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Analyticity of global solutions to the non-gauge invariant nonlinear Schrödinger equations

By

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Abstract

In this paper, we study the global Cauchy problem for the non-gauge invariant quadratic nonlinear Schrödinger equations in the framework of the scale critical Sobolev space $\dot{H}^{n/2-2}$ with the space dimensions $n \geq 4$. In particuler, we study analyticity of global solutions in the sense of analytic Hardy space via the phase modulation operator to the non-gauge invariant nonlinear Schrödinger equations with data which decays exponentially in spatial infinity.

§ 1. Introduction

In this paper, we study the Cauchy problem for the nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \mathcal{N}(u,\overline{u}), & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0) = \phi, & x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where space dimensions $n \geq 4$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$, unknown functions $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ and the non-gauge invariant quadratic nonlinearity

$$\mathcal{N}(u,\overline{u}) = \lambda u^2 + \mu |u|^2 \tag{1.2}$$

with $\lambda, \mu \in \mathbb{C}$. In particular, main purpose of this study is to consider analyticity of solutions in the sense of analytic Hardy space via the phase modulation operator to the non-gauge invariant nonlinear Schrödinger equations (1.1)-(1.2) with data which decays

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exponentially in spatial infinity.

There are many papers study on the Cauchy problem for the nonlinear Scgrödinger equations (see [2, 3, 26, 21, 30] and reference therein). In particular, the Cauchy problem for the nonlinear Schrödinger equations in the scale critical setting has been studied in [3, 26] (see also [2]). The analytic smoothing effect for the gauge invariant nonlinear Schrödinger equations with data which decays exponentially has been studied in many papers ([5, 8, 9, 10, 14, 15, 16, 17, 25, 28]), by applying the following two types of operators

$$A_\delta = e^{it\Delta/2} e^{\delta\cdot x} e^{-it\Delta/2},\ \delta \in \mathbb{R}^n$$

or

$$J = x + it\nabla$$

based on the Galilei invariant property.

On the other hand, the operators A_δ and J do not work well for the non-gauge invariant nonlinearity, for example:

$$A_\delta(u^2) = (A_\delta u) e^{it\delta\cdot\nabla} u$$

where $e^{iy\cdot\nabla} = \mathcal{F}^{-1} e^{-y\cdot\xi}\mathcal{F}$, and

$$J(u^2) = uJu + uit\nabla u.$$

The operator $e^{it\delta\cdot\nabla}$ gives analytic continuation and the term $t\delta$ expands radius of convergence for large time $t \in \mathbb{R}$.

Therefore, we assume that the data satisfy exponentially decaying condition in the Fourier-Lebesgue type space such as

$$\sup_{0\le|\delta|<\gamma} \left\| e^{\delta\cdot x}\phi \right\|_{L^2} + \sup_{0\le|y|<a+\varepsilon} \left\| e^{-y\cdot\xi}\widehat{\phi} \right\|_{\dot{H}^{n/2-2}} < \infty$$

for some $a, \gamma, \varepsilon > 0$ and to control the term $t\delta$, we need the following cut-off function

$$\mathbf{I}_{\Omega_\gamma^a}(t,\delta) = \begin{cases} 1,\ (t,\delta) \in \Omega_\gamma^a, \\ 0,\ (t,\delta) \notin \Omega_\gamma^a \end{cases}$$

with the open set $\Omega_\gamma^a = \{t \in \mathbb{R},\ 0 \le |\delta| < \gamma;\ 0 \le |t\delta| < a\}$ (see (1.3)-(1.4) for precisely). Then the corresponding solutions of (1.1)-(1.2) satisfy the following property:

$$\sup_{0\le|y|<\min\{\gamma|t|,a\}} \left\| e^{iy\cdot\nabla} e^{-i|x|^2/2t} u(t) \right\|_{L^2} < \infty,\ t \neq 0.$$

