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Some remarks on Volume 17 of the *Taisei Sankei*

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Abstract

We shall see how the determinant was introduced in Volume 17 of the *Taisei Sankei* (1711), where the full expansion formulas of determinant of orders 2, 3, 4 and 5 were given in connection with the elimination theory of the system of algebraic equations with several unknowns. We shall examine, among others, the exactitude of the description of determinants.

§ 1. About the *Taisei Sankei*

The *Taisei Sankei* 大成算経 is an encyclopedic work of mathematics written by three mathematicians Seki Takakazu 関孝和 (1640s−1708), Takebe Kata'akira 建部賢明 (1661−1716) and Takebe Katahiro 建部賢弘 (1664−1739) during 1683−1711. Although many English translations of the title have been proposed, e. g., *Accomplished Classic of Mathematics* [2], *Complete Book of Mathematics* [5], *Great Accomplished Mathematical Treatise* [4], *Summa of Mathematical Canon* (SK), etc., we designate it in this paper by the romanised Japanese title *Taisei Sankei*.

The work is composed of twenty volumes preceded by an introductory volume.

Volumes 16 − 20 treat the theory of equations: that is,

Volume 16 題術辯 Discussions of problems, procedures and extractions,

Volume 17 全題解 Types of problems (direct calculations, implicit problems, concealed problems, submerged problems),

Volume 18 病題議 Restoration of defective problems,

Volume 19 演段例上 Operational examples (implicit problems, concealed problems),

Volume 20 演段例下 Operational examples (submerged problems).

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The theory of resultants and determinants, which is one of the most remarkable achievements of Seki, is expounded in Volume 17, while Volume 19 presents 15 examples to show how to apply this theory to concrete problems. Recently, the speaker translated these two volumes into English (see [12] and [13]).

Parts of this paper were presented orally at the ``Mathematical Texts in East Asia Mathematical History'' Workshop, March 11 − 15, 2016, Hainan, China and at the Bumwoo Conference on the History of Asian Mathematics, August 23 − 26, 2016, Seoul, Korea (see [10]).

§ 2. Manuscripts of the *Taisei Sankei*

There are more than 20 manuscripts of the *Taisei Sankei* (see Komatsu [1]). But in this paper, we work with the following four manuscripts.

1. MS University of Tokyo: This manuscript is called the Kashū edition 霞州本 and dates back to the Takebe Katahiro's time (ca. 1710),
2. MS Tokyo Science University: This manuscript dates back to ca. 1780,
3. MS Kyoto University B: This manuscript dates back to ca. 1850,
4. MS Kyoto University A: This is another manuscript conserved in the Library of Kyoto University (date unknown).

Note that the pages in MSS are numbered 1r (1 recto), 1v (1 verso), 2r (2 recto), 2v (2 vecto), etc. In MS U. Tokyo, each page has 12 lines of 20 characters with no punctuation. In MS Tokyo Science U., each page has 10 lines of 20 characters with no punctuation. In MS Kyoto U. B, each page has 12 lines of 20 characters with vermilion punctuation. In MS Kyoto U. A, each page has 12 lines of 20 characters with vermilion punctuation.

§ 3. Three rules

Now we use the notation of modern mathematics. Let n be an integer.

§ 3.1. Rule of cross multiplications

Consider the simultaneous system of equations:

$$\begin{cases} P_{0,0} & +P_{0,1}x & +\cdots & +P_{0,n-1}x^{n-1} & =0 \\ P_{1,0} & +P_{1,1}x & +\cdots & +P_{1,n-1}x^{n-1} & =0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_{n-1,0} & +P_{n-1,1}x & +\cdots & +P_{n-1,n-1}x^{n-1} & =0 \end{cases} \tag{3.1}$$

Then we have the equation

$$\det \begin{pmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,n-2} & P_{0,n-1} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,n-2} & P_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots & \\ P_{n-1,0} & P_{n-1,1} & \cdots & P_{n-1,n-2} & P_{n-1,n-1} \end{pmatrix} = 0 \tag{3.2}$$

This theorem is called the rule of cross multiplication 交乗法 in the *Taisei Sanklei*, where the determinant is defined recursively in n by the Van der Monde formula.

§ 3.2. Rule of varied multiplications

The coefficient matrix in (3.2) is called symmetric (with respect to the anti-diagonal) if

$$\begin{aligned} &P_{0.j} = P_{n-1-j,n-1}, \quad j = 0, 1, \cdots, n-2 \\ &P_{1,j} = P_{n-1-j,n-2}, \quad j = 0, 1, \cdots, n-3 \\ &\ldots \\ &P_{n-2,0} = P_{n-1,1}. \end{aligned}$$

The rule of cross multiplication for a symmetric coefficient matrix is called the rule of varied multiplication 変乗法 in the *Taisei Sankei*. Note that we transform a pair or two equations of degree n, then the transformed simultaneous system of equations is symmetric. Because of this fact, the symmetric case has a different name.

§ 3.3. Rule of deletion/generation

Problem: Let n be an integer. Consider the system of equations:

$$R_0 + R_1 x + R_2 x^2 + \cdots + R_{n-1} x^{n-1} = 0, \quad x^n = Q^n \tag{3.3}$$

where R_i, $i = 0, 1, 2, \cdots, n-1$ and Q are given constants. Eliminate the unknown x.

The equation (3.3) implies the simultaneous equation (3.4):

$$\begin{cases} R_0 & +R_1 x & +R_2 x^2 & + \cdots +R_{n-1} x^{n-1} = 0, \\ R_{n-1} Q^n & +R_0 x & +R_1 x^2 & + \cdots +R_{n-2} x^{n-1} = 0, \\ R_{n-2} Q^n & +R_{n-1} Q^n & +R_0 x & + \cdots +R_{n-3} x^{n-1} = 0, \\ \ldots \\ R_1 Q^n & +R_2 Q^n x & +R_3 Q^n x^2 & + \cdots +R_0 x^{n-1} \quad = 0 \end{cases} \tag{3.4}$$

By the rule of cross multiplication, we can eliminate x as follows:

$$\det \begin{pmatrix} R_0 & R_1 & R_2 & \cdots & R_{n-1} \\ R_{n-1} Q^n & R_0 & R_1 & \cdots & R_{n-2} \\ R_{n-2} Q^n & R_{n-1} Q^n & R_0 & \cdots & R_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \\ R_1 Q^n & R_2 Q^n & R_3 Q^n & \cdots & R_0 \end{pmatrix} = 0 \tag{3.5}$$

In fact, we used the rule of varied multiplications because the coefficient matrix in (3.4) is symmetric. This is the modern interpretation of the rule of deletion/generation 消長法.

