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Relations of multiple zeta values: from the viewpoint of some special functions

By

Zhonghua Li*

Abstract

We review some relations of multiple zeta values and of multiple zeta-star values that can be deduced by using some special functions.

§ 1. Introduction

For a sequence $\mathbf{k}=(k_1,k_2,\ldots,k_n)$ of positive integers with $k_1>1$, the multiple zeta value (abbreviated as MZV) $\zeta(\mathbf{k})$ and the multiple zeta-star value (abbreviated as MZSV) $\zeta^\star(\mathbf{k})$ are defined by the following convergent series

$$\zeta(\mathbf{k})=\zeta(k_1,k_2,\ldots,k_n):=\sum_{m_1>m_2>\cdots>m_n>0}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}},$$
$$\zeta^\star(\mathbf{k})=\zeta^\star(k_1,k_2,\ldots,k_n):=\sum_{m_1\geqslant m_2\geqslant\cdots\geqslant m_n\geqslant 1}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}},$$

respectively. When $n=1$, both values reduce to the special values of the Riemann zeta function at positive integer arguments, which we call zeta values.

There are many elegant algebraic relations among these values. In this paper, we review some relations of MZV's and of MZSV's that can be deduced by using some special functions. This paper is not served as a survey paper. Here we just recall the relations that we are familiar with or we have some contributions to.

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*Department of Mathematics, Tongji University, Shanghai 200092, China.
e-mail: `zhonghua_li@tongji.edu.cn`

This paper is organized as follows. In Section 2, we recall some special functions that will appear here. In Section 3, we illustrate two simple examples of evaluations of MZV's deduced by using the sine function. In Section 4, we review some results about some special sums of MZV's and of MZSV's which related to the generalized hypergeometric functions. In Section 5, we explain a proof of the Zagier's evaluation formula of the MZV's $\zeta(2,\dots,2,3,2,\dots,2)$ by using the generalized hypergeometric function ${}_3F_2$.

§ 2. Some special functions

We recall some special functions that will appear in this paper. The first one is the gamma function $\Gamma(z)$. When the real part of z is positive, the gamma function can be defined by an integral

$$\Gamma(z):=\int_0^\infty t^{z-1}e^{-t}dt.$$

The property of gamma function that we most frequently use is the Taylor series of $\Gamma(1-z)$ in 0

$$\Gamma(1-z)=\exp\left(\gamma z+\sum_{n=2}^\infty \frac{\zeta(n)}{n}z^n\right),\quad |z|<1, \tag{2.1}$$

where γ is the Euler's constant.

The second one is the digamma function, which is

$$\psi(z):=\frac{d}{dz}\log\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)}=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right)dt,\quad \Re(z)>0.$$

The Taylor series of $\psi(1-z)$ in 0 is

$$\psi(1-z)=-\gamma-\sum_{n=1}^\infty \zeta(n+1)z^n,\quad |z|<1.$$

The third one is the generalized hypergeometric function (For more details, see for example [4]). Let p be a nonnegative integer and $a_1,\dots,a_{p+1},b_1,\dots,b_p$ be complex numbers with $b_1,\dots,b_p$ not nonpositive integers. Then we have the generalized hypergeometric function

$$\begin{aligned}{}_{p+1}F_p\left({a_1,\dots,a_{p+1}\atop b_1,\dots,b_p};z\right):=&1+\frac{a_1\cdots a_{p+1}}{b_1\cdots b_p}z+\frac{a_1(a_1+1)\cdots a_{p+1}(a_{p+1}+1)}{2!b_1(b_1+1)\cdots b_p(b_p+1)}z^2+\cdots\\ =&\sum_{n=0}^\infty\frac{(a_1)_n\cdots(a_{p+1})_n}{n!(b_1)_n\cdots(b_p)_n}z^n,\end{aligned}$$

with the Pochhammer symbol $(a)_n$ defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n=1, \\ a(a+1)\cdots(a+n-1), & \text{if } n>1. \end{cases}$$

It is known that this formal power series converges absolutely for $|z|<1$, and it also converges absolutely for $|z|=1$ if $\Re(\sum b_i - \sum a_i) > 0$. Furthermore, this function is a solution to the differential equation

$$[\mathcal{D}(\mathcal{D}+b_1-1)\cdots(\mathcal{D}+b_p-1) - z(\mathcal{D}+a_1)(\mathcal{D}+a_2)\cdots(\mathcal{D}+a_{p+1})]y = 0 \tag{2.2}$$

with $\mathcal{D} = z\frac{d}{dz}$.

