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A small remark on finite multiple zeta values and p-adic multiple zeta values

By

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Abstract

We show a relation between finite multiple zeta values of depth 2 and p-adic multiple zeta values of depth 1.

Recently, Kaneko-Zagier defined finite multiple zeta values (i.e., a finite analogue of multiple zeta values), and they studied a (possible) relationship between the finite multiple zeta values and the (classical) multiple zeta values. In this short paper, we study a relationship between the finite multiple zeta values and the p-adic multiple zeta values in a little bit. More concretely, we show a relation between finite multiple zeta values of depth 2 and p-adic multiple zeta values of depth 1. This is a small partial answer to the question which the author was privately asked by Francis Brown in 2012 and by Masanobu Kaneko in 2013 independently (the question is on the relationship between finite multiple zeta values and p-adic multiple zeta values). The result comes from just a combination of calculations of Hoffman ([H]) and calculations in Washington's book ([W1997]), however, it seems to the author that no one has pointed out such relation yet and that no concrete relation between finite multiple zeta values and p-adic multiple zeta values is found yet within his best knowledge.

For $k_1, \cdots, k_d \in \mathbb{Z}$, we define *mod p multiple zeta value* $\zeta_{\mathrm{mod}\, p}(k_1, \ldots, k_d) \in \mathbb{F}_p$ to be

$$\zeta_{\mathrm{mod}\, p}(k_1, \ldots, k_d) := \sum_{0<n_1<\cdots<n_d<p} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} \mod p.$$

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Put $\mathcal{A} := (\prod_{p:\mathrm{prime}} \mathbb{F}_p)/(\oplus_{p:\mathrm{prime}}\mathbb{F}_p)$. Then, Kaneko-Zagier defined *finite multiple zeta value* $\zeta_{\mathcal{A}}(k_1, \ldots, k_d) \in \mathcal{A}$ to be

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_d) := (\zeta_{\mathrm{mod}\, p}(k_1, \ldots, k_d))_{p:\mathrm{prime}} \in \mathcal{A}.$$

On the other hand, Furusho defined *p-adic multiple zeta value* $\zeta_p(k_1, \cdots, k_d)$ for $k_1, \cdots, k_{d-1} \geq 1$, and $k_d > 1$ by using Coleman's p-adic integration theory (See [F2004], [Y2012] for the definition, [Y2010] for a fundamental conjecture and related theorem). (He also partially defined in the case where $k_d = 1$. However, we do not use this fact in this paper.)

We will show the following small remark:

Proposition 0.1. *Let $p > 2$ be a prime number, $n > 1$ a natural number such that $p-1$ does not divide $n-1$. Let B_m denote the m-th Bernoulli number (or Bernoulli-Seki number) for $m \in \mathbb{Z}_{\geq 0}$, i.e., the rational number defined by $\frac{t}{e^t-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}$. Set* ord *to be the p-adic valuation normalised by* $\mathrm{ord}(p) = 1$. *Then we have*

1. $\mathrm{ord}(\zeta_p(n)) \geq n$, *and*
2. *For any $i > 0$ such that $p-1$ does not divide $i-1$, we have*

$$\frac{1}{p^n}\zeta_p(n) \equiv \frac{(-1)^{n-i+1}}{\binom{n'}{i'}}\zeta_{\mathrm{mod}\, p}(i, n-i) \equiv \frac{B_m}{m} \mod p,$$

where $m > 1$ is a natural number such that $m \equiv 1 - n \pmod{p-1}$ (e.g. $m = p - n$ if $n < p - 1$), and i', n' are natural numbers such that $0 < i', n' < p - 1$ and $i' \equiv i \pmod{p-1}$, $n' \equiv n \pmod{p-1}$.

Remark. By the Kummer congruence, $\frac{B_m}{m}$ modulo p is independent of the choice of m.

Corollary 0.2. *For $n \in \mathbb{Z}_{>1}$ and $0 < i < n$, we have*

$$\zeta_{\mathcal{A}}(i, n-i) = \left[(*)_{p\leq \max(2,n+1)}, \left((-1)^{n-i+1}\frac{\binom{n}{i}}{p^n}\zeta_p(n) \bmod p \right)_{p>\max(2,n+1)} \right] \text{ in } \mathcal{A},$$

where $*$ is any element in $\mathbb{F}_p$, and [ ] means modulo $\oplus_p\mathbb{F}_p$ (note that the class does not depend the choices of $*$ in $\mathbb{F}_p$ for $p \leq \max(2, n+1)$ (i.e., finitely many p)). In particular, if $\zeta_{\mathcal{A}}(i, n-i)$ does not vanish, then $\zeta_p(n)$ does not vanish for infinitely many p.

