

Sharp L^p – L^q estimates for the spherical harmonic projection

By

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Abstract

We consider L^p – L^q estimates for the spherical harmonic projection operators and obtain sharp bounds on a certain range of p, q . As an application, we provide a proof of off-diagonal Carleman estimates for the Laplacian, which extends the earlier results due to Jerison and Kenig [22], and Stein [34].

§ 1. Introduction

Spherical harmonics and spectral projection Let \mathbb{S}^d be the d -dimensional unit sphere contained in \mathbb{R}^{d+1} and \mathcal{H}_n^d be the space of spherical harmonic polynomials of degree n defined on \mathbb{S}^d . It is well known that $L^2(\mathbb{S}^d) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^d$, and \mathcal{H}_n^d is of dimension $\sim n^{d-1}$ and the eigenspace of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^d}$ with the eigenvalue $-n(n+d-1)$. See [37] for further details. Let us denote by H_n^d the projection operator to \mathcal{H}_n^d . Then, for any $f \in L^2(\mathbb{S}^d)$, we may write

$$f = \sum_{n=0}^{\infty} H_n^d f.$$

The optimal L^p – L^q bound for H_n^d in terms of n has drawn interest being related to applications to various problems, for example, convergence of Riesz means on the sphere [29], Carleman estimates in connection with unique continuation problems ([30, 31, 21,

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24]). Even though the bounds for H_n^d have a wide range of (and frequent) applications, as far as the authors are aware, it seems that the optimal L^p - L^q bound for H_n^d has not been considered for general p, q . In this paper we attempt to obtain a complete characterization of L^p - L^q estimates for the spherical harmonic projection for a certain range of p, q , and make clear the connection between these bounds and estimates for the Carleson-Sjölin type oscillatory integral operator.

Let $p, q \in [1, \infty]$ and $p \leq q$. We define

$$\|H_n^d\|_{p,q} := \sup_{\|f\|_{L^p(\mathbb{S}^d)} \leq 1} \|H_n^d f\|_{L^q(\mathbb{S}^d)}.$$

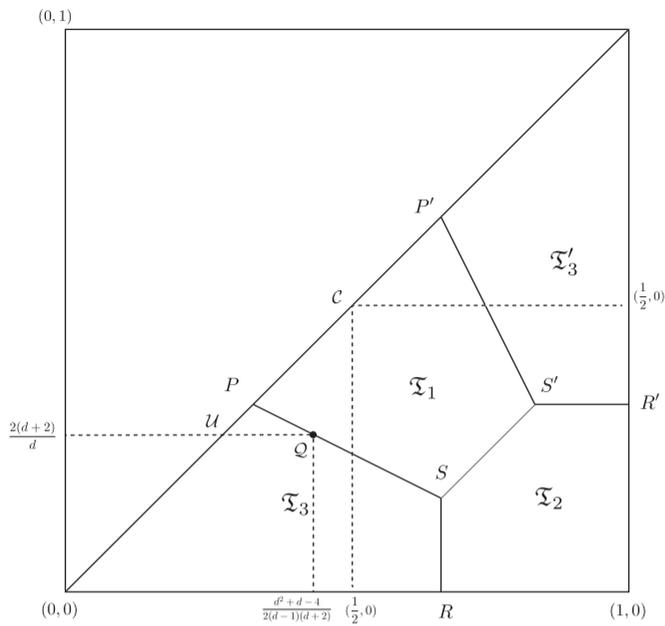
There are easy bounds which are basically consequences of Parseval's identity. Trivially, $\|H_n^d\|_{2,2} = 1$. This combined with the Cauchy-Schwarz inequality yields the bound $\|H_n^d\|_{2,\infty} \lesssim n^{\frac{d-1}{2}}$ since $\dim(\mathcal{H}_n^d) \sim n^{d-1}$. Duality argument gives $\|H_n^d\|_{1,\infty} \lesssim n^{d-1}$. Moreover, it turns out that all those bounds are optimal. (For the meaning of the standard notations \lesssim and \sim , see the end of this section.) However, for the other p, q , it is no longer trivial to obtain the optimal bound. As was observed in [29], for general p, q , the problem of proving the optimal bound for $\|H_n^d\|_{p,q}$ is closely tied to the optimal decay estimates for the Carleson-Sjölin type oscillatory integral operators [35, 18, 5, 6, 27], which are again related to the outstanding conjectures in harmonic analysis such as the Bochner-Riesz conjecture and the Fourier restriction conjecture on the sphere ([42, 44, 43, 40, 41, 26]). For most recent developments see Bourgain and Guth [6] and Guth, Hickman and Iliopoulou [17]. These results are respectively based on multilinear estimates due to Bennett, Carbery and Tao [2] and the method of polynomial partitioning due to Guth [16, 15].

Motivated by Stanton and Weinstein [33], Sogge [29] obtained optimal bounds for $\|H_n^d\|_{p,q}$ with $p \leq 2$, $q = 2$ for any dimensions $d \geq 2$. Especially, when $d = 2$, thanks to the well established 2-dimensional oscillatory integral estimates for Carleson-Sjölin type operators ([9, 18]) he also obtained bounds for p, q in a wider range. See [29] for details. Also, Sogge extended his result to the spectral projection operator on compact manifold [30]. There are also results concerning more specialized bounds such as $\sup_{\|f\|_{L^p(\mathbb{S}^d)}=1, f \in \mathcal{H}_n^d} \|H_n^d f\|_{L^q(\mathbb{S}^d)}$. See, for example, Dai, Feng and Tikhonov [11], De Carli and Grafakos [12].

In order to state our result we need to introduce some notations. Let I be the interval $[0, 1]$, and define points $P = P(d)$, $R = R(d)$, $S = S(d) \in I^2$ by setting

$$P = \left(\frac{d-1}{2d}, \frac{d-1}{2d} \right), \quad R = \left(\frac{d+1}{2d}, 0 \right), \quad S = \left(\frac{d+1}{2d}, \frac{(d-1)^2}{2d(d+1)} \right),$$

and we also define P', R', S' by $(x, y)' = (1-y, 1-x)$. Let $\mathfrak{T}_1 = \mathfrak{T}_1(d)$, $\mathfrak{T}_2 = \mathfrak{T}_2(d) \subset I^2$

Figure 1. The points P, P', R, R' and S, S'

be given by

$$\mathfrak{T}_1 = \left\{ (x, y) \in I^2 : y \leq x, y \geq \frac{d-1}{d+1}(1-x), \frac{d+1}{d-1}(1-x) \geq y, x-y < \frac{2}{d+1} \right\},$$

$$\mathfrak{T}_2 = \left\{ (x, y) \in I^2 : x \geq \frac{d+1}{2d}, x-y \geq \frac{2}{d+1}, y \leq \frac{d-1}{2d} \right\}.$$

We also define $\mathfrak{T}_3 = \mathfrak{T}_3(d)$, and $\mathfrak{T}'_3 = \mathfrak{T}'_3(d) \subset I^2$ by setting

$$\mathfrak{T}_3 = \left\{ (x, y) \in I^2 : y \leq x, y < \frac{d-1}{d+1}(1-x), x < \frac{d+1}{2d} \right\},$$

$$\mathfrak{T}'_3 = \left\{ (x, y) \in I^2 : (1-y, 1-x) \in \mathfrak{T}_3 \right\}.$$

Note that \mathfrak{T}_1 is the closed trapezoid with vertices P, S, S' , and P' from which the closed line segment $[S, S']$ is removed, and \mathfrak{T}_2 is the closed pentagon with vertices R, S, S', R' , and $(1, 0)$. We also note that the sets $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$, and \mathfrak{T}'_3 are mutually disjoint and $(\cup_{i=1}^3 \mathfrak{T}_i) \cup \mathfrak{T}'_3 = \{(x, y) \in I^2 : y \leq x\}$. See Figure 1.

Let us set

$$\gamma = \gamma(p, q) := \max \left\{ \frac{d-1}{2} \left(\frac{1}{p} - \frac{1}{q} \right), d \left(\frac{1}{p} - \frac{1}{q} \right) - 1, \frac{d-1}{2} - \frac{d}{q}, -\frac{d+1}{2} + \frac{d}{p} \right\}.$$

Theorem 1.1. *Let $1 \leq p \leq q \leq \infty$. Then*

$$(1.1) \quad \|H_n^d\|_{p,q} \gtrsim n^{\gamma(p,q)}.$$

Futhermore, if $(\frac{1}{p}, \frac{1}{q}) \in [S, R] \cup [S', R']$,

$$(1.2) \quad \sup_n n^{-\gamma(p,q)} \|H_n^d\|_{p,q} = \infty.$$

In view of Fefferman's disproof of the disk multiplier conjecture [13], the bound for $(1/p, 1/q) \in [P, P']$ is not likely to be uniformly bounded except the case $p = q = 2$, and it seems possible that the lower bounds can be improved by making use of a type of Besicovitch set. It is convenient to notice that

$$(1.3) \quad \gamma(p, q) = \begin{cases} \frac{d-1}{2}(\frac{1}{p} - \frac{1}{q}) & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathfrak{T}_1, \\ d(\frac{1}{p} - \frac{1}{q}) - 1 & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathfrak{T}_2, \\ \frac{d-1}{2} - \frac{d}{q} & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathfrak{T}_3, \\ -\frac{d+1}{2} + \frac{d}{p} & \text{if } (\frac{1}{p}, \frac{1}{q}) \in \mathfrak{T}'_3. \end{cases}$$

In the following theorem we show that, for $p \leq q$, these lower bounds are also the upper bounds on a certain range of p, q , and hence prove the optimal bounds. Combined with Theorem 1.1, the following provide a complete characterization of the bounds for $\|H_n^d\|_{p,q}$. We set

$$\mathcal{Q} = \left(\frac{d^2 + d - 4}{2(d-1)(d+2)}, \frac{d}{2(d+2)} \right), \quad \mathcal{U} = \left(\frac{d}{2(d+2)}, \frac{d}{2(d+2)} \right), \quad \mathcal{C} = \left(\frac{1}{2}, \frac{1}{2} \right),$$

and define \mathcal{Q}' and \mathcal{U}' as before. For given vertices $A, B, C \in I^2$ let us denote by $[A, B, C]$ the convex hulls of vertices A, B, C .

Theorem 1.2. *Let $n \geq 1$ and p, q satisfy that $1 \leq p \leq q \leq \infty$ and $(\frac{1}{p}, \frac{1}{q}) \notin ([\mathcal{Q}, \mathcal{U}, \mathcal{C}] \cup [\mathcal{Q}', \mathcal{U}', \mathcal{C}]) \setminus \mathcal{C}$. Then, if $(\frac{1}{p}, \frac{1}{q}) \notin [S, R] \cup [S', R']$, then*

$$(1.4) \quad \|H_n^d\|_{p,q} \lesssim n^{\gamma(p,q)}.$$

Futhermore, if $(\frac{1}{p}, \frac{1}{q}) = S$ or S' , $\|H_n^d f\|_{L^{q,\infty}(\mathbb{S}^d)} \lesssim n^{1-\frac{2}{d+1}} \|f\|_{L^{p,1}(\mathbb{S}^d)}$, and if $(\frac{1}{p}, \frac{1}{q}) \in (S', R']$, $\|H_n^d f\|_{L^{q,\infty}(\mathbb{S}^d)} \lesssim n^{-\frac{d+1}{2} + \frac{d}{p}} \|f\|_{L^p(\mathbb{S}^d)}$. Here $L^{r,s}(\mathbb{S}^d)$ denotes the Lorentz space.

