\mathfrak{A} = \{a(n)\}_{n\geq 1}$ be a sequence of complex numbers. In [4], the authors considered the double series

$$L_2(s_1, s_2; \mathfrak{A}) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a(n)}{m^{s_1}(m+n)^{s_2}}. \tag{5.1}$$

The original motivation of [4] is to study the case when $a(n)$'s are Fourier coefficients of modular forms. In [16] [17], Manin extended the theory of periods of modular forms replacing integration along geodesics by iterated integrations. In this context some kind of multiple Dirichlet series with Fourier coefficients on the numerator naturally appears (cf. [3]). Multiple series with Fourier coefficients is also expected to be useful to evaluate certain multiple sums of Fourier coefficients (analogously to de la Bretèche [2], Ishikawa [10], Essouabri [6] etc).

In the present paper we consider a slight generalization of (5.1), that is

$$L_2(s_1, s_2; \alpha; \omega; \mathfrak{A}) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{a(n)}{(\alpha+m)^{s_1}(\alpha+m+n\omega)^{s_2}}, \tag{5.2}$$

where $\omega \in \mathbb{C}$ with $|\arg\omega| < \pi/2$. This is a generalization of both (2.1) and (5.1). (The exponential factor $e^{2\pi i n\beta}$ on the numerator of (2.1) is to be included in $a(n)$.) The following two theorems (Theorems 5.1 and 5.2) are the main results of the present paper. When $\alpha = \omega = 1$, those theorems are Theorem 1.1 and Theorem 1.5 in [4], respectively.

Concerning the order of the coefficients $a(n)$, we assume:

(i) $a(n) \ll n^{(\kappa-1)/2+\varepsilon}$ with a certain constant $\kappa > 1$, where ε is an arbitrarily small positive number.

Then, the Dirichlet series

$$L(s, \mathfrak{A}) = \sum_{n=1}^{\infty} a(n) n^{-s} \tag{5.3}$$

is absolutely convergent in the region $\Re s > (\kappa + 1)/2$, and we further assume

(ii) (5.3) can be continued meromorphically to the whole complex plane.

Under these two assumptions, we see that (5.2) is absolutely convergent in the region

$$\sigma_2 > \frac{\kappa+1}{2}, \quad \sigma_1 + \sigma_2 > \frac{\kappa+3}{2} \tag{5.4}$$

(using (i) and [20, Theorem 3]). As a generalization of (2.3), define

$$F_{\pm}(s_1, s_2; \alpha; \omega; \mathfrak{A}) = \sum_{l=1}^{\infty} A_{s_1+s_2-1}(l; \pm\alpha; \mathfrak{A}) \Psi\left(s_2, s_1 + s_2; \pm 2\pi i \omega l\right), \tag{5.5}$$

where

$$A_c(l; \pm\alpha; \mathfrak{A}) = \sum_{mn=l} e^{\pm 2\pi i m\alpha} a(n) n^c. \tag{5.6}$$

Similarly to the case of (2.3), we can show that the series (5.5) is absolutely convergent in the region $\sigma_1 < -(\kappa - 1)/2$, $\sigma_2 > 1$. (This time ω is complex, but $|\arg \omega| < \pi/2$, so the argument after (2.3) works.) Corresponding to Theorem 2.1, we have

Theorem 5.1. *Assume* (i) *and* (ii). *Then the functions $L_2(s_1, s_2; \alpha; \omega; \mathfrak{A})$ and $F_{\pm}(s_1, s_2; \alpha; \omega; \mathfrak{A})$ can be continued meromorphically to the whole space $\mathbb{C}^2$, and the functional equation*

$$\begin{aligned} &L_2(s_1, s_2; \alpha; \omega; \mathfrak{A}) \\ &= \frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\Gamma(s_2)\omega^{s_1+s_2-1}} L(s_1+s_2-1, \mathfrak{A}) \\ &+ \frac{\Gamma(1-s_1)}{\omega^{s_1+s_2-1}} \left\{ F_{+}(1-s_2, 1-s_1; \alpha; \omega; \mathfrak{A}) + F_{-}(1-s_2, 1-s_1; \alpha; \omega; \mathfrak{A}) \right\} \end{aligned} \tag{5.7}$$

holds.