Notes: This problem was very popular in the early days of *wasan*. The cases $n = 2$ (quadratic) and $n = 3$ (cubic) were quite well-known in the beginning stage of *wasan*. In the *Hatsubi Sanpō* 発微算法 (1674), Seki Takakazu treated the problems of elimination only for these cases.

Later, the octic case ($n = 8$) was stated and published by Nakane Genkei 中根元圭 (1662 -- 1733) in the *Shichijōbeki Enshiki* 七乗冪演式 (Procedures for the Octic Power, 1691).

In the *Taisei Sankei* (1711) the exposition was only up to the cases $n = 4$ (quartic) and $n = 5$ (quintic) but the problem is clearly stated along with the theory of determinant (resultant).

§ 4. Statements in the *Taisei Sankei*

§ 4.1. Problem 17-56 (English translation)

Suppose given a pair of two quadratic formulas and transform it to obtain[1]

(4.1)
$$\begin{array}{lll} \text{First formula} & \boxed{[b, a]} & [b + ax = 0] \\ \text{Second fo r mula} & \boxed{[d, c]} & [d + cx = 0] \end{array}$$

(c is regarded as b in the symmetric case.)
Formula to be folded by the quadratic power

(4.2)
$$\boxed{[R_0, R_1]} \quad [R_0 + R_1 x = 0]$$

Three rules for the quadratic case

平方交乗法 Quadratic rule of cross multiplications

乙丙 (加) $[+1bc]$ 甲丁 (減) $[-1ad]$

変乗法 Rule of varied multiplications

乙冪 (一段)(加) $[+1b^2]$ 甲丁 (一段)(減) $[-1ad]$

消長法 Rule of deletion/generation

実冪 (一段)(長) $[+1R_0^2]$ 方冪商冪 (一段)(消) $[-1R_1^2Q^2]$

In the notation of modern mathematics, the three rules can be stated as follows:

Quadratic rule of cross multiplication: (4.1) implies

$$\det \begin{pmatrix} b\ a \\ d\ c \end{pmatrix} = bc - ad = 0.$$

[1] To translate this text into English, we transcribe the ten branches by alphabet as follows:

甲	乙	丙	丁	戊	己	庚	辛	壬	癸
a	b	c	d	e	f	g	h	i	j

In the translation a formula in the bracket [ ] is the transcription in modern mathematical notation.

Rule of varied multiplication: If $c = b$ (symmetric), then

$$\det\begin{pmatrix} b & a \\ d & b \end{pmatrix} = b^2 - ad = 0.$$

Rule of deletion/generation: If (4.2) and $x^2 = Q^2$, then

$$\det\begin{pmatrix} R_0 & R_1 \\ R_1Q^2 & R_0 \end{pmatrix} = R_0^2 - R_1^2Q^2 = 0.$$

§ 4.2. Problem 17--57 (English translation)

Suppose given a pair of two cubic formulas and transform it to obtain

$$\begin{array}{llll} & \text{First formula} & \boxed{[c,b,a]} & [c + bx + ax^2 = 0] \\ (4.3) & \text{Second formula} & \boxed{[f,e,d]} & [f + ex + dx^2 = 0] \\ & \text{Third formula} & \boxed{[i,h,g]} & [i + hx + gx^2 = 0] \end{array}$$

(If these formulas are symmetric, d is regarded as b, g as c and h as f.)
Formula to be folded by the cubic power[2]

$$(4.4) \qquad \boxed{[R_0, R_1, R_2]} \quad [R_0 + R_1x + R_2x^2 = 0]$$

Three rules for the cubic case

立方交乘法 Cubic rule of cross multiplication

甲己辛 (加) $[+1afh]$ 乙丁壬 $[+1bdi]$ 丙戊庚 $[+ceg]$
甲戊壬 (減) $[-1aei]$ 乙己庚 $[-1bfg]$ 丙丁辛 $[-1cdh]$

変乗法 Rule of varied multiplication

甲己冪 (一段)(加) $[+1af^2]$ 乙冪壬 (一段) $[+1b^2i]$
丙冪戊 (一段) $[+1c^2e]$
甲戊壬 (一段)(減) $[-1aei]$ 乙丙己 (二段) $[-2bcf]$

消長法 Rule of deletion/generation

実再 (一段)(長) $[+1R_0^3]$ 方再商再 (一段) $[+1R_1^3Q^3]$
廉再商五 (一段) $[+1R_2^3Q^6]$
実方廉商再 (三段)(消) $[-3R_0R_1R_2Q^3]$

[2]In this problem, we transcribe the Reality 実, the Square 方, the Side 廉 and the quotient 商 by R_0, R_1, R_2 and Q, respectively.

Restate the problem in modern mathematics. If we have (4.3), then

$$\det\begin{pmatrix} c & b & a \\ f & e & d \\ i & h & g \end{pmatrix} = c\det\begin{pmatrix} e & d \\ h & g \end{pmatrix} - b\det\begin{pmatrix} f & d \\ i & g \end{pmatrix} + a\det\begin{pmatrix} f & e \\ i & h \end{pmatrix}$$
$$= afh + bdi + ceg - aei - bfg - cdh = 0.$$

This is called the rule of cross multiplications.

In the symmetric case, that is, $d = b$, $g = c$, $h = f$, the rule of cross multiplications is called the rule of varied multiplications:

$$\det\begin{pmatrix} c & b & a \\ f & e & b \\ i & f & c \end{pmatrix} = af^2 + b^2 i + c^2 e - aei - 2bcf = 0.$$

Suppose (4.4) and $-Q^3 + x^3 = 0$, where R_0, R_1, R_2 and Q^3 are known. Because we have

$$\begin{aligned} R_0 + R_1 x + R_2 x^2 &= 0, \\ R_2 Q^3 + R_0 x + R_1 x^2 &= 0, \\ R_1 Q^3 + R_2 Q^3 x + R_0 x^2 &= 0, \end{aligned}$$

the rule of varied multiplications implies the rule of deletion/generation:

$$\det\begin{pmatrix} R_0 & R_1 & R_2 \\ R_2 Q^3 & R_0 & R_1 \\ R_1 Q^3 & R_2 Q^3 & R_0 \end{pmatrix} = R_0^3 + R_1^3 Q^3 - 3R_0 R_1 R_2 Q^3 + R_2^3 Q^6 = 0.$$

