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Remarks on the global dynamics for solutions with an infinite group invariance to the nonlinear Schrödinger equation

By

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Abstract

We consider the focusing mass-supercritical and energy-subcritical nonlinear Schrödinger equation (NLS). The global dynamics below the ground state standing waves is known (see [6, 1, 9]). Recently, the author [12] gave the global dynamics above the ground state standing waves for finite group invariant solutions. In the present paper, we are interested in the global dynamics for the solutions with an infinite group invariance.

§ 1. Introduction

§ 1.1. Background

We consider the following nonlinear Schrödinger equation:

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta u + |u|^{p-1}u = 0, (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \qquad\qquad x \in \mathbb{R}^d, \end{cases}$$

where $d \in \mathbb{N}$ and $1 + 4/d < p < 1 + 4/(d-2)$. We regard $1 + 4/(d-2)$ as ∞ if $d = 1, 2$. It is well known that (NLS) is locally well-posed in $H^1(\mathbb{R}^d)$ and the energy, the mass, and the momentum are conserved (see [8] and the standard texts [2, 18, 14]). Here, the

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energy, the mass, and the momentum are defined as follows:

$$\text{(Energy)} \qquad E(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1},$$

$$\text{(Mass)} \qquad M(u) := \|u\|_{L^2}^2,$$

$$\text{(Momentum)} \qquad P(u) := \operatorname{Im}\int_{\mathbb{R}^d} \overline{u(x)}\nabla u(x)dx.$$

Since a pioneer work by Kenig and Merle [13], many researchers have studied the global dynamics for (NLS). For the 3d cubic Schrödinger equation, Holmer and Roudenko [10] obtained the following two statements if the initial data $u_0 \in H^1$ is radially symmetric and satisfies the mass-energy condition $M(u_0)E(u_0) < M(Q)E(Q)$, where Q is the ground state solutions.

- $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2} \Rightarrow$ the solution scatters.
- $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|Q\|_{L^2}\|\nabla Q\|_{L^2} \Rightarrow$ the solution blows up in finite time.

For the non-radial solutions, Duyckaerts, Holmer, and Roudenko [5] obtained the scattering result and Holmer and Roudenko [11] proved that the solutions in the above blow-up region blow up in finite time or grow up at infinite time. Fang, Xie, and Cazenave [6] proved the scattering result and Akahori and Nawa [1] and Guevara [9] proved both the scattering and the blow-up result for (NLS).

These results mean that the ground state standing waves are thresholds to classify the scattering and blow-up. However, if we consider odd solutions, the ground state standing waves are no longer thresholds since they are not odd. More generally, we expect that we can classify the solutions with a symmetry above the ground state standing waves to scatter or blow up.

Recently, the author considered the global dynamics for group invariant solutions to (NLS) in [12]. To state this result, we introduce some notations.

Let $O(d)$ denote the set of $d \times d$ orthogonal matrices. Let G be a subgroup in $O(d)$. We only consider the subgroups in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ denoted by $\{(\theta(\mathcal{G}), \mathcal{G}) : \mathcal{G} \in G\}$ for some group homomorphism $\theta : G \to \mathbb{R}/2\pi\mathbb{Z}$. We denote this subgroup by G for simplicity although this is determined by G and θ. And we also use the notation $\mathcal{G}$ without confusion to denote not only a matrix but also an element of a subgroup G in $\mathbb{R}/2\pi\mathbb{Z}\times O(d)$. For a subgroup G of $\mathbb{R}/2\pi\mathbb{Z}\times O(d)$, we say that a function φ is G-invariant (or with G-invariance) if $\varphi = \mathcal{G}\varphi$ for all $\mathcal{G} \in G$, where $\mathcal{G}\varphi(x) := e^{-i\theta}(\varphi \circ \mathcal{G}^{-1})(x) = e^{-i\theta}\varphi(\mathcal{G}^{-1}x)$ for $\mathcal{G} = (\theta, \mathcal{G}) \in \mathbb{R}/2\pi\mathbb{Z} \times O(d)$. We define the Sobolev space with G-invariance by

$$H_G^1 := \{\varphi \in H^1(\mathbb{R}^d) : \varphi = \mathcal{G}\varphi, \forall \mathcal{G} \in G\}.$$

If the initial data u_0 belongs to H_G^1, then the corresponding solution to (NLS) also belongs to H^1_G since the Laplacian Δ is invariant for group actions by $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and (NLS) is gauge invariant.

Let ω be a positive number. We define the action S_ω by

$$S_\omega(\varphi) := E(\varphi) + \frac{\omega}{2}M(\varphi).$$

Moreover, let K denote the functional which appears in the virial identity, that is,

$$K(\varphi) := \partial_\lambda(S_\omega(\varphi^\lambda))|_{\lambda=0} = \frac{2}{d}\|\nabla\varphi\|_{L^2}^2 - \frac{p-1}{p+1}\|\varphi\|_{L^{p+1}}^{p+1},$$

where $\varphi^\lambda(x) := e^\lambda \varphi(e^{\frac{2}{d}\lambda}x)$. For a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, we consider the restricted minimizing problem

$$l_\omega^G := \inf\{S_\omega(\varphi) : \varphi \in H^1_G \setminus \{0\}, K(\varphi) = 0\}.$$

We say that the solution u to (NLS) scatters if there exist $\varphi_\pm \in H^1(\mathbb{R}^d)$ such that

$$\left\|u(t) - e^{it\Delta}\varphi_\pm\right\|_{H^1} \to 0 \text{ as } t \to \pm\infty,$$

where $e^{it\Delta}$ denotes the free propagator of the Schrödinger equation.

In [12], we prove the following theorem.

Theorem 1.1 ([12]). *Let $\omega > 0$, G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z}\times O(d)$, $u_0 \in H^1_G(\mathbb{R}^d)$ satisfy $S_\omega(u_0) < l_\omega^G$, and u be the solution of* (NLS) *with the initial data u_0. Then, the following statements hold.*

(1) We assume either that (i) G is a finite group or (ii) G is an infinite group such that the embedding $H^1_G \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is compact. Then, if $K(u_0) \geq 0$, the solution u scatters.

(2) If $K(u_0) < 0$, then the solution u blows up in finite time or grows up at infinite time. More precisely, one of the following four cases occurs.

(a) u blows up in finite time in both directions.

(b) u blows up in positive finite time and u is global in the negative time direction and $\limsup_{t\to-\infty}\|\nabla u(t)\|_{L^2} = \infty$.

(c) u blows up in negative finite time and u is global in the positive time direction and $\limsup_{t\to\infty}\|\nabla u(t)\|_{L^2} = \infty$.

(d) u is global in both time directions and $\limsup_{t\to\pm\infty}\|\nabla u(t)\|_{L^2} = \infty$.

If G is the unit group, then Theorem 1.1 coincides with Theorems 1.1 and 1.2 in [1]. We remark that the blow-up result does not require any assumptions for the group. We need to assume the finiteness or the compactness of embedding to prove scattering.

We are interested in the global dynamics for G-invariant solutions when G is infinite and the embedding $H_G^1 \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is not compact. For example, we treat an embedded vortex solution in 3D. See Section 5 (2). (See Fibich's textbook [7, Section 15] for vortex solutions in 2D, which can be treated by Theorem 1.1.) In the present paper, we will give global dynamics for solutions with an infinite group invariance.

§ 1.2. Main result

For $k \in \{0, 1, 2, \cdots, d\}$ and subgroups $M \subset O(k)$ and $N \subset O(d-k)$, we define a group $M \times N$ in $O(d)$ by

$$M \times N := \left\{ \begin{pmatrix} \mathcal{M} & 0 \\ 0 & \mathcal{N} \end{pmatrix} : \mathcal{M} \in M, \mathcal{N} \in N \right\}.$$

Let G be a finite subgroup in $O(d-k)$. We consider the subgroup in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ denoted by $\{(\theta(\mathcal{G}), \mathcal{G}) : \mathcal{G} \in O(k) \times G\}$ for some group homomorphism $\theta : O(k) \times G \to \mathbb{R}/2\pi\mathbb{Z}$. As stated before, we denote this subgroup by $O(k) \times G$. And we set $G_k := O(k) \times G$ for simplicity. Then G_k is infinite and the embedding $H_{G_k} \hookrightarrow L^{p+1}(\mathbb{R}^d)$ is not compact when $d \geq 3$ and $k \in \{2, \cdots, d-1\}$. Then, we have the following main theorem for the G_k-invariant solutions.

Theorem 1.2. *Let $d \geq 3$ and $k \in \{2, \cdots, d-1\}$, and $\omega > 0$. Let G be a finite group in $O(d-k)$ and G_k be the subgroup in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ defined above. Let $u_0 \in H_{G_k}^1$ and u be the solution of* (NLS) *with the initial data u_0. Then, if $S_\omega(u_0) < l_\omega^{G_k}$ and $K(u_0) \geq 0$, then the solution u scatters.*

Remark.

(1). If $k = 0, 1$, or d, then the scattering result follows from Theorem 1.1 since G_k is finite if $k = 0, 1$ and $G_d = O(d)$.

(2). If $u_0 \in H_{G_k}^1$ satisfies $S_\omega(u_0) < l_\omega^{G_k}$ and $K(u_0) < 0$, then the solution u blows up in finite time or grows up at infinite time by Theorem 1.1.

(3). See Section 5 for the applications of Theorem 1.2.

To show Theorem 1.2, we prepare a proposition. Before stating the proposition, we introduce some notations.

For a subgroup G in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, we define subsets $\mathscr{K}_{G,\omega}^{\pm}$ in $H^1(\mathbb{R}^d)$ by

$$\mathscr{K}_{G,\omega}^{+} := \{\varphi \in H_G^1 : S_\omega(\varphi) < l_\omega^G, K(\varphi) \geq 0\},$$
$$\mathscr{K}_{G,\omega}^{-} := \{\varphi \in H_G^1 : S_\omega(\varphi) < l_\omega^G, K(\varphi) < 0\},$$

and we say that a subgroup G' of G satisfies $(*)$ if there exists a sequence $\{x_n\} \subset \mathbb{R}^d$ such that

$$\begin{cases} \{x_n - \mathcal{G}' x_n\} \text{ is bounded for all } \mathcal{G}' \in G', \\ |x_n - \mathcal{G} x_n| \to \infty \text{ as } n \to \infty, \text{ for all } \mathcal{G} \in G \setminus G'. \end{cases}$$

For a subgroup G in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, we define a critical action for the data with G-invariance by

$$\mathbf{S}_\omega^G := \sup\{\mathbf{S} \in (-\infty, l_\omega^G] : \forall \varphi \in \mathscr{K}_{G,\omega}^+, S_\omega(\varphi) < \mathbf{S} \\ \Rightarrow \text{ the solution to (NLS) with the initial data } \varphi \text{ belongs to } L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))\}.$$

See (2.3) below for the definition of α and r. We remark that $u \in L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$ implies that the solution u scatters (see Proposition 2.5).

For a finite group $G \subset O(d-k)$, we denote the subgroup $\{\mathcal{I}_k\} \times G$ of G_k in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ by G for simplicity, where $\mathcal{I}_k$ is the $k \times k$ identity matrix. We define

$$m_\omega^{G_k} := \min_{G' \subsetneq G \text{ satisfying } (*)} \frac{\#G}{\#G'} \mathbf{S}_\omega^{G'_k},$$

where $\#X$ denotes the number of the elements in a set X.

Proposition 1.3. *Let $d \geq 3$ and $k \in \{2, \cdots, d-1\}$, and $\omega > 0$. Let G be a finite group in $O(d-k)$ and G_k be the subgroup in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ defined above. Let $u_0 \in H^1_{G_k}$ and u be the solution of* (NLS) *with the initial data u_0. If $S_\omega(u_0) < m_\omega^{G_k}$ and $K(u_0) \geq 0$, then the solution u scatters.*

To prove Theorem 1.2, we combine Proposition 1.3 with the Neotherian induction argument. The proof of Proposition 1.3 is based on the method of Kenig and Merle [13]. However, we need to improve Linear Profile Decomposition (LPD). In [12], we obtained LPD for the finite group invariant data (see Proposition 4 in [12]). To obtain the LPD for G_k-invariant data, we combine the proof of LPD for the finite group invariant data with that of LPD for the radial data. See Proposition 3.1. Once getting LPD for G_k-invariant data, the construction of a critical element and the rigidity argument work in the similar way to those in [12].

The rest of the present paper is organized as follows. In Section 2.1, we reorganize variational argument for the data with G-invariance for general subgroup G in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and we also refer to the blow-up result. We prepare some lemmas to prove scattering in Section 2.2. Section 3 is devoted to prove Theorem 1.2. In Section 3.1, we give LPD for partially radial data Proposition 3.1, which is a key ingredient. In Section 3.2, we show Proposition 1.3 by constructing a critical element and the rigidity argument. In Section 3.3, we derive Theorem 1.2 from Proposition 1.3 by the Noetherian induction argument. We collect some lemmas in Section 4. In Section 5, we state the applications of Theorem 1.2.

§ 2. Variational structure and Preliminaries

This section is same as that in author's paper [12]. However, we give some proofs for the reader's convenience. Let G denote an arbitrary subgroup in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ in this section.

§ 2.1. Variational structure

We discuss the variational structure and refer to the blow-up result.

