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Remarks on scattering problem for a class of nonlinear Schrödinger equations

By

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Abstract

We study the scattering problem in the energy space for nonlinear Schrödinger equations with general nonlinearities. Employing the idea of Kenig and Merle [16], we show that any solution behaves asymptotically free as $t \to \pm\infty$.

§ 1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation:

$$i\frac{\partial \psi}{\partial t}(x,t) + \Delta\psi(x,t) + f(\psi(x,t)) = 0, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \tag{1.1}$$

where $i := \sqrt{-1}, d \geq 1, \psi$ is a complex-valued function on $\mathbb{R}^d \times \mathbb{R}, \Delta$ is the Laplace operator on $\mathbb{R}^d$ and $f : \mathbb{C} \to \mathbb{C}$ is a continuously differentiable function in $\mathbb{R}^2$-sense to be specified later. Our purpose is to show that any solution scatters in the energy space under suitable conditions on nonlinearities in the spirit of Kenig-Merle [16].

Now, we state our basic assumption on the nonlinearity f:

(N1) The origin is always a fixed point, that is,

$$f(0) = 0.$$

This allows (1.1) to have a trivial solution.

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(N2) We impose an H^1-subcritical growth condition such that there exist p_1 and p_2 with $2+\frac{4}{d}<p_1+1<p_2+1<2^*$, and a constant $C_f>0$, such that

$$\left|\frac{\partial f}{\partial z}(z)\right|+\left|\frac{\partial f}{\partial \bar{z}}(z)\right|\le C_f\left(|z|^{p_1-1}+|z|^{p_2-1}\right)\qquad \text{for any } z\in\mathbb{C},$$

where

$$\frac{\partial f}{\partial z}:=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right),\qquad \frac{\partial f}{\partial \bar{z}}:=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right), \tag{1.2}$$

and

$$2^*=\begin{cases}\infty & \text{if } d=1,2,\\ \frac{2d}{d-2} & \text{if } d\ge 3.\end{cases} \tag{1.3}$$

From this condition together with (N1), we have:

$$|f(z)|\le C_f(|z|^{p_1}+|z|^{p_2})\qquad \text{for any } z\in\mathbb{C}. \tag{1.4}$$

(N3) L^2-norm conservation law is guaranteed by

$$\Im f(z)\bar{z}=0.$$

(N4) Our equation (1.1) is governed by a Hamiltonian under the following condition that there exists a real-valued function $F\in C^2(\mathbb{C},\mathbb{R})$ such that

$$F(0)=0,\qquad 2\frac{\partial F}{\partial \bar{z}}=f.$$

In this case, the Hamiltonian $\mathcal{H}\colon H^1(\mathbb{R}^d)\longrightarrow\mathbb{R}$ is given by

$$\mathcal{H}(u):=\frac{1}{2}\|\nabla u\|_{L^2}^2-\int_{\mathbb{R}^d}F(u(x))\,dx\qquad \text{for } u\in H^1(\mathbb{R}^d). \tag{1.5}$$

For F, we have from (N1) and (N2) that

$$|F(z)|\le C_f(|z|^{p_1+1}+|z|^{p_2+1})\qquad \text{for any } z\in\mathbb{C}. \tag{1.6}$$

We associate the equation (1.1) with an initial datum from $H^1(\mathbb{R}^d)$ at $t=0$. We put:

$$\psi(\cdot,0)=\psi_0\in H^1(\mathbb{R}^d). \tag{1.7}$$

Here we summarize the basic properties of the Cauchy problem (1.1) and (1.7). Under the conditions (N1)–(N4), we know that for any $\psi_0\in H^1(\mathbb{R}^d)$, there exists a unique

solution ψ in $C(I_{\max}; H^1(\mathbb{R}^d))$ for some interval $I_{\max} = (-T^-_{\max}, T^+_{\max}) \subset \mathbb{R}$: maximal existence interval including 0. If $I_{\max} \subsetneqq \mathbb{R}$, then we have

$$\lim_{t \to * T^*_{\max}} \|\nabla \psi(t)\|_{L^2} = \infty \quad \text{(blowup)}, \tag{1.8}$$

provided that $T^*_{\max} < \infty$, where $*$ stands for $+$ or $-$. The solution ψ satisfies the following conservation laws of L^2-norm (or mass), the Hamiltonian (or energy), and the momentum in this order : for any $t \in I_{\max}$,

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}\,, \tag{1.9}$$

$$\mathcal{H}(\psi(t)) = \mathcal{H}(\psi_0), \tag{1.10}$$

$$\mathcal{P}(\psi(t)) := \Im \int_{\mathbb{R}^d} \nabla \psi(x,t) \overline{\psi(x,t)}\, dx = \mathcal{P}(\psi_0) \tag{1.11}$$

(see, e.g., [5, 10, 13, 14, 15, 21]). If, in addition, $\psi_0 \in L^2(\mathbb{R}^d, |x|^2 dx)$, then the corresponding solution ψ also belongs to $C(I_{\max}; L^2(\mathbb{R}^d, |x|^2 dx))$ and satisfies the so-called virial identity (see [10]):

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2 \left|\psi(x,t)\right|^2 dx = & \int_{\mathbb{R}^d} |x|^2 \left|\psi_0(x)\right|^2 dx + 2t\Im \int_{\mathbb{R}^d} x \cdot \nabla \psi_0(x) \overline{\psi_0(x)}\, dx \\ & + \int_0^t \int_0^{t'} \mathcal{K}(\psi(t''))\, dt'' dt' \qquad \text{for any } t \in I_{\max}, \end{aligned} \tag{1.12}$$

where

$$\mathcal{K}(u) := \|\nabla u\|^2_{L^2} - \frac{d}{2} \int_{\mathbb{R}^d} \left\{ f(u(x))\overline{u}(x) - 2F(u(x)) \right\} dx. \tag{1.13}$$

We note that the functional $\mathcal{K}$ is expressed by the Hamitonian $\mathcal{H}$ as follows:

$$\mathcal{K}(u) = \frac{d}{d\lambda} \bigg|_{\lambda=1} \mathcal{H}\left(T_\lambda u\right),$$

where $T_\lambda u(x) := \lambda^{\frac{d}{2}} u(\lambda x)$ $(\lambda > 0)$.

The virial identity (1.12) gives us a kind of propagation or concentration estimates for solutions. We emphasize that we treat any $H^1(\mathbb{R}^d)$ function as initial data, so that we need some modification on (1.12) by truncating the weight functions $|x|^2$ and x. Thus, in the course of our analysis in this paper, we shall employ a generalized version of the virial identity (see Appendix A below).