When $p=1$, we get the classical Gaussian hypergeometric function

$$_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n.$$

If $\Re(c-a-b) > 0$, we have the Gaussian summation formula

$$_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2.3}$$

which relates Gaussian hypergeometric function to gamma functions, and then to zeta values. When $p=2$, we get the generalized hypergeometric function

$$_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n}{n!(b_1)_n(b_2)_n} z^n.$$

At this case, we have the Dixon's theorem

$$_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}; 1\right) = \frac{\Gamma(1+a/2)\Gamma(1+a/2-b-c)\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a/2-b)\Gamma(1+a/2-c)},$$

provided that $\Re(1+a/2-b-c) > 0$. In [9], J. L. Lavoie et al. generalized Dixon's theorem. They showed that for some integers i, j (23 pairs), the specialization

$$_3F_2\left(\begin{matrix} a, b, c \\ 1+i+a-b, 1+i+j+a-c \end{matrix}; 1\right)$$

can be represented by gamma functions, and for some integers m, k, l (26 triples), the specialization

$$_3F_2\left(\begin{matrix} a, b, m \\ a-b+l-k+m+1, a+l+1 \end{matrix}; 1\right)$$

can be represented by digamma functions. In the case $m=1$, $k=l=0$, the summation formula is

$$ {}_3F_2\left(\begin{matrix} a,b,1 \\ a-b+2,a+1 \end{matrix};1\right) = -\frac{a(1+a-b)}{2(b-1)} \times (\psi((a+1)/2)-\psi(a/2)+\psi(a/2-b+1)-\psi((a+1)/2-b+1)). \tag{2.4} $$

§ 3. Evaluations of some MZV's

We recall two simple evaluation formulas of MZV's, which can be deduced from the infinite product expansion of the sine function

$$ \frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right). \tag{3.1} $$

Since the Taylor series of $\sin x$ at $x=0$ is

$$ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, $$

we get from equation (3.1) that

$$ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi z)^{2n} = \sum_{n=0}^{\infty} (-1)^n \zeta(\underbrace{2,\ldots,2}_{n}) z^{2n}. $$

Hence one get the evaluation formula of $\zeta(2,\ldots,2)$.

Theorem 3.1 ([6]). *For any nonnegative integer n, we have*

$$ \zeta(\underbrace{2,\ldots,2}_{n}) = \frac{1}{(2n+1)!} \pi^{2n}. $$

Applying the operator $\frac{d}{dz}\log$ to equation (3.1), we get

$$ \sum_{k=1}^{\infty} 2\zeta(2k) z^{2k-1} = \frac{1}{z} - \pi \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z} - \pi\sqrt{-1} - \frac{2\pi\sqrt{-1}}{e^{2\pi\sqrt{-1}z}-1}. $$

Then one obtain the evaluation formula of zeta values at even arguments.

Theorem 3.2 (Euler). *For any positive integer k, we have*

$$ \zeta(2k) = -\frac{B_{2k}}{2(2k)!} (2\pi\sqrt{-1})^{2k}, $$

where $\{B_n\}_{n=0}^{\infty}$ are the Bernoulli numbers defined by

$$ \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t-1}. $$

§ 4. Sum of MZ(S)V's

In this section, we review some results about the sum of MZV's (and of MZSV's) that can be represented as a polynomial of zeta values with rational coefficients.

