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Scattering for the defocusing, cubic nonlinear wave equation

By

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Abstract

The defocusing, nonlinear wave equation is globally well - posed and scattering for radial initial data lying close to the critical Sobolev space. We also prove scattering for data lying in a scale - invariant Besov space. This talk was an introduction to the papers [6], [7], and [8].

§ 1. Introduction

The defocusing, nonlinear wave equation

$$u_{tt} - \Delta u + u^3 = 0, \qquad u : \mathbf{R} \times \mathbf{R}^3 \to \mathbf{R}, \qquad u(0,x) = u_0, \qquad u_t(0,x) = u_1, \tag{1.1}$$

has the scaling symmetry

$$u(t,x) \mapsto \lambda u(\lambda t, \lambda x), \qquad \lambda > 0. \tag{1.2}$$

The $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ norm of the initial data is preserved under (1.2). It is conjectured

Conjecture 1.1. (1.1) *is globally well - posed and scattering for initial data* $(u_0, u_1) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$.

[12] proved that (1.1) is locally well - posed for initial data lying in $\dot{H}^{1/2} \times \dot{H}^{-1/2}$. This fact, combined with the fact that there are no known explicit soliton solutions or blowup solutions, motivates conjecture 1.1.

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Also, [11] proved that (1.1) is globally well - posed for $u : \mathbf{R} \times \mathbf{R}^4 \to \mathbf{R}$. [2] proved space - time estimates for this solution, and then [1] proved scattering. In that case, the $\dot{H}^1 \times L^2$ norm is the critical norm. This quantity is controlled by the conserved energy

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t,x)|^2 dx + \frac{1}{2}\int |u_t(t,x)|^2 dx + \frac{1}{4}\int |u(t,x)|^4 dx. \tag{1.3}$$

This fact represents the key difficulty in moving from dimension four to dimension three. Indeed,

Theorem 1.1. *There exists a function $f : [0,\infty) \to [0,\infty)$ such that, if (1.1) has a solution on interval $I \subset \mathbf{R}$, then for any compact $J \subset I$,*

$$\|u\|_{L^4_{t,x}(J\times\mathbf{R}^3)} \leq f(\|u\|_{L_t^\infty \dot{H}^{1/2}(J\times\mathbf{R}^3)} + \|u_t\|_{L_t^\infty \dot{H}^{-1/2}(J\times\mathbf{R}^3)}). \tag{1.4}$$

Thus if one can prove a uniform bound on $\|u\|_{L_t^\infty \dot{H}^{1/2}(J\times\mathbf{R}^3)} + \|u_t\|_{L_t^\infty \dot{H}^{-1/2}(J\times\mathbf{R}^3)}$ then scattering would follow by the results of [10] and [16].

Remark. Observe that the analogous result for $d = 4$ combined with (1.3) immediately implies scattering.

Here we obtain the following estimates on the growth of the critical Sobolev norm.

Theorem 1.2. *For any $\epsilon > 0$, if u_0 and u_1 are radial functions and $u_0 \in \dot{H}^{1/2}(\mathbf{R}^3) \cap \dot{H}^{1/2+\epsilon}(\mathbf{R}^3)$ and $u_1 \in \dot{H}^{-1/2}(\mathbf{R}^3) \cap \dot{H}^{-1/2+\epsilon}(\mathbf{R}^3)$, then (1.1) is globally well - posed.*

Theorem 1.3. *Suppose u is radial and there exists a positive constant $\epsilon > 0$ such that*

$$\|u_0\|_{\dot{H}^{1/2+\epsilon}(\mathbf{R}^3)} + \||x|^{2\epsilon} u_0\|_{\dot{H}^{1/2+\epsilon}(\mathbf{R}^3)} \leq A < \infty, \tag{1.5}$$

and

$$\|u_1\|_{\dot{H}^{-1/2+\epsilon}(\mathbf{R}^3)} + \||x|^{2\epsilon} u_1\|_{\dot{H}^{-1/2+\epsilon}(\mathbf{R}^3)} \leq A < \infty. \tag{1.6}$$

Then (1.1) has a global solution and there exists some $C(A,\epsilon) < \infty$ such that

$$\int_{\mathbf{R}}\int (u(t,x))^4 dx dt \leq C(A,\epsilon). \tag{1.7}$$

Theorem 1.4. *The initial value problem* (1.1) *is globally well - posed and scattering for* $u_0 \in B^2_{1,1}(\mathbf{R}^3)$*, radial, and* $u_1 \in B^1_{1,1}(\mathbf{R}^3)$*, radial. Moreover, there exists a function* $f : [0,\infty) \to [0,\infty)$ *which gives a uniform bound*

$$\|u\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \leq f(\|u_0\|_{B^2_{1,1}} + \|u_1\|_{B^1_{1,1}}). \tag{1.8}$$