This gives an almost complete characterization of the bounds for the 2-dimensional projection operator H_n^2 when $p \leq q$ except some endpoint cases.

Remark 1. The region for sharp boundedness of Theorem 1.2, as well as that of Theorem 1.4 and Proposition 2.4 below, can be further extended by making use of improved estimates in [17]. But we do not intend to pursue it here. Related results will appear elsewhere.

As a consequence of Theorem 1.2 we have, for $(\frac{1}{p}, \frac{1}{q}) \in (S, S')$,

$$(1.5) \quad \|H_n^d f\|_{L^q(\mathbb{S}^d)} \lesssim n^{1-\frac{2}{d+1}} \|f\|_{L^p(\mathbb{S}^d)},$$

which is crucial for the proof of the Carleman estimate (1.12). (1.2) shows the natural bound $\|H_n^d\|_{p,q} \lesssim n^{d(\frac{1}{p}-\frac{1}{q})-1}$ does not hold for (p, q) with $(\frac{1}{p}, \frac{1}{q}) \in [S, R] \cup [S', R']$. For these p, q we can get $\|H_n^d\|_{p,q} \lesssim (\log n)n^{\gamma(p,q)}$ by direct summation. This was observed in Sogge [31] for $(\frac{1}{p}, \frac{1}{q}) = S, S'$. However, Theorem 1.2 provides weaker substitutes without logarithmic loss.

For $d \geq 3$, Sogge [29] obtained (1.4) for $p = \frac{2(d+1)}{d+3}$, $q = \frac{2(d+1)}{d-1}$ using a T^*T argument and, since H_n^d is self adjoint, interpolation between trivial estimates gives the optimal bound (1.4) for $(p, q) \in [\frac{2(d+1)}{d+3}, 2] \times [2, \frac{2(d+1)}{d-1}]$. For the special case $\frac{1}{p} - \frac{1}{q} = \frac{2}{d}$ and $\min(|\frac{1}{2} - \frac{1}{p}|, |\frac{1}{2} - \frac{1}{q}|) > \frac{1}{2d}$, the estimate was obtained by Huang and Sogge [19] with sharp bound n , and recently, this was extended by Ren [28] to the range $\frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{d}$ and $\min(|\frac{1}{2} - \frac{1}{p}|, |\frac{1}{2} - \frac{1}{q}|) > \frac{1}{2d}$, which is contained in \mathfrak{T}_2 . It should be mentioned that interpolation between this estimate and the previously known bounds does not give the optimal bounds in Theorem 1.2 when $\frac{1}{p} - \frac{1}{q} > \frac{2}{d}$.

Oscillatory integral estimate $H_n^d f$ is given by the zonal convolution with the zonal spherical harmonic function \mathbf{Z}_n of degree n . In fact,

$$(1.6) \quad H_n^d f(\zeta) = \int_{\mathbb{S}^d} \mathbf{Z}_n(\xi \cdot \zeta) f(\xi) d\sigma(\xi).$$

Moreover, \mathbf{Z}_n can be explicitly expressed by the Jacobi polynomial $P_n^{\alpha, \beta}$:

$$(1.7) \quad \mathbf{Z}_n(t) = C(d, n) P_n^{\left(\frac{d-2}{2}, \frac{d-2}{2}\right)}(t), \quad t \in [-1, 1],$$

where $C(d, n) = \left(\frac{2n}{d-1} + 1\right) \frac{\Gamma(d/2)\Gamma(d+n-1)}{\Gamma(d-1)\Gamma(n+d/2)}$. After several steps of reduction which makes use of the asymptotic expansion of the Jacobi polynomials (see Theorem 2.1 below) it will be seen that the heart of matter is to obtain sharp bound for an oscillatory operator which is very similar to a simpler model operator

$$T_\lambda f(x) = \lambda^{\frac{d-1}{2}} \int_{\mathbb{R}^d} (1 + \lambda|x-y|)^{-\frac{d-1}{2}} a(x, y) e^{i\lambda|x-y|} f(y) dy$$

where a is a smooth function with compact support. This kind of oscillatory integral operator appears in the studies of the Bochner-Riesz multiplier operator ([9]). Since this operator has singularity near the diagonal, we are naturally led to consider dyadic decomposition away from it. This will reduce the problem to obtaining sharp bounds for each operator which results from decomposition. As is well known, to obtain the optimal bounds it is important to exploit the decay in L^p - L^q bound due to the oscillatory kernel.

This leads us to consider the oscillatory integral operators which satisfy the Carleson-Sjölin condition.

Carleson-Sjölin condition and oscillatory integral Let $\lambda \geq 1$, ψ be a smooth function, and a be a smooth function with compact support, which are defined on $\mathbb{R}^d \times \mathbb{R}^{d-1}$. Then we set

$$S_\lambda f(x) = \int e^{i\lambda\psi(x,z)} a(x,z) f(z) dz, \quad (x,z) \in \mathbb{R}^d \times \mathbb{R}^{d-1}.$$

Suppose that on the support of a

$$(1.8) \quad \text{rank } \partial_z \partial_x \psi = d - 1$$

and, if $\pm v \in \mathbb{S}^{d-1}$ is the unique vector such that $\partial_z(v \cdot \partial_x \psi) = 0$, then

$$(1.9) \quad \text{rank } \partial_z^2(v \cdot \partial_x \psi) = d - 1.$$

The conditions (1.8) and (1.9) are equivalently stated as follows: The map $z \rightarrow \nabla_x \psi(x,z)$ defines a family of smooth immersed surfaces with nonvanishing Gaussian curvature. The following is due to Stein [35] (also, see [36, Ch.9]).

Theorem 1.3. *Suppose ψ satisfies (1.8) and (1.9) in the support of a . Then, for p, q satisfying $q \geq \frac{2(d+1)}{d-1}$ and*

$$(1.10) \quad \frac{d+1}{q} \leq (d-1)\left(1 - \frac{1}{p}\right),$$

the estimate $\|S_\lambda f\|_q \lesssim \lambda^{-\frac{d}{q}} \|f\|_p$ holds.

When $d = 2$ the estimate was shown to be true by Hörmander [18] for the optimal range of p, q ((1.10) and $q > 4$). In higher dimensions it was shown by Bourgain [5] that the estimate generally fails if $q < \frac{2(d+1)}{d-1}$ whenever $d \geq 3$ is odd. However, it was observed in [27, Theorem 1.3] that the range can be improved under the stronger assumption that

$$(1.11)$$

the surface $z \rightarrow \nabla_x \psi(x,z)$ has $d - 1$ nonzero principal curvatures of the same sign,

equivalently, the matrix $\partial_z^2(v \cdot \partial_x \psi)$ is positive definite or negative definite. Under the assumption (1.11) the range can be improved to $q > \frac{2(d+2)}{d}$. When $d = 3$, it is known that the range is sharp. In [3, Remark 3.4] (also see Theorem 2.1 in [3]), by removing ϵ -loss of the estimate due to the second author [27] they obtained the bound

$$\left\| \sum_{(\nu, \nu') \in \Xi_j} T h_\nu^j T h_{\nu'}^j \right\|_{L^{q/2}} \lesssim 2^{2j \left(\frac{d+1}{q} - (d-1)\left(1 - \frac{1}{p}\right) \right)} \lambda^{-\frac{2d}{q}} \|h\|_p \|h\|_p$$

for $q > \frac{2(d+2)}{d}$ and $p \geq 2$. Here we keep using the notations from [3]. By summation along j this gives the estimate $\|S_\lambda f\|_q \lesssim \lambda^{-\frac{d}{q}} \|f\|_p$ for p, q satisfying (1.10) with strict inequality. However, this can be combined with a simple summation trick (for example, see Lemma 3.1 below and [25, Lemma 2.6]) to give restricted weak type estimate for p, q satisfying (1.10) and $q > \frac{2(d+2)}{d}$. These estimates can be (real) interpolated to yield the strong type bound. Hence, we have the following.

Theorem 1.4. *Suppose ψ satisfies (1.8) and (1.11). Then, for p, q satisfying $q > \frac{2(d+2)}{d}$ and (1.10), the estimate $\|S_\lambda f\|_q \lesssim \lambda^{-\frac{d}{q}} \|f\|_p$ holds.*

Carleman estimate Let $d \geq 3$ and $1 \leq p \leq q \leq \infty$. For $\tau \in \mathbb{R}$, we consider the estimate

$$(1.12) \quad \| |x|^{-\tau} u \|_{L^q(\mathbb{R}^d)} \leq C \| |x|^{-\tau} \Delta u \|_{L^p(\mathbb{R}^d)}, \quad u \in C_c^\infty(\mathbb{R}^d \setminus \{0\}).$$

Jerison-Kenig [22] showed that, for $p = \frac{2d}{d+2}$, $q = \frac{2d}{d-2}$, (1.12) holds with C independent of τ under the condition $\text{dist}(\tau, \mathbb{Z} + \frac{d-2}{2}) > 0$. Their proof is based on analytic interpolation between L^2 - L^2 and L^1 - L^∞ estimates for an analytic family of operators, which gives the $L^{\frac{2d}{d+2}}$ - $L^{\frac{2d}{d-2}}$ estimate on the line of duality. Bounds for (p, q) other than $(\frac{2d}{d+2}, \frac{2d}{d-2})$ were also obtained by Stein [34]. He obtained (1.12) for p, q satisfying $\frac{1}{2} + \frac{d-2}{d(d-1)} < \frac{1}{p} < \frac{1}{2} + \frac{1}{d-1}$ and (1.13). See [34, Proof of Theorem 1]. Compared to Theorem 1.5 below, the range obtained by Stein is smaller when $d > 4$, coincides when $d = 4$, and is larger when $d = 3$. Jerison [21] later provided an alternative proof which is based on the bound $\|H_n^{d-1}\|_{\frac{2d}{d+2}, \frac{2d}{d-2}}$ (see Section 5).

Homogeneity dictates that (1.12) is possible only if

$$(1.13) \quad \frac{1}{p} - \frac{1}{q} = \frac{2}{d}.$$

If $\tau \in (-d, d)$, it is easy to show that (1.12) holds with C depending on τ for $1 < p \leq q < \infty$ satisfying (1.13). However, the uniformity of the bound usually makes the range of admissible (p, q) smaller. An easy argument using the spherical harmonics shows that the uniform estimate (1.12) can not be true for all $1 < p \leq q < \infty$ satisfying (1.13) (see Remark 2 in the last section). Similar phenomena were also observed in Kenig-Ruiz-Sogge [23] where uniform Sobolev estimates for second order differential operators are obtained (also see [20]).

Following Jerison's argument and making use of (1.5) we have the following.