Besides the conditions (N1)–(N4) we assumed the following structural condition on the functional $\mathcal{K}$ in our previous result [2]:

(N5) For any $u \in H^1(\mathbb{R}^d) \setminus \{0\}$, there exists a unique number $\lambda(u) > 0$ such that

$$\mathcal{K}(T_\lambda u) \begin{cases} > 0 & \text{if} \quad 0 < \lambda < \lambda(u), \\ = 0 & \text{if} \quad \lambda = \lambda(u), \\ < 0 & \text{if} \quad \lambda(u) < \lambda. \end{cases}$$

We note that the focusing pure power nonlinearity $f(u) = |u|^{p-1}u$ satisfies the above condition. Under the above condition, it was shown in [2] that there exists a ground state for any frequency $\omega > 0$. Then, we define two invariant regions $\mathcal{A}_{\omega,+}$ and $\mathcal{A}_{\omega,-}$ for each $\omega > 0$ in $H^1(\mathbb{R}^d)$. We show that any solution starting from $\mathcal{A}_{\omega,+}$ behaves asymptotically free as $t \to \pm\infty$ and one from $\mathcal{A}_{\omega,-}$ blows up and grows up and the ground state belongs to $\overline{\mathcal{A}_{\omega,+}} \cap \overline{\mathcal{A}_{\omega,-}}$. Therefore, under the conditions, there exist three different kinds of solution (see also [1, 7, 8, 11, 12]).

Thus, we are interested in the case where the nonlinearity f does not satisfies the above condition. Besides the conditions (N1)–(N4), we assume the following condition:

(N6) For any $u \in H^1(\mathbb{R}^d)$, we have

$$\mathcal{K}(u) \geq 0.$$

We can easily find that the defocusing pure power nonlinearity $f(u) = -|u|^{p-1}u$ satisfies the condition (N5).

Now, we are in a position to state our main results. Our main theorem is the following:

Theorem 1.1 (Scattering). *Assume that $d \geq 1$ and* (N1)–(N4) *and* (N6). *Then, we have:*

If $\psi_0 \in H^1(\mathbb{R}^d)$, then the corresponding solution ψ exists globally in time, i.e., $I_{\max} = \mathbb{R}$. Furthermore, there exist $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t\to+\infty} \left\|\psi(t) - e^{it\Delta}\phi_+\right\|_{H^1} = \lim_{t\to-\infty} \left\|\psi(t) - e^{it\Delta}\phi_-\right\|_{H^1} = 0. \tag{1.14}$$

Using the Morawetz-type estimate, Nakanishi [19] showed the scattering result (1.14) in the energy space under the following condition:

$$\partial_{|z|}\left(\frac{F(z)}{|z|^2}\right) \leq 0. \tag{1.15}$$

In fact, we can show that the condition (1.15) is equivalent to (N6) under the conditions (N1)–(N4) (see [3, Section 2]). Thus, we can obtain Theorem 1.1 by applying the result of Nakanishi [19]. In this paper, we shall give an alternative proof of Theorem 1.1 by employing the variational method which was also used in our previous result [2]. The

advantage to employ the variational method is that we can treat not only the focusing nonlinearities but also the defocusing ones.

We now give an outline of the proof of Theorem 1.1. A key step to prove Theorem 1.1 is to show that any solution ψ belongs to an appropriate function space $X(\mathbb{R})$ over the space-time $\mathbb{R}^d \times \mathbb{R}$ (see (4.12) below) such that

$$\|\psi\|_{X(\mathbb{R})} < \infty. \tag{1.16}$$

To obtain such kind of boundedness, the Morawetz-type estimate is often employed (see [19, Lemma 2.6]). Here, instead of the Morawetz-type estimate, we can obtain the same conclusion following Kenig and Merle [16]. We sketch our strategy to prove (1.16) briefly. We show this by contradiction: Suppose the contrary that (1.16) fails to be valid. Then, we can find a solution called "critical element", which behaves like a one-soliton. However, with the aid of the fact that $\inf_{t>0} \mathcal{K}(\psi(t)) > 0$ for any solution ψ with $\psi(0) = \psi_0 \in H^1(\mathbb{R}^d)$ (see Theorem 3.1 below), the generalized virial identity does not allow a critical element to exist, which yields the contradiction. Therefore, we see that (1.16) holds.

This paper is organized as follows. In Section 2, we give several preliminary results. Especially, we shall show the coerciveness of the Hamiltonian functional. In Section 3, we obtain an uniform lower bound of the functional $\mathcal{K}$ of a solution. In Section 4, we recall a sufficient condition for the scattering. In Section 5, we give a proof of Theorem 1.1. Appendix A is devoted to an auxiliary result.

Notation.

1. Let A and B be two positive quantities. Then, the symbol $A \lesssim B$ means that there exists a constant $C > 0$, which depends only on d, p_1, p_2 and C_f (see (N2)), such that $A \leq CB$.

2. $|\nabla|^s$ and $\langle\nabla\rangle$ are the Fourier multiplier operators associated to the symbols $|\xi|^s$ and $\sqrt{1+|\xi|^2}$, respectively.

3. For $\lambda > 0$, we define a scaling operator T_λ by

$$(T_\lambda u)(x) := \lambda^{\frac{d}{2}} u(\lambda x) \quad \text{for a function } u \text{ on } \mathbb{R}^d. \tag{1.17}$$

4. For a point $Q \in [0,1] \times [0,1]$ with the coordinate $\left(\frac{1}{q}, \frac{1}{r}\right)$ and an interval I, $L(Q;I)$ denotes the Bochner space of $L^r(I, L^q(\mathbb{R}^d))$.

5. For $q \in [1,\infty]$, q' denotes the Hölder conjugate of q, i.e., $\frac{1}{q'} = 1 - \frac{1}{q}$.

6. For a point $Q = \left(\frac{1}{q}, \frac{1}{r}\right) \in [0,1] \times [0,1]$, we put $Q' := \left(\frac{1}{q'}, \frac{1}{r'}\right)$.