In particular, applying Theorem 5.1 to the special case $a(n) = 1$ for only one fixed n, and $a(n) = 0$ for all other n, we obtain a functional equation for the single series

$$\sum_{m=0}^{\infty} \frac{1}{(\alpha+m)^{s_1}(\alpha+m+n\omega)^{s_2}}, \tag{5.8}$$

which gives the "refinement" (or the "decomposition") of Theorem 5.1 in the sense of [4, Remark 2.4]. Note that (5.8) is a generalization of the series

$$\sum_{m=0}^{\infty} \frac{1}{(\alpha+m)^{s_1}(\beta+m)^{s_2}}$$

($\beta \geq \alpha > 0$) which was used in [21] and [18]. Also Ram Murty and Sinha [26] studied analytic properties of this type of function and its generalizations.

Now we consider the case when $a(n)$'s are Fourier coefficients of modular forms. Let $\mathcal{H}$ be the complex upper half plane, and

$$f(\tau) = \sum_{n=1}^{\infty} a(n) e^{2\pi i \tau n}, \tag{5.9}$$

where $\tau \in \mathcal{H}$. When $f(\tau)$ is a holomorphic cusp form of even weight κ with respect to the Hecke congruence subgroup $\Gamma_0(N)$, it is well known that assumptions (i) and (ii) are satisfied. Therefore, obviously, Theorem 5.1 can be applied to this case. In this case, however, we can show a different type of functional equation, using the Fricke involution. Let

$$\widetilde{f}(\tau) = (\sqrt{N}\tau)^{-\kappa} f\left(-\frac{1}{N\tau}\right) \tag{5.10}$$

and denote its Fourier expansion by $\widetilde{f}(\tau) = \sum_{n\geq 1} \widetilde{a}(n) e^{2\pi i \tau n}$. Define

$$\begin{aligned} &H_{2,N}^{\pm}(s_1, s_2; \alpha; \omega; \widetilde{f}) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \widetilde{a}(n) e^{\mp 2\pi i m\alpha} m^{-s_1-s_2} \Psi\left(s_1+s_2, s_2; \pm\frac{2\pi i n}{N\omega m}\right). \end{aligned} \tag{5.11}$$

This series is absolutely convergent in the region $\sigma_1 > 0$, $\sigma_2 > (\kappa+3)/2$ ([4, (8.2)]). Write $L(s, f)$ and $L_2(s_1, s_2; \alpha; \omega; f)$ instead of $L(s, \mathfrak{A})$ and $L_2(s_1, s_2; \alpha; \omega; \mathfrak{A})$, respectively. Then we have

Theorem 5.2. *Assume* (i) *and* (ii)*, and* $f(\tau)$ *is as above. Then the functions* $H_{2,N}^{\pm}(s_1, s_2; \alpha; \omega; \widetilde{f})$ *can be continued meromorphically to the whole space* $\mathbb{C}^2$*, and the* *functional equation*

$$\begin{aligned}(5.12)\qquad \frac{\Gamma(s_2)}{\Gamma(1-s_1)}&L_2(s_1,s_2;\alpha;\omega;f)\\ &= \frac{\Gamma(s_1+s_2-1)}{\omega^{s_1+s_2-1}}L(s_1+s_2-1,f)\\ &+(2\pi)^{s_1+s_2-1}N^{-\kappa/2}\omega^{-\kappa}\Gamma(\kappa-s_1-s_2+1)\\ &\times\Big\{e^{\pi i(1-s_1-s_2)/2}H^{+}_{2,N}(-s_1,\kappa-s_2+1;\alpha;\omega;\widetilde{f})\\ &\qquad +e^{\pi i(s_1+s_2-1)/2}H^{-}_{2,N}(-s_1,\kappa-s_2+1;\alpha;\omega;\widetilde{f})\Big\}\end{aligned}$$

holds.