Lemma 2.1. *If $K(\varphi) \geq 0$, then we have*

$$S_\omega(\varphi) \leq \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{\omega}{2}\|\varphi\|_{L^2}^2 \leq \frac{d(p-1)}{d(p-1)-4}S_\omega(\varphi). \tag{2.1}$$

Proof. The left inequality is trivial. We show the right inequality. We have $0 \leq K(\varphi) = \left(\frac{2}{d} - \frac{p-1}{2}\right)\|\nabla\varphi\|_{L^2}^2 + (p-1)E(\varphi)$. Adding $\omega(p-1)M(\varphi)/2$, we obtain $\left(\frac{p-1}{2} - \frac{2}{d}\right)\|\nabla\varphi\|_{L^2}^2 + \frac{\omega}{2}(p-1)M(\varphi) \leq (p-1)S_\omega(\varphi)$. Therefore, we get

$$\left(p - 1 - \frac{4}{d}\right)\left\{\frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{\omega}{2}M(\varphi)\right\} \leq (p-1)S_\omega(\varphi).$$

This completes the proof. □

Lemma 2.2. *If $u_0 \in \mathscr{K}_{G,\omega}^+$, then the corresponding solution $u(t)$ belongs to $\mathscr{K}_{G,\omega}^+$ for all existence time t. Moreover, if $u_0 \in \mathscr{K}_{G,\omega}^-$, then the corresponding solution $u(t)$ belongs to $\mathscr{K}_{G,\omega}^-$ for all existence time t.*

Proof. We prove the second statement. Let $u_0 \in \mathscr{K}_{G,\omega}^-$. Since the energy and the mass are conserved and the solution belongs to H_G^1, we have $u(t) \in \mathscr{K}_{G,\omega}^+ \cup \mathscr{K}_{G,\omega}^-$ for all existence time t. We assume that there exists $t_1 > 0$ such that $u(t_1) \in \mathscr{K}_{G,\omega}^+$. By the continuity of the solution in $H^1(\mathbb{R}^d)$, there exists $t_0 \in (0, t_1)$ such that $K(u(t_0)) = 0$. By the definition of l_ω^G, if $u(t_0) \neq 0$, then we see that

$$l_\omega^G > E(u_0) + \frac{\omega}{2}M(u_0) = E(u(t_0)) + \frac{\omega}{2}M(u(t_0)) \geq l_\omega^G.$$

This is a contradiction. Thus, $u(t_0) = 0$. By the uniqueness of the solution, $u = 0$ for all time. However, this contradicts $u_0 \in \mathscr{K}_{G,\omega}^-$. Thus, we see that $u(t) \in \mathscr{K}_{G,\omega}^-$ for all t. The first statement follows from the same argument (see also Lemma 2.2 in [12]). □

By Lemmas 2.1 and 2.2, we get an apriori estimate and thus the solution to (NLS) exists globally in time if the initial data u_0 belongs to $\mathscr{K}_{G,\omega}^+$.

Lemma 2.3. *Let $\varphi \in H_G^1$ satisfy $S_\omega(\varphi) < l_\omega^G$. Then, one of the following holds.*

$$K(\varphi) \geq \min\{4(l_\omega^G - S_\omega(\varphi))/d, \delta\|\nabla\varphi\|_{L^2}^2\}, \text{ or } K(\varphi) \leq -4(l_\omega^G - S_\omega(\varphi))/d, \tag{2.2}$$

for some $\delta > 0$.

Proof. We give the sketch of the proof. See Lemma 2.3 in [12] for details. We may assume that $\varphi \neq 0$. Let $s(\lambda) := S_\omega(\varphi^\lambda)$, where $\varphi^\lambda(x) = e^\lambda \varphi(e^{\frac{2}{d}\lambda})$. Then, $s(0) = S_\omega(\varphi)$ and $s'(0) = K(\varphi)$. By direct calculations, we have $s'' \leq 4s'/d$. First, we consider the case of $K < 0$. Then, there exists $\lambda_0 < 0$ such that $s'(\lambda_0) = 0$ since $K < 0$. Integrating $s'' \leq 4s'/d$ on $[\lambda_0, 0]$, we obtain $s'(0) - s'(\lambda_0) \leq 4(s(0) - s(\lambda_0))/d$. This completes the proof in the case of $K < 0$. Next, we consider the case of $K > 0$. Then, there exists λ_1 such that $s''(\lambda_1) + 4s'(\lambda_1)/d = 0$ and $s''(\lambda) + 4s'(\lambda)/d < 0$ for all $\lambda > \lambda_1$. If $\lambda_1 \geq 0$, then we obtain $K(\varphi) \geq \delta \|\nabla\varphi\|_{L^2}^2$ where $\delta := 2(p-1-4/d)/\{d(p-1+4/d)\}$. If $\lambda_1 < 0$, then $s''(\lambda) < -4s'(\lambda)/d$ for $\lambda \in [0, \lambda_0]$, where we note that $\lambda_0 > 0$ since $K \geq 0$. Integrating the inequality $s''(\lambda) < -4s'(\lambda)/d$ on $[0, \lambda_0]$, this completes the proof. □

By Lemmas 2.2 and 2.3, if $u_0 \in \mathscr{K}_{G,\omega}^-$, then the solution u satisfies $K(u(t)) < -4(l_\omega^G - S_\omega(u_0))/d < 0$ for all existence time t. Therefore, the blow-up result (Theorem 1.1 (2)) follows directly from Theorem 2.1 in [4].

§ 2.2. Preliminaries

We show some basic lemmas, which are used to prove scattering. Their proofs can be found in [6] and [12]. Let

$$\begin{aligned} &\alpha := \tfrac{2(p-1)(p+1)}{4-(d-2)(p-1)}, \beta := \tfrac{2(p-1)(p+1)}{d(p-1)^2+(d-2)(p-1)-4}, \gamma := \tfrac{2(d+2)}{d} \\ &q := \tfrac{4(p+1)}{d(p-1)}, \qquad r := p+1, \qquad s := \tfrac{d}{2} - \tfrac{2}{p-1}. \end{aligned} \tag{2.3}$$

Let β' and r' denote the Hölder conjugate exponents of the exponent β and r, respectively.

Lemma 2.4 (Strichartz estimates). *The following estimates are vaild.*

$$\left\|e^{it\Delta}\varphi\right\|_{L^q(\mathbb{R}:L^r)} + \left\|e^{it\Delta}\varphi\right\|_{L^\gamma(\mathbb{R}:L^\gamma)} \lesssim \|\varphi\|_{L^2}, \tag{2.4}$$

$$\left\|e^{it\Delta}\varphi\right\|_{L^\alpha(\mathbb{R}:L^r)} \lesssim \|\varphi\|_{\dot{H}^s}, \tag{2.5}$$

$$\left\|\int_0^t e^{i(t-t')\Delta} f(t')dt'\right\|_{L^\alpha(I:L^r)} \lesssim \|f\|_{L^{\beta'}(I:L^{r'})}, \tag{2.6}$$

where I is a time interval and the implicit constant is independent of I.

See Theorem 2.3.3 and Proposition 2.4.1 in [2].

Proposition 2.5. *Let $u_0 \in H^1(\mathbb{R}^d)$ and u be the solution to* (NLS) *with the initial data u_0. If the solution u is positively global and $u \in L^\alpha((0,\infty) : L^r(\mathbb{R}^d))$, then the solution scatters in the positive time direction. Moreover, the same statement holds in the negative case.*

See Proposition 2.3 in [3] and Theorem 7.8.1 in [2] for the proof.

Proposition 2.6. *There exists $\varepsilon_{sd} > 0$ satisfying the following. If $u_0 \in H^1(\mathbb{R}^d)$ and $\left\|e^{it\Delta}u_0\right\|_{L^\alpha((0,\infty):L^r)} \leq \varepsilon_{sd}$, then the solution u of* (NLS) *with the initial data u_0 is positively global and we have*

$$\|u\|_{L^\alpha((0,\infty):L^r)} \lesssim \varepsilon_{sd}. \tag{2.7}$$

In particular, if $\|u_0\|_{H^1} \leq \varepsilon_{sd}$, then the solution u is global and we have

$$\|u\|_{L^\gamma(\mathbb{R}:L^\gamma)} + \|u\|_{L^\alpha(\mathbb{R}:L^r)} + \|u\|_{L^\infty(\mathbb{R}:H^1)} \lesssim \|u_0\|_{H^1}. \tag{2.8}$$

See Proposition 2.4 in [3] or Proposition 4.3 in [6].

Lemma 2.7. *If $\psi \in H^1_G$ satisfies $\|\nabla\psi\|_{L^2}^2/2 + \omega M(\psi)/2 < l_\omega^G$, then there exists a global solution U_+ to* (NLS) *such that $U_+(0) \in \mathscr{K}_{G,\omega}^+$ and $\left\|U_+(t) - e^{it\Delta}\psi\right\|_{H^1} \to 0$ as $t \to \infty$. Moreover, the same statement holds in the negative case.*

Proof. We may assume that $\psi \neq 0$ since the statement is true if $\psi = 0$. It is known in [17, Theorem 17] (see also [16, Theorem 8]) that there exist $T \in \mathbb{R}$ and a unique solution $U_+ \in C((T,\infty) : H^1(\mathbb{R}^d))$ of (NLS) such that

$$\left\|U_+(t) - e^{it\Delta}\psi\right\|_{H^1} \to 0 \text{ as } t \to \infty. \tag{2.9}$$

The uniqueness and the assumption that ψ is G-invariant imply that the solution U_+ is also G-invariant. By the triangle inequality, the Sobolev embedding, (2.9), and $\left\|e^{it\Delta}\psi\right\|_{L^{p+1}} \to 0$ as $t \to \infty$ (see [2, Corollary 2.3.7]), we have

$$\|U_+(t)\|_{L^{p+1}} \lesssim \left\|U_+(t) - e^{it\Delta}\psi\right\|_{H^1} + \left\|e^{it\Delta}\psi\right\|_{L^{p+1}} \to 0,$$

as $t \to \infty$. Therefore, by the conservation laws and the assumption, we obtain

$$S_\omega(U_+) = \lim_{t\to\infty} S_\omega(U_+(t)) = \frac{1}{2}\|\nabla\psi\|_{L^2}^2 + \frac{\omega}{2}M(\psi) < l_\omega^G$$

and

$$\lim_{t\to\infty} K(U_+(t)) = \frac{2}{d}\|\nabla\psi\|_{L^2}^2 > 0.$$

Thus, $U_+(t)$ belongs to $\mathscr{K}_{G,\omega}^+$ for large $t > T$. This statement, Lemmas 2.1, and 2.2, imply that U_+ is global in both time directions and $U_+(0) \in \mathscr{K}_{G,\omega}^+$. □

Lemma 2.8 (Perturbation Lemma). *Given $A \geq 0$, there exist $\varepsilon(A) > 0$ and $C(A) > 0$ with the following property. If $u \in C([0,\infty) : H^1(\mathbb{R}^d))$ is a solution of* (NLS), *if* $\tilde{u} \in C([0,\infty) : H^1(\mathbb{R}^d))$ *and* $e \in L^1_{loc}([0,\infty) : H^{-1}(\mathbb{R}^d))$ *satisfy* $i\partial_t \tilde{u} + \Delta\tilde{u} + |\tilde{u}|^{p-1}\tilde{u} = e$*, for a.e.* $t > 0$*, and if*

$$\|\tilde{u}\|_{L^\alpha([0,\infty):L^r)} \leq A, \tag{2.10}$$

$$\|e\|_{L^{\beta'}([0,\infty):L^{r'})} \leq \varepsilon(A), \tag{2.11}$$

$$\left\|e^{it\Delta}(u(0) - \tilde{u}(0))\right\|_{L^\alpha([0,\infty):L^r)} \leq \varepsilon \leq \varepsilon(A), \tag{2.12}$$

then $u \in L^\alpha((0,\infty) : L^r(\mathbb{R}^d))$ *and* $\|u - \tilde{u}\|_{L^\alpha([0,\infty):L^r)} \leq C\varepsilon$.

See Proposition 4.7 in [6] for the proof.

§ 3. Proof of Scattering

§ 3.1. Linear Profile Decomposition for G_k-invariant functions

In this section, let G be a finite group in $O(d-k)$. We recall that G_k denote the subgroup $\{(\theta(\mathcal{G}), \mathcal{G}) : \mathcal{G} \in O(k) \times G\}$ in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ for some group homomorphism $\theta : O(k) \times G \to \mathbb{R}/2\pi\mathbb{Z}$. We assume that $d \geq 3$ and $k \in \{2, 3, \cdots, d-1\}$. We prove a linear profile decomposition for G_k-invariant functions. Let $\tau_y\varphi(x) = \varphi(x-y)$ throughout this paper. We note that $\mathcal{G}\tau_y\varphi = \tau_{\mathcal{G}y}\mathcal{G}\varphi$ for all $y \in \mathbb{R}^d$ and $\mathcal{G} \in O(d)$.