Which fact says that the corresponding solutions belong to analytic Hardy space via the phase modulation operator $e^{-i|x|^2/2t}$, $t \neq 0$. Here the analytic Hardy space is defined as (see Chapter III of [29])

$$\mathcal{H}^p(\Omega) = \left\{\varphi : \text{analytic on } \mathbb{R}^n + i\Omega;\ \|\varphi\|_{\mathcal{H}^p(\Omega)} = \sup_{y\in\Omega} \|\varphi(\cdot + iy)\|_{L^p} < \infty\right\},\ 1 \leq p \leq \infty$$

for domain $\Omega \subset \mathbb{R}^n$. It is difficult to obtain the analyticity in the sense of analytic Hardy space via the phase modulation operator for solutions to the non-gauge invariant nonlinear Schrödinger equations without exponentially decaying condition in the Fourier space, even if data satisfy exponentially decaying condition in the real space.

The Cauchy problem and asymptotic behavior of solutions to the non-gauge invariant nonlinear Schrödinger equations have been studied in many papers (see [6, 7, 18, 19, 20, 27] for instance). The analyticity of solutions in the sense of analytic Hardy space via the phase modulation operator $e^{-i|x|^2/2t}$, $t \neq 0$ to the non-gauge invariant nonlinear Schrödinger equations, locally in time has been studied in [13]. In [11], the author has improved method introduced in [13] and has shown analyticity of solutions in the sense of analytic Hardy space via the phase modulation operator to a system of nonlinear Schödinger equations without mass resonance condition. Also in [12], the author has shown analyticity of solutions in the sense of analytic Hardy space via the phase modulation operator to the non-gauge invariant cubic nonlinear Schrödinger equations. The present paper is a sequel of studies [11, 12, 13].

$L^p = L^p(\mathbb{R}^n)$ is the usual Lebesgue space for $1 \leq p \leq \infty$. The map $\mathcal{F} : \varphi \mapsto \widehat{\varphi}$ is the Fourier transform defined by

$$\widehat{\varphi}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx,\ \xi \in \mathbb{R}^n$$

and $\mathcal{F}^{-1} : \varphi \mapsto \check{\varphi}$ is the inverse Fourier transform defined by

$$\check{\varphi}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(\xi)d\xi,\ x \in \mathbb{R}^n.$$

The homogeneous Sobolev space is defined by

$$\dot{H}^s_p = \left\{\varphi \in \mathcal{S}'/\mathcal{P};\ \|\varphi\|_{\dot{H}^s_p} = \left\|(-\Delta)^{s/2}\varphi\right\|_{L^p} < \infty\right\}$$

for $s \geq 0$ and $1 \leq p \leq \infty$, where $(-\Delta)^{s/2} = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$. The homogeneous Besov space is defined by

$$\dot{B}^s_{p,q} = \left\{\varphi \in \mathcal{S}'/\mathcal{P};\ \|\varphi\|_{\dot{B}^s_{p,q}} = \left\|\mathcal{F}^{-1}\left[2^{sj}\eta_j\widehat{\varphi}\right]\right\|_{l^q_j(\mathbb{Z};L^p)} < \infty\right\}$$

for $s \geq 0$ and $1 \leq p, q \leq \infty$, where the Littlewood-Paley's unit decomposition $\eta_j \in \mathcal{S}$ (see [2] for precisely). We use the notation $\dot{B}^s_{p,2} = \dot{B}^s_p$, shortly. We denote the open ball with radius $r > 0$ and center at $x_0 \in \mathbb{R}^n$ by

$$B_{x_0}(r) = \{x \in \mathbb{R}^n;\ |x - x_0| < r\}.$$

The free Schrödinger propagator is defined by $U(t) = e^{it\Delta/2} = \mathcal{F}^{-1}e^{-it|\xi|^2/2}\mathcal{F}$, $t \in \mathbb{R}$. We often denote $U^{-1}(t) = U(-t)$, $t \in \mathbb{R}$. The phase modulation operator is $M(t) = e^{i|x|^2/2t}$, $t \neq 0$. We define an operator

$$A_\delta(t) = U(t)e^{\delta\cdot x}U(-t),\ t \in \mathbb{R}$$

for $\delta \in \mathbb{R}^n$, where $\delta \cdot x = \sum_{j=1}^n \delta_j x_j$. A_δ has another representation