Remark 3. There is only one factor $a(n)$ on the numerator of (5.1), (5.2). This is an unsatisfactory point; it is desirable to study the double series with two factors (such as $a(m)b(n)$) on the numerator. However our present method cannot be applied to such a situation.

In the remaining sections we give the proofs of Theorems 5.1 and 5.2. Since the proofs are straightforward generalization of the proofs described in [4], we just outline the argument briefly. Denote $\theta=\arg\omega$.

§ 6. Sketch of the Proof of Theorem 5.1

Step 1. The integral

$$\begin{aligned}(6.1)\qquad &\Lambda(s_1,s_2;\alpha;\omega;\mathfrak{A})\\ &=\int_0^\infty f(i\omega y)\int_0^\infty \frac{e^{2\pi(1-\alpha)(x+y)}}{e^{2\pi(x+y)}-1}x^{s_1-1}y^{s_2-1}dxdy\end{aligned}$$

converges in the region $\sigma_1>0$, $\sigma_2>(\kappa+1)/2$, $\sigma_1+\sigma_2>(\kappa+3)/2$, and

$$(6.2)\qquad L_2(s_1,s_2;\alpha;\omega;\mathfrak{A})=\frac{(2\pi)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)}\Lambda(s_1,s_2;\alpha;\omega;\mathfrak{A}).$$

This can be shown by using (5.9) and the Taylor expansion

$$(6.3)\qquad \frac{e^{2\pi(1-\alpha)(x+y)}}{e^{2\pi(x+y)}-1}=\sum_{m=0}^{\infty}e^{-2\pi m(x+y)-2\pi\alpha(x+y)}.$$

Step 2. Let

$$(6.4)\qquad h(z,\alpha)=\frac{e^{2\pi(1-\alpha)z}}{e^{2\pi z}-1}-\frac{1}{2\pi z},$$

and divide the integral $\Lambda(s_1, s_2; \alpha; \omega; \mathfrak{A})$ as

$$\begin{aligned}\Lambda(s_1, s_2; \alpha; \omega; \mathfrak{A}) &\\ = \int_0^\infty f(i\omega y) \int_0^\infty h(x+y, \alpha) x^{s_1-1} y^{s_2-1} dxdy &\\ + \int_0^\infty f(i\omega y) \int_0^\infty \frac{1}{2\pi(x+y)} x^{s_1-1} y^{s_2-1} dxdy &\\ = I_1 + I_2, &\end{aligned} \tag{6.5}$$

say. The important point here is that the function $h(z, \alpha)$ is holomorphic at $z = 0$. The idea of this type of decomposition goes back to Motohashi's short note [25], which influences [12] (see Remark 1).

Step 3. By the beta integral formula, we find that

$$I_2 = \frac{\Gamma(s_1)\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{(2\pi)^{s_1+s_2}\omega^{s_1+s_2-1}} L(s_1+s_2-1, \mathfrak{A}) \tag{6.6}$$

in the region

$$0 < \sigma_1 < 1, \ \sigma_1 + \sigma_2 > (\kappa+3)/2. \tag{6.7}$$

Step 4. Using estimates of $h(z, \alpha)$ ([14, p.1454]) and $f(i\omega y)$ ([24, Lemma 4.3.3]), we find that the integral I_1 is convergent in the region $\sigma_2 > (\kappa+1)/2$. Since (6.7) implies $\sigma_2 > (\kappa+1)/2$, now we know that the decomposition (6.5) is valid in the region (6.7).