Theorem 1.5. *Let $d \geq 3$. Then, the estimate (1.12) holds uniformly in τ whenever $\text{dist}(\tau, \mathbb{Z} + \frac{d}{q}) > 0$ if p, q satisfy (1.13) and $\frac{2(d-1)d}{d^2+2d-4} < p < \frac{2(d-1)}{d}$.*

It is well known [22] that under suitable conditions on u the inequality (1.12) implies strong unique continuation property for the inequality $|\Delta u(x)| \leq V(x)|u(x)|$ when

$V \in L_{loc}^{d/2}(\mathbb{R}^d)$. As was shown by Stein [34] real interpolation between estimates in Theorem 1.5 yields a refined estimate, for the same p, q as above, $\| |x|^{-\tau} u \|_{L^{q,p}(\mathbb{R}^d)} \leq C \| |x|^{-\tau} \Delta u \|_{L^p(\mathbb{R}^d)}$ provided $u \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$, which extends the strong unique continuation property to $V \in L_{loc}^{d/2, \infty}(\mathbb{R}^d)$ under smallness condition.

In Sogge [31, final remark] it was briefly (without proof) mentioned that the method therein can be applied to obtain (1.12) for p, q on a certain range. The argument was based on the weaker estimate $\| H_n^{d-1} \|_{p,q} \lesssim (\log n) n^{1-\frac{2}{d}}$ for some p, q satisfying (1.13). However, the proof there doesn't seem easily accessible.¹ So we decided to include Theorem 1.5 and its proof which is based on the spectral projection estimate (1.5). In view of Stein's result when $d = 3$ and the range in Remark 2 below, the range in Theorem 1.5 is unlikely to be optimal even in higher dimensions. The problem of characterizing p, q for which (1.12) holds remains open.

Notations For positive real numbers X and Y , we use the notation $X \lesssim Y$ (or $Y \gtrsim X$) to say that there is a C that $X \leq CY$. Particularly, $\| H_n^d \|_{p,q} \lesssim n^{\gamma(p,q)}$ means that $\| H_n^d \|_{p,q} \leq C n^{\gamma(p,q)}$, where C is a constant independent of n , although it may depend on p, q and d . We also use the notation $X \sim Y$ which denotes $X \lesssim Y$ and $Y \lesssim X$.

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§ 2. Preliminaries

Asymptotic expansion of the Jacobi polynomials For our purpose we need asymptotic expansion of the polynomial $P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\cos \theta)$. We use the following theorem from Frenzen and Wong [14]. A similar result was also obtained in Szegő [39].

Theorem 2.1. [14, Main Theorem] Let $N = n + (\alpha + \beta + 1)/2$. For $\alpha > -1/2$, $\alpha - \beta > -2m$, $\alpha + \beta \geq -1$, and $\theta \in (0, \pi)$, we have

$$(2.1) \quad P_n^{(\alpha, \beta)}(\cos \theta) = \frac{\Gamma(n + \alpha + 1)}{n!} \left(\sin \frac{\theta}{2} \right)^{-\alpha} \left(\cos \frac{\theta}{2} \right)^{-\beta} \left(\frac{\theta}{\sin \theta} \right)^{\frac{1}{2}} \\ \times \left[\sum_{l=0}^{m-1} A_l(\theta) N^{-\alpha-l} J_{\alpha+l}(N\theta) + \theta^\alpha O(N^{-m}) \right],$$

¹Numerology in [31] are not correct and should be corrected as those in this paper.

where A_l are analytic functions on $[0, \pi)$ and the O -term is uniform with respect to $\theta \in [0, \pi - \epsilon]$, ϵ being an arbitrary positive number. Here, J_ν denotes the Bessel function of order $\nu > -1/2$.

In particular, $A_0 = 1$. (See [14, p. 994].) Recall the symmetric property of Jacobi polynomials: $P_n^{(\alpha, \beta)}(-z) = (-1)^n P_n^{(\beta, \alpha)}(z)$. In particular, we have

$$(2.2) \quad P_n^{\left(\frac{d-2}{2}, \frac{d-2}{2}\right)}(\cos(\pi - \theta)) = (-1)^n P_n^{\left(\frac{d-2}{2}, \frac{d-2}{2}\right)}(\cos \theta).$$

By this symmetry we may get around the uniformity issue related to ϵ , in the above theorem.

Oscillatory intergral estimate In what follows we obtain oscillatory integral estimates which are needed later.

Lemma 2.2. *Let $\epsilon > 0$ and set $\psi_\epsilon(\theta) = \epsilon^{-1} \arccos(1 - \epsilon^2 \theta)$. Then $\psi_\epsilon \rightarrow \sqrt{2}\theta^{\frac{1}{2}}$ as $\epsilon \rightarrow 0$ in $C^N([2^{-8}, 2^8])$ for any N .*

Proof. Since $(\arccos \theta)' = -1/\sqrt{1 - \theta^2}$ and $\arccos 1 = 0$, we have

$$\arccos(1 - \theta) = \int_{1-\theta}^1 \frac{d\tau}{\sqrt{1 - \tau^2}} = \int_0^\theta \frac{d\tau}{\sqrt{2\tau - \tau^2}} = \sqrt{2}\theta^{\frac{1}{2}} + O(\theta^{\frac{3}{2}}).$$

Hence, $\psi_\epsilon(\theta) = \sqrt{2}\theta^{\frac{1}{2}} + R_\epsilon(\theta)$ with $\|R_\epsilon\|_{C^N([2^{-8}, 2^8])} = O(\epsilon)$. This gives the convergence in $C^N([2^{-8}, 2^8])$. \square

Lemma 2.3. *Let us set*

$$\Phi_\epsilon(x, y) = \epsilon^{-1} \arccos(\epsilon^2 x \cdot y + \sqrt{(1 - \epsilon^2 |x|^2)(1 - \epsilon^2 |y|^2)}).$$

Then, for $|x - y| \in [2^{-3}, 2^3]$, $\Phi_\epsilon(x, y) \rightarrow |x - y|$. Furthermore, the convergence holds in $C^N(K_1 \times K_2)$ for any large N as long as K_1, K_2 are compact subsets of \mathbb{R}^d satisfying $2^{-3} \leq |x - y| \leq 2^3$ for $(x, y) \in K_1 \times K_2$.

Proof. We write

$$\epsilon^2 x \cdot y + \sqrt{(1 - \epsilon^2 |x|^2)(1 - \epsilon^2 |y|^2)} = 1 - \epsilon^2 \left(\frac{|x|^2 + |y|^2 - \epsilon^2 |x|^2 |y|^2}{1 + \sqrt{(1 - \epsilon^2 |x|^2)(1 - \epsilon^2 |y|^2)}} - x \cdot y \right),$$

and set

$$\mu_\epsilon(x, y) = \frac{|x|^2 + |y|^2 - \epsilon^2 |x|^2 |y|^2}{1 + \sqrt{(1 - \epsilon^2 |x|^2)(1 - \epsilon^2 |y|^2)}} - x \cdot y.$$

Clearly $\mu_\epsilon(x, y) \rightarrow |x - y|^2/2$ as $\epsilon \rightarrow 0$ and $\Phi_\epsilon = \psi_\epsilon \circ \mu_\epsilon$. Hence, with sufficiently small $\epsilon > 0$ which gives $\mu_\epsilon(x, y) \in [2^{-8}, 2^8]$, Lemma 2.2 shows the convergence. \square

Let us define the oscillatory integral operator T_λ^ϵ by

$$(2.3) \quad T_\lambda^\epsilon f(x) = \int a(x, y) e^{i\lambda\Phi_\epsilon(x, y)} f(y) dy$$

where $a \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\text{supp } a \subset \{(x, y) : 2^{-2} \leq |x - y| \leq 2^2\}$.

Since $|x - y| \geq 2^{-2}$ by decomposing the support of a we may assume $|x_i - y_i| \geq (4d)^{-1}$ for some $i = 1, \dots, d$. In particular, let us assume $|x_d - y_d| \geq (4d)^{-1}$ on the support of a and set $\Phi_0(x, y) = |x - y|$, and $\Phi_0^{y_d}(x, z) = \sqrt{|\bar{x} - z|^2 + (x_d - y_d)^2}$, where $x = (\bar{x}, x_d)$ and $y = (z, y_d)$. It is easy to show that $\Phi_0^{y_d}$ satisfies (1.8) and (1.11) since $|x_d - y_d| \sim (4d)^{-1}$. Freezing the variable y_d and using Minkowski's inequality and Theorem 1.4, we see that

$$\|T_\lambda^0 f\|_q \leq C \int \left\| \int a(x, z, y_d) e^{i\lambda\Phi_0^{y_d}(x, z)} f(z, y_d) dz \right\|_q dy_d \lesssim \lambda^{-\frac{d}{q}} \|f\|_p.$$

for p, q satisfying (1.10) and $q > \frac{2(d+2)}{d}$. From Lemma 2.3, we also see that the same argument also works with $\Phi_\epsilon^{y_d}(x, z) = \Phi_\epsilon(x, z, y_d)$ as long as ϵ is small enough. Furthermore, since the bound for S_λ in Theorem 1.3 is stable under smooth small perturbation of the phase, we see that the oscillatory integral operators defined by $\Phi_\epsilon^{y_d}$ have uniform bounds. Hence, from the above argument we get the following.

Proposition 2.4. *For p, q satisfying (1.10) and $q > \frac{2(d+2)}{d}$, there are constants $\epsilon_0 > 0$ and $C > 0$ such that*

$$(2.4) \quad \|T_\lambda^\epsilon f\|_q \leq C \lambda^{-\frac{d}{q}} \|f\|_p$$

provided that $0 < \epsilon \leq \epsilon_0$. Here C is independent of $\epsilon \in (0, \epsilon_0]$ and $\lambda \geq 1$.

§ 3. Spectral projection operator: Proof of Theorem 1.2

Fix $0 < r \leq 10^{-2}\epsilon_0$. Here ϵ_0 is the one in Proposition 2.4. In order to prove Theorem 1.2, by rotational symmetry and finite decomposition of \mathbb{S}^d it is sufficient to show that

$$(3.1) \quad \|H_n^d f\|_{L^q(\mathcal{C}(e_{d+1}, r))} \lesssim n^{\gamma(p, q)} \|f\|_p.$$

Here $\mathcal{C}(e, \rho)$ is the geodesic ball centered at $e \in \mathbb{S}^d$ with radius $\rho > 0$.

We distinguish three cases in which f is supported near the north pole e_{d+1} , near the south pole $-e_{d+1}$, and away from both the north pole e_{d+1} and the south pole $-e_{d+1}$, respectively:

$$(3.2) \quad \text{supp } f \subset \{x \in \mathbb{S}^d : x_{d+1} \in (1 - 100r^2, 1]\},$$

$$(3.3) \quad \text{supp } f \subset \{x \in \mathbb{S}^d : x_{d+1} \in [-1, -1 + 100r^2]\},$$

$$(3.4) \quad \text{supp } f \subset \{x \in \mathbb{S}^d : x_{d+1} \in [-1 + 100r^2, 1 - 100r^2]\}.$$

Recalling (1.6) and (1.7) we note that $\arccos(\zeta \cdot \xi) \in (\pi - 100r, \pi]$ if $\zeta \in \mathcal{C}(e_{d+1}, r)$ and $\xi \in \{x \in \mathbb{S}^d : x_{d+1} \in [-1, -1 + 100r^2]\}$. Since the asymptotic expansion (2.1) is uniform only for $\theta \in [0, \pi - \epsilon]$, we can not make use of it directly when we handle the second case (3.3). However, by (2.2) we may again use (2.1) after reflection. In this manner the case (3.3) can be handled in the same way as the case (3.2). Therefore it is sufficient to consider the first and the third cases only.