Proposition 3.1 (Linear Profile Decomposition). *Let* $\{\varphi_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $H^1_{G_k}$. *Then, after replacing a subsequence, for* $j \in \mathbb{N}$ *there exist a subgroup* G^j *of* G, $\psi^j \in H^1_{G^j_k}$, $\{W^j_n\} \subset H^1_{G^j_k}$, $\{t^j_n\} \subset \mathbb{R}$, *and* $\{x^j_n\} \subset \mathbb{R}^d$ *such that*

$$\varphi_n = \sum_{j=1}^{J} e^{it^j_n\Delta} \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x^j_n}\psi^j)}{\#G} + \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W^J_n}{\#G} \tag{3.1}$$

for every $J \in \mathbb{N}$*, and the following statements hold.*

(1) For any fixed j, $\{t^j_n\}$ *satisfies either* $t^j_n = 0$ *or* $t^j_n \to \pm\infty$ *as* $n \to \infty$,

(2) For any fixed j, $\{x^j_n\}$ *satisfies that 1st, 2nd,* $\cdots$*, and* k*th components of* x^j_n *are zero for all* n, j *and that* $x^j_n = \mathcal{G}x^j_n$ *for all* $\mathcal{G} \in G^j$ *and* $|x^j_n - \mathcal{G}x^j_n| \to \infty$ *for all* $\mathcal{G} \in G \setminus G^j$. *In other words,* $x^j_n = \mathcal{G}x^j_n$ *for all* $\mathcal{G} \in G^j_k$ *and* $|x^j_n - \mathcal{G}x^j_n| \to \infty$ *for all* $\mathcal{G} \in G_k \setminus G^j_k$

(3) We have the orthogonality of the parameters: for $j \neq h$,

$$\lim_{n\to\infty} |t^j_n - t^h_n| = \infty \text{ or } \lim_{n\to\infty} |\mathcal{G}x^j_n - \mathcal{G}'x^h_n| = \infty \text{ for all } \mathcal{G}, \mathcal{G}' \in G.$$

(4) We have smallness of the remainder:

$$\limsup_{n\to\infty} \left\| e^{it\Delta} \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n^J}{\#G} \right\|_{L^\alpha(\mathbb{R}:L^r)} \to 0 \text{ as } J \to \infty.$$

(5) We have the orthogonality in norms: for all $\lambda \in [0,1]$,

$$\|\varphi_n\|_{\dot{H}^\lambda}^2 = \sum_{j=1}^J \left\| \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G} \right\|_{\dot{H}^\lambda}^2 + \left\| \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n^J}{\#G} \right\|_{\dot{H}^\lambda}^2 + o_n(1), \tag{3.2}$$

$$\|\varphi_n\|_{L^{p+1}}^{p+1} = \sum_{j=1}^J \left\| e^{it_n^j\Delta} \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G} \right\|_{L^{p+1}}^{p+1} + \left\| \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n^J}{\#G} \right\|_{L^{p+1}}^{p+1} + o_n(1) \tag{3.3}$$

and, in particular,

$$S_\omega(\varphi_n) = \sum_{j=1}^J S_\omega\left(e^{it_n^j\Delta} \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G} \right) + S_\omega\left(\sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n^J}{\#G} \right) + o_n(1), \tag{3.4}$$

$$K(\varphi_n) = \sum_{j=1}^J K\left(e^{it_n^j\Delta} \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G} \right) + K\left(\sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n^J}{\#G} \right) + o_n(1). \tag{3.5}$$

This lemma can be obtained by combining an induction argument and Lemma 3.2 below. We only give the proof of Lemma 3.2 and omit the proof of Proposition 3.1 (see [12, Proposition 4] and [6, Theorem 5.1] for details).

Remark. An anonymous referee gave me another simple proof. In the proof, the linear profile decomposition for general functions, which is obtained by [6, 1], is applied to the group invariant setting. In the present paper, for beginners, we show Proposition 3.1 by repeating the usual proof of the linear profile decomposition under group invariant setting, which may be lengthy for experts.

Lemma 3.2. *Let $a > 0$ and $\{\varphi_n\} \subset H_{G_k}^1$ satisfy $\limsup_{n\to\infty} \|\varphi_n\|_{H^1} \leq a < \infty$. If $\|e^{it\Delta}\varphi_n\|_{L^\infty(\mathbb{R}:L^{p+1})} \to A$ as $n \to \infty$, then there exist a subsequence, which is still denoted by $\{\varphi_n\}_{n\in\mathbb{N}}$, a subgroup G' of G, $\psi \in H_{G_k'}^1$, sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$, and $\{W_n\}_{n\in\mathbb{N}} \subset H_{G_k'}^1$ such that*

$$\varphi_n = e^{it_n\Delta} \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n}\psi)}{\#G} + \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n}{\#G}, \tag{3.6}$$

and the following hold.

(1) $e^{-it_n\Delta}\tau_{-\mathcal{G}x_n}\varphi_n \rightharpoonup \mathcal{G}\psi/(\#G/\#G')$ in $H^1(\mathbb{R}^d)$ and $e^{-it_n\Delta}\tau_{-\mathcal{G}x_n}\tilde{W}_n \rightharpoonup 0$ in $H^1(\mathbb{R}^d)$ for all $\mathcal{G} \in G$, where $\tilde{W}_n := \sum_{\mathcal{G}\in G} \mathcal{G}W_n/\#G$.

(2) The sequence $\{t_n\}$ satisfies either $t_n = 0$ or $t_n \to \pm\infty$ as $n \to \infty$.

(3) The sequence $\{x_n\}$ satisfies that 1st, 2nd, $\cdots$, and kth components of x_n^j are zero for all n, j and that $\mathcal{G}'x_n = x_n$ for all $\mathcal{G}' \in G'$ and $|x_n - \mathcal{G}x_n| \to \infty$ for all $\mathcal{G} \in G\backslash G'$.

(4) We have the orthogonality in norms:

$$\|\varphi_n\|_{\dot{H}^\lambda}^2 - \left\| \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n}\psi)}{\#G} \right\|_{\dot{H}^\lambda}^2 - \left\| \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n}{\#G} \right\|_{\dot{H}^\lambda}^2 \to 0 \text{ as } n \to \infty,$$

for all $0 \le \lambda \le 1$.

$$\|\varphi_n\|_{L^{p+1}}^{p+1} - \left\| e^{it_n\Delta} \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n}\psi)}{\#G} \right\|_{L^{p+1}}^{p+1} - \left\| \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n}{\#G} \right\|_{L^{p+1}}^{p+1} \to 0 \text{ as } n \to \infty.$$

(5) We have

$$\|\psi\|_{H^1} \ge \nu A^{\frac{d-2\Lambda^2}{2\Lambda(1-\Lambda)}} a^{-\frac{d-2\Lambda}{2\Lambda(1-\Lambda)}},$$

where $\Lambda := d(p-1)/\{2(p+1)\} \in (0, \min\{1, d/2\})$ and the constant $\nu > 0$ is independent of a, A, and $\{\varphi_n\}_{n\in\mathbb{N}}$.

(6) If $A = 0$, then for every sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$, and $\{W_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$ satisfying (3.6) and (1), we must have $\psi = 0$.

Proof. Let $\widehat{\chi} \in C_0^\infty(\mathbb{R}^d)$ satisfy $\widehat{\chi}(\xi) = 1$ if $|\xi| \le 1$, $\widehat{\chi}(\xi) = 0$ if $|\xi| \ge 2$, and $0 \le \widehat{\chi} \le 1$. Given $\rho > 0$, we set $\widehat{\chi_\rho}(\xi) := \widehat{\chi}(\xi/\rho)$. Since $\Lambda < d/2$, we have

$$|\chi_\rho * u(x)| \le \kappa \rho^{\frac{d-2\Lambda}{2}} \|u\|_{\dot{H}^\Lambda} \text{ for any } u \in H^1(\mathbb{R}^d), \tag{3.7}$$

where κ is a constant independent of ρ and u.

First, we consider the case of $A > 0$. Then, we have, for large n,

$$\left\| e^{it\Delta}(\chi_\rho * \varphi_n) \right\|_{L^\infty(\mathbb{R}:L^\infty)} \ge (2a)^{-\frac{d-2\Lambda}{2\Lambda}} \left(\frac{A}{4}\right)^{\frac{d}{2\Lambda}}. \tag{3.8}$$

(See Lemma 4.1 in [12] or Lemma 5.2 in [6] for proofs of (3.7) and (3.8).) Therefore, there exist $\{T_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{X_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\left| e^{-iT_n\Delta}(\chi_\rho * \varphi_n)(X_n) \right| \ge (4a)^{-\frac{d-2\Lambda}{2\Lambda}} \left(\frac{A}{4}\right)^{\frac{d}{2\Lambda}}, \tag{3.9}$$

for large n.

We show that $\{\tilde{X}_n\}_n \subset \mathbb{R}^k$ is bounded, where $\tilde{X}_n = (X_n^1, X_n^2, \cdots, X_n^k)$ and X_n^j denotes a jth component of X_n. We suppose that $\{\tilde{X}_n\}_n \subset \mathbb{R}^k$ is unbounded. By taking a subsequence, we may assume that $|\tilde{X}_n| \to \infty$ as $n \to \infty$. Since $\{\varphi_n\}_n$ is bounded in H^1, there exist $\tilde{\psi} \in H^1_{G_k}$ and $\psi \in H^1(\mathbb{R}^d)$ such that

$$\begin{aligned}
&e^{-iT_n\Delta}\tau_{-X_n'}\varphi_n \rightharpoonup \tilde{\psi} \text{ weakly in } H^1,\\
&e^{-iT_n\Delta}\tau_{-\tilde{X}_n}\tau_{-X_n'}\varphi_n \rightharpoonup \psi \text{ weakly in } H^1,
\end{aligned}$$

where we regard $\tilde{X}_n$ as $(X_n^1, \cdots, X_n^k, 0 \cdots, 0) \in \mathbb{R}^d$ and $X_n' = (0 \cdots, 0, X_n^{k+1}, \cdots, X_n^d)$. For $R > 0$, we define a cylinder set by $C^R := \{(\tilde{x}, x') \in \mathbb{R}^k \times \mathbb{R}^{d-k} : |x'| < R\}$. It follows from Lemma 4.2 that $e^{-iT_n\Delta}\tau_{-X_n'}\varphi_n \to \tilde{\psi}$ in $L^{p+1}(C^R)$ for any $R > 0$. On the other hand, for any $R > 0$, we have $e^{-iT_n\Delta}\tau_{-\tilde{X}_n}\tau_{-X_n'}\varphi_n \to \psi$ in $L^{p+1}(B_R)$, where B_R is the ball of radius R centered at the origin. Therefore, we obtain

$$\begin{aligned}
&\left\|\tau_{-\tilde{X}_n}\tilde{\psi} - \psi\right\|_{L^{p+1}(B_R)}\\
&\leq \left\|\tau_{-\tilde{X}_n}\tilde{\psi} - e^{-iT_n\Delta}\tau_{-\tilde{X}_n}\tau_{-X_n'}\varphi_n\right\|_{L^{p+1}(C^R)} + \left\|e^{-iT_n\Delta}\tau_{-\tilde{X}_n}\tau_{-X_n'}\varphi_n - \psi\right\|_{L^{p+1}(B_R)}\\
&\leq \left\|\tilde{\psi} - e^{-iT_n\Delta}\tau_{-X_n'}\varphi_n\right\|_{L^{p+1}(C^R)} + \left\|e^{-iT_n\Delta}\tau_{-\tilde{X}_n}\tau_{-X_n'}\varphi_n - \psi\right\|_{L^{p+1}(B_R)}\\
&\to 0.
\end{aligned}$$

Since $|\tilde{X}_n| \to \infty$ as $n \to \infty$, we have $\left\|\tau_{-\tilde{X}_n}\tilde{\psi}\right\|_{L^{p+1}(B_R)} \to 0$. Combining them, we obtain

$$\|\psi\|_{L^p(B_R)} \leq \left\|\tau_{-\tilde{X}_n}\tilde{\psi} - \psi\right\|_{L^p(B_R)} + \left\|\tau_{-\tilde{X}_n}\tilde{\psi}\right\|_{L^p(B_R)}.$$

This means that $\psi = 0$. On the other hand, it follows from (3.9) and $e^{-iT_n\Delta}\tau_{-X_n}\varphi_n \rightharpoonup \psi$ weakly in H^1 that

$$0 < (4a)^{-\frac{d-2\Lambda}{2\Lambda}}\left(\frac{A}{4}\right)^{\frac{d}{2\Lambda}} \leq |(\chi_\rho * \psi)(0)|.$$

This is a contradiction. Thus, $\{\tilde{X}_n\}_n \subset \mathbb{R}^k$ is bounded. By taking a subsequence, we may assume that $\{\tilde{X}_n\}_n$ converges.

We consider the following two cases.

Case1 $\{T_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is unbounded.

Case2 $\{T_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded.

