Multi-agent Coordination via Distributed Pattern Matching

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***Abstract*—This paper addresses a distributed formation control problem of multiple mobile agents using relative positions and local bearings. First, a formation control method via distributed pattern matching is proposed, which is executed over each clique (i.e. complete induced subgraph) of a network. It is shown that this method achieves the best control performance for a given desired formation and network topology in the following sense: the closest formation to the desired one is achieved among all formations achievable by distributed and relative control over the network. Next, a necessary and sufficient network condition is derived under which the desired formation can be obtained. It turns out that a new concept of connectivity, called clique-rigidity, plays a crucial role. Finally, the effectiveness of the proposed method is illustrated through simulations in both two and three-dimensional spaces.**

***Index Terms*—Multi-agent systems, distributed control, formation control, relative measurements.**

I. Introduction

Multi-agent systems have attracted a lot of attention in the field of the control engineering [1]. A multi-agent system consists of a large number of components, called agents, which interact with each other through communication and/or sensing [2]. To reduce computational and sensing burdens on agents, distributed control based on limited information is important [3]. Actually, various problems of multi-agent systems have been investigated based on distributed control, e.g. consensus [4], [5], flocking [6], [7], coverage [8], pursuit [9], [10] and attitude synchronization [11], [12]. Formation control of multiple mobile agents [13], [14], [15], [16], [17], [18], [19], [20] is one of the most fundamental problems since it can be applied to various missions, e.g. monitoring and surveillance by multiple unmanned aerial vehicles (UAVs) [21], ocean sampling and mapping by multiple autonomous underwater vehicles (AUVs) [22], and so forth.

This paper particularly focuses on distributed formation control of mobile agents using relative positions and local bearings. With this approach, agents can avoid using sensors for absolute measurements, e.g. global positioning systems (GPSs). Let $x_i(t) \in \mathbb{R}^d$ be the current position of agent $i \in \{1, 2, \ldots, n\}$ in the global frame, and $x_{*i} \in \mathbb{R}^d$ be its position in a desired formation. Then, the control objective is described as follows:

$$\lim_{t\to\infty}(x_i(t) - (R(t)x_{*i} + \tau(t))) = 0 \tag{1}$$

for all i with some rotation matrix $R(t) \in \mathbb{R}^{d\times d}$ and translation vector $\tau(t) \in \mathbb{R}^d$ from the global frame. The parameters $R(t)$ and $\tau(t)$ have to be determined by agents in a distributed manner under the situation that only relative positions and local bearings are measurable. Fig. 1 illustrates this problem: in (a), the squares denote the desired positions x_{*i} and the edges do mutual observations between agents; in (b), the squares represent the desired positions $\hat{x}_{*i}(t) = R(t)x_{*i} + \tau(t)$ with some rotation and translation and the circles denote the agent positions $x_i(t)$, expected to converge to $\hat{x}_{*i}(t)$.

There are several possible approaches to this problem. One way is to apply attitude synchronization [11], [12] to obtain $R(t)$ and $\tau(t)$ in (1) by consensus of agent-dependent rotations $R_i(t)$ and translations $\tau_i(t)$ from the global frame. However, this approach requires exchanging information on $R_i(t)$ and $\tau_i(t)$ between agents, not based on relative positions or local bearings. Another way would be to employ distance-based formation control [16], [17], [18], [19], formulated as

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = \|x_{*i} - x_{*j}\| \tag{2}$$

for any i, j connected over the observation network. Thanks to the expression with the relative distance in (2), this approach uses only relative positions and local bearings. However, this problem is just a relaxed version of (1); even if (2) is achieved, the original problem (1) is not necessarily solved. Actually, an unexpected formation can be obtained as illustrated in Fig. 2, where the desired formation in (a) looks fairly different from the possible resultant formation in (b) because of the reflection of agent 3 though (2) is satisfied.