Step 5. Changing the path of the inner integral of I_1 by the contour $\mathcal{C}$ consisting of the half-line on the positive real axis from infinity to a small positive number, a small circle counterclockwise round the origin, and the other half-line on the positive real axis back to infinity. Then

$$I_1 = \frac{1}{e^{2\pi i s_1} - 1} I_3, \tag{6.8}$$

where

$$I_3 = \int_0^\infty f(i\omega y) y^{s_2-1} \int_{\mathcal{C}} h(x+y, \alpha) x^{s_1-1} dxdy. \tag{6.9}$$

Since I_3 is convergent in the region

$$\sigma_1 < 1, \ \sigma_2 > (\kappa+1)/2, \tag{6.10}$$

and the right-hand side of (6.6) can be continued to the whole space $\mathbb{C}^2$ (by assumption (ii)), now we see by (6.5) that $\Lambda(s_1, s_2; \alpha; \omega; \mathfrak{A})$ is (and hence $L_2(s_1, s_2; \alpha; \omega; \mathfrak{A})$ is) continued meromorphically to the region (6.10).

Step 6. Assume

$$\sigma_1 < 0,\ \sigma_2 > (\kappa+1)/2 \tag{6.11}$$

(a little smaller than (6.10)), and change the path $\mathcal{C}$ on the right-hand side of (6.10) by

$$\mathcal{C}_R = \{x = -y + (R+1/2)e^{i\varphi} \mid 0 \le \varphi < 2\pi\}$$

($R \in \mathbb{N}$), and let $R \to \infty$. Counting the residues, we have

$$\begin{aligned} I_3 &= -i \int_0^\infty f(i\omega y) y^{s_2-1} \sum_{m\neq 0} e^{-2\pi i m\alpha} (-y+im)^{s_1-1}\, dy \\ &= -i(I_{31}+I_{32}), \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} I_{31} = e^{\pi i(s_1-s_2-1)/2} &\sum_{m=1}^\infty e^{-2\pi i m\alpha} m^{s_1+s_2-1} \\ &\times \int_0^{e^{i\pi/2}\infty} f\left(\omega m z\right) z^{s_2-1}(z+1)^{s_1-1} dz, \end{aligned} \tag{6.13}$$

$$\begin{aligned} I_{32} = e^{\pi i(3s_1+s_2-3)/2} &\sum_{m=1}^\infty e^{2\pi i m\alpha} m^{s_1+s_2-1} \\ &\times \int_0^{e^{-i\pi/2}\infty} f\left(-\omega m z\right) z^{s_2-1}(z+1)^{s_1-1} dz. \end{aligned} \tag{6.14}$$

Step 7. Applying (5.9) to the right-hand side of (6.13) and putting $mn = l$, we obtain

$$\begin{aligned} I_{31} = &e^{\pi i(s_1-s_2-1)/2}\Gamma(s_2) \\ &\times \sum_{l=1}^\infty A^0_{s_1+s_2-1}(l;-\alpha;\mathfrak{A}) \Psi\left(s_2, s_1+s_2; -2\pi i\omega l\right), \end{aligned} \tag{6.15}$$

where

$$A^0_c(l;\alpha;\mathfrak{A}) = \sum_{mn=l} e^{2\pi i m\alpha} m^c a(n). \tag{6.16}$$

Similarly we have

$$\begin{aligned} I_{32} = &e^{\pi i(3s_1+s_2-3)/2}\Gamma(s_2) \\ &\times \sum_{l=1}^\infty A^0_{s_1+s_2-1}(l;\alpha;\mathfrak{A}) \Psi\left(s_2, s_1+s_2; 2\pi i\omega l\right). \end{aligned} \tag{6.17}$$