Remark. For odd n, the non-vanishing of $\zeta_p(n)$ is equivalent to the higher Leopoldt conjecture (See [F2004, Examples 2.19 (b)]), which is currently known in the case where p is regular or $p-1$ divides $n-1$. Note also that it is not known yet that there is infinitely many regular primes.

Remark. For fixed $k \in \mathbb{Z}$, if $p-1$ does not divide k, then we have $\zeta_{\mathrm{mod}\, p}(k) = 0$. Therefore, we have $\zeta_{\mathcal{A}}(k) = 0$ for any $k \in \mathbb{Z}$. It is a general expectation of Kaneko-Zagier that the finite multiple zeta values of depth $d+1$ behave like the multiple zeta values of depth d. The above proposition and corollary show a relation with the p-adic ones for $d = 1$.

Remark. Recently [IKT2014, p.11] shows that $\zeta_{\mathcal{A}}(1, \ldots (n-2\,\text{times}\,) \ldots, 1, 2) = (B_{p-n} \bmod p)_p$. By this result, we also deduce that if $\zeta_{\mathcal{A}}(1, \ldots (n-2\,\text{times}\,) \ldots, 1, 2)$ does not vanish, then $\zeta_p(n)$ does not vanish for infinitely many p.

Proof of the proposition. Take i' and n' as in the proposition. Then we have $\zeta_{\mathrm{mod}\, p}(i, n-i) = \zeta_{\mathrm{mod}\, p}(i', n'-i')$. We also have (see [H, Theorem 6.1])

$$\zeta_{\mathrm{mod}\, p}(i', n'-i') = (-1)^{n'-i'} \binom{n'}{i'} \frac{B_{p-n'}}{n'} = (-1)^{n-i+1} \binom{n'}{i'} \frac{B_{p-n'}}{p-n'}$$

in $\mathbb{F}_p$, by using the formula of Bernoulli-Seki ([S1712], [B1713])

$$\sum_{0<a<n} a^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j}.$$

Let $\omega : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ be the Teichmüller character. Then, we have

$$\zeta_p(n) = \frac{p^n}{p^n - 1} L_p(n, \omega^{1-n})$$

for $n > 1$ by [F2004, Examples 2.19]. On the other hand, we have

$$L_p(a, \omega^{1-n}) \in \mathbb{Z}_p$$

for $a, n \in \mathbb{Z}$ such that $n - 1 \not\equiv 0 \mod p-1$, by [W1997, Theorem 5.12], and we have

$$L_p(a, \omega^{1-n}) \equiv L_p(b, \omega^{1-n}) \mod p$$

for $a, b, n \in \mathbb{Z}$ such that $n-1 \not\equiv 0 \mod p-1$, by [W1997, Corollary 5.13]. Thus, we have

$$\mathrm{ord}(\zeta_p(n)) = n + \mathrm{ord}(L_p(n, \omega^{1-n})) \geq n,$$

and

$$\frac{1}{p^n}\zeta_p(n) \equiv -L_p(n, \omega^{1-n}) = -L_p(n, \omega^m) \equiv -L_p(1-m, \omega^m) = (1-p^{m-1})\frac{B_m}{m} \equiv \frac{B_m}{m} \bmod p$$

for $m > 1$ such that $m \equiv 1-n \mod p-1$. $\square$

Remark. By a repeated use of the formula of Bernoulli-Seki, we can show that $\zeta_{\mathrm{mod}\,p}(k_1, \dots, k_{d+1})$ is a linear combination of products of d Bernoulli numbers by the same argument of the calculation of $\zeta_{\mathrm{mod}\,p}(i, k-i)$ in [H].

We conjecture that $\mathrm{ord}(\zeta_p(k_1, \dots, k_d)) \geq k_1 + \cdots + k_d$ and $\frac{1}{p^{k_1+\cdots+k_d}}\zeta_p(k_1, \dots, k_d)$ mod p should be related to the finite multiple zeta values of depth $d+1$. The author was informed by Seidai Yasuda that Kentaro Ihara and he had formulated a precise conjectural relation between the finite multiple zeta values and the p-adic multiple zeta values for arbitrary depth.

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