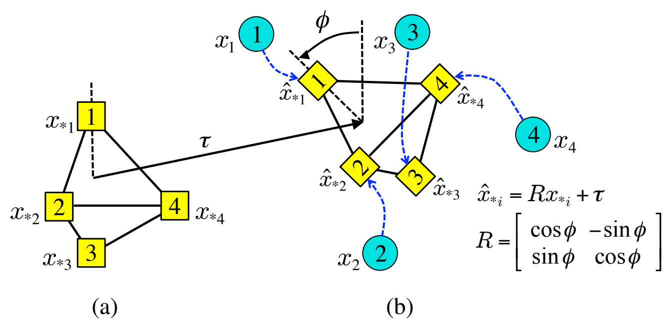

Fig. 1. Multi-agent coordination: (a) desired positions, (b) desired positions with rotation and translation (squares) and agent positions (circles).

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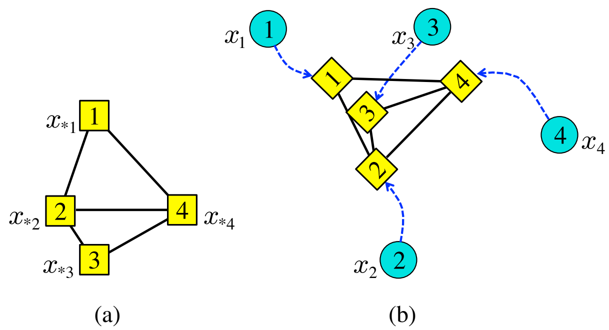

Fig. 2. Distance-based formation: (a) desired positions, (b) possible resultant positions (squares) and agent positions (circles).

Hence, we try to solve the formation control problem (1) directly. For this approach, pattern matching of image data [23] is promising. Actually, (1) can be regarded as an optimization problem to find a rotation matrix $R(t)$ and translation vector $\tau(t)$ to best match the desired positions x_{*i} to the current agent positions $x_i(t)$. Since pattern matching algorithms suppress the reflections of matching data, we can expect to avoid unexpected formations due to the reflections of agents. This approach however involves a serious issue when applied to formation control: these algorithms are executed in a centralized manner. In fact, existing papers do not resolve this issue [24]. To the best of the authors' knowledge, there is no solution to this issue so far.

In this paper, we propose a formation control method using relative positions and local bearings via *distributed pattern matching*. The designed distributed controller executes the pattern matching over each clique (i.e. complete induced subgraph) of networks. We show that it exhibits the best control performance for a given desired formation and network topology in the following sense: agents attain (1) the closest among all formations achievable by distributed control over the network. Note that it depends on networks whether the desired formation is achievable by distributed control or not. Thus, next, we derive a necessary and sufficient condition of network topologies under which the desired formation is realizable. Here, a new concept of connectivity, called *clique-rigidity*, plays a crucial role, which is different from the conventional rigidity. Finally, the effectiveness of the proposed method is illustrated through simulations in both two and three-dimensional (2D and 3D) spaces.

The existing papers [25], [26], [27], [28] have addressed this problem based on the distance-based formation control. They imposed additional constraints such as signed angles or volumes of formations, and guaranteed the achievement of the desired formation under particular network topologies. Therefore, it was not clear what is the best possible performance for a given network, or what kind of condition is essentially necessary for the network to achieve the desired formation. Moreover, discontinuous dynamics were not investigated though they are inevitable to the formation control problem for guaranteeing convergence. This is because there are the positions which can be undesired equilibria under continuous dynamics (e.g. when all agents stay at the same point or collinear (coplaner) in $\mathbb{R}^2$ ($\mathbb{R}^3$) [29]). In contrast, this paper investigates the convergence properties over such dynamics via differential inclusion.

This paper is based on the author's conference paper [30]. The update points from [30] are as follows. (i) The detailed discussions including the proofs of all theorems and lemmas are added. (ii) It is shown that the proposed controller uses only relative positions and local bearings. (iii) A network condition to achieve the desired formation is derived. (iv) The convergence properties are investigated. (v) The simulation in the 3D space is included.