For a sequence $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ of positive integers with $k_1 > 1$, as in [18], the weight, depth and height of $\mathbf{k}$ are defined by

$$\mathrm{wt}(\mathbf{k}) := k_1 + \cdots + k_n, \quad \mathrm{dep}(\mathbf{k}) := n, \quad \mathrm{ht}(\mathbf{k}) := \sharp\{l \mid 1 \leqslant l \leqslant n, k_l \geqslant 2\},$$

respectively. Note that $\mathrm{ht}(\mathbf{k}) = 1$ if and only if $\mathbf{k}$ has the form $\mathbf{k} = (k-n+1, \underbrace{1, \ldots, 1}_{n-1})$. Now for positive integers k, n, s with $k \geqslant n+s$ and $n \geqslant s$, we define the sums

$$\begin{aligned} X_0(k, n, s) &:= \sum_{\mathrm{wt}(\mathbf{k})=k, \mathrm{dep}(\mathbf{k})=n, \mathrm{ht}(\mathbf{k})=s} \zeta(\mathbf{k}), \\ X_0^\star(k, n, s) &:= \sum_{\mathrm{wt}(\mathbf{k})=k, \mathrm{dep}(\mathbf{k})=n, \mathrm{ht}(\mathbf{k})=s} \zeta^\star(\mathbf{k}). \end{aligned}$$

For example, the sums in the height one case are just one term

$$X_0(k, n, 1) = \zeta(k-n+1, \underbrace{1, \ldots, 1}_{n-1}), \quad X_0^\star(k, n, 1) = \zeta^\star(k-n+1, \underbrace{1, \ldots, 1}_{n-1}).$$

In [18], Y. Ohno and D. Zagier found that some type generating function of sums $X_0(k, n, s)$ is related to the Gaussian hypergeometric function

$$\sum_{k \geqslant n+s, n \geqslant s \geqslant 1} X_0(k, n, s) u^{k-n-s} v^{n-s} t^{s-1} = \frac{1}{uv - t} \left\{ 1 - {}_2F_1 \left(\begin{matrix} \alpha - u, \beta - u \\ 1 - u \end{matrix} ; 1 \right) \right\}$$

with α and β determined by $\alpha + \beta = u + v$ and $\alpha\beta = t$. Then by the Gaussian summation formula (2.3) and the Taylor series of gamma function (2.1), one get the so-called Ohno-Zagier relation.

Theorem 4.1 ([18]). *Let u, v and t be formal variables. We have*

$$\begin{aligned} &\sum_{k \geqslant n+s, n \geqslant s \geqslant 1} X_0(k, n, s) u^{k-n-s} v^{n-s} t^{s-1} \\ =& \frac{1}{uv - t} \left\{ 1 - \exp \left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (u^n + v^n - \alpha^n - \beta^n) \right) \right\} \end{aligned}$$

with α and β determined by $\alpha + \beta = u + v$ and $\alpha\beta = t$. In particular, for any positive integers k, n, s with $k \geqslant n+s$ and $n \geqslant s$, the sum $X_0(k, n, s)$ is a polynomial of zeta values with rational coefficients.

In [1], a similar formula holds for the sums $X_0^\star(k,n,s)$. While this time we do not meet the Gaussian hypergeometric function but the generalized hypergeometric function ${}_3F_2$.

Theorem 4.2 ([1]). *Let u,v and t be formal variables. We have*

$$\begin{aligned}\Phi_0^\star(u,v,t):=&\sum_{k\geqslant n+s,n\geqslant s\geqslant 1}X_0^\star(k,n,s)u^{k-n-s}v^{n-s}t^{2s-2}\\=&\frac{1}{(1-v)(1-\beta)}\,{}_3F_2\left(\begin{matrix}1-\beta,1-\beta+u,1\\2-v,2-\beta\end{matrix};1\right)\end{aligned}$$

with α and β determined by $\alpha+\beta=u+v$ and $\alpha\beta=uv-t^2$.