§ 4.3. Problem 17−58 (English translation)

Suppose given the pair of two formulas of degree 4 and transform it to obtain:[3]

(4.5)

First formula	$\boxed{[\delta, \gamma, \beta, \alpha]}$	$[\delta + \gamma x + \beta x^2 + \alpha x^3 = 0]$
Second formula	$\boxed{[\theta, \eta, \zeta, \epsilon]}$	$[\theta + \eta x + \zeta x^2 + \epsilon x^3 = 0]$
Third formula	$\boxed{[\mu, \lambda, \kappa, \iota]}$	$[\mu + \lambda x + \kappa x^2 + \iota x^3 = 0]$
Fourth formula	$\boxed{[\pi, o, \xi, \nu]}$	$[\pi + o x + \xi x^2 + \nu x^3 = 0]$

[3] In order to translate the original text into English, we have to fix the rule of transcription of the twenty eight lodges by alphabets as follows:

角	亢	氐	房	心	尾	箕	斗	牛	女	虚	危	室	壁
α	β	γ	δ	ϵ	ζ	η	θ	ι	κ	λ	μ	ν	ξ
奎	婁	胃	昴	畢	觜	参	井	鬼	柳	星	張	翼	軫
o	π	ρ	σ	τ	υ	ϕ	χ	ψ	ω	Ω	張	翼	軫

Note that we need only $4^2 = 16$ ($5^2 = 25$) characters for a determinant of degree 4 (degree 5), respectively.

(In the symmetric case,we regard ϵ as β, ι as γ, κ as η, ν as δ, ξ as θ and o as μ.)

Formula to be folded by the (fourth) power: [4]

$$\boxed{[R_0, R_1, R_2, R_3]} \quad [R_0 + R_1x + R_2x^2 + R_3x^3 = 0] \tag{4.6}$$

Three rules for the quartic case

三乗方交乗法 Quartic rule of cross multiplication

角尾虚婁 (加)	$[+1\alpha\zeta\lambda\pi]$	角箕危壁	$[+1\alpha\eta\mu\xi]$	角斗女奎	$[+1\alpha\theta\kappa o]$
亢心危奎	$[+1\beta\epsilon\mu o]$	亢箕牛婁	$[+1\beta\eta\iota\pi]$	亢斗虚室	$[+1\beta\theta\lambda\nu]$
氐心女婁	$[+1\gamma\epsilon\kappa\pi]$	氐尾危室	$[+1\gamma\zeta\mu\nu]$	氐斗牛壁	$[+1\gamma\theta\iota\xi]$
房心虚壁	$[+1\delta\epsilon\lambda\xi]$	房尾牛奎	$[+1\delta\zeta\iota o]$	房箕女室	$[+1\delta\eta\kappa\nu]$
角尾危奎 (減)	$[-1\alpha\zeta\mu o]$	角箕女婁	$[-1\alpha\eta\kappa\pi]$	角斗虚壁	$[-1\alpha\theta\lambda\xi]$
亢心虚婁	$[-1\beta\epsilon\lambda\pi]$	亢箕危室	$[-1\beta\eta\mu\nu]$	亢斗牛奎	$[-1\beta\theta\iota o]$
氐心危壁	$[-1\gamma\epsilon\mu\xi]$	氐尾牛婁	$[-1\gamma\zeta\iota\pi]$	氐斗女室	$[-1\gamma\theta\kappa\nu]$
房心女奎	$[-1\delta\epsilon\kappa o]$	房尾虚室	$[-1\delta\zeta\lambda\nu]$	房箕牛壁	$[-1\delta\eta\iota\xi]$

変乗法 Rule of varied multiplication

角尾虚婁 (一段) (加)	$[+1\alpha\zeta\lambda\pi]$	角箕斗危 (二段)	$[+2\alpha\eta\theta\mu]$	亢冪危冪 (一段)	$[+1\beta^2\mu^2]$
亢氐箕婁 (二段)	$[+2\beta\gamma\eta\pi]$	亢房斗虚 (二段)	$[+2\beta\delta\theta\lambda]$	氐冪斗冪 (一段)	$[+1\gamma^2\theta^2]$
氐房尾危 (二段)	$[+2\gamma\delta\zeta\mu]$	房冪箕冪 (一段)	$[+1\delta^2\eta^2]$		
角尾危冪 (一段) (減)	$[-1\alpha\zeta\mu^2]$	角箕冪婁 (一段)	$[-1\alpha\eta^2\pi]$	角斗冪虚 (一段)	$[-1\alpha\theta^2\lambda]$
亢冪虚婁 (一段)	$[-1\beta^2\lambda\pi]$	亢氐斗危 (二段)	$[-2\beta\gamma\theta\mu]$	亢房箕危 (二段)	$[-2\beta\delta\eta\mu]$
氐冪尾婁 (一段)	$[-1\gamma^2\zeta\pi]$	氐房箕斗 (二段)	$[-2\gamma\delta\eta\theta]$	房冪尾虚 (一段)	$[-1\delta^2\zeta\lambda]$

消長法 Rule of deletion/generation

実三 (一段) (長)	$[+1R_0^4]$	実方冪上商三 (四段)	$[+4R_0R_1^2R_2Q^4]$
実上下冪商七 (四段)	$[+4R_0R_2R_3^2Q^8]$	方冪下冪商七 (二段)	$[+2R_1^2R_3^2Q^8]$
上三商七 (一段)	$[+1R_2^4Q^8]$		
実冪方下商三 (四段)(消) $*$	$[-4R_0^2R_1R_3Q^4]$	実冪上冪商三 (二段)	$[-2R_0^2R_2^2Q^4]$
方三商三 (一段)	$[-1R_1^4Q^4]$	方上冪下商七 (四段)	$[-4R_1R_2^2R_3Q^8]$
下三商一十一 (一段)	$[-1R_3^4Q^{12}]$		

Notes on MSS ($*$) The character 消 (deletion, negative) is missing in the original text and is supplemented here.