§ 2. Preliminaries

In this section, we give several preliminary results. We first define the action functional $\mathcal{S}$ on $H^1(\mathbb{R}^d)$ by

$$\mathcal{S}(u) := \frac{1}{2}\|u\|_{L^2}^2 + \mathcal{H}(u) \qquad \text{for } u \in H^1(\mathbb{R}^d). \tag{2.1}$$

We can easily verify that

$$\mathcal{K}(u) = \frac{d}{d\lambda}\bigg|_{\lambda=1} \mathcal{S}(T_\lambda u) = \frac{d}{d\lambda}\bigg|_{\lambda=1} \mathcal{H}(T_\lambda u) \qquad \text{for any } u \in H^1(\mathbb{R}^d). \tag{2.2}$$

Lemma 2.1. *Assume that $d \geq 1$ and* (N1)–(N4) *and* (N6)*. Then, we have the followings:*

(i) *For any function $u \in H^1(\mathbb{R}^d) \setminus \{0\}$, we have $\mathcal{K}(u) > 0$.*

(ii) *For any function $u \in H^1(\mathbb{R}^d) \setminus \{0\}$ and any $\lambda > 0$, it holds that*

$$\frac{d}{d\lambda}\mathcal{S}(T_\lambda u) = \frac{d}{d\lambda}\mathcal{H}(T_\lambda u) = \frac{1}{\lambda}\mathcal{K}(T_\lambda u). \tag{2.3}$$

(iii)

$$\mathcal{H}(T_\lambda u) > \mathcal{H}(T_{\lambda'} u) \qquad \textit{for } \lambda > \lambda'. \tag{2.4}$$

(iv) *For any function $u \in H^1(\mathbb{R}^d) \setminus \{0\}$, we have $\mathcal{H}(u) \geq \|\nabla u\|_{L^2}^2/2$.*

Proof. (i) Suppose that there exists $u_0 \in H^1(\mathbb{R}^d)\setminus\{0\}$ satisfying $\mathcal{K}(u_0) = 0$. Then, we put

$$a_0 := \|\nabla u_0\|_{L^2}^2, \qquad b_0 := \frac{d}{2}\int_{\mathbb{R}^d} \{f(u_0)\overline{u}_0 - 2F(u_0)\}\, dx. \tag{2.5}$$

Since $\mathcal{K}(u_0) = 0$, we obtain $a_0 = b_0$. It follows from the positivity of a_0 that

$$\mathcal{K}(u_0(\frac{1}{2}\cdot)) = 2^d\left\{\frac{a_0}{4} - b_0\right\} = 2^d \times (-\frac{3}{4}a_0) < 0, \tag{2.6}$$

which contradicts the condition (N6). Thus, we see that (i) holds.

(ii) is clear. (iii) was proved in [2, Lemma 2.1 (ii)].

(iv) First, we shall show that

$$\mathcal{H}(u) > 0 \qquad \text{for all } u \in H^1(\mathbb{R}^d) \setminus \{0\}. \tag{2.7}$$

It follows from (1.6) that for any $\varepsilon > 0$, there exists a sufficiently small $L > 0$ such that

$$|F(s)| \le \varepsilon s^{2+\frac{4}{d}} \qquad \text{for } 0 < s < L. \tag{2.8}$$

By (1.6) and (2.8), we have, for $0 < s < 1$, that

$$\begin{aligned}
\mathcal{H}(u) &\ge \mathcal{H}(T_s u) \\
&\ge \frac{s^2}{2}\|\nabla u\|_{L^2}^2 - s^{-d}\int_{\{s^{d/2}u \le L\}} F(s^{d/2}u)dx - s^{-d}\int_{\{s^{d/2}u > L\}} F(s^{d/2}u)dx \\
&\ge \frac{s^2}{2}\|\nabla u\|_{L^2}^2 - s^{-d}\varepsilon\int_{\{s^{d/2}u \le L\}} |s^{d/2}u|^{2+\frac{4}{d}}dx \\
&\qquad - s^{-d}C_f\int_{\{s^{d/2}u > L\}} \left\{|s^{d/2}u|^{2+\frac{4}{d}} + |s^{d/2}u|^{p_2+1}\right\}dx \\
&\ge \frac{s^2}{2}\|\nabla u\|_{L^2}^2 - \varepsilon s^2\|u\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} - C_f s^2\int_{\{s^{d/2}u > L\}} |u|^{2+\frac{4}{d}}dx \\
&\quad - C_f s^{\frac{d}{2}(p_2+1)-d}\int_{\{s^{d/2}u > L\}} |u|^{p_2+1}dx.
\end{aligned} \tag{2.9}$$

For each $\varepsilon > 0$, we can take $s > 0$ sufficiently small so that

$$\varepsilon > C_f\int_{\{s^{d/2}u > L\}} |u|^{2+\frac{4}{d}}dx + C_f s^{\frac{d}{2}(p_2+1)-d-2}\int_{\{s^{d/2}u > L\}} |u|^{p_2+1}dx. \tag{2.10}$$

This together with (2.9) gives us that

$$\mathcal{H}(u) > s^2\left\{\frac{1}{2}\|\nabla u\|_{L^2}^2 - \varepsilon\|u\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} - \varepsilon\right\} > 0 \tag{2.11}$$

for sufficiently small $\varepsilon > 0$ and $s > 0$. Thus, (2.7) holds.

Next, we claim that

$$\int_{\mathbb{R}^d} F(u)dx < 0 \qquad \text{for any } u \in H^1(\mathbb{R}^d)\setminus\{0\}. \tag{2.12}$$

Suppose that there exists $u_1 \in H^1(\mathbb{R}^d)\setminus\{0\}$ such that $\int_{\mathbb{R}^d} F(u_1)dx \ge 0$. For each $\lambda > 0$, we set $u_\lambda(\cdot) = u_1(\lambda\cdot)$. Then, it follows from (2.7) that $\mathcal{H}(u_\lambda) > 0$ for any $\lambda > 0$. On the other hand, we can take $\lambda > 0$ so that

$$\lambda^2\|\nabla u_1\|_{L^2}^2 < \frac{1}{2}\int_{\mathbb{R}^d} F(u_1)dx. \tag{2.13}$$

This yields that

$$\mathcal{H}(u_\lambda) = \lambda^{-d}\left\{\frac{\lambda^2}{2}\|\nabla u_1\|_{L^2}^2 - \int_{\mathbb{R}^d} F(u_1)dx\right\} \le -\frac{\lambda^{-d}}{2}\int_{\mathbb{R}^d} F(u_1)dx \le 0, \tag{2.14}$$

which is a contradiction. Thus, our claim (2.12) holds. (iv) immediately follows from (2.12). □