$$A_\delta(t) = M(t)e^{it\delta\cdot\nabla}M(-t),\ t \neq 0$$

for $\delta \in \mathbb{R}^n$ by the formula

$$U(t) = M(t)D(t)\mathcal{F}M(t)$$

with dilation $D(t)\varphi = (it)^{-n/2}\varphi\left(\frac{\cdot}{t}\right)$, $t \neq 0$. We define the cut-off function $\mathbf{I}_{\Omega^a_\gamma}$ by

$$\mathbf{I}_{\Omega^a_\gamma}(t,\delta) = \begin{cases} 1,\ (t,\delta) \in \Omega^a_\gamma, \\ 0,\ (t,\delta) \notin \Omega^a_\gamma \end{cases} \tag{1.3}$$

where the open set

$$\Omega^a_\gamma = \{(t,\delta) \in \mathbb{R} \times B_0(\gamma);\ t\delta \in B_0(a)\}. \tag{1.4}$$

We often denote $\mathbf{I}_{\Omega^a_\gamma}(\delta) = \mathbf{I}_{\Omega^a_\gamma}(\cdot,\delta)$, $\delta \in B_0(\gamma)$. We introduce the basic function spaces X and Y defined by

$$X = L^\infty\left(\mathbb{R}; L^2\right),$$
$$Y = L^\infty\left(\mathbb{R}; \dot{H}^{n/2-2}\right) \bigcap L^2\left(\mathbb{R}; \dot{B}^{n/2-2}_{2n/(n-2)}\right).$$

We set the following function spaces:

$$G^\gamma L^2 = \left\{\phi \in L^2;\ \|\phi\|_{G^\gamma L^2} \equiv \sup_{\delta \in B_0(\gamma)} \left\|e^{\delta\cdot x}\phi\right\|_{L^2} < \infty\right\},$$

$$A^{a+\varepsilon}\dot{H}^{n/2-2} = \left\{\phi \in \dot{H}^{n/2-2};\ \|\phi\|_{A^{a+\varepsilon}\dot{H}^{n/2-2}} \equiv \sup_{y \in B_0(a+\varepsilon)} \left\|e^{iy\cdot\nabla}\phi\right\|_{\dot{H}^{n/2-2}} < \infty\right\}$$

and

$$G^{a,\gamma}X = \left\{ u \in X;\ \|u\|_{G^{a,\gamma}X} \equiv \sup_{\delta \in B_0(\gamma)} \left\| \mathbf{I}_{\Omega_\gamma^a}(\delta) A_\delta u \right\|_X < \infty \right\},$$
$$A^{a+\varepsilon}Y = \left\{ u \in Y;\ \|u\|_{A^{a+\varepsilon}Y} \equiv \sup_{y \in B_0(a+\varepsilon)} \left\| e^{iy\cdot\nabla} u \right\|_Y < \infty \right\}.$$

We put

$$Z_0 = G^{\gamma} L^2 \bigcap A^{a+\varepsilon} \dot{H}^{n/2-2}$$

and

$$Z = G^{a,\gamma} X \bigcap A^{a+\varepsilon} Y.$$

Let two Banach spaces $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$. Then $\left(\mathcal{A} \bigcap \mathcal{B}, \|\cdot\|_{\mathcal{A}\bigcap\mathcal{B}}\right)$ is a Banach space with norm

$$\|\cdot\|_{\mathcal{A}\bigcap\mathcal{B}} = \max\left(\|\cdot\|_{\mathcal{A}}, \|\cdot\|_{\mathcal{B}}\right).$$

In this paper, analyticity of functions is based on the following proposition:

Proposition 1.1 ([16, 29]). *Let $1 \le p \le \infty$ and $\Omega \subset \mathbb{R}^n$ be domain. If $\varphi \in \mathcal{S}'$ satisfies*

$$e^{-\eta\cdot\xi}\widehat{\varphi} \in L^p$$

for all $\eta \in \Omega$. Then

$$e^{-y\cdot\xi}\widehat{\varphi} \in L^1$$

for all $y \in \Omega$ and

$$e^{iy\cdot\nabla}\varphi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(x+iy)\cdot\xi}\widehat{\varphi}(\xi)d\xi$$

is analytic on $\mathbb{R}^n + i\Omega$.