§ 2. Proof of theorem 1.2

Theorem 1.2 is proved using the I - method. The I - method was introduced to the cubic nonlinear Schrödinger equation in [3]. On the wave equation side, [13] used the I - method to prove global well - posedness for (1.1) when $s > \frac{13}{18}$ and to $s > \frac{7}{10}$ if u has radial symmetry.

The I - operator is the Fourier multiplier

$$\widehat{If}(\xi) = \hat{m}(\xi)\hat{f}(\xi), \tag{2.1}$$

where

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq N \\ (\frac{N}{|\xi|})^{1-s} & \text{if } |\xi| > 2N. \end{cases} \tag{2.2}$$

Then $I : \dot{H}^s \to \dot{H}^1$ and $I : \dot{H}^{s-1} \to L^2$. Set $s = \frac{1}{2} + \epsilon$. By the Sobolev embedding theorem,

$$E(Iu(0)) \lesssim_{\|u_0\|_{\dot{H}^{1/2}}, \|u_0\|_{\dot{H}^s}, \|u_1\|_{\dot{H}^{s-1}}} N^{2(1-s)}, \tag{2.3}$$

where $E(Iu(t))$ is the energy (1.3). If $E(Iu(t))$ was conserved by a solution to (1.1) then the local well - posedness result of [12] would imply theorem 1.2. $E(Iu(t))$ is called the modified energy.

By direct computation,

$$\frac{d}{dt}E(Iu(t)) = \int (Iu_t(t,x))((Iu)^3 - Iu^3)(t,x)dx. \tag{2.4}$$

This term is identically zero unless at least two terms in the quadrilinear expression (2.4) are supported at frequency $\geq \frac{N}{8}$. This fact implies that

$$\int_0^T \frac{d}{dt} E(Iu(t))dt \tag{2.5}$$

may be well controlled by the long time Strichartz estimates. Such estimates were first introduced in [4].

The long time Strichartz estimates were created to study almost periodic solutions to the mass - critical nonlinear Schrödinger equation. Suppose for example that the set

$$\{u(t) : t \in \mathbf{R}\} \tag{2.6}$$

is precompact in $L_x^2(\mathbf{R}^d)$. Then at higher and higher frequencies the solution to the mass - critical problem is dominated by a linear solution for longer and longer periods of time. The proof of theorem 1.2 follows the argument of [5], proving a similar result for the nonlinear Schrödinger equation.

First observe that by small data theory and finite propagation speed there exists $R < \infty$ such that if u solves (1.1),

$$\|u\|_{L_{t,x}^4(\mathbf{R} \times \{x : |x| > R + |t|\})} \lesssim 1. \tag{2.7}$$

Next, a solution to (1.1) satisfies the Duhamel formula

$$(u(t), u_t(t)) = S(t)(u_0, u_1) - \int_0^t S(t - \tau)(0, u^3) d\tau, \tag{2.8}$$

where $S(t)$ is the operator

$$S(t) = \begin{pmatrix} \cos(t\sqrt{-\Delta}) & \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \\ -\sqrt{-\Delta}\sin(t\sqrt{-\Delta}) & \cos(t\sqrt{-\Delta}) \end{pmatrix}. \tag{2.9}$$

The proof of the long time Strichartz estimate combines the local energy decay,

$$\sup_{R > 0} R^{-1/2} \|\nabla_{t,x} S(t)(u_0, u_1)\|_{L_{t,x}^2(\mathbf{R} \times \{x : |x| \leq R\})} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2}. \tag{2.10}$$

Since u is radially symmetric, (2.10) may be combined with the radial Sobolev embedding theorem

$$\| |x|^{1/2} u \|_{L_x^\infty(\mathbf{R}^3)} \lesssim \|u\|_{\dot{H}^1(\mathbf{R}^3)}, \tag{2.11}$$

the Strichartz estimate, and (2.7), to prove that for any M,

$$\begin{aligned} (\sup_{R>0} R^{-1/2} \|\nabla_{t,x} I P_{>M} u\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})}) &\lesssim E(Iu_0, Iu_1) \\ + \frac{\ln(T)}{M} (\sup_{R>0} R^{-1/2} \|\nabla_{t,x} I P_{>\frac{M}{8}} u\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})}) &\|Iu\|^2_{L_t^\infty \dot{H}^1([0,T]\times\mathbf{R}^3)} \\ + \| |\nabla|^{\frac{1}{2}} P_{>\frac{M}{8}} Iu\|_{L^4_{t,x}([0,T]\times\mathbf{R}^3)} &\|(1-I)u\|^2_{L^4_{t,x}([0,T]\times\mathbf{R}^3)}. \end{aligned} \tag{2.12}$$