Case1: Since $\{T_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is unbounded, we may assume $T_n \to \pm\infty$ as $n \to \infty$ taking a subsequence. Let $t_n := T_n$. Taking a subsequence and using Lemma 4.3, we obtain a subgroup G' of G such that a subsequence, which is still denoted by $\{X_n\}$, satisfies that

$$\begin{cases} X_n - \mathcal{G}'X_n \to \bar{x}_{\mathcal{G}'} \text{ as } n\to\infty, \forall \mathcal{G}' \in G', \\ |X_n - \mathcal{G}X_n| \to \infty \text{ as } n \to \infty, \forall \mathcal{G} \in G \setminus G', \end{cases}$$

for some $\bar{x}_{\mathcal{G}'} \in \mathbb{R}^d$. Using Lemma 4.4 and the convergence of $\{\tilde{X}_n\}$, we obtain a sequence $\{x_n\}$ such that

$$\begin{cases} x_n - \mathcal{G}'x_n = 0, & \forall \mathcal{G}' \in G', \\ |x_n - \mathcal{G}x_n| \to \infty \text{ as } n \to \infty, & \forall \mathcal{G} \in G \setminus G', \\ (x_n^1, x_n^2, \cdots, x_n^k) = (0, 0, \cdots, 0) & \text{for all } n \in \mathbb{N}, \end{cases}$$

and there exists $x_\infty \in \mathbb{R}^d$ such that

$$x_n - X_n \to x_\infty \text{ as } n \to \infty.$$

Since $\|\varphi_n\|_{H^1}$ is bounded, there exists $\psi \in H^1(\mathbb{R}^d)$ such that, after taking a subsequence, $e^{-it_n\Delta}\tau_{-x_n}\varphi_n \rightharpoonup \psi/(\#G/\#G')$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$. Here, we note that ψ is G'_k-invariant since φ_n is G_k-invariant and $x_n = \mathcal{G}'x_n$ for all $\mathcal{G}' \in G'_k$. We prove (5). Now, we have $e^{-iT_n\Delta}\tau_{-X_n}\varphi_n \rightharpoonup \tau_{x_\infty}\psi/(\#G/\#G')$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$. Since $e^{it\Delta}$ commutes with the convolution with χ_ρ, we find that $e^{-iT_n\Delta}(\chi_\rho * \varphi_n)(X_n) = \chi_\rho * (e^{-iT_n\Delta}\tau_{-X_n}\varphi_n)(0)$. By (3.7) and (3.9), we have

$$(4a)^{-\frac{d-2\Lambda}{2\Lambda}} \left(\frac{A}{4}\right)^{\frac{d}{2\Lambda}} \leq \left|\frac{\chi_\rho * \psi(-x_\infty)}{\#G/\#G'}\right| \leq \kappa\rho^{\frac{d-2\Lambda}{2}} \frac{\|\psi\|_{\dot{H}^\Lambda}}{\#G/\#G'} \leq \kappa\rho^{\frac{d-2\Lambda}{2}} \frac{\|\psi\|_{H^1}}{\#G/\#G'}.$$

Taking $\rho = (4Ca/A)^{\frac{1}{1-\Lambda}}$, we obtain the statement (5). We set $W_n := \varphi_n - e^{it_n\Delta}\tau_{x_n}\psi$. Since φ_n is G_k-invariant, we see that

$$\varphi_n = \sum_{\mathcal{G}\in G} \frac{\mathcal{G}\varphi_n}{\#G} = \sum_{\mathcal{G}\in G} \frac{\mathcal{G}(e^{it_n\Delta}\tau_{x_n}\psi + W_n)}{\#G} = \sum_{\mathcal{G}\in G} \frac{e^{it_n\Delta}\mathcal{G}(\tau_{x_n}\psi)}{\#G} + \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n}{\#G}.$$

This is the statement (3.6). Moreover, W_n is G'_k-invariant since φ_n and $\tau_{x_n}\psi$ are G'_k-invariant. We check the statement (1). The first statement $e^{-it_n\Delta}\tau_{-\mathcal{G}x_n}\varphi_n \rightharpoonup \mathcal{G}\psi/(\#G/\#G')$ in $H^1(\mathbb{R}^d)$ follows from the definition of ψ and the G_k-invariance of φ_n. We prove the second statement $e^{-it_n\Delta}\tau_{-\mathcal{G}x_n}\tilde{W}_n \rightharpoonup 0$ in $H^1(\mathbb{R}^d)$ for all $\mathcal{G} \in G$, where we recall that $\tilde{W}_n = \sum_{\mathcal{G}\in G} \mathcal{G}W_n/\#G$. Let $\{\mathcal{G}_m\}_{m=1}^{\#G/\#G'}$ be the set of left coset representatives, that is, we have

$$G = \sum_{m=1}^{\#G/\#G'} \mathcal{G}_m G'.$$

Since W_n is G'_k-invariant, we find that

$$\tilde{W_n} = \sum_{\mathcal{G}\in G} \frac{\mathcal{G}W_n}{\#G} = \sum_{m=1}^{\#G/\#G'} \frac{\mathcal{G}_m W_n}{\#G/\#G'}.$$

Let $\mathcal{G} = \mathcal{G}_l\mathcal{G}'$ for some $l \in \{1, 2, \cdots, \#G/\#G'\}$ and $\mathcal{G}' \in G'$. Then, by the definition of W_n and the first statement in (1), we obtain

$$\begin{aligned} e^{-it_n\Delta}\tau_{-\mathcal{G}x_n}\tilde{W_n} &= e^{-it_n\Delta}\tau_{-\mathcal{G}_l x_n}\varphi_n - \sum_{m=1}^{\#G/\#G'} \frac{\tau_{-\mathcal{G}_l x_n + \mathcal{G}_m x_n}\mathcal{G}_m\psi}{\#G/\#G'} \\ &\rightharpoonup \frac{\mathcal{G}_l\psi}{\#G/\#G'} - \frac{\mathcal{G}_l\psi}{\#G/\#G'} = 0, \end{aligned}$$

where we note that $|-\mathcal{G}_l x_n + \mathcal{G}_m x_n| = |-x_n + \mathcal{G}_l^{-1}\mathcal{G}_m x_n| \to \infty$ since $\mathcal{G}_l^{-1}\mathcal{G}_m \notin G'$ if $m \neq l$. Thus, we get the second statement in (1). Next, we prove (4). We set $\tilde{\psi}_n := \sum_{\mathcal{G}\in G} e^{it_n\Delta}\mathcal{G}(\tau_{x_n}\psi)/\#G$. We have

$$\begin{aligned} \|\varphi_n\|_{\dot{H}^\lambda}^2 &= \left\|\tilde{\psi}_n + \tilde{W_n}\right\|_{\dot{H}^\lambda}^2 = \left\|\tilde{\psi}_n\right\|_{\dot{H}^\lambda}^2 + \left\|\tilde{W_n}\right\|_{\dot{H}^\lambda}^2 + 2\left(\tilde{\psi}_n, \tilde{W_n}\right)_{\dot{H}^\lambda} \\ &= \left\|\tilde{\psi}\right\|_{\dot{H}^\lambda}^2 + \left\|\tilde{W_n}\right\|_{\dot{H}^\lambda}^2 + 2\left(\tilde{\psi}_n, \varphi_n - \tilde{\psi}_n\right)_{\dot{H}^\lambda}, \end{aligned}$$

where $(\cdot,\cdot)_{\dot{H}^\lambda}$ denotes the inner product in $\dot{H}^\lambda$. We calculate $\left(\tilde{\psi}_n, \varphi_n - \tilde{\psi}_n\right)_{\dot{H}^\lambda}$. Since $\tau_{x_n}\psi$ is G'_k-invariant, we observe that

$$\tilde{\psi}_n = \sum_{\mathcal{G}\in G} \frac{e^{it_n\Delta}\mathcal{G}(\tau_{x_n}\psi)}{\#G} = \sum_{m=1}^{\#G/\#G'} \frac{e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi)}{\#G/\#G'}.$$

By this observation, we have

$$\begin{aligned} &\left(\tilde{\psi}_n, \varphi_n - \tilde{\psi}_n\right)_{\dot{H}^\lambda} \\ &= \left(\sum_{m=1}^{\#G/\#G'} \frac{e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi)}{\#G/\#G'}, \varphi_n - \sum_{l=1}^{\#G/\#G'} \frac{e^{it_n\Delta}\mathcal{G}_l(\tau_{x_n}\psi)}{\#G/\#G'}\right)_{\dot{H}^\lambda} \\ &= \frac{1}{(\#G/\#G')^2} \sum_{m=1}^{\#G/\#G'} \sum_{l=1}^{\#G/\#G'} \left(e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi), \varphi_n - e^{it_n\Delta}\mathcal{G}_l(\tau_{x_n}\psi)\right)_{\dot{H}^\lambda} \\ &= \frac{1}{(\#G/\#G')^2} \sum_{m,l=1}^{\#G/\#G'} \left\{\left(e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi), \varphi_n\right)_{\dot{H}^\lambda} - \left(e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi), e^{it_n\Delta}\mathcal{G}_l(\tau_{x_n}\psi)\right)_{\dot{H}^\lambda}\right\} \end{aligned}$$

For the first term, we find that, for all $m \in \{1, 2, \cdots, \#G/\#G'\}$,

$$\left(e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi), \varphi_n\right)_{\dot{H}^\lambda} = \left(\psi, e^{-it_n\Delta}\tau_{-x_n}\mathcal{G}_m^{-1}\varphi_n\right)_{\dot{H}^\lambda} \to \frac{\|\psi\|_{\dot{H}^\lambda}^2}{(\#G/\#G')} \tag{3.10}$$

since φ_n is G_k-invariant and $e^{-it_n\Delta}\tau_{-x_n}\varphi_n$ weakly converges to $\psi/(\#G/\#G')$ as $n\to\infty$ in $H^1(\mathbb{R}^d)$. For the second term, we obtain

$$\begin{aligned}(3.11)\qquad \left(e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi), e^{it_n\Delta}\mathcal{G}_l(\tau_{x_n}\psi)\right)_{\dot{H}^\lambda} &= \left(\psi, \tau_{-x_n}\mathcal{G}_m^{-1}\mathcal{G}_l(\tau_{x_n}\psi)\right)_{\dot{H}^\lambda}\\ &= \left(\psi, \tau_{-x_n+\mathcal{G}_m^{-1}\mathcal{G}_l x_n}\mathcal{G}_m^{-1}\mathcal{G}_l\psi)\right)_{\dot{H}^\lambda}\\ &\to \begin{cases}\|\psi\|_{\dot{H}^\lambda}^2, & \text{if } m=l,\\ 0, & \text{if } m\neq l.\end{cases}\end{aligned}$$

Combining (3.10) with (3.11), we get

$$\begin{aligned}&\sum_{m,l=1}^{\#G/\#G'}\left\{\left(e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi),\varphi_n\right)_{\dot{H}^\lambda} - \left(e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi), e^{it_n\Delta}\mathcal{G}_l(\tau_{x_n}\psi)\right)_{\dot{H}^\lambda}\right\}\\ &\qquad\to \sum_{m,l=1}^{\#G/\#G'}\frac{\|\psi\|_{\dot{H}^\lambda}^2}{(\#G/\#G')} - \sum_{m=1}^{\#G/\#G'}\|\psi\|_{\dot{H}^\lambda}^2 = 0.\end{aligned}$$

This implies the first statement of (4). We set

$$f_n := \left|\|\varphi_n\|_{L^{p+1}}^{p+1} - \left\|\tilde{\psi}_n\right\|_{L^{p+1}}^{p+1} - \left\|\tilde{W}_n\right\|_{L^{p+1}}^{p+1}\right|.$$

Since we have

$$||z_1+z_2|^{p+1} - |z_1|^{p+1} - |z_2|^{p+1}| \leq C|z_1||z_2|(|z_1|^{p-1}+|z_2|^{p-1}),$$

for $z_1, z_2\in\mathbb{C}$, letting $g_n = |\tilde{\psi}_n|^{p-1} + |\tilde{W}_n|^{p-1}$, we get

$$\begin{aligned}f_n &\leq C\int_{\mathbb{R}^d}\left|\tilde{\psi}_n(x)\right|\left|\tilde{W}_n(x)\right| g_n(x)dx\\ &\leq C\int_{\mathbb{R}^d}\left|\sum_{m=1}^{\#G/\#G'}\frac{e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi)(x)}{\#G/\#G'}\right|\left|\tilde{W}_n(x)\right| g_n(x)dx\\ &\leq C\sum_{m=1}^{\#G/\#G'}\int_{\mathbb{R}^d}\left|e^{it_n\Delta}\mathcal{G}_m(\tau_{x_n}\psi)(x)\right|\left|\tilde{W}_n(x)\right| g_n(x)dx\\ &\leq C\sum_{m=1}^{\#G/\#G'}\int_{\mathbb{R}^d}\left|e^{it_n\Delta}\psi(x)\right|\left|\tau_{-x_n}\mathcal{G}_m^{-1}\tilde{W}_n(x)\right|\tau_{-x_n}\mathcal{G}_m^{-1}g_n(x)dx.\end{aligned}$$

Note that, by the triangle inequality and the Sobolev embedding,

$$\begin{aligned}\left\|\tau_{-x_n}\mathcal{G}_m^{-1}g_n\right\|_{L^{\frac{p+1}{p-1}}} = \|g_n\|_{L^{\frac{p+1}{p-1}}} &\leq \left\||\tilde{\psi}_n|^{p-1}\right\|_{L^{\frac{p+1}{p-1}}} + \left\||\tilde{W}_n|^{p-1}\right\|_{L^{\frac{p+1}{p-1}}}\\ &= \left\|\tilde{\psi}_n\right\|_{L^{p+1}}^{p-1} + \left\|\tilde{W}_n\right\|_{L^{p+1}}^{p-1} \lesssim \left\|\tilde{\psi}_n\right\|_{H^1}^{p-1} + \left\|\tilde{W}_n\right\|_{H^1}^{p-1}\\ &\lesssim \|\psi\|_{H^1}^{p-1} + \|W_n\|_{H^1}^{p-1} < C\end{aligned}$$

where we use $\{W_n\}$ is bounded in H^1 since $\{\varphi_n\}$ is bounded. And $\left\|\tau_{-x_n}\mathcal{G}_m^{-1}\tilde{W}_n\right\|_{L^{p+1}} = \left\|\tilde{W}_n\right\|_{L^{p+1}} < C$. Now, $\|e^{it_n\Delta}\psi\|_{L^{p+1}} \to 0$ as $n \to \infty$ since $t_n \to \pm\infty$ (see [2, Corollary 2.3.7]). Therefore, by the Hölder inequality, we get

$$f_n \lesssim \|e^{it_n\Delta}\psi\|_{L^{p+1}} \left\|\tau_{-x_n}\mathcal{G}_m^{-1}\tilde{W}_n\right\|_{L^{p+1}} \|\tau_{-x_n}\mathcal{G}_m^{-1}g_n\|_{L^{\frac{p+1}{p-1}}} \lesssim \|e^{it_n\Delta}\psi\|_{L^{p+1}} \to 0.$$

This means the second statement of (4).