The interchanges of summation and integration in Steps 6 and 7 can be verified similarly to the argument in [4, Section 5], in the region (6.11). Substituting (6.15) and (6.17) into (6.12) (and then (6.8)), we obtain

$$\frac{(2\pi)^{s_1+s_2}}{\Gamma(s_1)\Gamma(s_2)} I_1 = (2\pi)^{s_1+s_2-1}\Gamma(1-s_1) \times \left\{ e^{\pi i(1-s_1-s_2)/2} F_-^0(s_1,s_2;\alpha;\omega;\mathfrak{A}) + e^{\pi i(s_1+s_2-1)/2} F_+^0(s_1,s_2;\alpha;\omega;\mathfrak{A}) \right\}, \tag{6.18}$$

where

$$F_\pm^0(s_1,s_2;\alpha;\omega;\mathfrak{A}) = \sum_{l=1}^{\infty} A^0_{s_1+s_2-1}(l;\pm\alpha;\mathfrak{A})\Psi(s_2, s_1+s_2; \pm 2\pi i\omega l). \tag{6.19}$$

Step 8. Using (2.9) and noting

$$A_c^0(l;\pm\alpha;\mathfrak{A})l^{-c} = A_{-c}(l;\pm\alpha;\mathfrak{A}), \tag{6.20}$$

we find that

$$F_\pm^0(s_1,s_2;\alpha;\omega;\mathfrak{A}) = (\pm 2\pi i\omega)^{1-s_1-s_2} F_\pm(1-s_2, 1-s_1;\alpha;\omega;\mathfrak{A}). \tag{6.21}$$

This is a generalization of (2.8). Substituting this into (6.18), and combining with (6.6), we now arrive at formula (5.7), in the region (6.11).

Step 9. The final task is to show the meromorphic continuation of the functions $L_2(s_1,s_2;\alpha;\omega;\mathfrak{A})$ and $F_\pm(s_1,s_2;\alpha;\omega;\mathfrak{A})$. The continuation of $F_\pm$ can be done by using the asymptotic expansion (2.7) of the confluent hypergeometric function. Then (5.7) gives the continuation of L_2 to the whole space $\mathbb{C}^2$. This completes the proof of Theorem 5.1.

§ 7. Sketch of the proof of Theorem 5.2

Let $f(\tau)$ be as in the statement of Theorem 5.2. Then $f(i\omega y)$ is of exponential decay as $y \to 0$, and hence the integral (6.9) is convergent in the region $\sigma_1 < 1$ (wider than (6.10)). Therefore, the formulas stated in Steps 6 and 7 in the preceding section are now valid under the simple assumption $\sigma_1 < 0$.

Applying the modular relation (5.10) to (6.13) and (6.14), and changing the order of integration and summation, we obtain

$$I_{31} = N^{-\kappa/2}\omega^{-\kappa} e^{\pi i(s_1-s_2-1)/2} \times \Gamma(\kappa - s_1 - s_2 + 1) H^+_{2,N}(-s_1, \kappa - s_2 + 1; \alpha; \omega; \widetilde{f}), \tag{7.1}$$

$$I_{32} = N^{-\kappa/2}\omega^{-\kappa}e^{\pi i(3s_1+s_2-3)/2} \times \Gamma(\kappa - s_1 - s_2 + 1)H^{-}_{2,N}(-s_1, \kappa - s_2 + 1; \alpha; \omega; \widetilde{f}). \tag{7.2}$$

Similarly to [4, Section 7], we can verify the above argument in the region

$$\sigma_1 < 0,\ \sigma_2 < (\kappa - 1)/2. \tag{7.3}$$

From (7.1), (7.2) and (6.6), we obtain (5.12) in the region (7.3). Therefore the only remaining task is to prove the meromorphic continuation of $H^{\pm}_{2,N}$. We only discuss the case of $H^{+}_{2,N}$.

Substituting the expression (2.2) of the confluent hypergeometric function into the right-hand side of (5.11), putting $y = -i\eta$ and changing the order of summation and integration, we obtain

$$H^{+}_{2,N}(s_1, s_2; \alpha; \omega; \widetilde{f}) = \frac{-i}{\Gamma(s_1+s_2)}\int_0^{e^{i(\varphi+\pi/2)}\infty}\sum_{m=1}^{\infty} e^{-2\pi i m\alpha}m^{-s_1-s_2} \times \widetilde{f}\left(\frac{i\eta}{N\omega m}\right)(-i\eta)^{s_1+s_2-1}(1-i\eta)^{-s_1-1}d\eta, \tag{7.4}$$