$P_{>M}$ is the Littlewood - Paley cutoff that restricts to frequencies $|\xi| > M$.

$$P_{>M} = \sum_{j:2^j \geq M} P_j. \tag{2.13}$$

P_j is the usual Littlewood - Paley multiplier that restricts to frequencies $|\xi| \sim 2^j$. Then assume that $[0,T]$ is an interval on which

$$\sup_{t\in[0,T]} E(Iu(t)) \leq CN^{2(1-s)}. \tag{2.14}$$

Therefore,

$$\begin{aligned} &\frac{\ln(T)}{M} (\sup_{R>0} R^{-1/2} \|\nabla_{t,x} I P_{>\frac{M}{8}} u\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})}) \|Iu\|^2_{L_t^\infty \dot{H}^1([0,T]\times\mathbf{R}^3)} \\ &\leq \frac{C^2 \ln(T) N^{2(1-s)}}{M} (\sup_{R>0} R^{-1/2} \|\nabla_{t,x} I P_{>\frac{M}{8}} u\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})}). \end{aligned} \tag{2.15}$$

Then since $s > \frac{1}{2}$ there exists $\delta > 0$ such that $2(1-s) < 1-\delta$. Choosing $M > N^{1-\frac{\delta}{2}}$,

$$(2.15) \lesssim C^2 \ln(T) N^{-\frac{\delta}{2}} (\sup_{R>0} R^{-1/2} \|\nabla_{t,x} I P_{>\frac{M}{8}} u\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})}). \tag{2.16}$$

Then arguing by induction, starting with the base case (which follows from (2.14)),

$$(\sup_{R>0} R^{-\frac{1}{2}} \|\nabla_{t,x} Iu\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})}) \leq CT^{1/2} N^{1-s}, \tag{2.17}$$

by (2.12), for $N(T)$ sufficiently large there exists $c(\delta) > 0$ such that

$$(\sup_{R>0} R^{-\frac{1}{2}} \|\nabla_{t,x} I P_{> \frac{N}{8}} u\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})}) \lesssim N^{1-s} + CN^{-c\ln(N)} N^{1-s} T^{1/2}. \tag{2.18}$$

Therefore,

$$\sup_{R>0} R^{-\frac{1}{2}} \|\nabla_{t,x} I P_{> \frac{N}{8}} u\|_{L^2_{t,x}([0,T]\times\{x:|x|\leq R\})} \lesssim N^{1-s}. \tag{2.19}$$

Then plugging (2.19) into (2.4), by Bernstein's inequality and the fact that at least two terms of (2.4) are at frequency $> \frac{N}{8}$, and making a computation similar to (2.12),

$$\int_0^T |\frac{d}{dt} E(Iu(t))| dt \lesssim \frac{1}{N} N^{4(1-s)} \ln(T) << N^{2(1-s)}. \tag{2.20}$$

Then by (2.3), the bootstrap is closed. Therefore, for any T there exists $N(T)$ such that

$$\sup_{t\in[0,T]} E(Iu(t)) \lesssim N^{2(1-s)}, \tag{2.21}$$

where N is implicit in the I - operator (2.2). This proves global well - posedness.

§ 3. Proof of theorem 1.3

The proof of theorem 1.3 uses a conformal change of coordinates. See [15] and [17] for prior work on the wave equation using a conformal change of coordinates. Shift the time coordinates of $t = 0$ to $t = 1$. Also, after rescaling suppose

$$\|S(t)(u_0, u_1)\|_{L^4_{t,x}([1,\infty)\times\{x:|x|>|t|-\frac{1}{2}\})} \leq \epsilon. \tag{3.1}$$

Then by finite propagation speed and small data arguments

$$\|u\|_{L^4_{t,x}([1,\infty)\times\{x:|x|>|t|-\frac{1}{2}\})} \lesssim \epsilon. \tag{3.2}$$