§ 2. Main results

We now state our main result:

Theorem 2.1. *Assume that $n \geq 4$, $a, \gamma, \varepsilon > 0$. Then there exists $\rho > 0$ such that; if $\phi \in Z_0$ satisfying $\|\phi\|_{A^{a+\varepsilon}\dot{H}^{n/2-2}} \leq \rho$. Then (1.1)-(1.2) has a unique solution $u \in Z$. Furthermore u satisfies*

$$\sup_{y \in B_0(\min\{\gamma|t|, a\})} \left\| e^{iy\cdot\nabla} M(-t) u(t) \right\|_{L^2} < \infty \tag{2.1}$$

for $t \neq 0$.

We give the following some remarks:

Remark 2.2. (2.1) is equivalent to the following property

$$\sup_{\delta \in B_0(\min\{\gamma, a|t|^{-1}\})} \left\| e^{\delta\cdot x} e^{it\delta\cdot\nabla} u(t) \right\|_{L^2} < \infty \tag{2.2}$$

for $t \neq 0$.

Remark 2.3. We do not need the smallness condition to the $G^\gamma L^2$-norm on the data (see also Lemma 3.1, below).

Remark 2.4. The space F of test functions for Fourier hyperfunctions is defined as follows (see [4] for instance):

$$F = \Big\{ \varphi \in C^\infty(\mathbb{R}^n);\ \text{there exist } k, h > 0 \text{ such that;}$$
$$\sup_{x\in\mathbb{R}^n} e^{k|x|}|\varphi(x)| < \infty \text{ and } \sup_{\xi\in\mathbb{R}^n} e^{h|\xi|}|\widehat{\varphi}(\xi)| < \infty \Big\}.$$

Therefore by the Hölder inequality, we see that the following including relationship

$$F \subset G^\gamma L^2 \bigcap A^{a+\varepsilon}\dot{H}^{n/2-2}$$

with some $a, \gamma > 0$ and $\varepsilon > 0$. According to [22] (see also [4]), we see that the space F is isomorphic to the following space

$$P_* = \Big\{ \varphi : \text{analytic on } \mathbb{R}^n + iB_0(r) \text{ for some } r > 0;$$
$$\sup_{z\in\mathbb{R}^n+iB_0(r)} e^{k|z|}|\varphi(z)| < \infty \text{ for some } k > 0 \Big\}.$$

Remark 2.5. If $v : \mathbb{R}^n \to \mathbb{C}$ has an analytic continuation $\widetilde{v} : \mathbb{R}^n + iB_0(r) \to \mathbb{C}$ for some $r > 0$. Then the complex-valued functions $x \mapsto v^2(x)$, $x \mapsto |v|^2(x)$ and $x \mapsto \overline{v}^2(x)$ have analytic continuation to $\mathbb{R}^n + iB_0(r)$, because analytic continuation of $\overline{v}(x)$ is $\overline{\widetilde{v}(\overline{z})}$. On the other hand the operator

$$A_\delta(t) = U(t)e^{\delta\cdot x}U(-t) = M(t)e^{it\delta\cdot\nabla}M(-t),\ t \neq 0$$

act on the nonlinearity $\overline{u}^2$ such as

$$\begin{aligned} A_\delta(t)\overline{u}^2(t) &= M(t)e^{it\delta\cdot\nabla}\left(M(-t)\overline{u}^2(t)\right) \\ &= M(t)e^{it\delta\cdot\nabla}\left(\overline{M(t)u(t)u(t)}\right) \\ &= \left(\overline{M(-t)e^{-it\delta\cdot\nabla}M(t)u(t)}\right)\left(\overline{e^{-it\delta\cdot\nabla}u(t)}\right) \\ &= \left(\overline{A_\delta(-t)u(t)}\right)\left(\overline{e^{-it\delta\cdot\nabla}u(t)}\right). \end{aligned}$$