Case2: Since $\{T_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded, we may assume $T_n \to \bar{t} \in \mathbb{R}$ as $n \to \infty$ taking a subsequence. Let $t_n := 0$ for all n. Minor modifications imply the statements, (1)–(3) and the first statement of (4). See the argument below (5.22) in [6] for the second statement of (4) .

At last, we consider the case of $A = 0$. Then we have $\|e^{-it_n\Delta}\tau_{-x_n}\varphi_n\|_{L^{p+1}} = \|e^{-it_n\Delta}\varphi_n\|_{L^{p+1}} \leq \|e^{-it\Delta}\varphi_n\|_{L^\infty(\mathbb{R}:L^{p+1})} \to 0$. On the other hand, if $e^{-it_n\Delta}\tau_{-x_n}\varphi_n \rightharpoonup \psi/(\#G/\#G')$ as $n \to \infty$ in $H^1(\mathbb{R}^d)$, then $e^{-it_n\Delta}\tau_{-x_n}\varphi_n \to \psi/(\#G/\#G')$ as $n \to \infty$ in $L^{p+1}(B_R)$ for any $R > 0$ by a compactness argument. Combining them, we get $\psi = 0$. $\square$

Lemma 3.3. *Let G be an arbitrary subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$. Let m be a nonnegative integer and $\varphi_j \in H_G^1$ for $j \in \{1, 2, \cdots, m\}$ satisfy*

$$\begin{array}{ll} S_\omega(\sum_{j=1}^m \varphi_j) \leq l_\omega^G - \delta, & S_\omega(\sum_{j=1}^m \varphi_j) \geq \sum_{j=1}^m S_\omega(\varphi_j) - \varepsilon, \\ K(\sum_{j=1}^m \varphi_j) \geq -\varepsilon, & K(\sum_{j=1}^m \varphi_j) \leq \sum_{j=1}^m K(\varphi_j) + \varepsilon, \end{array}$$

for δ, ε satisfying $(1 + d/2)\varepsilon < \delta$. Then we have $0 \leq S_\omega(\varphi_j) < l_\omega^G$ and $K(\varphi_j) \geq 0$ for all $j \in \{1, 2, \cdots, m\}$. Namely, we see that $\varphi_j \in \mathscr{K}_{G,\omega}^+$ for all $j \in \{1, 2, \cdots, m\}$.

Proof. We assume that there exists $j \in \{1, 2, \cdots, m\}$ such that $K(\varphi_j) < 0$. Let $J_\omega := S_\omega - dK/4$. Since we have

$$l_\omega^G = \inf\left\{J_\omega(\varphi) : \varphi \in H_G^1 \setminus \{0\}, K(\varphi) \leq 0\right\}$$

and J_ω is positive, we obtain

$$\begin{aligned} l_\omega^G &\leq \sum_{j=1}^m J_\omega(\varphi_j) = \sum_{j=1}^m S_\omega(\varphi_j) - \sum_{j=1}^m \frac{d}{4}K(\varphi_j) \\ &\leq S_\omega\left(\sum_{j=1}^m \varphi_j\right) + \varepsilon - \frac{d}{4}\left(K\left(\sum_{j=1}^m \varphi_j\right) - \varepsilon\right) \\ &\leq l_\omega^G - \delta + \varepsilon + \frac{d}{2}\varepsilon < l_\omega^G. \end{aligned}$$

This is a contradiction. So, $K(\varphi_j) \geq 0$ for all $j \in \{1, 2, \cdots, m\}$. Moreover, for any $j \in \{1, 2, \cdots, m\}$, we have $S_\omega(\varphi_j) = J_\omega(\varphi_j) + \frac{d}{4}K(\varphi_j) \geq 0$ and

$$S_\omega(\varphi_j) \leq \sum_{j=1}^{m} S_\omega(\varphi_j) \leq S_\omega\left(\sum_{j=1}^{m} \varphi_j\right) + \varepsilon \leq l_\omega^G - \delta + \varepsilon < l_\omega^G.$$

This completes the proof. □

Lemma 3.4. *Let $\{x_n\}$ be a sequence, $\psi \in H^1$, and U be a solution of* (NLS) *with the initial data ψ. Then, we have*

$$U_n(t) = e^{it\Delta}\tau_{x_n}\psi + i\int_0^t e^{i(t-s)\Delta}(|U_n(s)|^{p-1}U_n(s))ds,$$

where $U_n(t,x) := U(t, x - x_n)$.

Lemma 3.4 follows from the space translation invariance of the equation (NLS).

Lemma 3.5. *Let $\{t_n\}$ satisfy $t_n \to \pm\infty$, $\{x_n\}$ be a sequence, $\psi \in H^1$, and U be a solution of* (NLS) *satisfying*

$$\left\|U_\pm(t) - e^{it\Delta}\psi\right\|_{H^1} \to 0 \text{ as } t \to \pm\infty$$

Then, we have

$$U_{\pm,n}(t) = e^{it\Delta}e^{it_n\Delta}\tau_{x_n}\psi + i\int_0^t e^{i(t-s)\Delta}(|U_{\pm,n}(s)|^{p-1}U_{\pm,n}(s))ds + e_{\pm,n}(t),$$

where $U_{\pm,n}(t,x) := U_\pm(t + t_n, x - x_n)$ and $\|e_{\pm,n}\|_{L^\alpha(\mathbb{R}:L^r)} \to 0$ as $n \to \infty$.

Proof. Since $U_{\pm,n}$ is a solution of (NLS) with the initial data $\tau_{x_n}U_\pm(t_n)$ by the time and space translation invariance, we have

$$\begin{aligned} e_{\pm,n}(t) &= U_{\pm,n}(t) - e^{it\Delta}e^{it_n\Delta}\tau_{x_n}\psi - i\int_0^t e^{i(t-s)\Delta}(|U_{\pm,n}(s)|^{p-1}U_{\pm,n}(s))ds \\ &= e^{it\Delta}\tau_{x_n}U_\pm(t_n) - e^{it\Delta}e^{it_n\Delta}\tau_{x_n}\psi. \end{aligned}$$

By the Strichartz estimate,

$$\|e_{\pm,n}\|_{L^\alpha(\mathbb{R}:L^r)} \lesssim \left\|U_\pm(t_n) - e^{it_n\Delta}\psi\right\|_{H^1} \to 0 \text{ as } n \to \infty.$$

This completes the proof. □

§ 3.2. Construction of a critical element and Rigidity

By the definition of $\mathbf{S}_\omega^G$, we have $\mathbf{S}_\omega^{G_k} \leq l_\omega^{G_k}$. Lemma 2.1 and Proposition 2.6 give $\mathbf{S}_\omega^{G_k} > 0$. We prove $\mathbf{S}_\omega^{G_k} = \min\{m_\omega^{G_k}, l_\omega^{G_k}\}$ by contradiction argument so that we suppose $\mathbf{S}_\omega^{G_k} < \min\{m_\omega^{G_k}, l_\omega^{G_k}\}$.

Proposition 3.6. *Assume $\mathbf{S}_\omega^{G_k} < \min\{m_\omega^{G_k}, l_\omega^{G_k}\}$. Then, there exists a global solution u^c to* (NLS) *with G_k-invariance such that $S_\omega(u^c) = \mathbf{S}_\omega^G$ and $\|u^c\|_{L^\alpha(\mathbb{R}:L^r)} = \infty$.*

We call u^c a critical element.

Proof. By the definition of $\mathbf{S}_\omega^G$ and the assumption of $\mathbf{S}_\omega^{G_k} < \min\{m_\omega^{G_k}, l_\omega^{G_k}\}$, there exists a sequence $\{\varphi_n\} \in \mathscr{K}_{G_k,\omega}^+$ satisfying $\mathbf{S}_\omega^{G_k} < S_\omega(\varphi_n) < \min\{m_\omega^{G_k}, l_\omega^{G_k}\}$, $S_\omega(\varphi_n) \searrow \mathbf{S}_\omega^{G_k}$, and $u_n \notin L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$, where u_n is a global solution with the initial data φ_n. Since $\{\varphi_n\}$ is bounded in $H^1(\mathbb{R}^d)$, we apply the linear profile decomposition with G_k-invariance, Proposition 3.1, to the sequence $\{\varphi_n\}$ and then we obtain

$$\varphi_n = \sum_{j=1}^{J} \tilde{\psi}_n^j + \tilde{W}_n^J,$$

where we recall that $\tilde{\psi}_n^j = \sum_{\mathcal{G}\in G} e^{it_n^j\Delta}\mathcal{G}(\tau_{x_n^j}\psi^j)/\#G$ and $\tilde{W}_n^J = \sum_{\mathcal{G}\in G} \mathcal{G}W_n^J/\#G$. We also see that

$$S_\omega(\varphi_n) = \sum_{j=1}^{J} S_\omega\left(\tilde{\psi}_n^j\right) + S_\omega\left(\tilde{W}_n^J\right) + o(1),$$
$$K(\varphi_n) = \sum_{j=1}^{J} K\left(\tilde{\psi}_n^j\right) + K\left(\tilde{W}_n^J\right) + o(1),$$

where $o(1) \to 0$ as $n \to \infty$. By these decompositions, we have

$$\begin{aligned}
&S_\omega\left(\varphi_n\right) \le l_\omega^{G_k} - \delta,\\
&S_\omega\left(\varphi_n\right) \ge \sum_{j=1}^{J} S_\omega\left(\tilde{\psi}_n^j\right) + S_\omega\left(\tilde{W}_n^J\right) - \varepsilon,\\
&K\left(\varphi_n\right) \ge 0 > -\varepsilon,\\
&K\left(\varphi_n\right) \le \sum_{j=1}^{J} K\left(\tilde{\psi}_n^j\right) + K\left(\tilde{W}_n^J\right) + \varepsilon,
\end{aligned}$$

for large n where $\delta = l_\omega^{G_k} - S_\omega(\varphi_1)$ and $\varepsilon > 0$ satisfies $(1 + d/2)\varepsilon < \delta$. Therefore, 3.3 gives us that $\tilde{\psi}_n^j \in \mathscr{K}_{G_k,\omega}^+$ for all $j \in \{1, 2, \cdots, J\}$ and $\tilde{W}_n^J \in \mathscr{K}_{G_k,\omega}^+$. Thus, for any J, we obtain

$$\mathbf{S}_\omega^{G_k} = \lim_{n\to\infty} S_\omega(\varphi_n) \ge \sum_{j=1}^{J} \limsup_{n\to\infty} S_\omega\left(\tilde{\psi}_n^j\right).$$

We prove $\mathbf{S}_\omega^{G_k} = \limsup_{n\to\infty} S_\omega(\tilde{\psi}_n^j)$ for some j by a contradiction argument. We suppose that $\mathbf{S}_\omega^{G_k} = \limsup_{n\to\infty} S_\omega(\tilde{\psi}_n^j)$ fails for all j. Namely, we assume that

$\limsup_{n\to\infty} S_\omega(\tilde{\psi}_n^j) < \mathbf{S}_\omega^{G_k}$ for all j. By reordering, we can choose $0 \le J_1 \le J_2 \le J$ such that

$$\begin{array}{cc} 1 \le j \le J_1 : & t_n^j = 0,\ \forall n \\ J_1+1 \le j \le J_2 : & \lim_{n\to\infty} t_n^j = -\infty, \\ J_2+1 \le j \le J : & \lim_{n\to\infty} t_n^j = +\infty. \end{array}$$

Above we are assuming that if $a > b$ then there is no j such that $a \le j \le b$.

For $j \in [0, J_1]$, by the assumption of the contradiction argument and $t_n^j = 0$, we have $0 < \limsup_{n\to\infty} S_\omega(\sum_{\mathcal{G}\in G} \mathcal{G}(\tau_{x_n^j}\psi^j)/\#G) < \mathbf{S}_\omega^{G_k}$. By the choice of $\{x_n^j\}$ and Lemma 4.5,

$$\frac{\#G}{\#G^j} S_\omega\left(\frac{\psi^j}{\#G/\#G^j}\right) = \limsup_{n\to\infty} S_\omega\left(\sum_{\mathcal{G}\in G} \frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G}\right) < \mathbf{S}_\omega^{G_k} < m_\omega^{G_k}.$$

Therefore, $S_\omega\left(\psi^j/(\#G/\#G^j)\right) < \mathbf{S}_\omega^{G_k^j}$. By the definition of $\mathbf{S}_\omega^{G_k^j}$, the solution U^j to (NLS) with the initial data $\psi^j/(\#G/\#G^j)$ belongs to $L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$.