Lemma 2.2. *Assume that $d \geq 1$, and* (N1)–(N4) *and* (N6). *Let $\psi_0 \in H^1(\mathbb{R}^d)$ and let ψ be the corresponding solution to* (1.1) *with $\psi|_{t=0} = \psi_0$. Then, ψ exists globally and there exists a constant $K > 0$ such that*

$$\sup_{t\in\mathbb{R}} \|\nabla\psi(t)\|_{L^2} \leq K. \tag{2.15}$$

Proof. It follows from the conservation laws (1.10) and Lemma 2.1 (iv) that

$$\frac{1}{2}\|\nabla\psi(t)\|_{L^2}^2 \leq \mathcal{H}(\psi(t)) = \mathcal{H}(\psi_0), \tag{2.16}$$

which yields (2.15). This completes the proof. □

§ 3. Uniform lower bound of the functional $\mathcal{K}$

In this section, we give an important property of solutions. We obtain an uniform lower bound of the functional $\mathcal{K}$ of a solution starting from any initial data.

Theorem 3.1. *Assume that $d \geq 1$, and* (N1)–(N4) *and* (N6). *Let $\psi_0 \in H^1(\mathbb{R}^d)$, and let ψ be the corresponding solution to (1.1). Then, we have that*

$$\inf_{t\in\mathbb{R}} \mathcal{K}(\psi(t)) > 0. \tag{3.1}$$

In order to prove Theorem 3.1, we prepare the following Brezis-Lieb [4] type lemma.

Lemma 3.2. *Let $u \in H^1(\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that*

$$\lim_{n\to\infty} u_n = u \qquad \text{weakly in } H^1(\mathbb{R}^d). \tag{3.2}$$

Then, passing to some subsequence, we have that

$$\lim_{n\to\infty} u_n = u \qquad \text{almost everywhere in } \mathbb{R}^d, \tag{3.3}$$

and that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left|F(u_n(x))\,dx - F(u_n(x) - u(x)) - F(u(x))\right| dx = 0, \tag{3.4}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Big|f(u_n(x))\overline{u_n(x)} - f(u_n(x) - u(x))(\overline{u_n(x) - u(x)}) - f(u(x))\overline{u(x)}\Big|\,dx = 0. \tag{3.5}$$

See page 1227 of [6] for the proof of Lemma 3.2.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We suppose the contrary that (3.1) fails. Then, there exists a sequence $\{t_n\}$ in $\mathbb{R}$ such that

$$\lim_{n\to\infty} t_n \in \{\pm\infty\}, \tag{3.6}$$

$$\lim_{n\to\infty} \mathcal{K}(\psi(t_n)) = 0. \tag{3.7}$$

Now, we suppose a dispersive situation

$$\liminf_{n\to\infty} \|\psi(t_n)\|_{L^{2+\frac{4}{d}}} = 0. \tag{3.8}$$

Then, passing to some subsequence, we see from (2.15) that

$$\lim_{n\to\infty} \|\psi(t_n)\|_{L^{p_1+1}} = \lim_{n\to\infty} \|\psi(t_n)\|_{L^{p_2+1}} = 0. \tag{3.9}$$

Combining (1.6) with (3.9), we obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |F(\psi(x,t_n))|\, dx = 0. \tag{3.10}$$

Similarly, we find that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left| f(\psi(x,t_n))\overline{\psi(x,t_n)} \right| dx = 0. \tag{3.11}$$

Using Lemma 2.1 (iv) and (3.10), we have that

(3.12)

$$\begin{aligned} 0 < 2\mathcal{H}(\psi_0) = \lim_{n\to\infty} 2\mathcal{H}(\psi(t_n)) &\le \limsup_{n\to\infty} \|\nabla\psi(t_n)\|_{L^2}^2 - \lim_{n\to\infty} \int_{\mathbb{R}^d} 2F(\psi(x,t_n))\, dx \\ &= \limsup_{n\to\infty} \|\nabla\psi(t_n)\|_{L^2}^2 . \end{aligned}$$

On the other hand, (3.7) together with (3.10) and (3.11), gives us that

$$\begin{aligned} &\limsup_{n\to\infty} \|\nabla\psi(t_n)\|_{L^2}^2 \\ &= \limsup_{n\to\infty} \|\nabla\psi(t_n)\|_{L^2}^2 - \lim_{n\to\infty} \frac{d}{2} \int_{\mathbb{R}^d} \left\{ f(\psi(x,t_n))\overline{\psi(x,t_n)} - 2F(\psi(x,t_n)) \right\} dx \\ &= \limsup_{n\to\infty} \mathcal{K}(\psi(t_n)) = 0, \end{aligned} \tag{3.13}$$

which contradicts (3.12). Thus, the dispersive situation (3.8) never happens and therefore, we have that

$$\liminf_{n\to\infty} \|\psi(t_n)\|_{L^{2+\frac{4}{d}}} > 0. \tag{3.14}$$

Then, combining with the result, we find by Lemma 2.1 of [9] and Lemma 6 of [17] that there exists a subsequence of $\{\psi(t_n)\}$ (still denoted by the same symbol) with the following property: There exists a sequence $\{y_n\}$ in $\mathbb{R}^d$, and a non-trivial function $u_\infty \in H^1(\mathbb{R}^d)$ such that, putting $u_n(x) := \psi(x + y_n, t_n)$, we have that

$$\lim_{n\to\infty} u_n = u_\infty \qquad \text{weakly in } H^1(\mathbb{R}^d), \tag{3.15}$$

$$\lim_{n\to\infty} u_n = u_\infty \qquad \text{almost everywhere in } \mathbb{R}^d, \tag{3.16}$$

$$\sup_{n\in\mathbb{N}} \|u_n\|_{H^1} < K_1, \tag{3.17}$$

$$\inf_{n\in\mathbb{N}} \mathcal{L}^d\left([|u_n| > \eta]\right) \geq C \qquad \text{for some constants } \eta > 0 \text{ and } C > 0. \tag{3.18}$$

Moreover, it follows from Lemma 3.2 that passing to subsequence, we have

$$\lim_{n\to\infty} \{\mathcal{K}(u_n) - \mathcal{K}(u_n - u_\infty) - \mathcal{K}(u_\infty)\} = 0. \tag{3.19}$$

In particular, we can verify that

$$\lim_{n\to\infty} \mathcal{K}(u_n - u_\infty) = -\mathcal{K}(u_\infty) \qquad \text{(by (3.7))}. \tag{3.20}$$

It follows from Lemma 2.1 (i) that

$$\mathcal{K}(u_\infty) > 0.$$

Then, (3.20) shows that, extracting some subsequence of $\{u_n\}$ (still denoted by the same symbol), we have that

$$\mathcal{K}(u_n - u_\infty) \leq -\frac{1}{2}\mathcal{K}(u_\infty) < 0 \qquad \text{for any } n \in \mathbb{N}, \tag{3.21}$$

which is a contradiction. Hence, Theorem 3.1 holds. □

§ 4. Sufficient condition for scattering

In this section, we shall give a sufficient condition for a solution to scatter in $H^1(\mathbb{R}^d)$. As a matter of fact, the condition is same as that of our previous paper [2, Section 4]. However, for the sake of completeness, we recall it.