Note also that $A_\delta(-t) = U(-t)e^{\delta\cdot x}U(t)$. However we only assume

$$\sup_{\delta\in B_0(\gamma)}\left\|\mathbf{I}_{\Omega^a_\gamma}(t,\delta)A_\delta(t)u(t)\right\|_{L^\infty_t(\mathbb{R};L^2)} < \infty$$

it is difficult to estimate the term

$$\mathbf{I}_{\Omega^a_\gamma}(t,\delta)A_\delta(-t)u(t).$$

Therefore, we assume that the non-gauge invariant nonlinearity $\mathcal{N}$ has the following form

$$\mathcal{N}(u,\overline{u}) = \lambda u^2 + \mu|u|^2$$

with $\lambda,\mu\in\mathbb{C}$.

§ 3. Preliminaries

To show main theorem by the contraction mapping theorem, we need the following completeness of a metric space.

Lemma 3.1. *Let $a,\gamma,\varepsilon > 0$. Then the metric space $(Z_{r,R}, d)$ defined by*

$$\begin{aligned} Z_{r,R} &= \left\{u\in Z;\ \|u\|_{G^{a,\gamma}X}\le R,\ \|u\|_{A^{a+\varepsilon}Y}\le r\right\}, \\ d(u,v) &= \|u-v\|_{X\bigcap A^{a+\varepsilon}Y} \end{aligned}$$

is a complete metric space.

Proof. Let $\{u_j\}_{j=1}^\infty$ be Cauchy sequence in $(Z_{r,R}, d)$. Then by the completeness of $X\bigcap A^{a+\varepsilon}Y$, we have $u\in X\bigcap A^{a+\varepsilon}Y$ such that

$$\lim_{j\to\infty}\|u_j-u\|_{X\bigcap A^{a+\varepsilon}Y} = 0.$$

L^2 is a reflexive Banach space and $\{e^{\delta\cdot x}U(-t)u_j(t)\}_{j=1}^{\infty}$ is a bounded sequence in L^2 for each $(t,\delta)\in\mathbb{R}\times B_0(\gamma)$. There exists a sub-sequence $\{e^{\delta\cdot x}U(-t)u_{k_j}(t)\}_{j=1}^{\infty}$ such that

$$\begin{aligned}(\mathbf{I}_{\Omega_\gamma^a}(t,\delta)e^{\delta\cdot x}U(-t)u_{k_j}(t),\varphi) &= (U(-t)u_{k_j}(t),\mathbf{I}_{\Omega_\gamma^a}(t,\delta)e^{\delta\cdot x}\varphi)\\ &\longrightarrow (U(-t)u(t),\mathbf{I}_{\Omega_\gamma^a}(t,\delta)e^{\delta\cdot x}\varphi),\ j\to\infty.\end{aligned}$$

Hence

$$(\mathbf{I}_{\Omega_\gamma^a}(t,\delta)A_\delta(t)u_{k_j}(t),\varphi)\longrightarrow(\mathbf{I}_{\Omega_\gamma^a}(t,\delta)A_\delta(t)u(t),\varphi),\ j\to\infty$$

for all $(t,\delta)\in\mathbb{R}\times B_0(\gamma)$ and all $\varphi\in C_0^\infty$, where $(\cdot,\cdot)$ is the scaler product in L^2. Therefore

$$\left\|\mathbf{I}_{\Omega_\gamma^a}(t,\delta)A_\delta(t)u(t)\right\|_{L^2}\le\liminf_{j\to\infty}\left\|\mathbf{I}_{\Omega_\gamma^a}(t,\delta)A_\delta(t)u_{k_j}(t)\right\|_{L^2}\le R$$

for all $(t,\delta)\in\mathbb{R}\times B_0(\gamma)$ and hence

$$\sup_{\delta\in B_0(\gamma)}\left\|\mathbf{I}_{\Omega_\gamma^a}(\delta)A_\delta u\right\|_X\le R.$$

□

The following lemma is the Strichartz estimate.