For $j \in [J_1+1, J_2]$, we have

$$\begin{aligned} m_\omega^{G_k} &> \mathbf{S}_\omega^{G_k} > \limsup_{n\to\infty} S_\omega\left(\tilde{\psi}_n^j\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{2}\left\|\sum_{\mathcal{G}\in G}\frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G}\right\|_{\dot{H}^1}^2 + \frac{\omega}{2}\left\|\sum_{\mathcal{G}\in G}\frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G}\right\|_{L^2}^2\right) \\ &\quad - \frac{1}{p+1}\lim_{n\to\infty}\left\|e^{it_n^j\Delta}\sum_{\mathcal{G}\in G}\frac{\mathcal{G}(\tau_{x_n^j}\psi^j)}{\#G}\right\|_{L^{p+1}}^{p+1} \\ &= \frac{\#G}{\#G^j}\left(\frac{1}{2}\left\|\frac{\psi^j}{\#G/\#G^j}\right\|_{\dot{H}^1}^2 + \frac{\omega}{2}\left\|\frac{\psi^j}{\#G/\#G^j}\right\|_{L^2}^2\right), \end{aligned}$$

where we use $\left\|e^{it_n\Delta}\phi\right\|_{L^{p+1}} \to 0$ as $n \to \infty$ (see [2, Corollary 2.3.7]) and Lemma 4.5. This inequality implies that $\psi^j/(\#G/\#G^j)$ satisfies the assumption of Lemma 2.7 as $G = G_k^j$, where we note that $\mathbf{S}_\omega^{G_k^j} \le l_\omega^{G_k^j}$. Thus, we obtain the global solution U_-^j to (NLS) such that $U_-^j(0) \in \mathscr{K}_{G_k^j,\omega}^+$ and

$$\left\|U_-^j(t) - e^{it\Delta}\frac{\psi^j}{\#G/\#G^j}\right\|_{H^1} \to 0 \text{ as } t \to -\infty.$$

Moreover, U_-^j belongs to $L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$ by the definition of $\mathbf{S}_\omega^{G_k^j}$ since we have

$$S_\omega(U_-^j) = \frac{1}{2}\left\|\frac{\psi^j}{\#G/\#G^j}\right\|_{\dot{H}^1}^2 + \frac{\omega}{2}\left\|\frac{\psi^j}{\#G/\#G^j}\right\|_{L^2}^2 < \mathbf{S}_\omega^{G^j}.$$

For $j \in [J_2+1, J]$, by the similar argument, we obtain a global solution U_+^j such that $U_+^j(0) \in \mathscr{K}^+_{G_k^j,\omega}$, $U_+^j \in L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$, and

$$\left\| U_+^j(t) - e^{it\Delta} \frac{\psi^j}{\#G/\#G^j} \right\|_{H^1} \to 0 \text{ as } t \to \infty.$$

We define

$$U^j := \begin{cases} U^1, \text{ if } j = 1, \\ U_-^j, \text{ if } j \in [2, J_2], \\ U_+^j, \text{ if } j \in [J_2+1, J], \end{cases} \qquad \text{and } U_n^j(t,x) := U^j(t+t_n^j, x - x_n^j).$$

Moreover, we define

$$u_n^J := \sum_{j=1}^{J} \sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)} U_n^j,$$

where $\{\mathcal{G}_m^{(j)}\}_{m=1}^{\#G/\#G^j}$ be the set of left coset representatives. Then u_n^J satisfies

$$i\partial_t u_n^J + \Delta u_n^J + \left|u_n^J\right|^{p-1} u_n^J = e_n^J,$$
$$e_n^J = \left|u_n^J\right|^{p-1} u_n^J - \sum_{j=1}^{J} \sum_{m=1}^{\#G/\#G^j} \left|\mathcal{G}_m^{(j)} U_n^j\right|^{p-1} \mathcal{G}_m^{(j)} U_n^j.$$

Moreover, we have

$$u_n(0) - u_n^J(0) = \sum_{j=1}^{J} \sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)} \left(e^{it_n^j\Delta} \tau_{x_n^j} \frac{\psi^j}{\#G/\#G^j} - \tau_{x_n^j} U^j(t_n^j) \right) + \tilde{W}_n^J.$$

To apply the perturbation lemma, Lemma 2.8, we prove the following inequalities hold for large n.

$$\left\|u_n^J\right\|_{L^\alpha(\mathbb{R};L^r)} \leq A, \tag{3.12}$$

$$\left\|e_n^J\right\|_{L^{\beta'}(R:L^{r'})} \leq \varepsilon(A), \tag{3.13}$$

$$\left\|e^{it\Delta}(u_n(0) - u_n^J(0))\right\|_{L^\alpha(\mathbb{R}:L^r)} \leq \varepsilon(A). \tag{3.14}$$

We prove (3.12). By the definition of U_n^j, we have

$$\begin{aligned} u_n^J(t) &= \sum_{j=1}^{J} \sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)} U_n^j(t) \\ &= \sum_{j=1}^{J_1} \sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)} (\tau_{x_n^j} U^j(t)) + \sum_{j=J_1+1}^{J_2} \sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)} (\tau_{x_n^j} U_-^j(t+t_n^j)) \\ &\quad + \sum_{j=J_2+1}^{J} \sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)} (\tau_{x_n^j} U_+^j(t+t_n^j)). \end{aligned}$$

Let v_n^j denote $\sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)}(\tau_{x_n^j} U^j(t))$ when $1 \leq j \leq J_1$, $\sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)}(\tau_{x_n^j} U_-^j(t+t_n^j))$ when $J_1+1 \leq j \leq J_2$, and $\sum_{m=1}^{\#G/\#G^j} \mathcal{G}_m^{(j)}(\tau_{x_n^j} U_+^j(t+t_n^j))$ when $J_2+1 \leq j \leq J$. Thus, we have

$$u_n^J = \sum_{j=1}^{J} v_n^j.$$

By (5) in Proposition 3.1 and Lemma 4.5, we have

$$\|\varphi_n\|_{H^1}^2 = \sum_{j=1}^{J} \frac{\#G}{\#G^j} \left\| \frac{\psi^j}{\#G/\#G^j} \right\|_{H^1}^2 + \left\| \tilde{W}_n^J \right\|_{H^1}^2 + o_n(1).$$

Therefore, $\sup_{n\in\mathbb{N}} \|\varphi_n\|_{H^1}^2 < \infty$ implies that there exists a finite set $\mathscr{J}$ such that $\left\|\psi^j/(\#G/\#G^j)\right\|_{H^1} < \varepsilon_{sd}$ for $j \notin \mathscr{J}$, where ε_{sd} is a constant appearing in Proposition 2.6. Thus, we get

$$\begin{aligned}
\limsup_{n\to\infty} \left\|u_n^J\right\|_{L^\alpha(\mathbb{R}:L^r)} &= \limsup_{n\to\infty} \left\| \sum_{j=1}^{J} v_n^j \right\|_{L^\alpha(\mathbb{R}:L^r)} \\
&\leq \limsup_{n\to\infty} \left\| \sum_{j\in\mathscr{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}:L^r)} + \limsup_{n\to\infty} \left\| \sum_{j\notin\mathscr{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}:L^r)} \\
&\leq \limsup_{n\to\infty} \sum_{j\in\mathscr{J}} \left\|v_n^j\right\|_{L^\alpha(\mathbb{R}:L^r)} + \limsup_{n\to\infty} \left\| \sum_{j\notin\mathscr{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}:L^r)}
\end{aligned}$$

Using $|t_n^j - t_n^h| \to \infty$ or $|\mathcal{G}x_n^j - \mathcal{G}'x_n^h| \to \infty$ for all $\mathcal{G}, \mathcal{G}' \in G$ if $j \neq h$ and $|\mathcal{G}_m^{(j)} x_n^j - \mathcal{G}_l^{(j)} x_n^j| \to \infty$ if $m \neq l$, the first term is estimated as follows.

$$\limsup_{n\to\infty} \sum_{j\in\mathscr{J}} \left\|v_n^j\right\|_{L^\alpha(\mathbb{R}:L^r)} = \sum_{j\in\mathscr{J}} \left\|U^j\right\|_{L^\alpha(\mathbb{R}:L^r)} < A_1 < \infty. \tag{3.15}$$

Next, we estimate the second term. By the Gagliardo–Nirenberg inequarity $\|f\|_{L^r} \leq C\|\nabla f\|_{L^2}^\eta \|f\|_{L^\gamma}^{1-\eta}$, where $\eta = n^2/4 - n(n+2)/\{2(p+1)\}$ and it satisfies $1-\eta > \gamma/a$, and the Sobolev embedding $\|f\|_{L^\gamma} \leq C\|f\|_{H^1}$, we obtain

$$\|f\|_{L^a(\mathbb{R}:L^r)} \leq C \|f\|_{L^a(\mathbb{R}:H^1)}^{\frac{a-\gamma}{a}} \|f\|_{L^\gamma(\mathbb{R}:L^\gamma)}^{\frac{\gamma}{a}}. \tag{3.16}$$

We have

$$\begin{aligned}
\limsup_{n\to\infty} \left\| \sum_{j\notin\mathscr{J}} v_n^j \right\|_{L^\gamma(\mathbb{R}:L^\gamma)}^\gamma &\leq \limsup_{n\to\infty} \sum_{j\notin\mathscr{J}} \left\|v_n^j\right\|_{L^\gamma(\mathbb{R}:L^\gamma)}^\gamma \\
&\quad + C_J \limsup_{n\to\infty} \sum_{j\notin\mathscr{J},\ j\neq h} \int_{\mathbb{R}\times\mathbb{R}^d} |v_n^j||v_n^h|^{\gamma-1} dxdt
\end{aligned}$$

By Proposition 2.6 and $\gamma > 2$, we get

$$\limsup_{n\to\infty} \sum_{j\notin \mathscr{J}} \left\|v_n^j\right\|_{L^\gamma(\mathbb{R}:L^\gamma)}^\gamma \leq C \limsup_{n\to\infty} \sum_{j\notin \mathscr{J}} \left\|U^j(t_n^j)\right\|_{H^1}^2 \leq C\varepsilon_{sd}.$$

And we have

$$\int_{\mathbb{R}\times\mathbb{R}^d} |v_n^j||v_n^h|^{\gamma-1} dxdt \to 0 \text{ as } n \to \infty,$$

where we use $|t_n^j - t_n^h| \to \infty$ or $|\mathcal{G}x_n^j - \mathcal{G}'x_n^h| \to \infty$ for all $\mathcal{G}, \mathcal{G}' \in G$ if $j \neq h$ and $|\mathcal{G}_m^{(j)}x_n^j - \mathcal{G}_l^{(j)}x_n^j| \to \infty$ if $m \neq l$ (See [6, Lemma 4.5 and (6.38)]). Thus, we obtain

$$\limsup_{n\to\infty} \left\| \sum_{j\notin \mathscr{J}} v_n^j \right\|_{L^\gamma(\mathbb{R}:L^\gamma)} < C < \infty.$$

Moreover, we have

$$\begin{aligned}\limsup_{n\to\infty} \left\| \sum_{j\notin \mathscr{J}} v_n^j \right\|_{L^\infty(\mathbb{R}:H^1)}^2 &\leq \limsup_{n\to\infty} \sum_{j\notin \mathscr{J}} \left\|v_n^j\right\|_{H^1}^2 \\ &\quad + 2 \limsup_{n\to\infty} \sum_{j\notin \mathscr{J},\ j\neq k} \left\langle v_n^j, v_n^k \right\rangle_{H^1}\end{aligned}$$

The second term tends to 0 as $n \to \infty$ and, by Proposition 2.6, the first term is estimated as follows:

$$\limsup_{n\to\infty} \sum_{j\notin \mathscr{J}} \left\|v_n^j\right\|_{H^1}^2 \leq C\varepsilon_{sd}.$$

And thus, we get

$$\limsup_{n\to\infty} \left\| \sum_{j\notin \mathscr{J}} v_n^j \right\|_{L^\infty(\mathbb{R}:H^1)} < C < \infty.$$

Therefore, by (3.16), we have

$$\limsup_{n\to\infty} \left\| \sum_{j\notin \mathscr{J}} v_n^j \right\|_{L^\alpha(\mathbb{R}:L^r)} < A_2 < \infty. \tag{3.17}$$

Combining (3.15) and (3.17), we get

$$\limsup_{n\to\infty} \left\|u_n^J\right\|_{L^\alpha(\mathbb{R}:L^r)} < A_1 + A_2 =: A < \infty.$$

We prove (3.14). By the triangle inequality, the Strichartz estimate, the definition of U^j, (4) in Proposition 3.1, and Lemmas 3.4 and 3.5, we have

$$\begin{aligned}
&\left\|e^{it\Delta}(u_n(0)-u_n^J(0))\right\|_{L^\alpha(\mathbb{R}:L^r)}\\
&\leq \sum_{j=1}^{J}\sum_{k=1}^{\#G/\#G^j}\left\|e^{it\Delta}\left(\tau_{x_n^j}U^j(t_n^j)-e^{it_n^j\Delta}\tau_{x_n^j}\frac{\psi^j}{\#G/\#G^j}\right)\right\|_{L^\alpha(\mathbb{R}:L^r)}+\left\|e^{it\Delta}\tilde{W}_n^J\right\|_{L^\alpha(\mathbb{R}:L^r)}\\
&\leq \sum_{j=1}^{J}\sum_{k=1}^{\#G/\#G^j}\left\|\tau_{x_n^j}U^j(t_n^j)-e^{it_n^j\Delta}\tau_{x_n^j}\frac{\psi^j}{\#G/\#G^j}\right\|_{H^1}+\left\|e^{it\Delta}\tilde{W}_n^J\right\|_{L^\alpha(\mathbb{R}:L^r)}\\
&\leq \varepsilon\leq\varepsilon(A),
\end{aligned}$$

for large n and J. We prove (3.13). In general, the following inequality holds.

$$\left|\left|\sum_{j=1}^{J}z^j\right|^{p-1}\sum_{j=1}^{J}z^j-\sum_{j=1}^{J}\left|z^j\right|^{p-1}z^j\right|\leq C_J\sum_{1\leq j\neq h\leq J}|z^j|^{p-1}|z^h|.$$

This implies that

$$\left\|e_n^J\right\|_{L^{\beta'}(\mathbb{R}:L^{r'})}\leq C_J\sum_{1\leq j\neq h\leq J}\left\||U_n^j|^{p-1}|U_n^h|\right\|_{L^{\beta'}(\mathbb{R}:L^{r'})}.$$