We shall restate three rules in language of modern mathematics. If we have (4.5). then we

[4] In this problem, we transcribe the Reality 実, the Square 方, the Upper Side 上簾, the Lower Side 下簾 and the quotient 商 by R_0, R_1, R_2, R_3 and Q, respectively.

have

$$\det\begin{pmatrix} \delta & \gamma & \beta & \alpha \\ \theta & \eta & \zeta & \epsilon \\ \mu & \lambda & \kappa & \iota \\ \pi & o & \xi & \nu \end{pmatrix} = 0.$$

The quartic rule of cross multiplications is given as the expansion formula of the determinant of a general matrix of degree 4:

In the symmetric case, we have the rule of varied multiplications:

$$\det\begin{pmatrix} \delta & \gamma & \beta & \alpha \\ \theta & \eta & \zeta & \beta \\ \mu & \lambda & \eta & \gamma \\ \pi & \mu & \theta & \delta \end{pmatrix} = 0.$$

If we have (4.6) and $Q^4 - x^4 = 0$, where R_0, R_1, R_2, R_3 and Q^4 are known, then we have

$$\det\begin{pmatrix} R_0 & R_1 & R_2 & R_3 \\ R_3Q^4 & R_0 & R_1 & R_2 \\ R_2Q^4 & R_3Q^4 & R_0 & R_1 \\ R_1Q^4 & R_2Q^4 & R_3Q^4 & R_0 \end{pmatrix} = 0$$

Expand this determinate amd we have the rule of deletion/generation:

§ 4.4. Problem 17−59 (English translation)

Suppose given the pair of formula of degree 5. Transform it:

(4.7)

First formula	$\boxed{[\epsilon, \delta, \gamma, \beta, \alpha]}$	$[\epsilon + \delta x + \gamma x^2 + \beta x^3 + \alpha x^4 = 0]$
Second formula	$\boxed{[\kappa, \iota, \theta, \eta, \zeta]}$	$[\kappa + \iota x + \theta x^2 + \eta x^3 + \zeta x^4 = 0]$
Third formula	$\boxed{[o, \xi, \nu, \mu, \lambda]}$	$[o + \xi x + \nu x^2 + \mu x^3 + \lambda x^4 = 0]$
Fourth formula	$\boxed{[\upsilon, \tau, \sigma, \rho, \pi]}$	$[\upsilon + \tau x + \sigma x^2 + \rho x^3 + \pi x^4 = 0]$
Fifth formula	$\boxed{[\Omega, \omega, \psi, \chi, \phi]}$	$[\Omega + \omega x + \psi x^2 + \chi x^3 + \phi x^4 = 0]$

In the symmetric case, we regard ζ as β, λ as γ, μ as θ, π as δ, ρ as ι, σ as ξ, ϕ as ϵ, χ as κ, ψ as o and ω as υ.

Formula to be folded by the (fifth) power:

(4.8) $\boxed{[R_0, R_1, R_2, R_3, R_4]}$ $[R_0 + R_1x + R_2x^2 + R_3x^3 + R_4x^4 = 0]$

Three rules for the quintic case

四乗方交乗法 Rule of cross multiplication

角箕室畢星 (加)	$[+1\alpha\eta\nu\tau\Omega]$	角箕壁觜鬼	$[+1\alpha\eta\xi\upsilon\psi]$	角箕奎昴柳	$[+1\alpha\eta o\sigma\omega]$
角斗危觜柳	$[+1\alpha\theta\mu\upsilon\omega]$	角斗壁胃星	$[+1\alpha\theta\xi\rho\Omega]$	角斗奎畢井	$[+1\alpha\theta o\tau\chi]$
角牛危昴星	$[+1\alpha\iota\mu\sigma\Omega]$	角牛室觜井	$[+1\alpha\iota\nu\upsilon\chi]$	角牛奎胃鬼	$[+1\alpha\iota o\rho\psi]$
角女危畢鬼	$[+1\alpha\kappa\mu\tau\psi]$	角女室胃柳	$[+1\alpha\kappa\nu\rho\omega]$	角女壁昴井	$[+1\alpha\kappa\xi\sigma\chi]$
亢尾壁昴星	$[+1\beta\zeta\xi\sigma\Omega]$	亢尾室觜柳	$[+1\beta\zeta\nu\upsilon\omega]$	亢尾奎畢鬼	$[+1\beta\zeta o\tau\psi]$
亢斗虚畢星	$[+1\beta\theta\lambda\tau\Omega]$	亢斗壁觜参	$[+1\beta\theta\xi\upsilon\phi]$	亢斗奎婁柳	$[+1\beta\theta o\pi\omega]$
亢牛虚觜鬼	$[+1\beta\iota\lambda\upsilon\psi]$	亢牛室婁星	$[+1\beta\iota\nu\pi\Omega]$	亢牛奎昴参	$[+1\beta\iota o\sigma\phi]$
亢女虚昴柳	$[+1\beta\kappa\lambda\sigma\omega]$	亢女室畢参	$[+1\beta\kappa\nu\tau\phi]$	亢女壁婁鬼	$[+1\beta\kappa\xi\pi\psi]$
氐尾危畢星	$[+1\gamma\zeta\mu\tau\Omega]$	氐尾壁觜井	$[+1\gamma\zeta\xi\upsilon\chi]$	氐尾奎胃柳	$[+1\gamma\zeta o\rho\omega]$
氐箕虚柳觜 (*)	$[+1\gamma\eta\lambda\omega\upsilon]$	氐箕壁婁星	$[+1\gamma\eta\xi\pi\Omega]$	氐箕奎畢参	$[+1\gamma\eta o\tau\phi]$
氐牛虚胃星	$[+1\gamma\iota\lambda\rho\Omega]$	氐牛危觜参	$[+1\gamma\iota\mu\upsilon\phi]$	氐牛奎婁井	$[+1\gamma\iota o\pi\chi]$
氐女虚畢井	$[+1\gamma\kappa\lambda\tau\chi]$	氐女危婁柳	$[+1\gamma\kappa\mu\pi\omega]$	氐女壁胃参	$[+1\gamma\kappa\xi\rho\phi]$
房尾危觜鬼	$[+1\delta\zeta\mu\upsilon\psi]$	房尾室胃星	$[+1\delta\zeta\nu\rho\Omega]$	房尾奎昴井	$[+1\delta\zeta o\sigma\chi]$
房箕虚昴星	$[+1\delta\eta\lambda\sigma\Omega]$	房箕室觜参	$[+1\delta\eta\nu\upsilon\phi]$	房箕奎婁鬼	$[+1\delta\eta o\pi\psi]$
房斗虚觜井	$[+1\delta\theta\lambda\upsilon\chi]$	房斗危婁星	$[+1\delta\theta\mu\pi\Omega]$	房斗奎胃参	$[+1\delta\theta o\rho\phi]$
房女虚胃鬼	$[+1\delta\kappa\lambda\rho\psi]$	房女危昴参	$[+1\delta\kappa\mu\sigma\phi]$	房女室婁井	$[+1\delta\kappa\nu\pi\chi]$
心尾危昴柳	$[+1\epsilon\zeta\mu\sigma\omega]$	心尾室畢井	$[+1\epsilon\zeta\nu\tau\chi]$	心尾壁胃鬼	$[+1\epsilon\zeta\xi\rho\psi]$
心箕虚畢鬼	$[+1\epsilon\eta\lambda\tau\psi]$	心箕室婁柳	$[+1\epsilon\eta\nu\pi\omega]$	心箕壁昴参	$[+1\epsilon\eta\xi\sigma\phi]$
心斗虚胃柳	$[+1\epsilon\theta\lambda\rho\omega]$	心斗危畢参	$[+1\epsilon\theta\mu\tau\phi]$	心斗壁婁井	$[+1\epsilon\theta\xi\pi\chi]$
心牛虚昴井	$[+1\epsilon\iota\lambda\sigma\chi]$	心牛危婁鬼	$[+1\epsilon\iota\mu\pi\psi]$	心牛室胃参	$[+1\epsilon\iota\nu\rho\phi]$