Lemma 3.2 ([2, 23, 30, 31]). *We have*

$$\left\|\int_0^{\cdot}U(\cdot-s)f(s)ds\right\|_X\le C\,\|f\|_{L^2\left(\mathbb{R};L^{2n/(n+2)}\right)}$$

and

$$\|U(\cdot)\phi\|_Y\le C\,\|\phi\|_{\dot H^{n/2-2}},\quad\left\|\int_0^{\cdot}U(\cdot-s)f(s)ds\right\|_Y\le C\,\|f\|_{L^2\left(\mathbb{R};\dot B^{n/2-2}_{2n/(n+2)}\right)}$$

with constant $C>0$.

We use the following lemma to estimate the nonlinear term in the homogeneous Besov spaces.

Lemma 3.3 ([1, 24]). *Let $s\ge0$ and let $1\le p,\nu\le\infty$ with $1/p=1/p_1+1/p_2$. Then*

$$\|uv\|_{\dot B^s_{p,\nu}}\le C_s\left(\|u\|_{\dot B^s_{p_1,\nu}}\|v\|_{L^{p_2}}+\|u\|_{L^{p_1}}\|v\|_{\dot B^s_{p_2,\nu}}\right),$$

with constant $C_s>0$.

The following lemma has an important role in the proof of main theorem.

Lemma 3.4 ([11]). *Let $a, \gamma, \varepsilon > 0$. Let $1 \le p \le q \le \infty$ and let $1 \le r \le \infty$. Then*

$$\sup_{\delta \in B_0(\gamma)} \left\| \mathbf{I}_{\Omega_\gamma^a}(\delta) e^{it\delta\cdot\nabla} u \right\|_{L^r(\mathbb{R};L^q)} \le C_{n,a,p,q,\varepsilon} \sup_{y \in B_0(a+\varepsilon)} \left\| e^{iy\cdot\nabla} u \right\|_{L^r(\mathbb{R};L^p)}$$

where the constant $C_{n,a,p,q,\varepsilon} > 0$ satisfying $\lim_{\varepsilon\to 0} C_{n,a,p,q,\varepsilon} = \infty$.

Proof. We see that $e^{it\delta\cdot\nabla}u(t,x) = u(t, x+it\delta)$, because $\widehat{u}(t)$ decays exponentially. Let $(t,\delta_0) \in \Omega_\gamma^a$ and $x_0 \in \mathbb{R}^n$. Let $R = \frac{\varepsilon}{2\sqrt{n}} > 0$ and put $z_0 = x_0 + it\delta_0$. By the mean-value theorem of harmonic functions, we have

$$u(t, x_0 + it\delta_0) = \frac{1}{(\pi R^2)^n} \int_{\prod_{j=1}^n \{|x_j + iy_j - z_{0,j}| < R\}} u(t, x+iy) dx dy.$$

Hence

$$\begin{aligned}
\|u(t, x_0 + it\delta_0)\|_{L^q_{x_0}} &= \frac{1}{(\pi R^2)^n} \left\| \int_{\prod_{j=1}^n \{|x_j + iy_j - z_{0,j}| < R\}} u(t, x+iy) dx dy \right\|_{L^q_{x_0}} \\
&\le \frac{1}{(\pi R^2)^n} \left\| \int_{\prod_{j=1}^n \{|x_j + iy_j - z_{0,j}| < R\}} |u(t, x+iy)| dx dy \right\|_{L^q_{x_0}} \\
&\le \frac{1}{(\pi R^2)^n} \left\| \int_{B_{x_0}(\varepsilon) \times B_{t\delta_0}(\varepsilon)} |u(t, x+iy)| dx dy \right\|_{L^q_{x_0}} \\
&\le \frac{1}{(\pi R^2)^n} \int_{B_{t\delta_0}(\varepsilon)} \left\| \int_{B_{x_0}(\varepsilon)} |u_y(t,x)| dx \right\|_{L^q_{x_0}} dy \\
&= \frac{1}{(\pi R^2)^n} \int_{B_{t\delta_0}(\varepsilon)} \left\| \int_{\mathbb{R}^n} \chi_{B_0(\varepsilon)}(x) |u_y(t, x_0 - x)| dx \right\|_{L^q_{x_0}} dy \\
&= \frac{1}{(\pi R^2)^n} \int_{B_{t\delta_0}(\varepsilon)} \left\| \chi_{B_0(\varepsilon)} * |u_y(t)| \right\|_{L^q} dy \\
&\le C \frac{1}{(\pi R^2)^n} \int_{B_{t\delta_0}(\varepsilon)} \left\| \chi_{B_0(\varepsilon)} \right\|_{L^{\tilde{p}}} \|u_y(t)\|_{L^p} dy \\
&\le C \varepsilon^{n\left(\frac{1}{q} - \frac{1}{p} - 1\right)} \int_{B_0(a+\varepsilon)} \|u_y(t)\|_{L^p} dy
\end{aligned}$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{\tilde{p}} - 1$ and $B_{t\delta_0}(\varepsilon) \subset B_0(a+\varepsilon)$ because $(t,\delta_0) \in \Omega_\gamma^a$, where we have used the notation $u_y(t,x) = u(t, x+iy)$. By taking L^r-norm, we have