To prove Theorem 4.1 and Theorem 4.2, the authors considered multiple polylogarithms

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z):=\sum_{m_1>m_2>\cdots>m_n>0}\frac{z^{m_1}}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}},$$
$$\mathrm{Li}^\star_{k_1,k_2,\ldots,k_n}(z):=\sum_{m_1\geqslant m_2\geqslant\cdots\geqslant m_n\geqslant 1}\frac{z^{m_1}}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}},$$

which tends to MZV $\zeta(k_1,k_2,\ldots,k_n)$ and MZSV $\zeta^\star(k_1,k_2,\ldots,k_n)$ when z goes to 1 provided that $k_1>1$, respectively. Then they considered similar sums of multiple polylogarithms and their generating function, built the differential equation satisfied by this generating function, obtained the representation of the generating function via (generalized) hypergeometric function by comparing the differential equation with (2.2), and finally got the desired formula by setting $z=1$. We would like to mention that in [14], we found that the Ohno-Zagier relation can be deduced from the regularized double shuffle relation([8]), which provides a pure algebraic proof of the Ohno-Zagier relation.

By the representation in Theorem 4.2, we could say that in general the sum $X_0^\star(k,n,s)$ is not a polynomial of zeta values with rational coefficients. For example, by [10, Eq. (2.2) and (2.3)], we have

$$X_0^\star(8,6,1)=\zeta^\star(3,\underbrace{1,\ldots,1}_{5})=-\zeta(6,2)-6\zeta(7,1)+\zeta(2)\zeta(6).$$

But it is conjectured that the value $\zeta(6,2)$ can not be reduced. Hence $X_0^\star(8,6,1)$ can not be a polynomial of zeta values with rational coefficients. However, in [10, Theorem 1], M. Kaneko and Y. Ohno showed that for any positive integers k and n, the difference

$$(-1)^k\zeta^\star(k+1,\underbrace{1,\ldots,1}_{n})-(-1)^n\zeta^\star(n+1,\underbrace{1,\ldots,1}_{k})$$

is indeed a polynomial of zeta values with rational coefficients. They proved this by using the special values of the zeta function

$$\xi_k(s) := \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \operatorname{Li}_k(1 - e^{-t}) dt$$

defined in [3] and Ohno's relation ([17]). In [20], C. Yamazaki gave another proof of this result by using the representation of the generating function $\Phi_0^\star(u, v, t)$ obtained in [1]. Then M. Kaneko and Y. Ohno conjectured in [10] that this result can be generalized from height one to general height. In [13], we modified the proof of C. Yamazaki, used Theorem 4.2, borrowed two transformation formulas of ${}_3F_2$ from [4], and proved this conjecture.

Theorem 4.3 ([13]). *We have*

$$\begin{aligned} u\Phi_0^\star(-u, v, t) - v\Phi_0^\star(-v, u, t) = & \frac{u - v}{\alpha\beta} + \frac{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(\beta)\Gamma(1-\beta)\Gamma(u+\beta)}{\Gamma(u)\Gamma(v)\Gamma(1-u-\alpha)} \\ & \times \left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(\alpha)\Gamma(1-\alpha)} + \frac{\Gamma(u)\Gamma(1-u)}{\Gamma(\beta)\Gamma(1-\beta)} \right) \end{aligned}$$

with α and β determined by $\alpha + \beta = -u + v$ and $\alpha\beta = -uv - t^2$. In particular, for any positive integers m, n, s with $m, n \geqslant s$, the difference

$$(-1)^m X_0^\star(m + n + 1, n + 1, s) - (-1)^n X_0^\star(m + n + 1, m + 1, s)$$

is a polynomial of zeta values with rational coefficients.