Hence, for the rest of this section, we assume $\theta := \arccos(\zeta \cdot \xi) \in [0, \pi - r]$. For simplicity we set $\nu = \frac{d-2}{2}$ and

$$\mathcal{A}_\nu(\theta) = \frac{\Gamma(n + \nu + 1)}{n!} \left(\sin \frac{\theta}{2}\right)^{-\nu} \left(\cos \frac{\theta}{2}\right)^{-\nu} \left(\frac{\theta}{\sin \theta}\right)^{\frac{1}{2}}.$$

From (2.1) we may write

$$(3.5) \quad P_n^{(\nu, \nu)}(\cos \theta) = \mathcal{A}_\nu(\theta) \left[\sum_{l=0}^{m-1} A_l(\theta) N^{-\nu-l} J_{\nu+l}(N\theta) \right] + \mathcal{E}_m(\theta),$$

where $\mathcal{E}_m(\theta) = O(N^{-m+\nu})$. We fix m large enough so that $\mathcal{E}_m(\theta) = O(N^{-\frac{d}{2}})$. Hence the contribution of $\mathcal{E}_m(\theta)$ to the convolution kernel (1.7) is $O(1)$, and is negligible since $\gamma(p, q) \geq 0$.

§ 3.1. Away from the north and the south poles

In this case the bound is much better than what we need to show. In fact, assuming (3.4) we show, for p, q satisfying (1.10) and $q > \frac{2(d+2)}{d}$,

$$(3.6) \quad \|H_n^d f\|_{L^q(\mathcal{C}(e_{d+1}, r))} \lesssim n^{\frac{d-1}{2} - \frac{d}{q}} \|f\|_p.$$

Since f is supported in $\{x \in \mathbb{S}^d : x_{d+1} \in [-1 + 100r^2, 1 - 100r^2]\}$, $\xi \cdot \zeta = \cos \theta \in [-1 + 85r^2, 1 - 85r^2]$ in (1.6). Hence, we only need to consider the case $\theta \in [10r, \pi - 10r]$.

From (1.7) and (3.5) we only consider the contribution from the main term

$$\mathcal{A}_\nu(\theta) A_0(\theta) N^{-\nu} J_\nu(N\theta)$$

while $\theta \in [10r, \pi - 10r]$. The contribution from the other terms $\mathcal{A}_\nu(\theta) A_l(\theta) N^{-\nu-l} J_{\nu+l}(N\theta)$, $1 \leq l \leq m - 1$ can be handled similarly but these give smaller bounds $n^{\frac{d-1}{2} - \frac{d}{q} - l}$. We recall the asymptotic expansion of Bessel function J_ν for large arguments

$$J_\nu(r) = r^{-\frac{1}{2}} e^{ir} \left(\sum_{0 \leq j \leq \frac{d+1}{2}} a_j r^{-j} \right) + r^{-\frac{1}{2}} e^{-ir} \left(\sum_{0 \leq j \leq \frac{d+1}{2}} b_j r^{-j} \right) + O(r^{-\frac{d+1}{2}}), \quad r \geq 1.$$

Inserting this in the above, we see that the main term is given by

$$\mathcal{K}_\pm(\theta) = N^{-\frac{1}{2}} \tilde{A}_\pm(\theta) e^{\pm iN\theta}$$

with smooth \tilde{A}_\pm which is supported in $[10r, \pi - 10r]$. As before, contribution from the lower order terms and $O(r^{-\frac{d+1}{2}})$ are less significant since these give smaller bounds. Since $C(d, n) \sim n^{\frac{d}{2}}$, combining the above with (1.7), for (3.6) it suffices to show that

$$(3.7) \quad \left\| \int_{\mathbb{S}^d} \tilde{A}_\pm(\arccos \zeta \cdot \xi) e^{\pm iN \arccos \zeta \cdot \xi} f(\xi) d\xi \right\|_{L^q(\mathcal{C}(e_{d+1}, r))} \lesssim N^{-\frac{d}{q}} \|f\|_p,$$

whenever f satisfies (3.4).

Let $x = (\bar{x}, v)$, $y = (\bar{y}, u) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Again by decomposition and rotation in \bar{y} we may assume f is supported in the set

$$\left\{ (\bar{y}, \sqrt{1 - |\bar{y}|^2 - u^2}, u) : |\bar{y}| \leq \frac{1}{2} \sqrt{1 - u^2}, u \in [100r^2 - 1, 1 - 100r^2] \right\}.$$

Hence, using the above parametrization and parameterizing near the north pole with

$$(\bar{x}, v, \sqrt{1 - |\bar{x}|^2 - v^2}), \quad |(\bar{x}, v)| \leq r,$$

and ignoring the harmless smooth factors resulted from parametrization, we are reduced to showing

$$(3.8) \quad \left\| \iint a(\bar{x}, v, \bar{y}, u) e^{iN \arccos \phi(\bar{x}, v, \bar{y}, u)} h(\bar{y}, u) d\bar{y} du \right\|_{L^q(\mathbb{R}^d)} \lesssim N^{-\frac{d}{q}} \|h\|_{L^p(\mathbb{R}^d)},$$

where a is a smooth function supported in the set $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |\bar{y}| \leq \frac{1}{2} \sqrt{1 - u^2}, u \in [100r^2 - 1, 1 - 100r^2], |(\bar{x}, v)| \lesssim r\}$ and

$$\phi(x, y) = \bar{x} \cdot \bar{y} + v \sqrt{1 - |\bar{y}|^2 - u^2} + u \sqrt{1 - |\bar{x}|^2 - v^2}.$$

Fixing $u \in (100r^2 - 1, 1 - 100r^2)$, let us set

$$\phi_u(\bar{x}, v, \bar{y}) = \phi(\bar{x}, v, \bar{y}, u).$$

By Minkowski's inequality, in order to show (3.8), it is sufficient to show that, for p, q satisfying (1.10) and $q > \frac{2(d+2)}{d}$,

$$(3.9) \quad \left\| \int a(\bar{x}, v, \bar{y}, u) e^{iN \arccos \phi_u(\bar{x}, v, \bar{y})} g(\bar{y}) d\bar{y} \right\|_{L^q(\mathbb{R}^d)} \leq CN^{-\frac{d}{q}} \|g\|_{L^p(\mathbb{R}^{d-1})}$$

with C independent of u .

To show (3.9) by Theorem 1.4 it is sufficient to check that the phase function $\arccos \phi_u$ satisfies the conditions (1.8), (1.9) and (1.11). We first notice that

$$\nabla_{\bar{x}, v} \arccos \phi_u = \frac{-1}{\sqrt{1 - (\phi_u)^2}} \left(\bar{y} - \frac{u\bar{x}}{\sqrt{1 - |\bar{x}|^2 - v^2}}, \sqrt{1 - |\bar{y}|^2 - u^2} - \frac{uv}{\sqrt{1 - |\bar{x}|^2 - v^2}} \right).$$

Also note that $\sqrt{1 - (\phi_u)^2} \geq r$ since $u \in (100r^2 - 1, 1 - 100r^2)$. Clearly, the map $\bar{y} \rightarrow \nabla_{\bar{x}, v} \arccos \phi_u(\bar{x}, v, \bar{y})$ defines an immersed surface. In order to show that it

has nonvanishing gaussian curvature everywhere, by rotational symmetry of the phase function (rotation and horizontal rotation on \mathbb{S}^d) it is enough to check this by assuming

$$(\bar{x}, v) = 0, (\bar{y}, u) = (0, u), u \in (100r^2 - 1, 1 - 100r^2).$$

Note that $\nabla_{\bar{y}}\phi_u = \bar{x} - \frac{v\bar{y}}{\sqrt{1-|\bar{y}|^2-u^2}}$. Hence $\nabla_{\bar{y}}\phi_u(0,0,0) = 0$. Using this and straightforward computation give

$$\nabla_{\bar{y}}\nabla_{\bar{x},v} \arccos \phi_u(0,0,0) = -\frac{1}{\sqrt{1-u^2}} \begin{pmatrix} I_{d-1} \\ 0 \end{pmatrix}.$$

So, the unique vector \mathbf{v} satisfying $\nabla_{\bar{y}}(\mathbf{v} \cdot \nabla_{\bar{x},v} \arccos \phi_u)(0,0,0) = 0$ is $\pm e_d$. We now consider

$$\begin{aligned} \Phi_u(\bar{x}, v, \bar{y}) &:= e_d \cdot \nabla_{\bar{x},v} \arccos \phi_u \\ &= \frac{-1}{\sqrt{1-(\phi_u)^2}} \times \left(\sqrt{1-|\bar{y}|^2-u^2} - \frac{uv}{\sqrt{1-|\bar{x}|^2-v^2}} \right) \\ &=: A \times B. \end{aligned}$$

Now it remains to show that the Hessian matrix $\partial_{\bar{y}}^2\Phi_u(0,0,0)$ is positive definite. This is easy to verify. Observe $\nabla_{\bar{y}}A(0,0,0) = \nabla_{\bar{y}}B(0,0,0) = 0$. Thus we get

$$\partial_{\bar{y}}^2\Phi_u(0,0,0) = A\partial_{\bar{y}}^2B(0,0,0) = \frac{1}{1-u^2}I_{d-1}.$$

This verifies that the surface $\bar{y} \rightarrow \nabla_x \arccos \phi_u(\bar{x}, v, \bar{y})$ has positive definite fundamental form. Hence we get the bound (3.9) and we see that the constant C in (3.9) can be taken to be uniform because the Hessian matrix $\partial_{\bar{y}}^2\Phi_u(0,0,0)$ can be controlled uniformly for $u \in (100r^2 - 1, 1 - 100r^2)$.

§ 3.2. Near the north pole

We now consider the case that f satisfies (3.2). For the rest of this section f is assumed to satisfy (3.2).