角箕室觜柳 (減)	$[-1\alpha\eta\nu\upsilon\omega]$	角箕壁昴星	$[-1\alpha\eta\xi\sigma\Omega]$	角箕奎畢鬼	$[-1\alpha\eta o\tau\psi]$
角斗危畢星	$[-1\alpha\theta\mu\tau\Omega]$	角斗壁觜井	$[-1\alpha\theta\xi\upsilon\chi]$	角斗奎胃柳	$[-1\alpha\theta o\rho\omega]$
角牛危觜鬼	$[-1\alpha\iota\mu\upsilon\psi]$	角牛室胃星	$[-1\alpha\iota\nu\rho\Omega]$	角牛奎昴井	$[-1\alpha\iota o\sigma\chi]$
角女危昴柳	$[-1\alpha\kappa\mu\sigma\omega]$	角女室畢井	$[-1\alpha\kappa\nu\tau\chi]$	角女壁胃鬼	$[-1\alpha\kappa\xi\rho\psi]$
亢尾室畢星	$[-1\beta\zeta\nu\tau\Omega]$	亢尾壁觜鬼	$[-1\beta\zeta\xi\upsilon\psi]$	亢尾奎昴柳	$[-1\beta\zeta o\sigma\omega]$
亢斗虚觜柳	$[-1\beta\theta\lambda\upsilon\omega]$	亢斗壁婁星	$[-1\beta\theta\xi\pi\Omega]$	亢斗奎畢参	$[-1\beta\theta o\tau\phi]$
亢牛虚昴星	$[-1\beta\iota\lambda\sigma\Omega]$	亢牛室觜参	$[-1\beta\iota\nu\upsilon\phi]$	亢牛奎婁鬼	$[-1\beta\iota o\pi\psi]$
亢女虚畢鬼	$[-1\beta\kappa\lambda\tau\psi]$	亢女室婁柳	$[-1\beta\kappa\nu\pi\omega]$	亢女壁昴参	$[-1\beta\kappa\xi\sigma\phi]$
氐尾危觜柳	$[-1\gamma\zeta\mu\upsilon\omega]$	氐尾壁胃星	$[-1\gamma\zeta\xi\rho\Omega]$	氐尾奎畢井	$[-1\gamma\zeta o\tau\chi]$
氐箕虚畢星	$[-1\gamma\eta\lambda\tau\Omega]$	氐箕壁觜参	$[-1\gamma\eta\xi\upsilon\phi]$	氐箕奎婁柳	$[-1\gamma\eta o\pi\omega]$
氐牛虚觜井	$[-1\gamma\iota\lambda\upsilon\chi]$	氐牛危婁星	$[-1\gamma\iota\mu\pi\Omega]$	氐牛奎胃参	$[-1\gamma\iota o\rho\phi]$
氐女虚胃柳	$[-1\gamma\kappa\lambda\rho\omega]$	氐女危畢参	$[-1\gamma\kappa\mu\tau\phi]$	氐女壁婁井	$[-1\gamma\kappa\xi\pi\chi]$
房尾危昴星	$[-1\delta\zeta\mu\sigma\Omega]$	房尾室觜井	$[-1\delta\zeta\nu\upsilon\chi]$	房尾奎胃鬼	$[-1\delta\zeta o\rho\psi]$
房箕虚觜鬼	$[-1\delta\eta\lambda\upsilon\psi]$	房箕室婁星	$[-1\delta\eta\nu\pi\Omega]$	房箕奎昴参	$[-1\delta\eta o\sigma\phi]$
房斗虚胃星	$[-1\delta\theta\lambda\rho\Omega]$	房斗危觜参	$[-1\delta\theta\mu\upsilon\phi]$	房斗奎婁井	$[-1\delta\theta o\pi\chi]$
房女虚昴井	$[-1\delta\kappa\lambda\sigma\chi]$	房女危婁鬼	$[-1\delta\kappa\mu\pi\psi]$	房女室胃参	$[-1\delta\kappa\nu\rho\phi]$
心尾危畢鬼	$[-1\epsilon\zeta\mu\tau\psi]$	心尾室胃柳	$[-1\epsilon\zeta\nu\rho\omega]$	心尾壁昴井	$[-1\epsilon\zeta\xi\sigma\chi]$
心箕虚昴柳	$[-1\epsilon\eta\lambda\sigma\omega]$	心箕室畢参	$[-1\epsilon\eta\nu\tau\phi]$	心箕壁婁鬼	$[-1\epsilon\eta\xi\pi\psi]$
心斗虚畢井	$[-1\epsilon\theta\lambda\tau\chi]$	心斗危婁柳	$[-1\epsilon\theta\mu\pi\omega]$	心斗壁胃参	$[-1\epsilon\theta\xi\rho\phi]$
心牛虚胃鬼	$[-1\epsilon\iota\lambda\rho\psi]$	心牛危昴参	$[-1\epsilon\iota\mu\sigma\phi]$	心牛室婁井	$[-1\epsilon\iota\nu\pi\chi]$

Notes on MSS $(*)$ 氐箕虚柳觜 should be 氐箕虚觜柳 if we follow the order of the 28 lodges. But this term is wrongly written in the all MSS. The last line (the 12th line) of 32v is missing in MS Kyoto U. A.