where φ satisfies the conditions $-\pi < \varphi < \pi$ and $|\varphi + (\pi/2) - \theta| < \pi/2$. Putting

$$\widetilde{\mathcal{F}}(\tau, s, \alpha) = \sum_{m=1}^{\infty} e^{-2\pi i m\alpha}m^{-s}\widetilde{f}\left(\frac{\tau}{Nm}\right), \tag{7.5}$$

we obtain

$$\begin{aligned} H^{+}_{2,N}&(s_1, s_2; \alpha; \omega; \widetilde{f}) \\ &= \frac{-i}{\Gamma(s_1+s_2)}\int_0^{e^{i(\varphi+\pi/2)}\infty}\widetilde{\mathcal{F}}\left(\frac{i\eta}{\omega}, s_1+s_2, \alpha\right)(-i\eta)^{s_1+s_2-1}(1-i\eta)^{-s_1-1}d\eta \\ &= \frac{e^{i\varphi(s_1+s_2)}}{\Gamma(s_1+s_2)}\int_0^{\infty}\widetilde{\mathcal{F}}\left(-\frac{1}{\omega}e^{i\varphi}\xi, s_1+s_2, \alpha\right)\xi^{s_1+s_2-1}(1+e^{i\varphi}\xi)^{-s_1-1}d\xi, \end{aligned} \tag{7.6}$$

where $\xi = e^{-i(\varphi+\pi/2)}\eta$.

As a generalization of [4, Lemma 8.1], we can show

$$\int_0^{\infty}\widetilde{\mathcal{F}}\left(-\frac{1}{\omega}e^{i\varphi}\xi, s, \alpha\right)\xi^{u-1}d\xi = \Gamma(u)\left(\frac{N\omega}{2\pi i}e^{-i\varphi}\right)^{u}\phi(s-u, -\alpha)L(u, \widetilde{f}) \tag{7.7}$$

for $u \in \mathbb{C}$ with $(\kappa+1)/2 < \Re u < \sigma - 1$. Therefore by the Mellin inversion formula we have

$$\widetilde{\mathcal{F}}\left(-\frac{1}{\omega}e^{i\varphi}\xi, s, \alpha\right) = \frac{1}{2\pi i}\int_{(c)} \xi^{-u}\Gamma(u)\left(\frac{N\omega}{2\pi i}e^{-i\varphi}\right)^{u} \phi(s-u,-\alpha)L(u,\widetilde{f})du, \tag{7.8}$$

where $(\kappa+1)/2 < c < \sigma - 1$ and the path of integration is the vertical line $\Re u = c$. Substituting (7.8) into (7.6) and changing the order of integration we obtain

$$H^{+}_{2,N}(s_1, s_2; \alpha; \omega; \widetilde{f}) = \frac{e^{i\varphi(s_1+s_2)}}{2\pi i\Gamma(s_1+s_2)}\int_{(c)} \Gamma(u)\left(\frac{N\omega}{2\pi i}e^{-i\varphi}\right)^{u} \times \phi(s_1+s_2-u,-\alpha)L(u,\widetilde{f})\int_0^\infty \xi^{s_1+s_2-1-u}(1+e^{i\varphi}\xi)^{-s_1-1}d\xi du. \tag{7.9}$$

Using the beta integral formula we can show that the inner integral on the right-hand side of (7.9) is

$$= (e^{-i\varphi})^{s_1+s_2-u}\frac{\Gamma(u-s_2+1)\Gamma(s_1+s_2-u)}{\Gamma(s_1+1)},$$

and hence

$$H^{+}_{2,N}(s_1, s_2; \alpha; \omega; \widetilde{f}) = \frac{1}{2\pi i\Gamma(s_1+s_2)\Gamma(s_1+1)}\int_{(c)} \Gamma(u)\Gamma(u-s_2+1)\Gamma(s_1+s_2-u) \times \left(\frac{N\omega}{2\pi i}\right)^{u} \phi(s_1+s_2-u,-\alpha)L(u,\widetilde{f})du. \tag{7.10}$$

Finally, modifying the path of integration on the right-hand side of (7.10) in the same way as in [4, Section 8], we can show the meromorphic continuation of $H^{+}_{2,N}$. The case of $H^{-}_{2,N}$ is similar. The proof of Theorem 5.2 is complete.