We start with observing

$$|\mathbf{Z}_n(\cos \theta)| \lesssim n^{d-1},$$

which is easy to see using (1.7) and (2.1). This gives the sharp L^1 to L^∞ bound for H_n^d . Using this, the contribution from the part of kernel $\theta \lesssim N^{-1}$ is easy to handle. In fact, let ψ be a smooth function supported in $[-2, 2]$ such that $\psi = 1$ on $[-1, 1]$. Then we have

$$\left| \int_{\mathbb{S}^d} \mathbf{Z}_n(\zeta \cdot \xi) \psi\left(\frac{n^2(1-\xi \cdot \zeta)}{100}\right) f(\xi) d\sigma(\xi) \right| \lesssim n^{d-1} \int_{|\xi-\zeta| \leq 20n^{-1}} |f(\xi)| d\sigma(\xi).$$

From a simple computation (Young's convolution inequality) we get

$$(3.10) \quad \left\| n^{d-1} \int_{|\xi-\zeta| \leq 20n^{-1}} |f(\xi)| d\sigma(\xi) \right\|_{L^q(\mathcal{C}(e_{d+1}, r))} \lesssim n^{-1+d(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{S}^d)}.$$

Since the bound $n^{-1+d(\frac{1}{p}-\frac{1}{q})}$ is acceptable, now we only need to consider the case $10n^{-1} \leq \theta \leq 50r$.

For the range $10n^{-1} \leq \theta \leq 50r$, we use the asymptotic expansion (2.1). Hence, as before, it is enough to control the leading term $\mathcal{A}_\nu(\theta)A_0(\theta)N^{-\nu}J_\nu(N\theta)$ in (3.5) as the lower order terms can be handled in the same way and these give smaller bounds. Combining this, the asymptotic expansion for the Bessel function and (1.7), it suffices to consider zonal convolution with

$$\mathcal{Z}(\cos \theta) = N^{\frac{d-1}{2}} \mathcal{A}_\pm(\theta) e^{\pm iN\theta},$$

where \mathcal{A}_\pm is smooth and supported in $(10n^{-1}, 50r)$ with $\frac{d^k}{d\theta^k} \mathcal{A}_\pm(\theta) = O(\theta^{-\frac{d-1}{2}-k})$. Thus, using the typical dyadic partition of unity, we may break $\mathcal{Z}(\cos \theta)$ dyadically such that

$$\mathcal{Z}(\cos \theta) = \sum_{j: n^{-1} \leq 2^{-j} \leq 50r} \mathcal{Z}_j(\cos \theta) := N^{\frac{d-1}{2}} \sum_{j: n^{-1} \leq 2^{-j} \leq 50r} 2^{\frac{d-1}{2}j} \psi_j(\theta) e^{\pm iN\theta},$$

where $\psi_j(\theta)$ is supported in $[2^{-j-1}, 2^{-j+1}]$ and $\|\psi_j(2^{-j}\cdot)\|_{C^l([2^{-1}, 2^2])}$ is uniformly bounded for any l . Let us set

$$T_j f(\zeta) = \int \mathcal{Z}_j(\zeta \cdot \xi) f(\xi) d\sigma(\xi).$$

We claim that, for p, q satisfying (1.10) (and f satisfying (3.2)) and $q > \frac{2(d+2)}{d}$,

$$(3.11) \quad \|T_j f\|_{L^q(\mathcal{C}(e_{d+1}, r))} \lesssim N^{\frac{d-1}{2}-\frac{d}{q}} 2^{(\frac{d}{p}-\frac{d+1}{2})j} \|f\|_{L^p(\mathbb{S}^d)}.$$

This gives, for p, q satisfying (1.10),

$$\left\| \sum_{j: n^{-1} \leq 2^{-j} \leq 50r} T_j f \right\|_{L^q(\mathcal{C}(e_{d+1}, r))} \lesssim \begin{cases} N^{\frac{d-1}{2}-\frac{d}{q}} \|f\|_{L^p(\mathbb{S}^d)}, & \frac{2d}{d+1} < p, q > \frac{2(d+2)}{d}, \\ N^{d(\frac{1}{p}-\frac{1}{q})-1} \|f\|_{L^p(\mathbb{S}^d)}, & 1 \leq p < \frac{2d}{d+1}. \end{cases}$$

For the critical case $p = \frac{2d}{d+1}$, we use a simple summation lemma which was implicit in [4]. A statement for a general multilinear setting can also be found in [10].

Lemma 3.1. *Let $\varepsilon_0, \varepsilon_1 > 0$, and let $\{\mathcal{T}_l : l \in \mathbb{Z}\}$ be a sequence of linear operators satisfying*

$$\|\mathcal{T}_l f\|_{q_0} \leq M_0 2^{\varepsilon_0 l} \|f\|_{p_0}, \quad \|\mathcal{T}_l f\|_{q_1} \leq M_1 2^{-\varepsilon_1 l} \|f\|_{p_1}$$

for some $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Then $\|\sum_l \mathcal{T}_l f\|_{q, \infty} \leq C M_0^\theta M_1^{1-\theta} \|f\|_{p, 1}$, where $\theta = \varepsilon_1/(\varepsilon_0 + \varepsilon_1)$, $1/q = \theta/q_0 + (1-\theta)/q_1$, $1/p = \theta/p_0 + (1-\theta)/p_1$. Additionally, if $q_0 = q_1$, then $\|\sum_l \mathcal{T}_l f\|_q \leq C M_0^\theta M_1^{1-\theta} \|f\|_{p, 1}$.

Hence, by Lemma 3.1 and (3.11) we have

$$\begin{aligned} \left\| \sum_{j:n^{-1} \leq 2^{-j} \leq 50r} T_j f \right\|_{L^q(\mathcal{C}(e_{d+1}, r))} &\lesssim N^{\frac{d-1}{2} - \frac{d}{q}} \|f\|_{L^{\frac{2d}{d+1}, 1}(\mathbb{S}^d)}, \quad q > \frac{2d(d+1)}{(d-1)^2}, \\ \left\| \sum_{j:n^{-1} \leq 2^{-j} \leq 50r} T_j f \right\|_{L^{q, \infty}(\mathcal{C}(e_{d+1}, r))} &\lesssim N^{\frac{d-1}{2} - \frac{d}{q}} \|f\|_{L^{\frac{2d}{d+1}, 1}(\mathbb{S}^d)}, \quad q = \frac{2d(d+1)}{(d-1)^2}. \end{aligned}$$

Now we can repeat the same argument with $\mathcal{A}_\nu(\theta)A_l(\theta)N^{-\nu-l}J_{\nu+l}(N\theta)$, $1 \leq l \leq m-1$, and the contributions from these terms are controlled by smaller norms. Hence combining this with (3.10) we conclude that for p, q satisfying (1.10),

$$(3.12) \quad \|H_n^d f\|_{L^q(\mathcal{C}(e_{d+1}, r))} \lesssim \begin{cases} N^{\frac{d-1}{2} - \frac{d}{q}} \|f\|_{L^p(\mathbb{S}^d)} & \frac{2d}{d+1} < p, q > \frac{2(d+2)}{d}, \\ N^{d(\frac{1}{p} - \frac{1}{q}) - 1} \|f\|_{L^p(\mathbb{S}^d)} & 1 \leq p < \frac{2d}{d+1}, \end{cases}$$

and

$$(3.13) \quad \begin{aligned} \|H_n^d f\|_{L^q(\mathcal{C}(e_{d+1}, r))} &\lesssim N^{\frac{d-1}{2} - \frac{d}{q}} \|f\|_{L^{\frac{2d}{d+1}, 1}(\mathbb{S}^d)}, \quad q > \frac{2d(d+1)}{(d-1)^2}, \\ \|H_n^d f\|_{L^{q, \infty}(\mathcal{C}(e_{d+1}, r))} &\lesssim N^{\frac{d-1}{2} - \frac{d}{q}} \|f\|_{L^{\frac{2d}{d+1}, 1}(\mathbb{S}^d)}, \quad q = \frac{2d(d+1)}{(d-1)^2}. \end{aligned}$$

Proof of Theorem 1.2. Finally, combining these estimates (3.12) and (3.13), with (3.6) and using rotational symmetry, we see that the same bounds are true for H_n^d by replacing $\mathcal{C}(e_{d+1}, r)$ with \mathbb{S}^d without restriction on f for p, q satisfying (1.10). Since the estimates for $(p, q) = (2, 2)$ and $(p, q) = (1, \infty)$ are trivially true, interpolation and duality yield all the bounds in Theorem 1.2. \square

Proof of (3.11). We now use the parametrization $\kappa(x) = (x, \sqrt{1 - |x|^2})$ of the sphere near e_{d+1} for both ζ and ξ . Let us set

$$\phi(x, y) = x \cdot y + \sqrt{(1 - |x|^2)(1 - |y|^2)}.$$

Then, we may write

$$T_j f(x) = N^{\frac{d-1}{2}} 2^{\frac{d-1}{2}j} \int \psi_j(\arccos \phi(x, y)) e^{iN \arccos \phi(x, y)} f(\kappa(y)) w(y) dy,$$

where $w(y) = \sqrt{1 + \frac{|y|^2}{1 - |y|^2}}$. By rescaling $(x, y) \rightarrow 2^{-j}(x, y)$ and recalling the definition of Φ_ϵ (in Lemma 2.3) we see that

$$T_j f(2^{-j}x) = N^{\frac{d-1}{2}} 2^{-\frac{d+1}{2}j} \int \psi_j(2^{-j}\Phi_{2^{-j}}(x, y)) e^{i2^{-j}N\Phi_{2^{-j}}(x, y)} f(\kappa(2^{-j}y)) w(2^{-j}y) dy.$$

By Lemma 2.3 we see that the functions $\psi_j(2^{-j}\Phi_{2^{-j}}(x, y))$ are nonzero only if $|x - y| \in [2^{-2}, 2^2]$ and are uniformly bounded in $C^M(\mathbb{R}^d \times \mathbb{R}^d)$ for any M provided that j is large enough so that $2^{-j} \leq 50r \leq \epsilon_0/2$. For $n^{-1} \leq 2^{-j} \leq 50r$, we set

$$\tilde{T}_j g(x) = \int \psi_j(2^{-j}\Phi_{2^{-j}}(x, y)) e^{i2^{-j}N\Phi_{2^{-j}}(x, y)} g(y) dy.$$

Discarding some harmless factors which arise from the parametrization, we see (3.11) is equivalent to the estimate

$$(3.14) \quad \|\tilde{T}_j f\|_q \lesssim (2^{-j}N)^{-\frac{d}{q}} \|f\|_p$$

for p, q satisfying (1.10). This follows from Proposition 2.4 and our choice of r . \square

§ 4. Lower bound for $\|H_n^d\|_{p,q}$: Proof of Theorem 1.1

In this section we prove the lower bound for $\|H_n^d\|_{p,q}$ in Theorem 1.1 by testing the equality $\|H_n^d f\|_{L^q(\mathbb{S}^d)} \leq C\|f\|_{L^p(\mathbb{S}^d)}$ with various input functions f . For p, q in a certain range, this can be done using spherical harmonic functions of degree n , but for general p, q we need to analyze the integral operator H_n^d directly using (1.7) and (2.1).

In order to show (1.1), by duality it is enough to show that

$$(4.1) \quad \|H_n^d\|_{p,q} \gtrsim n^{\frac{d-1}{2} - \frac{d}{q}},$$

$$(4.2) \quad \|H_n^d\|_{p,q} \gtrsim n^{\frac{d-1}{2}(\frac{1}{p} - \frac{1}{q})},$$

$$(4.3) \quad \|H_n^d\|_{p,q} \gtrsim n^{d(\frac{1}{p} - \frac{1}{q}) - 1}.$$

Proof of (4.1) On the limited range $p < \frac{2d}{d-1} < q$, (4.1) can be shown with the following estimate from Szegő [38, p. 391].