変乗法 Rule of varied multiplication

角箕室畢星 (一段) (加)	$[+1\alpha\eta\nu\tau\Omega]$	角箕壁奎觜 (二段)	$[+2\alpha\eta\xi o\upsilon]$	角斗冪觜冪 (一段)	$[+1\alpha\theta^2\upsilon^2]$
角斗牛壁星 (二段)	$[+2\alpha\theta\iota\xi\Omega]$	角斗女奎畢 (二段)	$[+2\alpha\theta\kappa o\tau]$	角牛冪奎冪 (一段)	$[+1\alpha\iota^2 o^2]$
角牛女室觜 (二段)	$[+2\alpha\iota\kappa\nu\upsilon]$	角女冪壁冪 (一段)	$[+\alpha\kappa^2\xi^2]$	亢冪室觜冪 (一段)	$[+1\beta^2\nu\upsilon^2]$
亢冪壁冪星 (一段)	$[+1\beta^2\xi^2\Omega]$	亢冪奎冪畢 (一段)	$[+1\beta^2 o^2\tau]$	亢氐斗畢星 (一段)	$[+1\beta\gamma\theta\tau\Omega]$
亢氐牛奎觜 (二段)	$[+2\beta\gamma\iota o\upsilon]$	亢氐女壁觜 (二段)	$[+2\beta\gamma\kappa\xi\upsilon]$	亢房斗奎觜 (二段)	$[+2\beta\delta\theta o\upsilon]$
亢房牛室星 (二段)	$[+2\beta\delta\iota\nu\Omega]$	亢房女壁奎 (二段)	$[+2\beta\delta\kappa\xi o]$	亢心斗壁觜 (二段)	$[+2\beta\epsilon\theta\xi\upsilon]$
亢心牛壁奎 (二段)	$[+2\beta\epsilon\iota\xi o]$	亢心女室畢 (二段)	$[+2\beta\epsilon\kappa\nu\tau]$	氐冪箕觜冪 (一段)	$[+1\gamma^2\eta\upsilon^2]$
氐冪牛冪星 (一段)	$[+1\gamma^2\iota^2\Omega]$	氐冪女冪畢 (一段)	$[+1\gamma^2\kappa^2\tau]$	氐房箕壁星 (二段)	$[+1\gamma\delta\eta\xi\Omega]$
氐房斗女觜 (二段)	$[+2\gamma\delta\theta\kappa\upsilon]$	氐房牛女奎 (二段)	$[+2\gamma\delta\iota\kappa o]$	氐心箕奎畢 (二段)	$[+2\gamma\epsilon\eta o\tau]$
氐心斗牛觜 (二段)	$[+2\gamma\epsilon\theta\iota\upsilon]$	氐心牛女壁 (二段)	$[+2\gamma\epsilon\iota\kappa\xi]$	房冪箕奎冪 (一段)	$[+2\delta^2\eta o^2]$
房冪斗冪星 (一段)	$[+1\delta^2\theta^2\Omega]$	房冪女冪室 (一段)	$[+1\delta^2\kappa^2\nu]$	房心箕室觜 (二段)	$[+2\delta\epsilon\eta\nu\upsilon]$
房心斗牛奎 (二段)	$[+2\delta\epsilon\theta\iota o]$	房心斗女壁 (二段)	$[+2\delta\epsilon\theta\kappa\xi]$	心冪箕壁冪 (一段)	$[+1\epsilon^2\eta\xi^2]$
心冪斗冪畢 (一段)	$[+1\epsilon^2\theta^2\tau]$	心冪牛冪室 (一段)	$[+1\epsilon^2\iota^2\nu]$		

角箕室觜冪 (一段) (減)	$[-1\alpha\eta\nu\upsilon^2]$	角箕壁冪星 (一段)	$[-1\alpha\eta\xi^2\Omega]$	角箕奎冪畢 (一段)	$[-1\alpha\eta o^2\tau]$
角斗冪畢星 (一段)	$[-1\alpha\theta^2\tau\Omega]$	角斗牛奎觜 (二段)	$[-2\alpha\theta\iota o\upsilon]$	角斗女壁觜 (二段)	$[-2\alpha\theta\kappa\xi\upsilon]$
角牛冪室星 (一段)	$[-1\alpha\iota^2\nu\Omega]$	角牛女壁奎 (二段)	$[-2\alpha\iota\kappa\xi o]$	角女冪室畢 (一段)	$[-1\alpha\kappa^2\nu\tau]$
亢冪室畢星 (一段)	$[-1\beta^2\nu\tau\Omega]$	亢冪壁奎觜 (二段)	$[-2\beta^2\xi o\upsilon]$	亢氐斗觜冪 (二段)	$[-2\beta\gamma\theta\upsilon^2]$
亢氐牛壁星 (二段)	$[-2\beta\gamma\iota\xi\Omega]$	亢氐女奎畢 (二段)	$[-2\beta\gamma\kappa o\tau]$	亢房斗壁星 (二段)	$[-2\beta\delta\theta\xi\Omega]$
亢房牛奎冪 (二段)	$[-2\beta\delta\iota o^2]$	亢房女室觜 (二段)	$[-2\beta\delta\kappa\nu\upsilon]$	亢心斗奎畢 (二段)	$[-2\beta\epsilon\theta o\tau]$
亢心牛室觜 (二段)	$[-2\beta\epsilon\iota\nu\upsilon]$	亢心女壁冪 (二段)	$[-2\beta\epsilon\kappa\xi^2]$	氐冪箕畢星 (一段)	$[-1\gamma^2\eta\tau\Omega]$
氐冪牛女觜 (二段)	$[-2\gamma^2\iota\kappa\upsilon]$	氐房箕奎觜 (二段)	$[-2\gamma\delta\eta o\upsilon]$	氐房斗牛星 (二段)	$[-2\gamma\delta\theta\iota\Omega]$
氐房女冪壁 (二段)	$[-2\gamma\delta\kappa^2\xi]$	氐心箕壁觜 (二段)	$[-2\gamma\epsilon\eta\xi\upsilon]$	氐心斗女畢 (二段)	$[-2\gamma\epsilon\theta\kappa\tau]$
氐心牛冪奎 (二段)	$[-2\gamma\epsilon\iota^2 o]$	房冪箕室星 (一段)	$[-1\delta^2\eta\nu\Omega]$	房冪斗女奎 (二段)	$[-2\delta^2\theta\kappa o]$
房心箕壁奎 (二段)	$[-2\delta\epsilon\eta\xi o]$	房心斗冪觜 (二段)	$[-2\delta\epsilon\theta^2\upsilon]$	房心牛女室 (二段)	$[-2\delta\epsilon\iota\kappa\nu]$
心冪箕室畢 (一段)	$[-1\epsilon^2\eta\nu\tau]$	心冪斗牛壁 (二段)	$[-2\epsilon^2\theta\iota\xi]$		