$$\begin{aligned}
\|u(t, \cdot + it\delta_0)\|_{L^r(\mathbb{R};L^q)} &\le C \varepsilon^{n\left(\frac{1}{q} - \frac{1}{p} - 1\right)} \int_{B_0(a+\varepsilon)} \|u_y(t)\|_{L^r(\mathbb{R};L^p)} dy \\
&\le C \varepsilon^{n\left(\frac{1}{q} - \frac{1}{p} - 1\right)} (a+\varepsilon)^n \sup_{y \in B_0(a+\varepsilon)} \|u_y(t)\|_{L^r(\mathbb{R};L^p)} .
\end{aligned}$$

The right hand side of the above estimate does not depend on δ_0. Therefore, we have

$$\sup_{\delta\in B_0(\gamma)}\left\|\mathbf{I}_{\Omega_\gamma^a}(\delta)e^{it\delta\cdot\nabla}u\right\|_{L^r(\mathbb{R};L^q)}\leq C_{n,a,p,q,\varepsilon}\sup_{y\in B_0(a+\varepsilon)}\left\|e^{iy\cdot\nabla}u\right\|_{L^r(\mathbb{R};L^p)}$$

with $C_{n,a,p,q,\varepsilon}=C\varepsilon^{n\left(\frac{1}{q}-\frac{1}{p}-1\right)}(a+\varepsilon)^n>0$ satisfying $\lim_{\varepsilon\to 0}C\varepsilon^{n\left(\frac{1}{q}-\frac{1}{p}-1\right)}(a+\varepsilon)^n=\infty$. □

§ 4. Proof of Theorem 1

We show Theorem 1, by the contraction mapping theorem. We consider the map $\Phi : u\mapsto \Phi u$ defined by

$$(\Phi u)(t)=U(t)\phi-i\int_0^t U(t-s)\mathcal{N}(u,\overline{u})(s)ds,\ t\in\mathbb{R}$$

in the complete metric space $(Z_{r,R},d)$ defined as Lemma 3.1:

$$\begin{aligned}&Z_{r,R}=\left\{u\in Z;\ \|u\|_{G^{a,\gamma}X}\leq R,\ \|u\|_{A^{a+\varepsilon}Y}\leq r\right\},\\&d(u,v)=\|u-v\|_{X\bigcap A^{a+\varepsilon}Y}.\end{aligned}$$

We see that

$$\begin{aligned}&A_\delta U(\cdot)\phi=U(\cdot)e^{\delta\cdot x}\phi,\ e^{iy\cdot\nabla}U(\cdot)\phi=U(\cdot)e^{iy\cdot\nabla}\phi,\\&A_\delta u^2=(A_\delta u)e^{it\delta\cdot\nabla}u,\ A_\delta|u|^2=(A_\delta u)\overline{e^{-it\delta\cdot\nabla}u}\end{aligned}$$

and

$$e^{iy\cdot\nabla}u^2=(e^{iy\cdot\nabla}u)^2,\ e^{iy\cdot\nabla}|u|^2=(e^{iy\cdot\nabla}u)\overline{e^{-iy\cdot\nabla}u},$$