The two transformation formulas ([4]) which we used in the proof are

$$\begin{aligned} {}_3F_2\left({\alpha_1, \alpha_2, \alpha_3 \atop \beta_1, \beta_2}; 1 \right) = & \frac{\Gamma(\beta_1)\Gamma(\beta_1 - \alpha_1 - \alpha_2)}{\Gamma(\beta_1 - \alpha_1)\Gamma(\beta_1 - \alpha_2)} \, {}_3F_2\left({\alpha_1, \alpha_2, \beta_2 - \alpha_3 \atop \alpha_1 + \alpha_2 - \beta_1 + 1, \beta_2}; 1 \right) \\ & + \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1 + \alpha_2 - \beta_1)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2 - \alpha_3)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)} \\ & \times {}_3F_2\left({\beta_1 - \alpha_1, \beta_1 - \alpha_2, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3 \atop \beta_1 - \alpha_1 - \alpha_2 + 1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2}; 1 \right), \end{aligned} \tag{4.1}$$

provided that $\Re(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\Re(\alpha_3 - \beta_1 + 1) > 0$, and

$${}_3F_2\left({\alpha_1, \alpha_2, \alpha_3 \atop \beta_1, \beta_2}; 1 \right) = \frac{\Gamma(\beta_2)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\beta_2 - \alpha_3)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)} \, {}_3F_2\left({\beta_1 - \alpha_1, \beta_1 - \alpha_2, \alpha_3 \atop \beta_1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2}; 1 \right), \tag{4.2}$$

provided that $\Re(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\Re(\beta_2 - \alpha_3) > 0$.

In [11], we studied a possible generalization of the Ohno-Zagier relation. In fact, for a sequence $\mathbf{k}=(k_1,k_2,\ldots,k_n)$ of positive integers with $k_1>1$ and for any positive integer i, we defined i-height of $\mathbf{k}$ to be

$$i\text{-}\operatorname{ht}(\mathbf{k}):=\sharp\{l\mid 1\leqslant l\leqslant n, k_l\geqslant i+1\}.$$

Note that 1-height is just height. Then for a positive integer r and integers $k,n,h_1,\ldots,h_r$, we set

$$X_0^{(r)}(k,n,h_1,\ldots,h_r):=\sum_{\substack{\operatorname{wt}(\mathbf{k})=k,\operatorname{dep}(\mathbf{k})=n,\\ 1\text{-}\operatorname{ht}(\mathbf{k})=h_1,\ldots,r\text{-}\operatorname{ht}(\mathbf{k})=h_r}}\zeta(\mathbf{k}).$$

By studying similar sums of multiple polylogarithms, building the differential equation satisfied by a generating function of these sums, and finally letting the variable z to be 1, we represent the generating function

$$\sum_{k,n,h_1,\ldots,h_r\geqslant 0}X_0^{(r)}(k,n,h_1,\ldots,h_r)u_1^{k-n-\sum h_j}u_2^{n-h_1}u_3^{h_1-h_2}u_4^{h_2-h_3}\cdots u_{r+1}^{h_{r-1}-h_r}u_{r+2}^{h_r}$$

by a sum of generalized hypergeometric functions ${}_{r+1}F_r\left({\cdots \atop \cdots};1\right)$'s. The general formula ([11, Theorem 1.1]) is too complicated to recall here. While for $r=1$, it is just Ohno-Zagier relation, and for $r=2$, the formula reads as

$$\begin{aligned}&\sum_{k,n,h_1,h_2\geqslant 0}X_0^{(2)}(k,n,h_1,h_2)u_1^{k-n-h_1-h_2}u_2^{n-h_1}u_3^{h_1-h_2}u_4^{h_2}\\ =&\frac{u_3}{1-u_1}\,{}_3F_2\left({a_1+1,a_2+1,a_3+1 \atop 2-u_1,2};1\right)+{}_3F_2\left({a_1,a_2,a_3 \atop 1-u_1,1};1\right)-1,\end{aligned}$$

where a_1,a_2,a_3 are determined by the conditions

$$\begin{cases}a_1+a_2+a_3=-u_1+u_2,\\ a_1a_2+a_2a_3+a_3a_1=u_3-u_1u_2,\\ a_1a_2a_3=u_4-u_1u_3.\end{cases}$$