消長法 Rule of deletion/generation

実四 (一段)(長)	$[+1R_0^5]$	実冪方上冪商四 (五段) (*6)	$[+5R_0^2R_1R_2^2Q^5]$
実冪方冪中商四 (五段)	$[+5R_0^2R_1^2R_3Q^5]$	実冪上下冪商九 (五段)	$[+5R_0^2R_2R_4^2Q^{10}]$
実冪中冪下商九 (五段)	$[+5R_0^2R_3^2R_4Q^{10}]$	実方冪下冪商九 (五段)	$[+5R_0R_1^2R_4^2Q^{10}]$
実方上中下商九 (五段) (*1)	$[+5R_0R_1R_2R_3R_4Q^{10}]$	実上冪中冪商九 (五段)	$[+5R_0R_2^2R_3^2Q^{10}]$
方四商四 (一段)	$[+1R_1^5Q^5]$	方冪上冪下商九 (五段)	$[+5R_1^2R_2^2R_4Q^{10}]$
方冪上中冪商九 (四段) (*2)	$[+4R_1^2R_2R_3^2Q^{10}]$	方中冪下冪商一十四 (六段) (*3)	$[+6R_1R_3^2R_4^2Q^{15}]$
上四商九 (一段) (*5)	$[+1R_2^5Q^{10}]$	上冪中下冪商一十四 (五段)	$[+5R_2^2R_3R_4^2Q^{15}]$
中四商一十四 (一段)	$[+1R_3^5Q^{15}]$	下四商一十九 (一段)	$[+1R_4^5Q^{20}]$
実再方下商四 (五段)(消)	$[-5R_0^3R_1R_4Q^5]$	実再上中商四 (五段)	$[-5R_0^3R_2R_3Q^5]$
実方再上商四 (六段) (*4)	$[-6R_0R_1^3R_2Q^5]$	実方中再商九 (五段)	$[-5R_0R_1R_3^3Q^{10}]$
実上再下商九 (五段)	$[-5R_0R_2^3R_4Q^{10}]$	実中下再商一十四 (五段)	$[-5R_0R_3R_4^3Q^{15}]$
方再中下商九 (五段)	$[-5R_1^3R_3R_4Q^{10}]$	方上再中商九 (五段)	$[-5R_1R_2^3R_3Q^{10}]$
方上下再商一十四 (五段)	$[-5R_1R_2R_4^3Q^{15}]$	上中再下商一十四 (五段)	$[-5R_2R_3^3R_4Q^{15}]$

Notes on MSS (*1) This term is not ``positive'' (generation 長) but ``negative'' (deletion 消). It should be move to the ``negative'' part. This is a common error of all the MSS. (*2) The coefficient 4 (四段) is an error. It should be 5 (五段). This is a common error of all the MSS. ($*3$ and $*4$) The coefficient 6 (六段) is an error. It should be 5 (五段). This is a common error of all the MSS. ($*5$) The coefficient 1 (一段) is correct in MSS U. Tokyo, Kyoto U. B and Kyoto U. A, but MS Tokyo Science U. is wrong here with the coefficient 2 (二段). ($*6$) The power 冪 is written with the formal style in MSS U. Tokyo and Tokyo Science U., while it is written with the simplified style 巾 in MSS Kyoto U. B and Kyoto U. A. This term is erroneously written as 実巾方上中商四 (五段) in MS Kyoto U. B. (The characters 巾 and 中 look similar.) MS Kyoto U. A is right at this term.

We shall restate three rules in the language of modern mathematics

Suppose (4.7). Then we have

$$\det\begin{pmatrix} \epsilon & \delta & \gamma & \beta & \alpha \\ \kappa & \iota & \theta & \eta & \zeta \\ o & \xi & \nu & \mu & \lambda \\ \upsilon & \tau & \sigma & \rho & \pi \\ \Omega & \omega & \psi & \chi & \phi \end{pmatrix} = 0.$$

This is called the quintic rule of cross multiplications.

The quintic rule of varied multiplications is the rule of cross multiplications for a ``symmetric'' coefficient matrix.

Suppose (4.8) and $Q^5 - x^5 = 0$, where R_0, R_1, R_2, R_3, R_4 and Q^5 are known. Then the rule of varied multiplications implies

$$\det\begin{pmatrix} R_0 & R_1 & R_2 & R_3 & R_4 \\ R_4Q^5 & R_0 & R_1 & R_2 & R_3 \\ R_3Q^5 & R_4Q^5 & R_0 & R_1 & R_2 \\ R_2Q^5 & R_3Q^5 & R_4Q^5 & R_0 & R_1 \\ R_1Q^5 & R_2Q^5 & R_3Q^5 & R_4Q^5 & R_0 \end{pmatrix} = 0$$

This is called the quintic rule of deletion/generation.

Around 2008, Komatsu Hikosaburo prepared a ``critical'' version of the *Taisei Sankei*. First four volumes were published by RIMS, Kyoto University but others are still circulating privately, for example., at the Nagoya seminar on the history of mathematics. In the Komatsu version, the rule of deletion/generation for the four multiplications is corrected accordingly.

§ 5. What is *shiki* 式 for Seki Takakazu?

Engaging ourselves in the mathematically faithful translation of Volume 17 of the *Taisei Sankei* 大成算経 into English, we want to transcribe a Japanese technical term into one specific English term. For example, *sū* 數 is transcribed into a ``number''. But sometimes a Japanese term has dual meanings and cannot be put into a single English term. This is the case for *shiki*

式. Although it was not clearly recognized by the authors of the *Taisei Sankei*, this term has two meanings: a ``formula" or an "equation".

tianyuanshu 天元術 (*tengenjutsu*)

Mathematics of Seki Takakazu and Takebe Katahiro stemmed from the Chinese mathematics. In fact, they studied Zhu Shijie's 朱世傑: the *Suanxue Qimeng* 算学啓蒙 (1299, Introduction to Computational Studies)and learned the *tianyuanshu* and extended this mathematical ideas into the *bōshohō* 傍書法, with which they could handle polynomials of several unknowns.