To this end, we first introduce Strichartz-type spaces which are convenient to control the long-time behavior of a solution to (1.1).

We note the integral equation associated with (1.1):

$$\psi(t) = e^{it\Delta}\psi(t_0) + i\int_{t_0}^{t} e^{i(t-t')\Delta} f(\psi(t'))dt'. \tag{4.1}$$

We also recall the Strichartz estimates: For any interval I and any admissible pairs[1] (q,r) and (q_1,r_1), we have

$$\left\|e^{it\Delta}u\right\|_{L^r(I,L^q)} \lesssim \|u\|_{L^2}\,, \tag{4.2}$$

$$\left\|\int_0^t e^{i(t-t')\Delta}v(t')\,dt'\right\|_{L^r(I,L^q)} \lesssim \|v\|_{L^{r_1'}(I,L^{q_1'})}\,, \tag{4.3}$$

where the implicit constants are independent of I.

Let p_1 and p_2 be numbers found in the assumption (N2), and fix a number σ satisfying $p_2+1<\sigma<2^*$. For $p>1$, we put

$$s_p := \frac{d}{2}-\frac{2}{p-1}. \tag{4.4}$$

It is easy to verify that $0<s_{p_1}\le s_{p_2}<1$.

We define $Q_0=\left(\frac{1}{q_0},\frac{1}{r_0}\right)$ by

$$q_0:=\sigma, \qquad \frac{1}{r_0}:=\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\sigma}\right), \tag{4.5}$$

so that the pair (q_0,r_0) is admissible. For any interval I, we define a usual Strichartz space $S(I)$ by

$$S(I):=L^\infty(I,L^2(\mathbb{R}^d))\cap L(Q_0;I). \tag{4.6}$$

Next, we shall introduce a Strichartz-type space $X(I)$. Define $Q_1=\left(\frac{1}{q_1},\frac{1}{r_1}\right)$ and $\widetilde{Q}_1=\left(\frac{1}{\widetilde{q}_1},\frac{1}{\widetilde{r}_1}\right)$ as points that

$$q_1:=\sigma, \qquad \frac{1}{r_1}:=\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\sigma}-\frac{s_{p_1}}{d}\right), \tag{4.7}$$

$$\frac{p_1-1}{\widetilde{q}_1}=1-\frac{2}{\sigma}, \qquad \frac{1}{\widetilde{r}_1}:=\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\widetilde{q}_1}-\frac{s_{p_1}}{d}\right). \tag{4.8}$$

We also define two points $Q_2=\left(\frac{1}{q_2},\frac{1}{r_2}\right)$ and $\widetilde{Q}_2=\left(\frac{1}{\widetilde{q}_2},\frac{1}{\widetilde{r}_2}\right)$ by

$$q_2:=\sigma, \qquad \frac{1}{r_2}:=\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\sigma}-\frac{s_{p_2}}{d}\right), \tag{4.9}$$

$$\frac{p_2-1}{\widetilde{q}_2}=1-\frac{2}{\sigma}, \qquad \frac{1}{\widetilde{r}_2}:=\frac{d}{2}\left(\frac{1}{2}-\frac{1}{\widetilde{q}_2}-\frac{s_{p_2}}{d}\right). \tag{4.10}$$

Besides these points, we define Q_1^* and Q_2^* by

$$Q_1^*:=Q_1+(p_1-1)\widetilde{Q}_1, \qquad Q_2^*:=Q_2+(p_2-2)\widetilde{Q}_2. \tag{4.11}$$

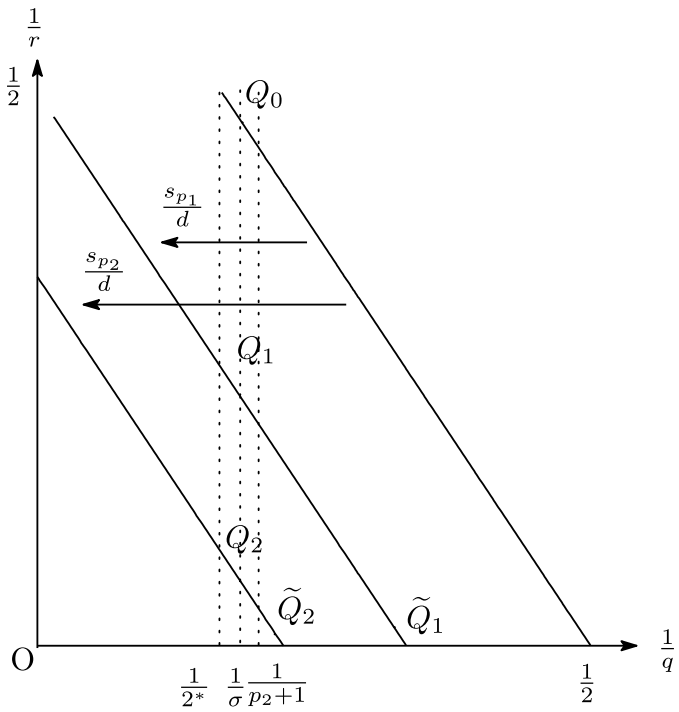

Figure 1. Strichartz-type spaces

We finally define our Strichartz-type space $X(I)$ by

$$X(I) := X_1(I) \cap X_2(I), \tag{4.12}$$

where

$$X_j(I) := L(Q_j; I) \cap L(\widetilde{Q}_j; I), \qquad j = 1, 2. \tag{4.13}$$

Then, we see from the Sobolev embedding and the Strichartz estimate (4.2) that

$$\|e^{it\Delta}u\|_{X(\mathbb{R})} \lesssim \||\nabla|^{s_{p_1}}u\|_{L^2} + \||\nabla|^{s_{p_2}}u\|_{L^2}. \tag{4.14}$$

Moreover, we can prove the following proposition in a way similar to [11].