We raised a problem in [11] that whether the sum $X_0^{(r)}(k,n,h_1,\ldots,h_r)$ is a polynomial of zeta values with rational coefficients. We have a positive answer in the case $r=1$ (by the Ohno-Zagier relation) and the case $n=h_1$ (by the symmetric sum formula ([6, Theorem 2.1])). While in the general case, J. Zhao found a counterexample ([21])

$$\begin{aligned}&X_0^{(2)}(10,4,3,2)=X_0^{(3)}(10,4,3,2,1)=\zeta(2,1,3,4)+\zeta(2,1,4,3)+\cdots\quad(18\ \text{ terms})\\ =&\frac{47}{2}\zeta(2)\zeta(3)\zeta(5)-\frac{4399}{770}\zeta(2)^5+\frac{91}{4}\zeta(5)^2+\frac{11}{20}\zeta(2)^2\zeta(3)^2-\frac{45}{4}\zeta(2)\zeta(6,2)-\frac{73}{16}\zeta(3)\zeta(7).\end{aligned}$$

Again, the double zeta value $\zeta(6,2)$ leads to the assertion that the above sum is not a polynomial of zeta values with rational coefficients.

In [2], T. Aoki, Y. Ohno and N. Wakabayashi studied the MZSV version of our result [11]. They represented a generating function of sums

$$\sum_{\substack{\mathrm{wt}(\mathbf{k})=k,\mathrm{dep}(\mathbf{k})=n,\\ 1\text{-}\mathrm{ht}(\mathbf{k})=h_1,\ldots,r\text{-}\mathrm{ht}(\mathbf{k})=h_r}} \zeta^\star(\mathbf{k})$$

by a sum of ${}_{r+2}F_{r+1}\left({\cdots \atop \cdots};1\right)$'s. And in [12], we studied the q-version of our result [11].

§ 5. Zagier's evaluation formula

In a recent paper [5], F. Brown made a great progress in the study of MZV's. He succeeded in proving the basis conjecture of M. Hoffman ([7]), which claims that every MZV is a rational combination of MZV's with all arguments 2 or 3. In his proof, an evaluation formula of the multiple zeta values $\zeta(2,\ldots,2,3,2,\ldots,2)$ proved by D. Zagier in [22] plays an important role.

Theorem 5.1 ([22]). *For any two nonnegative integers a and b, we have*

$$\zeta(\underbrace{2,\ldots,2}_{b},3,\underbrace{2,\ldots,2}_{a}) = \sum_{r=1}^{a+b+1} c_{a,b}^{r}\zeta(2r+1)\zeta(\underbrace{2,\ldots,2}_{a+b+1-r}), \tag{5.1}$$

$$\zeta^\star(\underbrace{2,\ldots,2}_{b},3,\underbrace{2,\ldots,2}_{a}) = \sum_{r=1}^{a+b+1} c_{a,b}^{\star,r}\zeta(2r+1)\zeta^\star(\underbrace{2,\ldots,2}_{a+b+1-r}), \tag{5.2}$$

where

$$c_{a,b}^{r} = 2(-1)^r\left\{\binom{2r}{2a+2} - \left(1-\frac{1}{2^{2r}}\right)\binom{2r}{2b+1}\right\},$$
$$c_{a,b}^{\star,r} = -2\left\{\binom{2r}{2a} - \delta_{r,a} - \left(1-\frac{1}{2^{2r}}\right)\binom{2r}{2b+1}\right\},$$

with $\delta_{r,a}$ the Kronecker's delta symbol.

Note that the above two formulas (5.1) and (5.2) are equivalent as shown in [22]. A pure algebraic proof of the equivalence is given in [15]. In this section, we want to talk about the proof of the evaluation formula (5.1). To prove this formula, D. Zagier considered the following two generating functions

$$F(x,y) := \sum_{a,b=0}^{\infty}(-1)^{a+b+1}\zeta(\underbrace{2,\ldots,2}_{b},3,\underbrace{2,\ldots,2}_{a})x^{2a+2}y^{2b+1},$$

$$\widehat{F}(x,y) := \sum_{a,b=0}^{\infty}(-1)^{a+b+1}\left(\sum_{r=1}^{a+b+1} c_{a,b}^{r}\zeta(2r+1)\zeta(\underbrace{2,\ldots,2}_{a+b+1-r})\right)x^{2a+2}y^{2b+1}.$$