Appendix

The contents of this Appendix was first suggested by the referee. From (5.10) we have

$$f(i\omega y) = \left(\frac{-1}{i\sqrt{N}\omega y}\right)^{\kappa}\widetilde{f}\left(\frac{-1}{iN\omega y}\right) = \left(\frac{i}{\sqrt{N}\omega y}\right)^{\kappa}\sum_{n=1}^{\infty}\widetilde{a}(n)e^{-2\pi n/N\omega y}.$$

Substituting this and (6.3) into (6.1), and changing the order of integration and summation (which can be verified by the absolute convergence), we obtain

$$\Lambda(s_1, s_2; \alpha, \omega; f) = \left(\frac{i}{\sqrt{N}\omega}\right)^{\kappa}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\widetilde{a}(n)\int_0^\infty y^{-\kappa}e^{-2\pi n/N\omega y}\int_0^\infty e^{-2\pi(m+\alpha)(x+y)}x^{s_1-1}y^{s_2-1}dxdy.$$

Denote the double integral on the right-hand side by J. Then

$$\begin{aligned} J = & \int_0^\infty e^{-2\pi(m+\alpha)x} x^{s_1-1} dx \int_0^\infty e^{-(2\pi n/N\omega y)-2\pi(m+\alpha)y} y^{s_2-\kappa-1} dy \\ = & J_1 \times J_2, \end{aligned}$$

say. Clearly $J_1 = (2\pi(m+\alpha))^{-s_1}\Gamma(s_1)$. As for J_2, first put $t = 2\pi(m+\alpha)y$ and $z = 4\pi((m+\alpha)n/N\omega)^{1/2}$. Then we obtain

$$J_2 = (2\pi(m+\alpha))^{-s_2+\kappa} \int_0^\infty e^{-t-(z^2/4t)} t^{s_2-\kappa-1} dt.$$

Since $\Re(z^2) > 0$ (because $|\arg \omega| < \pi/2$), we can use the integral expression of the K-Bessel function

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-(z^2/4t)} t^{-\nu-1} dt$$

([5, Section 7.12, formula (23)]) to obtain

$$J_2 = \frac{2}{(2\pi(m+\alpha))^{s_2-\kappa}} \left(\frac{z}{2}\right)^{s_2-\kappa} K_{\kappa-s_2}(z).$$

From the above calculations and (6.2), we arrive at the following

Theorem 7.1.

$$\begin{aligned} & L_2(s_1, s_2; \alpha, \omega; f) = \frac{2i^\kappa (2\pi)^{s_2}}{N^{s_2/2}\omega^{(s_2+\kappa)/2}\Gamma(s_2)} \\ & \times \sum_{m=0}^\infty \sum_{n=1}^\infty \frac{\widetilde{a}(n)}{(m+\alpha)^{s_1+s_2/2-\kappa/2} n^{-s_2/2+\kappa/2}} K_{\kappa-s_2}\left(4\pi\left(\frac{(m+\alpha)n}{N\omega}\right)^{1/2}\right). \end{aligned}$$

Since the K-Bessel function is of exponential decay, this formula gives a double series expression of $L_2(s_1, s_2; \alpha, \omega; f)$, which is valid for any $s_1, s_2 \in \mathbb{C}$. Therefore we can observe that $L_2(s_1, s_2; \alpha, \omega; f)$ is holomorphic in the whole space $\mathbb{C}^2$, and $s_2 = 0, -1, -2, \ldots$ are zero-divisors of it.

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