Proposition 4.1. *Assume that $d \geq 1$ and* (N1)–(N4)*. Let ψ be a global solution to (1.1). Suppose that*

$$\|\psi\|_{L^\infty(\mathbb{R},H^1)}, \ \|\psi\|_{X(\mathbb{R})} < \infty. \tag{4.15}$$

Then, there exist functions $\phi_+, \phi_- \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|\psi(t) - e^{it\Delta}\phi_\pm\|_{H^1} = 0. \tag{4.16}$$

[1] A pair (q, r) is said to be admissible, if $2 \leq q < 2^*$ and $\frac{1}{r} = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$.

The following proposition gives us a sufficient condition for a solution to be bounded in $X(I)$ and $\langle\nabla\rangle^{-1}S(I)$ for any given interval I:

Proposition 4.2 (Small data theory). *Assume that $d \geq 1$, and* (N1)–(N4). *Let I be an interval, and let $t_0 \in I$. Then, for any $A > 0$, there exists $\delta > 0$ with the following property: For any $\psi_0 \in H^1(\mathbb{R}^d)$ with*

$$\|\psi_0\|_{H^1} \leq A, \tag{4.17}$$

$$\left\|e^{i(t-t_0)\Delta}\psi_0\right\|_{X(I)} \leq \delta, \tag{4.18}$$

there exists a unique solution ψ in $C(I, H^1(\mathbb{R}^d))$ to (1.1) *with $\psi(t_0) = \psi_0$ such that*

$$\|\psi\|_{X(I)} < 2\left\|e^{i(t-t_0)\Delta}\psi_0\right\|_{X(I)}, \tag{4.19}$$

$$\|\langle\nabla\rangle\psi\|_{S(I)} \lesssim \|\psi_0\|_{H^1}, \tag{4.20}$$

where the implicit constant is independent of t_0 and I.

This proposition is essentially known (see e.g. [11, Proposition 2.1]). Hence, we omit the proof.

Propositions 4.1 and 4.2 together with the inequality (4.14) immediately show the conclusion.

Corollary 4.3. *There exists $\varepsilon > 0$ such that for any $\psi_0 \in H^1(\mathbb{R}^d)$ with $\|\psi_0\|_{H^1} < \varepsilon$, the corresponding solution ψ to* (1.1) *satisfies*

$$\|\psi\|_{X(\mathbb{R})}, \ \|\langle\nabla\rangle\psi\|_{S(\mathbb{R})} \lesssim \varepsilon, \tag{4.21}$$

and has asymptotic states in $H^1(\mathbb{R}^d)$ at $\pm\infty$.

§ 5. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In view of Lemma 2.2 and Proposition 4.1, it is sufficient to show that any solution ψ to (1.1) satisfies $\|\psi\|_{X(\mathbb{R})} < \infty$. To this end, we introduce the set

$$A_+(m) := \left\{u \in H^1(\mathbb{R}^d) \colon \mathcal{S}(u) < m\right\} \qquad \text{for } m > 0. \tag{5.1}$$

Moreover, we define a number m^* by

$$\begin{aligned} m^* &:= \sup\left\{m > 0 \colon \|\psi\|_{X(\mathbb{R})} < \infty \quad \text{for all } \psi_0 \in A_+(m)\right\} \\ &= \inf\left\{m > 0 \colon \|\psi\|_{X(\mathbb{R})} = \infty \quad \text{for some } \psi_0 \in A_+(m)\right\}, \end{aligned} \tag{5.2}$$

where ψ denotes the solution to (1.1) with $\psi(0) = \psi_0$.

Hence, our task is to show that

$$m^* = \infty. \tag{5.3}$$

Here, we can derive $m^* > 0$ from our small data theory (Proposition 4.2); Indeed, it follows from Lemma 2.1 (iv) that $\|u\|_{H^1}^2/2 \le \mathcal{S}(u)$. Therefore, for any $\psi_0 \in A_+(m)$ with $m \le m^*$ and $0 < s < 1$, we have

$$\||\nabla|^s \psi_0\|_{L^2} \le \|\psi_0\|_{L^2}^{1-s} \|\nabla \psi_0\|_{L^2}^s \lesssim m \to 0 \qquad \text{as } m \to 0, \tag{5.4}$$

which together with the inequality (4.14) shows that

$$\left\|e^{it\Delta}\psi_0\right\|_{X(\mathbb{R})} \to 0 \qquad \text{as } m \to 0. \tag{5.5}$$

Hence, our small data theory shows $m^* > 0$.

We shall show (5.3) by contradiction. Suppose the contrary that $m^* < \infty$. In this undesirable situation, we can find the so-called critical element:

Proposition 5.1 (Existence of a critical element in $H^1(\mathbb{R}^d)$)**.** *Suppose that $m^* < \infty$. Then, there exists a global solution $\Psi \in C(\mathbb{R}, H^1(\mathbb{R}^d))$ to* (1.1) *with the following property:*

$$\|\Psi\|_{X(\mathbb{R})} = \infty, \tag{5.6}$$

$$\mathcal{S}(\Psi(t)) = m^* \quad \textit{for any } t \in \mathbb{R}; \tag{5.7}$$

(ii)

$$\Im \int_{\mathbb{R}^d} \overline{\Psi}(x,t) \nabla \Psi(x,t)\, dx = 0 \qquad \textit{for any } t \in \mathbb{R}; \tag{5.8}$$

(iii) $\{\Psi(t)\}_{t \ge 0}$ *is tight in $H^1(\mathbb{R}^d)$ in the following sense: For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ and a continuous path $\gamma_\varepsilon \in C([0,+\infty), \mathbb{R}^d)$ with $\gamma_\varepsilon(0) = 0$ such that*

$$\int_{|x-\gamma_\varepsilon(t)|<R_\varepsilon} |\Psi(x,t)|^2\, dx > \|\Psi(0)\|_{L^2}^2 - \varepsilon \quad \textit{for any } t \in [0,+\infty) \tag{5.9}$$

and

$$\int_{|x-\gamma_\varepsilon(t)|<R_\varepsilon} |\nabla\Psi(x,t)|^2\, dx > \|\nabla\Psi(t)\|_{L^2}^2 - \varepsilon \quad \textit{for any } t \in [0,+\infty). \tag{5.10}$$

We can prove this proposition by the same argument of Proposition 5.1 in [2]. Thus, we omit it.