If $u(t,x)$ is a radial solution to (1.1) then

$$v(\tau, s) = \frac{e^\tau \sinh s}{s} u(e^\tau \cosh s, e^\tau \sinh s) \tag{3.3}$$

solves

$$\partial_{\tau\tau}v - \partial_{ss}v - \frac{2}{s}\partial_s v + (\frac{s}{\sinh s})^2 v^3 = 0. \tag{3.4}$$

(3.4) has the conserved energy

(3.5)
$$E(v(\tau)) = \frac{1}{2}\int |\partial_s v(\tau,s)|^2 s^2 ds + \frac{1}{2}\int |\partial_\tau v(\tau,s)|^2 s^2 ds + \frac{1}{4}\int |v(\tau,s)|^4 (\frac{s}{\sinh s})^2 s^2 ds.$$

A solution to (3.4) also has the Morawetz estimate

$$\int\int_0^\infty v(\tau,s)^4 (\frac{s}{\sinh s})^2 (\frac{\cosh s}{\sinh s}) s^2 ds d\tau \lesssim E(v). \tag{3.6}$$

Since $\frac{\cosh s}{\sinh s} \geq 1$, by a change of variables

$$\int_1^\infty \int_{|x|\leq|t|} u(t,x)^4 dx dt \leq \int\int_0^\infty v(\tau,s)^4 (\frac{s}{\sinh s})^2 (\frac{\cosh s}{\sinh s}) s^2 ds d\tau. \tag{3.7}$$

The proof of theorem 1.3 may then be split into four steps:

1. Prove that the initial data $(v(0), v_\tau(0))$ lies in $\dot{H}^s \times \dot{H}^{s-1}$ for some $s > \frac{1}{2}$.
2. Prove a long time Strichartz estimate on a solution to (3.4).
3. Prove an almost Morawetz estimate for Iv based on (3.6).
4. Prove that $E(Iv(t)) \lesssim E(Iv(0))$ for all t.

The estimates on the initial data rely on the fact that

$$v(\tau,s)|_{\tau=0} = \frac{\sinh s}{s} u(\cosh s, \sinh s), \tag{3.8}$$

combined with the fact that when $|\sin s - \cosh s + 1| \leq \frac{1}{2}$, (3.2) implies that the linear solution dominates u. Therefore, by the formula for a free wave equation in three dimensions, $s \cdot v(\tau,s)|_{\tau=0}$ is dominated by

$$\frac{1}{2}(e^s - 1)u_0(e^s - 1) + \frac{1}{2}(1 - e^{-s})u_0(1 - e^{-s}) + \frac{1}{2}\int_{1-e^{-s}}^{e^s - 1} u_1(r) r dr. \tag{3.9}$$

The appropriate norms may be explicitly calculated from (3.9).

The I - method arguments are quite similar to the argument in the proof of theorem 1.2. There are only two differences. The first is that the base case (2.17) does not depend on T, due to a bootstrap assumption on the Morawetz estimate.

The second difference is that the multiplier $(\frac{s}{\sinh s})^2$ in the nonlinearity gives some additional decay as $s \to \infty$, which removes the $\ln(T)$ that appears in (2.15).

§ 4. Proof of theorem 1.4

The proof of theorem 1.4 also uses conformal coordinates. Suppose $u_0 \in B^2_{1,1}$ and $u_1 \in B^1_{1,1}$, where $B^s_{p,q}$ is the Besov space given by

$$\|f\|_{B^s_{p,q}(\mathbf{R}^3)} = (\sum_j 2^{jsq} \|P_j f\|^q_{L^p})^{1/q}. \tag{4.1}$$

P_j is the Littlewood - Paley partition of unity. This norm is invariant under the scaling symmetry (1.2). For a radial function u_0 lying in $B^2_{1,1}$,

$$\int_{|x|>R} |\nabla u_0(x)|^2 dx \lesssim \frac{1}{R} \|u_0\|^2_{B^2_{1,1}}. \tag{4.2}$$

Moreover, by the dispersive estimate

$$\|S(t)(u_0, u_1)\|_{L^\infty} \lesssim \frac{1}{t}[\|u_0\|_{B^2_{1,1}} + \|u_1\|_{B^1_{1,1}}]. \tag{4.3}$$

(4.2) and (4.3) combine to imply

$$\| \int_0^t S(t-\tau)(0, \chi(\tau, x) u^3) d\tau \|_{\dot{H}^1 \times L^2} < \infty, \tag{4.4}$$

for $\chi \in C_0^\infty(\mathbf{R}^4)$ supported away from $t = 0$ and $x = 0$ and with support in time contained in the interval on which u is well - posed. More heuristically, this means that u is at its most singular at the origin in space and time.