With the *tianyuanshu* we can handle polynomials of one unknown: the unknown x is represented by the column vector $\begin{bmatrix} 0 \\ +1 \end{bmatrix}$ A polynomial, for example, $+1-2x+3x^2$ is represented by a column vector, in this case, $\begin{bmatrix} +1 \\ -2 \\ +3 \end{bmatrix}$

In general a polynomial in x is represented by a *shiki* 式, which is a column vector displayed on on the counting board 算盤 (*suanpan*) by means of two kinds of counting rods 算籌 (*suanchou*). (算木 (*sangi*) in Japanese). A red rod represents $+1$; a black rod -1. We propose to use the term ``configuration" to mean the column vector on the counting board.

The Method for Solving Hidden Problems

Seki Takakazu 関孝和 wrote the *Kai Indai no Hō* 解隠題之法 (ca. 1685) (The Method for Solving Hidden Problems) and expounded the theory of the algebraic equation of one unknown. This book follows the *Suanxue Qimeng* but the exposition becomes general and abstract.

There are 5 sections in the book.

1. 立元 (Placement of the celestial unit)

$$\begin{bmatrix} 0 \\ +1 \end{bmatrix}$$

(Introduction of the unknown x)

2. 加減 (Addition and Subtraction)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$$

(Addition and subtraction of polynomials in x.

$$(a+bx+cx^2)+(d+ex+fx^2)=(a+d)+(b+e)x+(c+f)x^2$$

The scalar multiplication is taken for granted.)

3. 相乘 (Multiplication)

$$\begin{bmatrix} 0 \\ +1 \end{bmatrix} \times \begin{bmatrix} 0 \\ +1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ +1 \end{bmatrix}$$

(Multiplication of two polynomials. Multiplication is assumed bi-linear.)

In Sections 1, 2 and 3 *shiki* 式 means a formula, a polynomial, or a configuration on the counting board. Note that a polynomial $P(x)$ is an element of the polynomial ring **Q**$[x]$.

shiki 式 (a formula) is also called a virtual number 假數, while an ordinary numbers si called a true number (眞數). This shows that the *shiki* is a mathematical object which can be added, subtracted and multiplied.

4. 相消 (Cancellation)

Place $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ on the counting board and move this configuration to the left. 寄左. Clear the board. Place $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$ on the board and cancel it with the left to obtain the equation to be extracted 開方式

$$\boxed{\begin{bmatrix} d-a \\ e-b \\ f-c \end{bmatrix}} \text{ or } \boxed{\begin{bmatrix} a-d \\ b-e \\ c-f \end{bmatrix}}$$

(We find the equation $(a-d)+(b-e)x+(c-f)x^2=0$)

5. 求数 (Finding the numerical solution)

Apply the so called Horner's method to find a numerical solution.
(A positive solution must exit. If not, the problem is ill-posed.)

In sections 4 and 5 *shiki* 式 (more explicitely *kaihōshiki* 開方式) is an equation or an equation to be extracted. The equation $P(x)=0$ can be considered as the *ideal* of the ring **Q**$[x]$ which is generated by $P(x)$.

N.B. Let R be a ring. A subset I of R is a sub-ring if $f,g \in I$ implies $f+g, f-g, fg \in I$. A sub-ring I of R is an ideal if $f \in I$ and $g \in R$ implies $fg \in I$.

Did Seki Takakazu discover the determinant?

Proposition 5.1. *Let R be a ring, for example* $\mathbf{Q}$, $\mathbf{Q}[y]$, *or* $Q[y,z]$. *Let* $a, b, \cdots$ *be in the ring* R. *If we have*

$$\begin{cases} a + bx + cx^2 = 0 \\ d + ex + fx^2 = 0 \\ g + hx + ix^2 = 0 \end{cases}$$

then we can eliminate the unknown x*:*

$$aei + dhc + gbf - afh - bdi - ceg = 0. \tag{5.1}$$

In the *Kaifukudai no Hō* 解伏題之法 (ca. 1685) (Methods for Solving Concealed Problems) Seki Takakazu stated Proposition 5.1 and called it the cubic method of cross multiplications 交乗法 (*kōjōhō*).

The left hand side of (5.1) is, in modern language, the full expansion formula of the determinant of degree 3. The full expansion formula is given for the determinant of degree 2, 3, 4, and 5 in the *Taisei Sankei*, Vol. 17.

There was no notation of det in the *Taisei Sankei*. If we dare to use the notation det, the determinant was defined in Volume 17 of the *Taisei Sankei* as follows

$$\begin{aligned} \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} b & c \\ h & i \end{pmatrix} + c\det\begin{pmatrix} b & c \\ e & f \end{pmatrix} \\ &= aei + dhc + gbf - afh - bdi - ceg. \end{aligned}$$

- There is no statement like

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -\det\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

- There is a statement like

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0 \quad \text{if and only if} \quad \det\begin{pmatrix} ka & b & c \\ kd & e & f \\ kg & h & i \end{pmatrix} = 0$$

- There is no mention to the multi-linearity of det.

§ 6. Concluding Remarks

In the *Taisei Sankei* the determinant is an *equation* and has not a specific name. It is not considered as a function of columns vectors (or row vectors) but rather as an equation.

In the *Taisei Sankei* (1711) the determinant 行列式 is defined inductively by the Van der Monde formula and the full expansion formulas of determinants of degree 2, 3, 4 and 5 are given correctly. But in the process of transcription, copiers did not recalculate the determinant of the fifth degree. One term in the full expansion formula of the determinant of degree 5 is in the wrong ordering. This error is conserved in all MSS which I examined.

According to the way of presentation of rules of cross multiplication, varied multiplication and deletion/generation, at least one of the authors realized the relation among them; that is, the rule of varied multiplications is a special case of the rule of cross multiplications and the rule of deletion/generation is a special case of the rule of varied multiplications. But the meaning of the rule of deletion/generation remained obscure to the author who made a final copy, Takebe Kata'akira. The given proof of the rule of deletion/generation is inconsistent of the arrangement of the three statements:

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