Since the momentum of the critical element is zero, we can expect that its center of mass does not move. Unfortunately, we just work in $H^1(\mathbb{R}^d)$ and therefore the notion of center of mass no longer has the meaning. However, we can find:

Lemma 5.2 (Almost center of mass). *Let Ψ be a global solution to* (1.1) *found in Proposition 5.1, and let R_ε denote the radius found in Proposition 5.1 for each $\varepsilon > 0$. We define the "almost center of mass" by*

$$\gamma^{ac}_{\varepsilon,R}(t) := \int_{\mathbb{R}^d} \nabla W_{20R}(x) \frac{|\Psi(x,t)|^2}{\|\Psi(0)\|^2_{L^2}}\, dx \tag{5.11}$$

for any $\varepsilon \in (0, \frac{1}{100})$ and $R > R_\varepsilon$, where W_R is the function defined by (Appendix A.7). Then, we have

$$\gamma^{ac}_{\varepsilon,R} \in C^1([0,\infty), \mathbb{R}^d). \tag{5.12}$$

Furthermore, there exists a constant $\alpha > 0$, depending only on d, p_1 and p_2, such that

$$\left|\gamma^{ac}_{\varepsilon,R}(t)\right| \le 20R, \tag{5.13}$$

$$\int_{|x-\gamma^{ac}_{\varepsilon,R}(t)|\le 4R} |\Psi(x,t)|^2 + |\nabla\Psi(x,t)|^2\, dx \ge \|\Psi(t)\|^2_{H^1} - \varepsilon \tag{5.14}$$

for any $t \in \left[0, \alpha\frac{R}{\sqrt{\varepsilon}}\right]$.

The proof of Lemma 5.2 is the same as that of Lemma 4.2 in [1]. Hence, we omit it.

Now, we give the proof of Theorem 1.1 (i):

Proof of Theorem 1.1 (i). The generalized virial identity (Appendix A.14) together with Lemma 3.1 and (Appendix A.10) yields that

$$\begin{aligned}
&\int_{\mathbb{R}^d} W_R|\Psi(t)|^2 \\
&\ge \int_{\mathbb{R}^d} W_R|\Psi(0)|^2 + t\Im\int_{\mathbb{R}^d} \nabla W_R\cdot\nabla\Psi(0)\overline{\Psi(0)} + C_1t^2 \\
&\qquad - C_2\int_0^t\int_0^{t'}\int_{|x|\ge R} |\nabla\Psi(x,t'')|^2\, dx\, dt''dt' \\
&\qquad - C_3\int_0^t\int_0^{t'}\int_{|x|\ge R} |\Psi(x,t'')|^{p_1+1} + |\Psi(x,t'')|^{p_2+1}\, dx\, dt'dt'' \\
&\qquad - C_4\frac{t^2}{R^2}\|\Psi(0)\|^2_{L^2} \qquad\qquad \text{for any } R > 0,
\end{aligned} \tag{5.15}$$

where $C_1 := \dfrac{1}{2}\inf_{t\in\mathbb{R}} \mathcal{K}(\Psi(t)) > 0$, and C_2, C_3, C_4 are some constants depending only on d, p_1, p_2, C_f and $\|\rho\|_{W^{2,\infty}}$. Using Lemma 5.2, for any $\varepsilon \in (0, \frac{1}{100})$, we can take $R_\varepsilon > 0$

with the property that for any $R \geq R_\varepsilon$ there exists $\gamma^{ac}_{\varepsilon,R} \in C^1([0,\infty),\mathbb{R}^d)$ such that

$$\left|\gamma^{ac}_{\varepsilon,R}(t)\right| \leq 20R, \tag{5.16}$$

$$\int_{|x-\gamma^{ac}_{\varepsilon,R}(t)|\geq 4R} |\nabla\Psi(x,t)|^2\,dx < \varepsilon, \tag{5.17}$$

$$\int_{|x-\gamma^{ac}_{\varepsilon,R}(t)|\geq 4R} |\Psi(x,t)|^{p_1+1} + |\Psi(x,t)|^{p_2+1}\,dx < \varepsilon \tag{5.18}$$

for any $t \in \left[\,0,\ \alpha\frac{R}{\sqrt{\varepsilon}}\,\right]$, where α is some constant depending only on p_1, p_2 and d. We see from (5.16) that

$$|x-\gamma^{ac}_{\varepsilon,R}(t)| \geq 4R \quad \text{for any } x \in \mathbb{R}^d \text{ with } |x| \geq 24R \text{ and } t \in \left[\,0,\ \alpha\frac{R}{\sqrt{\varepsilon}}\,\right]. \tag{5.19}$$

Hence, (5.15) together with (5.17) and (5.18) shows that

$$\begin{aligned}
&\int_{\mathbb{R}^d} W_{50R}|\Psi(t)|^2 \\
&\geq \int_{\mathbb{R}^d} W_{50R}|\Psi(0)|^2 + t\Im\int_{\mathbb{R}^d} \nabla W_{50R}\cdot\nabla\Psi(0)\overline{\Psi(0)} + C_1t^2 - (C_2+C_3)\varepsilon t^2 \\
&\qquad - C_4\frac{t^2}{(50R)^2}\left\|\Psi(0)\right\|^2_{L^2} \qquad \text{for any } R \geq R_\varepsilon \text{ and } t \in \left[\,0,\ \alpha\frac{R}{\sqrt{\varepsilon}}\,\right].
\end{aligned} \tag{5.20}$$

Choosing ε and R satisfying

$$\varepsilon < \min\left\{\frac{1}{100},\ \frac{C_1}{4(C_2+C_3)}\right\}, \quad R \geq \max\left\{R_\varepsilon,\ \left\|\Psi(0)\right\|_{L^2}\sqrt{\frac{C_4}{C_1}}\right\}, \tag{5.21}$$

we obtain from (5.20) that

$$\int_{\mathbb{R}^d} W_{50R}|\Psi(t)|^2 \geq \int_{\mathbb{R}^d} W_{50R}|\Psi(0)|^2 + t\Im\int_{\mathbb{R}^d} \nabla W_{50R}\cdot\nabla\Psi(0)\overline{\Psi(0)} + \frac{C_1}{2}t^2 \tag{5.22}$$

for any $t \in \left[\,0,\ \alpha\frac{R}{\sqrt{\varepsilon}}\,\right]$. Moreover, choosing $t = \alpha\frac{R}{\sqrt{\varepsilon}}$ in (5.22), we see from (Appendix A.8) and (Appendix A.9) that

$$(50R)^2\left\|\Psi(0)\right\|^2_{L^2} \gtrsim -(50R)^2\left\|\Psi(0)\right\|^2_{L^2} - 50\frac{\alpha}{\sqrt{\varepsilon}}R^2\left\|\Psi(0)\right\|_{L^2}\left\|\nabla\Psi(0)\right\|_{L^2} + \frac{C_1}{2}\alpha^2\frac{R^2}{\varepsilon}. \tag{5.23}$$