Lemma 4.1. For $\alpha, \beta, \mu > -1$, and $p > 0$,

$$\int_0^1 (1-t)^\mu |P_n^{(\alpha, \beta)}(t)|^p dt \sim \begin{cases} n^{\alpha p - 2\mu - 2}, & 2\mu < \frac{2\alpha+1}{2}p - 2, \\ n^{-\frac{p}{2}} \log n, & 2\mu = \frac{2\alpha+1}{2}p - 2, \\ n^{-\frac{p}{2}}, & 2\mu > \frac{2\alpha+1}{2}p - 2. \end{cases}$$

In particular, taking $\alpha = \beta = \mu = \frac{d-2}{2}$, we see that, for $e \in \mathbb{S}^d$,

$$\|P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\xi \cdot e)\|_{L^p(\mathbb{S}^d)} \sim \begin{cases} n^{\frac{d-2}{2} - \frac{d}{p}}, & p > \frac{2d}{d-1}, \\ n^{-\frac{1}{2}} (\log n)^{\frac{1}{p}}, & p = \frac{2d}{d-1}, \\ n^{-\frac{1}{2}}, & p < \frac{2d}{d-1}. \end{cases}$$

Testing the inequality with $f(\xi) = P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\xi \cdot e)$ which is in \mathcal{H}_n^d gives that, for $p < \frac{2d}{d-1} < q$, $\|H_n^d\|_{p,q} \gtrsim n^{\frac{d-1}{2} - \frac{d}{q}}$.

To get the condition on the full range, we need to consider the kernel of the operator H_n^d . For this, using (2.1) and the asymptotic expansion of the Bessel function it is easy to see that, for $N \gg 1$,

$$P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\cos \theta) = \frac{2^{\frac{d-1}{2}} \Gamma(n + \frac{d}{2})}{\sqrt{\pi n!}} N^{-\frac{d-1}{2}} (\sin \theta)^{-\frac{d-1}{2}} \left(\cos \left(N\theta - \frac{(d-1)\pi}{4} \right) + O(N^{-1}) \right)$$

provided that θ is away from zero, say, $\theta \in [\pi/6, \pi/2]$. Combining this with (1.7), we have that, if $0 \leq \zeta \cdot \xi \leq \sqrt{3}/2$,

$$(4.4) \quad \mathbf{Z}_n(\zeta \cdot \xi) = C(N)G(\arccos \zeta \cdot \xi) \left(\cos \left(N \arccos \zeta \cdot \xi - \frac{(d-1)\pi}{4} \right) + O(N^{-1}) \right)$$

with $C(N) \sim N^{\frac{d-1}{2}}$ and $G \sim 1$. Hence, if $|\zeta - e_{d+1}| \leq cN^{-1}$ for a small constant $c > 0$ and $1/10 \leq \zeta \cdot \xi \leq \sqrt{3}/2$, then

$$(4.5) \quad \mathbf{Z}_n(\zeta \cdot \xi) = C(N)G(\arccos \zeta \cdot \xi) \left(\cos \left(N \arccos e_{d+1} \cdot \xi - \frac{(d-1)\pi}{4} + O(c) \right) + O(N^{-1}) \right).$$

Let us set

$$\Sigma = \bigcup_{k \in \mathbb{Z}: \frac{N}{12} \leq k \leq \frac{N}{6}} \left\{ \xi \in \mathbb{S}^d : N \arccos(e_{d+1} \cdot \xi) - \frac{(d-1)\pi}{4} \in [2\pi k, \frac{\pi}{4} + 2\pi k] \right\}$$

and set $f(\xi) = \mathbf{1}_\Sigma(\xi)$. Then it is easy to see that $\|f\|_p \sim 1$ for any large N and $p \in [1, \infty]$. By (4.5), we see that $\mathbf{Z}_n(\zeta \cdot \xi) \sim N^{\frac{d-1}{2}}$ if $|\zeta - e_{d+1}| \leq cN^{-1}$ and $\xi \in \Sigma$. Hence, for $|\zeta - e_{d+1}| \leq cN^{-1}$,

$$H_n^d f(\zeta) = \int_{\mathbb{S}^d} \mathbf{Z}_n(\zeta \cdot \xi) f(\xi) d\sigma(\xi) \sim N^{\frac{d-1}{2}}.$$

This implies that $\|H_n^d f\|_{L^q(\mathbb{S}^d)} \gtrsim N^{\frac{d-1}{2} - \frac{d}{q}}$ while $\|f\|_{L^p(\mathbb{S}^d)} \sim 1$. So, we get $\|H_n^d\|_{p,q} \gtrsim N^{\frac{d-1}{2} - \frac{d}{q}}$.

Proof of (4.2) This can be shown by making use of Gaussian beam. Let us consider

$$h_n(\eta) = (\eta_1 + i\eta_2)^n, \quad \eta = (\eta_1, \eta_2, \eta') \in \mathbb{S}^d, \quad \eta_1, \eta_2 \in \mathbb{R}.$$

Then, clearly, $H_n^d h_n = h_n$ and it is well known that $\|h_n\|_{L^p(\mathbb{S}^d)} \sim n^{-\frac{d-1}{2p}}$. In fact, $|h_n(\eta)| = e^{n \ln \sqrt{\eta_1^2 + \eta_2^2}} = e^{\frac{n}{2} \ln(1 - |\eta'|^2)} \sim e^{-\frac{n}{2} \text{dist}(\gamma, \eta)^2}$. Here γ is the great circle which is contained in $\eta' = 0$. Hence it follows that $\|H_n^d\|_{p,q} \gtrsim n^{\frac{d-1}{2}(\frac{1}{p} - \frac{1}{q})}$.

Another proof of (4.2) It is also possible to show this directly without using special spherical harmonics. We make use of (4.4) and the parametrizations of \mathbb{S}^d near e_2 and e_1 , respectively,

$$\zeta = \left(x_1, \sqrt{1 - x_1^2 - |\tilde{x}|^2}, \tilde{x} \right), \quad \xi = \left(\sqrt{1 - y_2^2 - |\tilde{y}|^2}, y_2, \tilde{y} \right), \quad |x_1|, |y_2| \leq c, \quad |\tilde{x}|, |\tilde{y}| \leq cN^{-\frac{1}{2}}$$

with a small enough $c > 0$. Then, we have

$$\zeta \cdot \xi = x_1 \sqrt{1 - y_2^2} + y_2 \sqrt{1 - x_1^2} + O(c^2 N^{-1}).$$

Putting $x_1 = \cos \theta$ and $y_2 = \sin \theta'$ with $|\theta - \pi/2| \leq 2c$ and $|\theta'| \leq 2c$, we have from (4.4)

$$(4.6) \quad \mathbf{Z}_n(\zeta \cdot \xi) = C(N)G(\arccos \zeta \cdot \xi) \left(\cos \left(N(\theta - \theta') - \frac{(d-1)\pi}{4} + O(c^2) \right) + O(N^{-1}) \right)$$

because $\zeta \cdot \xi = \cos(\theta - \theta') + O(c^2 N^{-1})$ and $\arccos(s+t) = \arccos s + O(t)$. With a small enough $c > 0$, set

$$\Sigma = \bigcup_{l \in \mathbb{Z}: |l| \leq \frac{cN}{4\pi}} \left\{ (\theta', \tilde{y}) : \theta' \in \left[-\frac{c}{N}, \frac{c}{N} \right] + \frac{2\pi l}{N}, |\tilde{y}| \leq cN^{-\frac{1}{2}} \right\}.$$

Let us define a function f on the sphere by setting $f(\sqrt{\cos^2 \theta' - |\tilde{y}|^2}, \sin \theta', \tilde{y}) = \mathbf{1}_\Sigma(\theta', \tilde{y})$, and integrate $|H_n^d f(\cos \theta, \sqrt{\sin^2 \theta - |\tilde{x}|^2}, \tilde{x})|^q$ over $(\theta, \tilde{x}) \in \Sigma + \frac{\pi}{2}$. If we choose $N \gg 1$ so that $2N - d + 1 \in 4\mathbb{N}$ and $c > 0$ small enough, then in (4.6) we see that

$$\left| \cos \left(N(\theta - \theta') - \frac{(d-1)\pi}{4} + O(c^2) \right) + O(N^{-1}) \right| \gtrsim 1.$$

So, $\|H_n^d f\|_{L^q(\mathbb{S}^d)} \gtrsim N^{-\frac{d-1}{2q}}$ while $\|f\|_{L^p(\mathbb{S}^d)} \sim N^{-\frac{d-1}{2p}}$. Hence, the desired (4.2) follows.

Proof of (4.3) Using the fact that $\tilde{J}_k(r) = r^{-k} J_k(r)$ is a well defined analytic function and $\tilde{J}_k(0) > 0$, $k \geq 0$, it is easy to see from (2.1) that $P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\cos \theta) \gtrsim n^{\frac{d-2}{2}}$ if $0 \leq \theta < cN^{-1}$ with c small enough. Hence, by (1.7) we have that, for $0 < \theta \leq cN^{-1}$,

$$\mathbf{Z}_n(\theta) \gtrsim N^{d-1}.$$

Hence if we consider the function $f = \mathbf{1}_{|\xi - e_{d+1}| \ll N^{-1}}$, then $H_n^d f(\zeta) \gtrsim N^{-1}$ if $|\zeta - e_{d+1}| \ll N^{-1}$. This gives $\|H_n^d f\|_{L^q(\mathbb{S}^d)} / \|f\|_{L^p(\mathbb{S}^d)} \gtrsim N^{d(\frac{1}{p} - \frac{1}{q}) - 1}$ and (4.3).

We now show (1.2) when $(\frac{1}{p}, \frac{1}{q}) \in [S, R] \cup [S', R']$. For this we need the following which is an extension of a lemma in Sogge [29].

Lemma 4.2. *Let $p, q \in [1, \infty]$. Suppose that there exists a constant B such that*

$$(4.7) \quad \|H_n^d\|_{p,q} \leq B n^{d(\frac{1}{p} - \frac{1}{q}) - 1}.$$

Then we have, for $f \in \mathcal{S}(\mathbb{R}^d)$,

$$(4.8) \quad \left\| \int_{\mathbb{S}^{d-1}} e^{ix \cdot \eta} \widehat{f}(\eta) d\sigma(\eta) \right\|_{L^q(\mathbb{R}^d)} \lesssim B \|f\|_{L^p(\mathbb{R}^d)}.$$

The Bochner-Riesz operator R^α of order α is the multiplier operator defined by

$$\widehat{R^\alpha f}(\xi) = \frac{1}{\Gamma(\alpha+1)} (1 - |\xi|^2)_+^\alpha \widehat{f}(\xi), \quad \xi \in \mathbb{R}^d.$$

The definition $R^\alpha f$ can be extended to $\alpha \leq -1$ by analytic continuation from the above formula. L^p - L^q boundedness for the Bochner-Riesz operator of negative order has been studied by some authors [7, 29, 8, 1, 10], and it was shown by Börjeson [7] (also see [1]) that R^α , $\alpha < 0$, is bounded from L^p to L^q only if

$$\frac{1}{p} > \frac{d-1-2\alpha}{2d}, \quad \frac{1}{q} < \frac{d+1+2\alpha}{2d}, \quad \frac{1}{p} - \frac{1}{q} \geq \frac{-2\alpha}{d+1}.$$

The problem of L^p - L^q boundedness of R^α is completely settled in \mathbb{R}^2 and in higher dimensions the sharp boundedness is established for $\alpha < -\frac{d^2-d-2}{2(d^2+d-2)}$ (see [1, 10]).