Then it is enough to prove that $F(x,y)=\widehat{F}(x,y)$. D. Zagier showed that these two generating functions can be represented by special functions

$$\frac{\pi}{\sin \pi y}F(x,y)=\frac{d}{dz}\Big|_{z=0}\,{}_3F_2\left(\begin{matrix}x,-x,z\\1+y,1-y\end{matrix};1\right), \tag{5.3}$$

$$\begin{aligned}\frac{\pi}{\sin \pi y}\widehat{F}(x,y)=&\frac{1}{2}\cdot[2\psi(1+y)+2\psi(1-y)-\psi(1+x+y)-\psi(1-x-y)\\&-\psi(1+x-y)-\psi(1-x+y)]-\frac{\sin\pi x}{2\sin\pi y}\cdot[\psi(1+(x+y)/2)\\&+\psi(1-(x+y)/2)-\psi(1+(x-y)/2)-\psi(1-(x-y)/2)\\&-\psi(1+x+y)-\psi(1-x-y)+\psi(1+x-y)+\psi(1-x+y)].\end{aligned} \tag{5.4}$$

D. Zagier proved indirectly that $F(x,y)=\widehat{F}(x,y)$. In fact, he showed that $F(x,y)$ and $\widehat{F}(x,y)$ are entire functions on $\mathbb{C}\times\mathbb{C}$, and that they have good behaviors at infinity

$$F(x,y),\widehat{F}(x,y)=O(e^{\pi X}\log X),\quad X=\max(|x|,|y|)\to\infty.$$

Furthermore, he showed that for any $x\in\mathbb{C}$ and $k\in\mathbb{Z}$,

$$F(x,k)=\widehat{F}(x,k),\quad F(x,x)=\widehat{F}(x,x).$$

And then using the fact that an entire function $f:\mathbb{C}\longrightarrow\mathbb{C}$ that vanishes at all integers and satisfies $f(z)=O(e^{\pi|\Im(z)|})$ is a constant multiple of $\sin\pi z$, he got the equality $F(x,y)=\widehat{F}(x,y)$.

In [16], we gave another proof of the equality $F(x,y)=\widehat{F}(x,y)$. We showed that the right-hand sides of equations (5.3) and (5.4) are the same. In fact, noting that

$${}_3F_2\left(\begin{matrix}x,-x,z\\1+y,1-y\end{matrix};1\right)=\frac{1}{2}\,{}_3F_2\left(\begin{matrix}x,1-x,z\\1+y,1-y\end{matrix};1\right)+\frac{1}{2}\,{}_3F_2\left(\begin{matrix}1+x,-x,z\\1+y,1-y\end{matrix};1\right),$$

it is sufficient to prove

$$\begin{aligned}\frac{d}{dz}\Big|_{z=0}\,{}_3F_2\left(\begin{matrix}x,1-x,z\\1+y,1-y\end{matrix};1\right)&=\psi(1+y)+\psi(1-y)-\psi(1-x+y)-\psi(1-x-y)\\-\frac{\sin\pi x}{\sin\pi y}\cdot[\psi(1-x+y)&-\psi(1-x-y)-\psi(1-(x-y)/2)+\psi(1-(x+y)/2)].\end{aligned}$$

To prove the above formula, we used the transformation formula (4.1) to write the ${}_3F_2$-series occurred in the left-hand side as a sum of two ${}_3F_2$-series, and then applied the transformation formula (4.2) to the first ${}_3F_2$-series. Finally, we used the summation formula (2.4) after applying the operator $\frac{d}{dz}\big|_{z=0}$ to get the desired result. For more details, please see our paper [16].

We finally remark that in [19], T. Terasoma proved that the Zagier's evaluation formula holds for the coefficients of any associator.

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