An approximation argument and $|t_n^j-t_n^h|\to\infty$ or $|\mathcal{G}x_n^j-\mathcal{G}'x_n^h|\to\infty$ for all $\mathcal{G},\mathcal{G}'\in G$ if $j\neq h$ and also use $|\mathcal{G}_k^{(j)}x_n^j-\mathcal{G}_l^{(j)}x_n^j|\to\infty$ if $k\neq l$ give us $\left\||U_n^j|^{p-1}|U_n^h|\right\|_{L^{\beta'}(\mathbb{R}:L^{r'})}\to 0$ as $n\to\infty$. Thus, we obtain (3.13). Applying Lemma 2.8, we conclude that u_n scatters. However, this contradicts the definition of $\{\varphi_n\}$. Therefore, there exists j such that $\mathbf{S}_\omega^{G_k}=\limsup_{n\to\infty}S_\omega(\tilde{\psi}_n^j)$. We may assume $j=1$. The linear profile decomposition as $J=1$ and $\tilde{W}_n^1\in\mathscr{K}_{G,\omega}^{+}$ imply $\left\|\tilde{W}_n^1\right\|_{L^\infty(\mathbb{R}:H^1)}\to 0$ as $n\to\infty$ by Lemma 2.1. Therefore, we see that

$$\begin{aligned}
\varphi_n&=\tilde{\psi}_n^1+\tilde{W}_n^1,\\
\left\|\tilde{W}_n^1\right\|_{L^\infty(\mathbb{R}:H^1)}&\to 0,\\
\mathbf{S}_\omega^{G_k}&=\lim_{n\to\infty}S_\omega(\tilde{\psi}_n^1).
\end{aligned}$$

We assume that there exists $G^1\subsetneq G$ such that $x_n^1=\mathcal{G}^1x_n^1$ for all $\mathcal{G}^1\in G^1$ and $|x_n^1-\mathcal{G}x_n^1|\to\infty$ for all $\mathcal{G}\in G\setminus G^1$. Let U be a global solution of (NLS) with the initial data $\psi^1/(\#G/\#G^1)$ if $t_n^1=0$ or the final data $\psi^1/(\#G/\#G^1)$ if $|t_n^1|\to\infty$. Then, by the definition of $\mathbf{S}_\omega^{G_k^1}$, U belongs to $L^\alpha(\mathbb{R}:L^r(\mathbb{R}^d))$ since we have, by Lemma 4.5,

$$\lim_{n\to\infty}S_\omega(\tilde{\psi}_n^1)=\lim_{n\to\infty}\frac{\#G}{\#G^1}S_\omega\left(e^{it_n^1\Delta}\frac{\psi^1}{\#G/\#G^1}\right)=\mathbf{S}_\omega^{G_k}<m_\omega^{G_k}\leq\frac{\#G}{\#G^1}\mathbf{S}_\omega^{G_k^1}.$$

By Lemma 2.8 again, this contradicts that u_n does not belong to $L^\alpha(\mathbb{R} : L^r(\mathbb{R}^d))$. Thus, $G^1 = G$. This means that ψ^1 and W_n^1 are G_k-invariant, $x_n^1 = \mathcal{G}x_n^1$ for all $\mathcal{G} \in G_k$, and we see that

$$\varphi_n = e^{it_n^1\Delta}\tau_{x_n^1}\psi^1 + W_n^1.$$

Let u^c be a global solution of (NLS) with the initial data ψ^1 if $t_n^1 = 0$ or the final data ψ^1 if $|t_n^1| \to \infty$. Then, u^c is G_k-invariant. We prove $\|u^c\|_{L^\alpha(\mathbb{R}:L^r)} = \infty$. Suppose that $\|u^c\|_{L^\alpha(\mathbb{R}:L^r)} < \infty$. We observe that $\varphi_n - \tau_{x_n^1}u^c(t_n^1) = e^{it_n^1\Delta}\tau_{x_n^1}\psi^1 - \tau_{x_n^1}u^c(t_n^1) + W_n^1$, so that we have

$$\left\|e^{it\Delta}\left(\varphi_n - \tau_{x_n^1}u^c(t_n^1)\right)\right\|_{L^\alpha(\mathbb{R};L^r)} \to 0 \text{ as } n \to \infty.$$

By Lemma 2.8, we see that $u_n \in L^\alpha(\mathbb{R} : L^r(\mathbb{R}))$ for large n, which is absurd. Thus, we get $\|u^c\|_{L^\alpha(\mathbb{R}:L^r)} = \infty$. Moreover, we have $S_\omega(u^c) = \lim_{n\to\infty} S_\omega(e^{it_n^1}\psi^1) = \mathbf{S}_\omega^{G_k}$. Thus, we get a critical element u^c. □

We say that the solution u is a forward critical element if u is a critical element and satisfies $\|u\|_{L^\alpha([0,\infty):L^r)} = \infty$. In the same manner, we define a backward critical element. We only prove extinction of the forward critical element since that of the backward critical element can be obtained by the similar argument based on time reversibility. The extinction contradicts Proposition 3.6.

Lemma 3.7. *Let u be a forward critical element. There exists a continuous function $x : [0,\infty) \to \mathbb{R}^d$ such that $\mathcal{G}x(t) = x(t)$ for all $\mathcal{G} \in G_k$ and $\{u(t,\cdot - x(t)) : t \in [0,\infty)\}$ is precompact in $H^1(\mathbb{R}^d)$.*

The above lemma can be obtained by the same argument as in [5, Proposition 3.2] noting u is G_k-invariant and $\{x_n^1\}$, which appears in the profile decomposition, satisfies $\mathcal{G}x_n^1 = x_n^1$ for all $\mathcal{G} \in G_k$.

Lemma 3.8. *Let u be a solution to* (NLS) *satisfying that there exists a continuous function $x : [0,\infty) \to \mathbb{R}^d$ such that $\{u(t,\cdot - x(t)) : t \in [0,\infty)\}$ is precompact in $H^1(\mathbb{R}^d)$. Then, for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that*

$$\int_{|x+x(t)|>R} |\nabla u(t,x)|^2 + |u(t,x)|^2 + |u(t,x)|^{p+1}dx < \varepsilon \text{ for any } t \in [0,\infty). \tag{3.18}$$

It can be obtained by using directly the argument of [5, Corollary 3.3].

Lemma 3.9. *Let u be a forward critical element. Then, the momentum must be 0, i.e. $P(u) = 0$.*

Proof. First, we prove $\mathcal{G}P(u) = P(u)$ for all $\mathcal{G} \in G_k$. By the G_k-invariance of u, we see that

$$P(u) = P(\mathcal{G}^{-1}u) = \operatorname{Im} \int_{\mathbb{R}^d} \overline{e^{i\theta}u(\mathcal{G}x)} \nabla\{e^{i\theta}u(\mathcal{G}x)\}dx$$
$$= \mathcal{G} \operatorname{Im} \int_{\mathbb{R}^d} \overline{u(\mathcal{G}x)} \nabla u(\mathcal{G}x)dx = \mathcal{G} \operatorname{Im} \int_{\mathbb{R}^d} \overline{u(x)} \nabla u(x)dx = \mathcal{G}P(u).$$

Therefore, the Galilean transformation

$$u_{\xi_0}(t,x) := e^{i(x\cdot\xi_0 - |\xi_0|^2 t)} u(t, x - 2t\xi_0),$$

where $\xi_0 = -P(u)/M(u)$, conserves the G_k-invariance of the solution. The rest of the proof is same as in [5, Proposition 4.1] and [1, Proposition 4.1 (iii)]. □

We use the following lemma to prove the rigidity lemma, Lemma 3.11.

Lemma 3.10. *Let u be a solution to* (NLS) *on* $[0,\infty)$ *such that* $P(u) = 0$ *and there exists a continuous* $x : [0,\infty) \to \mathbb{R}^d$ *such that, for any* $\varepsilon > 0$*, there exists* $R = R(\varepsilon) > 0$ *such that*

$$\int_{|x+x(t)|>R} |\nabla u(t,x)|^2 + |u(t,x)|^2 + |u(t,x)|^{p+1} dx < \varepsilon \text{ for any } t \in [0,\infty).$$

Then, we have

$$\frac{x(t)}{t} \to 0 \text{ as } t \to \infty.$$

This follows from [5, Lemma 5.1], [6, Proof of Theorem 7.1, Step1].

Lemma 3.11 (Rigidity)**.** *Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$. If the solution u with G-invariance satisfies the following properties, then $u = 0$.*

1. $u_0 \in \mathcal{K}^+_{G,\omega}$.
2. $P(u) = 0$.
3. *There exists a continuous* $x : [0,\infty) \to \mathbb{R}^d$ *such that* $\mathcal{G}x(t) = x(t)$ *for all* $t \in [0,\infty)$ *and* $\mathcal{G} \in G$ *and, for any* $\varepsilon > 0$*, there exists* $R = R(\varepsilon) > 0$ *such that*

$$\int_{|x+x(t)|>R} |\nabla u(t,x)|^2 + |u(t,x)|^2 + |u(t,x)|^{p+1} dx < \varepsilon \text{ for any } t \in [0,\infty).$$

For the proof of Lemma 3.11, see [5, Theorem 6.1] and [6, Theorem7.1].

Combining Lemmas 3.7, 3.8, and 3.9, the forward critical element satisfies the assumption (1)–(3) in Lemma 3.11. The result by Lemma 3.11 contradicts $S_\omega(u) =$

$\mathbf{S}_\omega^{G_k} > 0$. Thus, we get $\mathbf{S}_\omega^{G_k} = \min\{m_\omega^{G_k}, l_\omega^{G_k}\}$, which completes the proof of Proposition 1.3.

§ 3.3. Proof of Theorem 1.2

First, we prove the following lemma.

Lemma 3.12. *Let G' be a subgroup of a finite group G in $O(d-k)$ satisfying $(*)$. Then, we have*

$$l_\omega^{G_k} \leq \frac{\#G}{\#G'} l_\omega^{G'_k}.$$

Proof. By the definition of $l_\omega^{G'_k}$, for large $N \in \mathbb{N}$, there exists $Q'_N \in H^1_{G'_k}$ such that

$$S_\omega(Q'_N) = l_\omega^{G'_k} + \frac{1}{N} \text{ and } K(Q'_N) = 0.$$

And there exists a sequence $\{x_n\} \subset \mathbb{R}^d$ such that

$$\begin{cases} \{x_n - \mathcal{G}' x_n\} \text{ is bounded for all } \mathcal{G}' \in G', \\ |x_n - \mathcal{G} x_n| \to \infty \text{ as } n \to \infty, \text{ for all } \mathcal{G} \in G \setminus G'. \end{cases}$$

We define

$$Q_n := \sum_{\mathcal{G} \in G} \frac{\mathcal{G}(\tau_{x_n} Q'_N)}{\#G'} \text{ and } \lambda_n := \frac{\frac{4}{d} \|\nabla Q_n\|_{L^2}^2}{\frac{2(p-1)}{p+1} \|Q_n\|_{L^{p+1}}^{p+1}}.$$

Then, $K(\lambda_n Q_n) = 0$ and $\lambda_n Q_n$ is G_k-invariant. Moreover, Lemma 4.5 implies that

$$\|Q_n\|_{\dot{H}^s}^2 \to \frac{\#G}{\#G'} \|Q'_N\|_{\dot{H}^s}^2 \text{ and } \|Q_n\|_{L^{p+1}}^{p+1} \to \frac{\#G}{\#G'} \|Q'_N\|_{L^{p+1}}^{p+1},$$

where $s = 0, 1$. Therefore, we obtain $\lambda_n \to 1$ as $n \to \infty$. This implies

$$S_\omega(\lambda_n Q_n) \to \frac{\#G}{\#G'} \left(l_\omega^{G'_k} + \frac{1}{N} \right).$$

This means that $l_\omega^{G_k} \leq \#G l_\omega^{G'_k} / \#G'$. □

Next, we prove $\mathbf{S}_\omega^{G_k} = l_\omega^{G_k}$ by the Noetherian induction argument.

Proof of Theorem 1.2. $\{O(k) \times G : G \text{ is a finite group in } O(d-k)\}$ is well-founded by the binary relation $\subset$ and the minimal element is $O(k) \times \{\mathcal{I}_{d-k}\}$.

Step1. By Proposition 1.3, we have $\mathbf{S}_\omega^G = l_\omega^G$ if $G = O(k) \times \{\mathcal{I}_{d-k}\}$.

Step2. Let G be a finite subgroup in $O(d-k)$. We assume that $\mathbf{S}_\omega^{G'_k} = l_\omega^{G'_k}$ for any subgroup G' of G. Then, by Lemma 3.12, we get

$$\begin{aligned} m_\omega^{G_k} &= \min_{G' \subsetneq G \text{ satisfying } (*)} \frac{\#G}{\#G'} \mathbf{S}_\omega^{G'_k} \\ &= \min_{G' \subsetneq G \text{ satisfying } (*)} \frac{\#G}{\#G'} l_\omega^{G'_k} \\ &\geq l_\omega^{G_k}. \end{aligned}$$

Therefore, by Proposition 1.3, we obtain $\mathbf{S}_\omega^{G_k} = \min\{m_\omega^{G_k}, l_\omega^{G_k}\} = l_\omega^{G_k}$. Thus, Noetherian induction implies that $\mathbf{S}_\omega^{G_k} = l_\omega^{G_k}$ for any finite group G in $O(d-k)$. This means that Theorem 1.2 holds. □

§ 4. Lemmas

We denote the Sobolev exponent by

$$2_d^* = \begin{cases} \infty & \text{if } d = 1, 2, \\ \frac{2d}{d-2} & \text{if } d \geq 3, \end{cases}$$

We define cylinder sets by

$$\begin{aligned} C_R &:= \{(\tilde{x}, x') \in \mathbb{R}^k \times \mathbb{R}^{d-k} : |\tilde{x}| < R\}, \\ C^R &:= \{(\tilde{x}, x') \in \mathbb{R}^k \times \mathbb{R}^{d-k} : |x'| < R\}, \end{aligned}$$

for $R > 0$ and we denote the complement of C_R by C_R^c.