Proof of (1.2) for $(\frac{1}{p}, \frac{1}{q}) \in [S, R] \cup [S', R']$ Since $\frac{1}{\Gamma(\alpha+1)} t_+^\alpha$ equals the delta distribution when $\alpha = -1$, it follows that $R^{-1}f(x) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} e^{ix \cdot \eta} \widehat{f}(\eta) d\sigma(\eta)$. Hence, from the above we see that (4.8) is possible only for p, q satisfying $\frac{1}{p} > \frac{d+1}{2d}$, $\frac{1}{q} < \frac{d-1}{2d}$, $\frac{1}{p} - \frac{1}{q} > \frac{2}{d+1}$. This is equivalent to $(\frac{1}{p}, \frac{1}{q}) \in \mathfrak{T}_2 \setminus ([S, R] \cup [S', R'])$. Now let $(\frac{1}{p}, \frac{1}{q}) \in ([S, R] \cup [S', R'])$ and suppose $n^{-\gamma(p,q)} \|H_n^d\|_{p,q} \leq B$ for some constant B . Then, by Lemma 4.2 the estimate (4.8) should also be true. This contradicts aforementioned necessary condition. Hence, there is no B such that (4.7) holds when $(\frac{1}{p}, \frac{1}{q}) \in [S, R] \cup [S', R']$. This gives (1.2). \square

Proof of Lemma 4.2. Let us define a map $\mu_n : \mathbb{R}^d \rightarrow n\mathbb{S}^d \setminus \{ne_{d+1}\}$ by

$$\mu_n(x) = \left(\frac{4n^2x}{|x|^2 + 4n^2}, \frac{n|x|^2 - 4n^3}{|x|^2 + 4n^2} \right).$$

If S_n is the stereographic projection of $n\mathbb{S}^d \setminus \{ne_{d+1}\}$ to $\mathbb{R}^d \times \{-n\}$, then $S_n(\mu_n(x)) = (x, -n)$.

Let $d\sigma_n$ and dx denote the surface measure on $n\mathbb{S}^d$ and the Lebesgue measure on \mathbb{R}^d , respectively. By using rotational symmetry and a computation it is easy to see that $\sqrt{\det((D\mu_n)^t D\mu_n)} = (\frac{|x|^2}{4n^2} + 1)^{-d}$. Thus we have

$$(4.9) \quad d\sigma_n(x) = \left(\frac{|x|^2}{4n^2} + 1 \right)^{-d} dx.$$

Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ and consider the integral

$$\begin{aligned} I_n &= n^{d+1} \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \mathbf{Z}_n(\xi \cdot \eta) f(\mu_n^{-1}(n\xi)) g(\mu_n^{-1}(n\eta)) d\sigma(\xi) d\sigma(\eta) \\ &= n^{d+1} \int_{\mathbb{S}^d} H_n^d(f(\mu_n^{-1}(n \cdot))) (\eta) g(\mu_n^{-1}(n\eta)) d\sigma(\eta). \end{aligned}$$

From (4.9) $\|f(\mu_n^{-1}(n\cdot))\|_{L^p(\mathbb{S}^d)} \lesssim n^{-\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}$ and $\|g(\mu_n^{-1}(n\cdot))\|_{L^{q'}(\mathbb{S}^d)} \lesssim n^{-\frac{d}{q'}} \|g\|_{L^{q'}(\mathbb{R}^d)}$. Hence, by (4.7) it is easy to see

$$(4.10) \quad |I_n| \lesssim n^{1-d(\frac{1}{p}-\frac{1}{q})} \|H_n^d\|_{p,q} \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{q'}(\mathbb{R}^d)} \lesssim B \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{q'}(\mathbb{R}^d)}.$$

On the other hand, by changes of variables,

$$\begin{aligned} I_n &= \int_{n\mathbb{S}^d} \int_{n\mathbb{S}^d} n^{1-d} \mathbf{Z}_n\left(\frac{\xi \cdot \eta}{n^2}\right) f(\mu_n^{-1}\xi) g(\mu_n^{-1}\eta) d\sigma_n(\xi) d\sigma_n(\eta) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} n^{1-d} \mathbf{Z}_n\left(\frac{\mu_n(x) \cdot \mu_n(y)}{n^2}\right) f(x) g(y) \frac{d\sigma_n}{dx} \frac{d\sigma_n}{dy} dx dy. \end{aligned}$$

We recall the identity known as Mehler-Heine type (see Szegő [38, p.192])

$$\lim_{n \rightarrow \infty} n^{1-d} \mathbf{Z}_n\left(\cos\left\{\frac{r}{n} + o\left(\frac{1}{n}\right)\right\}\right) = c_d r^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(r)$$

for some c_d . This can be easily shown by using (1.7) and (2.1). Also, note that

$$\frac{\mu_n(x) \cdot \mu_n(y)}{n^2} = 1 - \frac{|x-y|^2}{2n^2} + O\left(\frac{1}{n^3}\right) = \cos\left\{\frac{|x-y|}{n} + o\left(\frac{1}{n}\right)\right\}.$$

Hence, it follows that

$$\lim_{n \rightarrow \infty} n^{1-d} \mathbf{Z}_n\left(\frac{\mu_n(x) \cdot \mu_n(y)}{n^2}\right) = c_d |x-y|^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(|x-y|) = \tilde{c}_d \int_{\mathbb{S}^{d-1}} e^{i(x-y)\cdot\eta} d\sigma(\eta).$$

Therefore, recalling $\lim_{n \rightarrow \infty} \frac{d\sigma_n}{dx} = 1$ from (4.9), we get

$$\lim_{n \rightarrow \infty} I_n = \tilde{c}_d \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} e^{iy\cdot\eta} \hat{f}(\eta) d\sigma(\eta) g(y) dy.$$

Combining this with (4.10) and duality yield

$$\left| \tilde{c}_d \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} e^{iy\cdot\eta} \hat{f}(\eta) d\sigma(\eta) g(y) dy \right| \lesssim \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^{q'}(\mathbb{R}^d)}.$$

Duality gives (4.8). □

§ 5. Application to Carleman estimate

In this section we prove Theorem 1.5. For this it is sufficient to show the following since (5.1) holds as long as p, q satisfy (1.13) and $\frac{2(d-1)d}{d^2+2d-4} < p < \frac{2(d-1)}{d}$ by Theorem 1.2.

Proposition 5.1. *Let $p, q \in (1, \infty)$ and satisfy (1.13). Suppose that the estimate*

$$(5.1) \quad \|H_n^{d-1} f\|_{L^q(\mathbb{S}^{d-1})} \lesssim n^{1-\frac{2}{d}} \|f\|_{L^p(\mathbb{S}^{d-1})}$$

holds. Then, for the same p, q , (1.12) holds whenever $\text{dist}(\tau, \mathbb{Z} + \frac{d}{q}) \geq c$ for some $c > 0$.

Proof of Proposition 5.1. We follow Jerison's idea in [21].

First, using the spherical coordinates $(r, \omega) \in \mathbb{R}_+ \times \mathbb{S}^{d-1}$ and the identity $\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{1}{r^2}\Delta_{\mathbb{S}^{d-1}}$, we note that

$$|x|^{-\tau}\Delta|x|^\tau = \partial_r^2 + \frac{2\tau+d-1}{r}\partial_r + \frac{\tau^2+(d-2)\tau}{r^2} + \frac{1}{r^2}\Delta_{\mathbb{S}^{d-1}}.$$

Making the change of variables $r = e^{-t}$ gives $|x|^{-\tau}\Delta|x|^\tau = e^{2t}(P(\partial_t) + \Delta_{\mathbb{S}^{d-1}})$, where we set

$$P(t) = t^2 - (2\tau + d - 2)t + \tau(\tau + d - 2).$$

Hence, we see that (1.12) is equivalent to the estimate

$$\left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^{d-1}} |u(t, \omega)|^q d\omega e^{-dt} dt\right)^{\frac{1}{q}} \lesssim \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^{d-1}} |(P(\partial_t) + \Delta_{\mathbb{S}^{d-1}})u(t, \omega)|^p d\omega e^{(2p-d)t} dt\right)^{\frac{1}{p}}$$

for $u \in C_c^\infty((\mathbb{R} \setminus \{0\}) \times \mathbb{S}^{d-1})$.

By replacing $u \rightarrow e^{\frac{d}{q}t}u$ and then using the relations $\partial_t^m(e^{\frac{d}{q}t}u) = e^{\frac{d}{q}t}(\partial_t + \frac{d}{q})^m u$ and $\frac{d}{p} - \frac{d}{q} = 2$, it follows that the previous inequality is equivalent to

$$(5.2) \quad \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^{d-1}} |u(t, \omega)|^q d\omega dt\right)^{\frac{1}{q}} \lesssim \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^{d-1}} |(P(\partial_t + \frac{d}{q}) + \Delta_{\mathbb{S}^{d-1}})u(t, \omega)|^p d\omega dt\right)^{\frac{1}{p}}.$$

Thus we are reduced to showing

$$(5.3) \quad \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^{d-1}} |[P(\partial_t + \frac{d}{q}) + \Delta_{\mathbb{S}^{d-1}}]^{-1}u(t, \omega)|^q d\omega dt\right)^{\frac{1}{q}} \lesssim \left(\int_{-\infty}^{\infty} \int_{\mathbb{S}^{d-1}} |u(t, \omega)|^p d\omega dt\right)^{\frac{1}{p}}.$$

The operator $[P(\partial_t + \frac{d}{q}) + \Delta_{\mathbb{S}^{d-1}}]^{-1}$ can be expressed in terms of spherical harmonic projection operators. In fact, since $\Delta_{\mathbb{S}^{d-1}}H_n^{d-1} = -n(n+d-2)H_n^{d-1}$ and $P(\partial_t + \frac{d}{q}) - n(n+d-2) = (\partial_t + \frac{d}{q} - (\tau - n))(\partial_t + \frac{d}{q} - (\tau + n + d - 2))$, it follows that

$$(5.4) \quad [P(\partial_t + \frac{d}{q}) + \Delta_{\mathbb{S}^{d-1}}]H_n^{d-1} = (\partial_t - \tilde{\tau} + n)(\partial_t - \tilde{\tau} - n - d + 2)H_n^{d-1},$$

where we set $\tilde{\tau} = \tau - \frac{d}{q}$. Hence, taking Fourier transform in t , we have that

$$(5.5) \quad [P(\partial_t + \frac{d}{q}) + \Delta_{\mathbb{S}^{d-1}}]^{-1}H_n^{d-1}u(t, \omega) = \frac{1}{2\pi} \int \frac{e^{its}\mathcal{F}_t(H_n^{d-1}u(\cdot, \omega))(s) ds}{(is - \tilde{\tau} + n)(is - \tilde{\tau} - n - d + 2)} \\ = \frac{1}{2\pi(2n + d - 2)} \int e^{its} \left(\frac{1}{is - \tilde{\tau} - n - d + 2} - \frac{1}{is - \tilde{\tau} + n} \right) \mathcal{F}_t(H_n^{d-1}u(\cdot, \omega))(s) ds$$

if $2n \neq 2 - d$, which is always true because $d \geq 3$ and $n \geq 0$. Here \mathcal{F}_t denotes the Fourier transform with respect to t variable.