After rescaling, assume that [12] implies that (1.1) has a local solution on the interval $[0, 1]$. Moreover, by the radial Sobolev embedding theorem

$$\|u^3\|_{L^1_t L^2_x([0,1] \times \{x : |x| > R\})} \lesssim R^{-1/2}, \tag{4.5}$$

and by (4.3),

$$\|u^3\|_{L_t^1 L_x^2([R,1]\times\mathbf{R}^3)} \lesssim R^{-1/2}. \tag{4.6}$$

Again thinking heuristically, having weathered this initial "singularity storm" at the origin, there is good reason to think the solution should be global.

By the sharp Huygens principle combined with the radial Sobolev embedding theorem, for $t > 10R$,

$$\|\int_0^t S(t-\tau)(0,\chi u^3)d\tau\|_{L^\infty} \lesssim \frac{1}{t}, \tag{4.7}$$

where χ is a smooth function supported on $|x| \leq R$. This implies that $u(1) = v(1)+w(1)$, where

$$\|v(1)\|_{\dot{H}^1} + \|v_t(1)\|_{L^2} \lesssim 1, \tag{4.8}$$

and

$$\|S(t-1)(w(1), w_t(1))\|_{L^\infty} \lesssim \frac{1}{t}. \tag{4.9}$$

Meanwhile v solves

$$v_{tt} - \Delta v + (v+w)^3 = 0, \tag{4.10}$$

so then

$$\frac{d}{dt}E(v(t)) \lesssim \|v_t\|_{L^2}\|v\|_{L^4}^2\|w\|_{L^\infty} + \|v_t\|_{L^2}\|w\|_{L^6}^3. \tag{4.11}$$

This implies that $E(v(t))$ is bounded on any compact time interval, implying global well - posedness. Scattering may be proved using (3.3). Once again shift the time $t = 0$ to $t = 1$. Take

$$\tilde{v}(\tau, s) = \frac{e^\tau \sinh s}{s} v(e^\tau \cosh s, e^\tau \sinh s), \tag{4.12}$$

and

$$\tilde{w}(\tau, s) = \frac{e^{\tau}\sinh s}{s} w(e^{\tau}\cosh s, e^{\tau}\cosh s). \tag{4.13}$$

Recalling the conformal energy (3.5) and computing $\frac{d}{d\tau}E(\tilde{v}(\tau))$,

$$\begin{aligned} &\|\tilde{v}_{\tau}(\tau, s)(\frac{s}{\sinh s})^2 \tilde{v}(\tau, s)^2 \tilde{w}(\tau, s)\|_{L^1} \\ &\lesssim \|(\frac{s}{\sinh s})^{1/2}\tilde{v}(\tau, s)\|_{L^4}^2 \|\tilde{v}_{\tau}(\tau, s)\|_{L^2} \|e^{\tau}\tilde{w}(e^{\tau}\cosh s, e^{\tau}\sinh s)\|_{L^{\infty}} \\ &\lesssim E(\tilde{v}(\tau))\|e^{\tau}\tilde{w}(e^{\tau}\cosh s, e^{\tau}\sinh s)\|_{L^{\infty}}. \end{aligned} \tag{4.14}$$

By the radial Sobolev embedding theorem,

$$|e^{\tau}\tilde{w}(e^{\tau}\cosh s, e^{\tau}\sinh s)| \lesssim \frac{1}{\sinh s}. \tag{4.15}$$

Observe that by the sharp Huygens principle, w is supported on $||t-1| - |x|| \leq R$, where R is small. Then

$$e^{\tau}\cosh s - e^{\tau}\sinh s = e^{\tau - s} \sim 1, \tag{4.16}$$

which then implies $\frac{1}{\sinh s} \sim e^{-\tau}$ on the support of w. Therefore,

$$\|\tilde{v}_{\tau}(\tau, s)(\frac{s}{\sinh s})^2 \tilde{v}(\tau, s)^2 \tilde{w}(\tau, s)\|_{L^1} \lesssim e^{-\tau}E(\tilde{v}(\tau)). \tag{4.17}$$

Then by Gronwall's inequality, $E(\tilde{v}(\tau))$ is uniformly bounded. The error terms arising in the Morawetz estimate (3.6) from

$$3v^2 w + 3vw^2 + w^3 \tag{4.18}$$

may be handled in a similar manner, proving scattering. A profile decomposition argument then gives a uniform bound.

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