Lemma 4.1 (partially radial Sobolev inequality). *Let $d \geq 2$, $k \in \{2, 3, \cdots, d\}$, $2 < q < 2_d^*$. We assume that $f \in H^1(\mathbb{R}^d)$ is $O(k) \times \{\mathcal{I}_{d-k}\}$-invariant. Then,*

$$\|f\|_{L^q(C_R^c)} \lesssim R^{-\frac{(k-1)(q-2)}{2q}} \|f\|_{H^1}$$

holds for $R > 0$.

Proof. The radial Sobolev inequality is well known if $k = d$. We only consider the case of $k \in \{2, 3, \cdots, d-1\}$. Since $|f|$ is radial for $\tilde{x} \in \mathbb{R}^k$, we have the following inequality (see [15, Radial Lemma 1]).

$$|f(\tilde{x}, x')| \lesssim |\tilde{x}|^{-\frac{k-1}{2}} \|f(\cdot, x')\|_{L^2(\mathbb{R}^k)}^{1/2} \|f(\cdot, x')\|_{\dot{H}^1(\mathbb{R}^k)}^{1/2}.$$

Therefore, we get

$$|f(x)|^{q-2} \lesssim |\tilde{x}|^{-\frac{(k-1)(q-2)}{2}} \|f(\cdot, x')\|_{H^1(\mathbb{R}^k)}^{q-2}.$$

Multiplying $|f(x)|^2$ and integrating on C_R^c, we get

$$\begin{aligned}\int_{C_R^c} |f(x)|^q dx &\lesssim \int_{C_R^c} |\tilde{x}|^{-\frac{(k-1)(q-2)}{2}} \|f(\cdot,x')\|_{H^1(\mathbb{R}^k)}^{q-2} |f(x)|^2 dx \\ &\leq R^{-\frac{(k-1)(q-2)}{2}} \int_{\mathbb{R}^d} \|f(\cdot,x')\|_{H^1(\mathbb{R}^k)}^{q-2} |f(x)|^2 dx \\ &= R^{-\frac{(k-1)(q-2)}{2}} \int_{\mathbb{R}^{d-k}} \|f(\cdot,x')\|_{H^1(\mathbb{R}^k)}^{q-2} \left(\int_{\mathbb{R}^k} |f(\tilde{x},x')|^2 d\tilde{x}\right) dx' \\ &\leq R^{-\frac{(k-1)(q-2)}{2}} \int_{\mathbb{R}^{d-k}} \|f(\cdot,x')\|_{H^1(\mathbb{R}^k)}^{q} dx'.\end{aligned}$$

By the Minkowskii integral inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{d-k}} \|f(\cdot,x')\|_{H^1(\mathbb{R}^k)}^{q} dx' &= \int_{\mathbb{R}^{d-k}} \left(\|f(\cdot,x')\|_{H^1(\mathbb{R}^k)}^{2}\right)^{\frac{q}{2}} dx' \\ &= \int_{\mathbb{R}^{d-k}} \left(\|\nabla_{\tilde{x}} f(\cdot,x')\|_{L^2(\mathbb{R}^k)}^{2} + \|f(\cdot,x')\|_{L^2(\mathbb{R}^k)}^{2}\right)^{\frac{q}{2}} dx' \\ &\leq \int_{\mathbb{R}^{d-k}} \left(\|\nabla_{\tilde{x}} f(\cdot,x')\|_{L^2(\mathbb{R}^k)} + \|f(\cdot,x')\|_{L^2(\mathbb{R}^k)}\right)^{q} dx' \\ &\leq 2^{q-1} \int_{\mathbb{R}^{d-k}} \|\nabla_{\tilde{x}} f(\cdot,x')\|_{L^2(\mathbb{R}^k)}^{q} + \|f(\cdot,x')\|_{L^2(\mathbb{R}^k)}^{q} dx' \\ &\lesssim \int_{\mathbb{R}^{d-k}} \|\nabla_{\tilde{x}} f(\cdot,x')\|_{L^2(\mathbb{R}^k)}^{q} dx' + \int_{\mathbb{R}^{d-k}} \|f(\cdot,x')\|_{L^2(\mathbb{R}^k)}^{q} dx' \\ &\leq \left\| \|\nabla_{\tilde{x}} f\|_{L^q(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} + \left\| \|f\|_{L^q(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q}.\end{aligned}$$

We note that $2 < q < 2^*_{d-k}$. Indeed, when $d-k = 1, 2$, we have $2^*_{d-k} = \infty > q$ and when $d-k \geq 3$, we have

$$2^*_{d-k} = \frac{2(d-k)}{d-k-2} = 2 + \frac{4}{d-k-2} > 2 + \frac{4}{d-2} = 2^*_d > q.$$

Moreover, by the Sobolev embedding, we get

$$\begin{aligned}\left\| \|\nabla_{\tilde{x}} f\|_{L^q(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} &\lesssim \left\| \|\nabla_{\tilde{x}} f\|_{H^1(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} \\ &\lesssim \left\| \|\nabla_{\tilde{x}} f\|_{L^2(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} + \left\| \|\nabla f\|_{L^2(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} \\ &\lesssim \|f\|_{H^1(\mathbb{R}^d)}^{q}\end{aligned}$$

and

$$\begin{aligned}\left\| \|f\|_{L^q(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} &\lesssim \left\| \|f\|_{H^1(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} \\ &\lesssim \left\| \|f\|_{L^2(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} + \left\| \|\nabla_{x'} f\|_{L^2(\mathbb{R}^{d-k})} \right\|_{L^2(\mathbb{R}^k)}^{q} \\ &\lesssim \|f\|_{H^1(\mathbb{R}^d)}^{q}.\end{aligned}$$

Therefore, we have

$$\int_{C_R^c} |f(x)|^q dx \lesssim R^{-\frac{(k-1)(q-2)}{2}} \|f\|_{H^1(\mathbb{R}^d)}^q .$$

This completes the proof. □

Lemma 4.2. *Let $d \geq 2$, $k \in \{2, 3, \cdots, d\}$, $2 < q < 2_d^*$, and $R > 0$. Let $\{v_n\} \subset H^1_{O(k)\times\{\mathcal{I}_{d-k}\}}$ and $v \in H^1_{O(k)\times\{\mathcal{I}_{d-k}\}}$. If $v_n \rightharpoonup v$ weakly in H^1, then $v_{n_j} \to v$ in $L^q(C^R)$ by taking a subsequence.*

Proof. Since $v_n \rightharpoonup v$ weakly in H^1, we have $M := \sup_n \|v_n\|_{H^1} < \infty$. Using a diagonal argument and the Rellich–Kondrashov theorem, we can take a subsequence $\{v_{n_j}\}$ such that $v_{n_j} \to v$ in $L^q(B_N)$ for all $N \in \mathbb{N}$. We have

$$\begin{aligned}\left\|v_{n_i} - v_{n_j}\right\|_{L^q(C^R)} &\leq \left\|v_{n_i} - v_{n_j}\right\|_{L^q(B_N)} + \left\|v_{n_i} - v_{n_j}\right\|_{L^q(C^c_{N/2})} \\ &\leq \left\|v_{n_i} - v_{n_j}\right\|_{L^q(B_N)} + \|v_{n_i}\|_{L^q(C^c_{N/2})} + \left\|v_{n_j}\right\|_{L^q(C^c_{N/2})}\end{aligned}$$

for sufficiently large $N \in \mathbb{N}$ such that $N \gg R$. Take $\varepsilon > 0$ arbitrarily. By Lemma 4.1, there exists $N_\varepsilon = N(\varepsilon) \in \mathbb{N}$ such that for large $N \geq N_\varepsilon$, we have

$$\left\|v_{n_j}\right\|_{L^q(C^c_{N/2})} \lesssim N^{-\frac{(k-1)(q-2)}{2q}} \left\|v_{n_j}\right\|_{H^1} \leq MN^{-\frac{(k-1)(q-2)}{2q}} \leq \frac{\varepsilon}{4}$$

for any $j \in \mathbb{N}$ and

$$\|v\|_{L^q(C^c_{N/2})} \leq \frac{\varepsilon}{4}.$$

On the other hand, for fixed N, there exists $J_\varepsilon = J(\varepsilon, N) \in \mathbb{N}$ such that for $i, j > J_\varepsilon$ we have

$$\left\|v_{n_j} - v\right\|_{L^q(B_N)} < \frac{\varepsilon}{2}.$$

Therefore, for large $i, j \geq J(\varepsilon, N_\varepsilon)$, we obtain

$$\begin{aligned}\left\|v_{n_j} - v\right\|_{L^q(C^R)} &\leq \left\|v_{n_j} - v\right\|_{L^q(B_{N_\varepsilon})} + \left\|v_{n_j}\right\|_{L^q(C^c_{N_\varepsilon/2})} + \|v\|_{L^q(C^c_{N_\varepsilon/2})} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

This means that $v_{n_j} \to v$ in $L^q(C^R)$. □

The proof of the following lemmas can be found in [12, Appendix A].

Lemma 4.3. *Let G be a (possibly infinite) subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and $\{\tilde{x}_n\}$ be a sequence. Then, there exists a subgroup G' of G such that the sequence $\{\tilde{x}_n - \mathcal{G}'\tilde{x}_n\}$ is bounded for all $\mathcal{G}' \in G'$ and $|\tilde{x}_n - \mathcal{G}\tilde{x}_n| \to \infty$ as $n \to \infty$ for all $\mathcal{G} \in G \setminus G'$.*

Lemma 4.4. *Let $k \in \mathbb{N}$ and $\mathcal{A}$ be a $kd \times d$-matrix. We assume that a sequence $\{\tilde{x}_n\} \subset \mathbb{R}^d$ satisfies that there exists $\bar{x} \in \mathbb{R}^{kd}$ such that $\mathcal{I}\tilde{x}_n - \mathcal{A}\tilde{x}_n \to \bar{x}$ where $\mathcal{I}$ is a $kd \times d$-matrix such that*

$$\mathcal{I} = \left.\begin{pmatrix} \mathcal{I}_d \\ \mathcal{I}_d \\ \vdots \\ \mathcal{I}_d \end{pmatrix}\right\} k.$$

Then, there exist $\{x_n\} \subset \mathbb{R}^d$ and $x_\infty \in \mathbb{R}^d$ such that

$$\begin{cases} \mathcal{A}x_n = \mathcal{I}x_n, \\ x_n - \tilde{x}_n \to x_\infty. \end{cases}$$

Lemma 4.5. *Let G be a finite group in $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and G' be a subgroup of G. Let $f \in H^1_{G'}$ and $\{x_n\}$ satisfy $|x_n - \mathcal{G}x_n| \to \infty$ as $n \to \infty$ for $\mathcal{G} \in G \setminus G'$. We have the following identities.*

$$\left\| \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right\|_{\dot{H}^\lambda}^2 = \frac{\#G}{\#G'} \|\#G' f\|_{\dot{H}^\lambda}^2 + o(1), \tag{4.1}$$

$$\left\| \sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right\|_{L^p}^p = \frac{\#G}{\#G'} \|\#G' f\|_{L^p}^p + o(1) \tag{4.2}$$

where $\lambda \in [0,1]$, $p \geq 1$ and $o(1) \to 0$ as $n \to \infty$. In particular, the following identity holds for any $\omega > 0$.

$$S_\omega \left(\sum_{\mathcal{G} \in G} \mathcal{G}(\tau_{x_n} f) \right) = \frac{\#G}{\#G'} S_\omega (\#G' f) + o(1).$$

§ 5. Concluding remarks

(1) Our method can be applicable to the case of $O(k_1) \times O(k_2) \times \cdots \times O(k_n) \times G$ where G is a finite group in $O(d-k)$ and $k := \sum_{i=1}^n k_i < d$ and $k_i > 1$ for all $i \in \{1, 2, \cdots, n\}$.

(2) We show some applications of Theorem 1.2.

- Let $d = 3$. For $m \in \mathbb{Z}$, we define

$$G^1 = \left\{ \left(m\theta, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) : \theta \in [0, 2\pi) \right\}.$$

By Theorem 1.2, if $u_0 \in H^1_{G^1}$ satisfies $S_\omega(u_0) < l_\omega^{G^1}$ and $K(u_0) \geq 0$ then the solution u scatters.

- Let $d = 3$. For $m \in \mathbb{Z}$, we define

$$G^2 = \left\{ \left(m\theta, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right), \left(-m\theta, \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \right) : \theta \in [0, 2\pi) \right\}.$$

By Lemma 3.12 and an easy observation, we have $l_\omega^{G^2} = 2l_\omega^{G^1}$. Therefore, by Theorem 1.2, if $u_0 \in H^1_{G^2}$ satisfies $S_\omega(u_0) < 2l_\omega^{G^1}$ and $K(u_0) \geq 0$ then the solution u scatters. We note that $l_\omega^{G^1} \geq l_\omega^{\{\mathcal{I}_d\}}$, where $l_\omega^{\{\mathcal{I}_d\}}$ is the mass-energy of the usual ground state standing wave so that Theorem 1.2 means that we can determine the global behavior of the solutions above the ground state standing waves by the sign of the functional K when we assume the group invariance.

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