Lemma 5.2. *Let $\alpha \neq 0$. Then, for $g \in \mathcal{S}(\mathbb{R})$,*

$$\frac{1}{2\pi} \int \frac{e^{its}}{is + \alpha} \mathcal{F}_t g(s) ds = \begin{cases} \int_{-\infty}^t e^{-(t-s)\alpha} g(s) ds, & \alpha > 0 \\ -\int_t^{\infty} e^{-(t-s)\alpha} g(s) ds, & \alpha < 0. \end{cases}$$

For the proof of this lemma it is enough to show that

$$\frac{1}{2\pi} \int \frac{e^{its}}{is + \alpha} ds = \begin{cases} e^{-t\alpha} \mathbf{1}_{(0,\infty)}(t), & \alpha > 0 \\ -e^{-t\alpha} \mathbf{1}_{(-\infty,0)}(t), & \alpha < 0 \end{cases}$$

and this follows from an easy application of the residue theorem to the function e^z/z .

After applying spectral projection $u(t, \omega) = \sum_n H_n^{d-1} u(t, \omega)$, we see that the inverse of $P(\partial_t + \frac{d}{q}) + \Delta_{\mathbb{S}^{d-1}}$ is given as follows:

$$(5.6) \quad [P(\partial_t + \frac{d}{q}) + \Delta_{\mathbb{S}^{d-1}}]^{-1} u(t, \omega) = \sum_{k=0}^4 I_k u(t, \omega),$$

where

$$\begin{aligned} I_0 u(t, \omega) &= \frac{1}{2\pi(d-2)} \int \frac{e^{its} \mathcal{F}_t(H_0^{d-1} u(s, \cdot))(s)}{(is - \tilde{\tau})(is - \tilde{\tau} - d + 2)} ds, \\ I_1 u(t, \omega) &= \sum_{n > \tilde{\tau}, n \neq 0} \frac{-1}{2n + d - 2} \int_{-\infty}^t e^{-(n-\tilde{\tau})(t-s)} H_n^{d-1} u(s, \omega) ds, \\ I_2 u(t, \omega) &= \sum_{0 < n < \tilde{\tau}} \frac{1}{2n + d - 2} \int_t^{\infty} e^{-(n-\tilde{\tau})(t-s)} H_n^{d-1} u(s, \omega) ds, \\ I_3 u(t, \omega) &= \sum_{n < -\tilde{\tau} - d + 2, n \neq 0} \frac{1}{2n + d - 2} \int_{-\infty}^t e^{(\tilde{\tau} + n + d - 2)(t-s)} H_n^{d-1} u(s, \omega) ds, \\ I_4 u(t, \omega) &= \sum_{n > -\tilde{\tau} - d + 2, n \neq 0} \frac{-1}{2n + d - 2} \int_t^{\infty} e^{(\tilde{\tau} + n + d - 2)(t-s)} H_n^{d-1} u(s, \omega) ds. \end{aligned}$$

We see that $I_0 u$ is trivially bounded from L^p to L^q . Therefore, we are reduced to showing that, for $k = 1, 2, 3, 4$,

$$(5.7) \quad \|I_k u\|_{L^q(dt d\omega)} \lesssim \|u\|_{L^p(dt d\omega)}$$

with the implicit constant independent of τ . We shall only prove (5.7) for $k = 1, 2$, since the others can be handled similarly.

Proof of (5.7) for $k = 1$. We now use the spectral projection estimate (5.1), which is followed by Minkowski's inequality, to get

$$\|I_1 u(t, \cdot)\|_{L^q(\mathbb{S}^{d-1})} \lesssim \int_{-\infty}^t \sum_{n > \tilde{\tau}} n^{-\frac{2}{d}} e^{-(n-\tilde{\tau})(t-s)} \|u(s, \cdot)\|_{L^p(\mathbb{S}^{d-1})} ds.$$

Since $n - \tilde{\tau} \geq c := \text{dist}(\tau, \mathbb{Z} + \frac{d}{q}) > 0$, whenever $s > 0$, it is clear that $\sum_{n > \tilde{\tau}} n^{-\frac{2}{d}} e^{-(n-\tilde{\tau})s}$ is bounded by

$$\begin{aligned} \sum_{n > \tilde{\tau}} (n - \tilde{\tau})^{-\frac{2}{d}} e^{-(n-\tilde{\tau})s} &\leq \sum_{j=0}^{\infty} (c+j)^{-\frac{2}{d}} e^{-(c+j)s} \leq c^{-\frac{2}{d}} e^{-cs} + \int_0^{\infty} (c+u)^{-\frac{2}{d}} e^{-(c+u)s} du \\ &\leq e^{-cs} \left(c^{-\frac{2}{d}} + s^{\frac{2}{d}-1} \Gamma(1 - \frac{2}{d}) \right) \lesssim e^{-\tilde{c}s} s^{\frac{2}{d}-1} \end{aligned}$$

for some $\tilde{c} > 0$. In fact, we can take any $\tilde{c} < c$. Therefore, by Hardy-Littlewood-Sobolev inequality or Young's inequality, it follows that

$$\begin{aligned} \|I_1 u\|_{L^{\tilde{q}}(\mathbb{R}; L^q(\mathbb{S}^{d-1}))} &\lesssim \left\| \int_{-\infty}^t (t-s)^{-\frac{d-2}{2}(\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}})} e^{-\tilde{c}(t-s)} \|u(s, \cdot)\|_{L^p(dw)} ds \right\|_{L^{\tilde{q}}(\mathbb{R}, dt)} \\ &\lesssim \|u\|_{L^{\tilde{p}}(\mathbb{R}; L^p(\mathbb{S}^{d-1}))} \end{aligned}$$

provided that $1 < \tilde{p} \leq \tilde{q} < \infty$ and $\frac{1}{\tilde{p}} - \frac{1}{\tilde{q}} \leq \frac{2}{d}$. In particular, taking $\tilde{p} = p$, $\tilde{q} = q$, we get the desired estimate (5.7) for $k = 1$.

Proof of (5.7) for $k = 2$. In this case we are assuming that $\tilde{\tau} > 1$, because otherwise the summation is empty. As before Minkowski's inequality and the spectral projection estimate (5.1) give us

$$\|I_2 u(t, \cdot)\|_{L^q(\mathbb{S}^{d-1})} \lesssim \int_t^{\infty} \sum_{n=1}^{[\tilde{\tau}]} n^{-\frac{2}{d}} e^{(\tilde{\tau}-n)(t-s)} \|u(s, \cdot)\|_{L^p(\mathbb{S}^{d-1})} ds,$$

where $[\tilde{\tau}]$ is the largest integer less than $\tilde{\tau}$. We note that $\tilde{\tau} - [\tilde{\tau}] \geq \text{dist}(\tau, \mathbb{Z} + \frac{d}{q}) = c > 0$. To estimate the kernel let us fix $s < 0$. Then an elementary computation shows that

$$\begin{aligned} \sum_{n=1}^{[\tilde{\tau}]} n^{-\frac{2}{d}} e^{(\tilde{\tau}-n)s} &\leq e^{-|s|\tilde{\tau}} \left(\int_1^{[\tilde{\tau}]} (u-1)^{-\frac{2}{d}} e^{|s|u} du + e^{|s|[\tilde{\tau}]} \right) \\ (5.8) \quad &= e^{-|s|\tilde{\tau}} \left(|s|^{\frac{2}{d}-1} e^{|s|} \int_0^{|s|([\tilde{\tau}]-1)} u^{-\frac{2}{d}} e^u du + e^{|s|[\tilde{\tau}]} \right). \end{aligned}$$

If $1 \leq \tilde{\tau} < 2$, then (5.8) is bounded by $e^{-|s|(\tilde{\tau}-[\tilde{\tau}])} \leq e^{-c|s|}$, which is in $L^r((-\infty, 0), ds)$ for $1 \leq r \leq \infty$. If $\tilde{\tau} \geq 2$, then the integral in (5.8) is bounded by

$$\int_0^1 u^{-\frac{2}{d}} e^u du + \mathbf{1}_{(\frac{1}{[\tilde{\tau}]-1}, \infty)}(|s|) \int_1^{|s|([\tilde{\tau}]-1)} u^{-\frac{2}{d}} e^u du \lesssim 1 + \mathbf{1}_{(1/[\tilde{\tau}], \infty)}(|s|) e^{|s|([\tilde{\tau}]-1)}$$

with the implicit constant depends only on c and d . Hence the quantity (5.8) is bounded by

$$e^{-|s|(\tilde{\tau}-1)} |s|^{\frac{2}{d}-1} + \mathbf{1}_{(1/[\tilde{\tau}], \infty)}(|s|) e^{-|s|(\tilde{\tau}-[\tilde{\tau}])} |s|^{\frac{2}{d}-1} + e^{-|s|(\tilde{\tau}-[\tilde{\tau}])} \lesssim e^{-c|s|} |s|^{\frac{2}{d}-1} + e^{-c|s|}.$$

Therefore as in the proof for I_1 , we conclude that (5.7) is true for $k = 2$. This completes the proof. \square

Remark 2 (Failure of (1.12)). We show (1.12) does not hold if

$$(5.9) \quad \frac{1}{q} < \frac{d-4}{2(d-1)}, \quad \frac{1}{p} > \frac{d+2}{2(d-1)}.$$

We only need to consider p, q satisfying (1.13) because (1.12) implies the condition (1.13). Let $n \gg 1$ and choose $\tau \notin \mathbb{Z} + \frac{d}{q}$ so that $\tilde{\tau} = \tau - \frac{d}{q} = n + \frac{1}{2}$. Let h be a nontrivial smooth positive function supported in $[1/2, 2]$ and g be a spherical harmonic polynomial on \mathbb{S}^{d-1} of degree n . Let us consider

$$u(t, w) = h(t)g(w).$$

By (5.4) we have $(P(\partial_t + \frac{d}{q}) + \Delta_{S^{d-1}})u = O(n(|h(t)| + |h'(t)| + |h''(t)|)g(w))$. Since (1.12) and (5.2) are equivalent if (1.13) is satisfied, we apply (5.2) to the function u . Hence, from integration in t we have that

$$\|H_n^{d-1}\|_{p,q} \lesssim n.$$

whenever (1.12) holds. However, by Theorem 1.1 such bound is possible only if (5.9) is satisfied when p, q satisfy (1.13).

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