However, this leads us to a contradiction when ε tends to 0. Thus, we find $m^* = \infty$, so that the proof of Theorem 1.1 is completed. □

§ Appendix A. Generalized virial identity

In this appendix, we recall the generalized virial identity, which was used in [2]. Let ρ be a smooth function on $\mathbb{R}$ such that

(Appendix A.1) $$\rho(x) = \rho(4-x) \qquad \text{for all } x \in \mathbb{R},$$

(Appendix A.2) $$\rho(x) \geq 0 \qquad \text{for all } x \in \mathbb{R},$$

(Appendix A.3) $$\int_{\mathbb{R}} \rho(x)\, dx = 1,$$

(Appendix A.4) $$\operatorname{supp} \rho \subset (1,3),$$

(Appendix A.5) $$\rho'(x) \geq 0 \qquad \text{for all } x < 2.$$

We put

(Appendix A.6) $$w(r) := r - \int_0^r (r-s)\rho(s)\, ds \qquad \text{for } r \geq 0,$$

(Appendix A.7) $$W_R(x) := R^2 w\left(\frac{|x|^2}{R^2}\right) \qquad \text{for } x \in \mathbb{R}^d \text{ and } R > 0.$$

We easily verify that

(Appendix A.8) $$\|W_R\|_{L^\infty} \lesssim R^2,$$

(Appendix A.9) $$\|\nabla W_R\|_{L^\infty} \lesssim R,$$

(Appendix A.10) $$\left\|\Delta^2 W_R\right\|_{L^\infty} \lesssim \frac{1}{R^2} \|\rho\|_{W^{2,\infty}(\mathbb{R})}.$$

(Appendix A.11) $$|\nabla W_R(x)|^2 \leq 4 W_R(x) \qquad \text{for any } R > 0 \text{ and } x \in \mathbb{R}^d.$$

For a sufficiently regular solution ψ of the equation (1.1), we verify that

(Appendix A.12)

$$\frac{d}{dt}\int_{\mathbb{R}^d} W_R|\psi(t)|^2 = \Im\int_{\mathbb{R}^d} \nabla W_R\cdot\nabla\psi(t)\overline{\psi(t)},$$

(Appendix A.13)

$$\begin{aligned}\frac{d^2}{dt^2}\int_{\mathbb{R}^d} W_R|\psi(t)|^2 &= \mathcal{K}(\psi(t)) - \int_{\mathbb{R}^d}(2-H(W_R))\,\nabla\psi(t)\cdot\overline{\nabla\psi(t)}\\ &\quad+\int_{\mathbb{R}^d}\left(d-\frac{1}{2}\Delta W_R\right)\left\{f(\psi(t))\overline{\psi(t)}-F(\psi(t))\right\}\\ &\quad-\frac{1}{4}\int_{\mathbb{R}^d}\Delta^2 W_R|\psi(t)|^2,\end{aligned}$$

where $H(W_R)$ denotes the Hessian of W_R. These identities (Appendix A.12) and (Appendix A.13) together with a regularization argument, give us the following generalized version of the virial identity(cf. [18, 20]):

Lemma Appendix A.1 (Generalized virial identity and center of mass)**.** *Assume that $d\geq 1$. Let $\psi_0\in H^1(\mathbb{R}^d)$, and let ψ be the solution of the equation (1.1) with $\psi(t_0)=\psi_0$ for some $t_0\in I_{\max}$. Then, we have* (i) *the generalized virial identity:*

(Appendix A.14)

$$\begin{aligned}&\int_{\mathbb{R}^d} W_R|\psi(t)|^2\,dx\\ &=\int_{\mathbb{R}^d} W_R|\psi_0|^2\,dx+(t-t_0)\Im\int_{\mathbb{R}^d}\nabla W_R\cdot\nabla\psi_0\,\overline{\psi_0}\,dx+\int_{t_0}^{t}\int_{t_0}^{t'}\mathcal{K}(\psi(t''))\,dt''dt'\\ &\quad-\int_{t_0}^{t}\int_{t_0}^{t'}\int_{\mathbb{R}^d}\left\{2\int_0^{\frac{|x|^2}{R^2}}\rho(r)\,dr|\nabla\psi(t'')|^2+\frac{4|x|^2}{R^2}\rho\left(\frac{|x|^2}{R^2}\right)\left|\frac{x}{|x|}\cdot\nabla\psi(t'')\right|^2\right\}dxdt''dt'\\ &\quad+\int_{t_0}^{t}\int_{t_0}^{t'}\int_{\mathbb{R}^d}\left\{d\int_0^{\frac{|x|^2}{R^2}}\rho(r)\,dr+\frac{2|x|^2}{R^2}\rho\left(\frac{|x|^2}{R^2}\right)\right\}\left\{f(\psi(t))\overline{\psi(t)}-F(\psi(t))\right\}dxdt''dt'\\ &\quad-\frac{1}{4}\int_{t_0}^{t}\int_{t_0}^{t'}\int_{\mathbb{R}^d}\Delta^2 W_R\,|\psi(t'')|^2\,dxdt''dt' \qquad \textit{for any } R>0,\end{aligned}$$

and (ii) *the motion of the "almost center of mass":*

(Appendix A.15)

$$\int_{\mathbb{R}^d} \frac{\partial W_R}{\partial x_j} |\psi(t)|^2 \, dx$$

$$= \int_{\mathbb{R}^d} \frac{\partial W_R}{\partial x_j} |\psi_0|^2 \, dx + \Im \sum_{k=1}^{d} \int_{\mathbb{R}^d} \frac{\partial^2 W_R}{\partial x_j \partial x_k} \frac{\partial \psi}{\partial x_k}(t) \overline{\psi(t)} \, dx \quad \textit{for any } 1 \le j \le d \textit{ and } R > 0.$$

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