since $A_\delta(t)=M(t)e^{it\delta\cdot\nabla}M(-t)$, $t\neq 0$. Therefore

$$\mathbf{I}_{\Omega_\gamma^a}(\delta)A_\delta\Phi u=\mathbf{I}_{\Omega_\gamma^a}(\delta)\left(U(\cdot)e^{\delta\cdot x}\phi-i\int_0^\cdot U(\cdot-s)\left(\lambda A_\delta u(s)e^{is\delta\cdot\nabla}u(s)+\mu A_\delta u(s)\overline{e^{-is\delta\cdot\nabla}u(s)}\right)ds\right)$$

for $\delta\in B_0(\gamma)$ and

$$e^{iy\cdot\nabla}\Phi u=U(\cdot)e^{iy\cdot\nabla}\phi-i\int_0^\cdot U(\cdot-s)\left(\lambda\left(e^{iy\cdot\nabla}u(s)\right)^2+\mu e^{iy\cdot\nabla}u(s)\overline{e^{-iy\cdot\nabla}u(s)}\right)ds$$

for $y\in B_0(a+\varepsilon)$. By the Lemma 3.2, Lemma 3.3, the embedding $\dot{B}^{n/2-2}_{2n/(n-2)}\subset L^n$ and the Hölder estimate with $1/2=1/\infty+1/2$ and $(n+2)/2n=1/2+1/n$, we have

$$\begin{aligned}\|\Phi u\|_{A^{a+\varepsilon}Y}&\leq C\|\phi\|_{A^{a+\varepsilon}\dot{H}^{n/2-2}}+C\sup_{y\in B_0(a+\varepsilon)}\left\|\lambda\left(e^{iy\cdot\nabla}u\right)^2+\mu e^{iy\cdot\nabla}u\overline{e^{-iy\cdot\nabla}u}\right\|_{L^2\left(\mathbb{R};\dot{B}^{n/2-2}_{2n/(n+2)}\right)}\\&\leq C\|\phi\|_{A^{a+\varepsilon}\dot{H}^{n/2-2}}+C\|u\|^2_{A^{a+\varepsilon}Y}.\end{aligned}$$

Similarly by using Lemma 3.4, we have

$$\begin{aligned}
&\left\|\mathbf{I}_{\Omega_\gamma^a}(\delta)A_\delta \Phi u\right\|_{G^{a,\gamma}X}\\
&= \left\|U(\cdot)e^{\delta\cdot x}\phi - i\int_0^{\cdot} U(\cdot - s)\left(\lambda A_\delta u(s)e^{is\delta\cdot\nabla}u(s) + \mu A_\delta u(s)\overline{e^{-is\delta\cdot\nabla}u(s)}\right)ds\right\|_{L^\infty((-a/|\delta|,a/|\delta|);L^2)}\\
&\leq \|\phi\|_{G^\gamma L^2} + C_{n,a,p,q,\varepsilon}\|u\|_{G^{a,\gamma}X}\|u\|_{A^{a+\varepsilon}Y}
\end{aligned}$$

for $\delta \in B_0(\gamma)\setminus\{0\}$ and

$$\begin{aligned}
\left\|\mathbf{I}_{\Omega_\gamma^a}(0)A_0\Phi u\right\|_{G^{a,\gamma}X} &= \|\Phi u\|_X\\
&\leq \|\phi\|_{G^\gamma L^2} + C_{n,a,p,q,\varepsilon}\|u\|_{G^{a,\gamma}X}\|u\|_{A^{a+\varepsilon}Y}
\end{aligned}$$

for $\delta = 0$. By the above estimates, $\Phi : u \mapsto \Phi u$ is a contraction mapping in $(Z_{r,R}, d)$, if $\rho, r > 0$ and $R > 0$ satisfy the following conditions

$$\|\phi\|_{G^\gamma L^2} + CRr \leq R,\ C\rho + Cr^2 \leq r$$

and

$$Cr \leq \frac{1}{2}$$

where the universal constant $C > 0$.

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