The organization of this paper is as follows. Section II gives preliminaries of notations and definitions. Section III formulates the problems discussed in this paper. In Section IV, an optimal distributed and relative controller is designed via distributed patter matching. Section V gives simulation results. Sections VI and VII investigate necessary network topologies and convergence properties, respectively. Section VIII concludes this paper.

II. Preliminaries

A. Notations

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all the real numbers and non-negative real numbers, respectively. Let $\mathrm{SO}(d) \subset \mathbb{R}^{d\times d}$ denote the set of all the orthogonal matrices whose determinant is 1, and $\mathrm{SE}(d) = \mathrm{SO}(d) \times \mathbb{R}^d$ be the Euclidean group of dimension d. The notations $\mathbf{1}_n \in \mathbb{R}^n$ and $E_n \in \mathbb{R}^{n\times n}$ represent the vector all whose components are 1 and the n-dimensional identity matrix, respectively, and $e_{ni} \in \mathbb{R}^n$ denotes the n-dimensional unit vector whose i-th component is 1. Let $\mathrm{rank}(\cdot)$ and $\mathrm{ave}(\cdot)$ be the rank and the row-wise averages of a matrix, respectively. Let $\mathrm{tr}(\cdot)$ and $\det(\cdot)$ denote the trace and the determinant of a square matrix, respectively. The Frobenius norm of a matrix is given by $\|\cdot\|$.

Let $M_n \in \mathbb{R}^{n\times n}$ be the matrix

$$M_n = E_n - \frac{\mathbf{1}_n \mathbf{1}_n^\top}{n},$$

and the following expressions hold for any $X \in \mathbb{R}^{d\times n}$:

$$\mathbf{1}_n^\top M_n = 0 \tag{3}$$

$$X - \mathrm{ave}(X)\mathbf{1}_n^\top = X M_n. \tag{4}$$

The set $\mathcal{M}_{dn} \subset \mathbb{R}^{d\times n}$ is defined as

$$\mathcal{M}_{dn} = \{X \in \mathbb{R}^{d\times n} : \mathrm{rank}(XM_n) = \min\{d, n-1\}\}. \tag{5}$$

For a scalar function $v : \mathbb{R}^{d\times n} \to \mathbb{R}$ of a variable $X = [x_{ij}] \in \mathbb{R}^{d\times n}$, the gradient of $v(X)$ is defined as

$$\frac{\partial v}{\partial X}(X) = \begin{bmatrix} \frac{\partial v}{\partial x_{11}}(X) & \frac{\partial v}{\partial x_{12}}(X) & \cdots & \frac{\partial v}{\partial x_{1n}}(X) \\ \frac{\partial v}{\partial x_{21}}(X) & \frac{\partial v}{\partial x_{22}}(X) & \cdots & \frac{\partial v}{\partial x_{2n}}(X) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v}{\partial x_{d1}}(X) & \frac{\partial v}{\partial x_{d2}}(X) & \cdots & \frac{\partial v}{\partial x_{dn}}(X) \end{bmatrix}.$$

For vector-valued functions $f_1, f_2, \ldots, f_n : \mathbb{R}^{d\times n} \to \mathbb{R}^d$ and a set $\mathcal{I} \subset \{1, 2, \ldots, n\}$ of positive integers, let

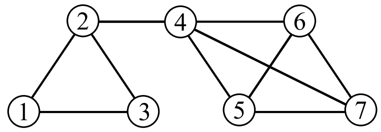

Fig. 3. Example of a graph.

$[f_j(X)]_{j\in\mathcal{I}} : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times|\mathcal{I}|}$ be the matrix-valued function consisting of the columns $f_j(X)$ for $j \in \mathcal{I}$ as

$$[f_j(X)]_{j\in\mathcal{I}} = [f_{i_1}(X)\ f_{i_2}(X)\ \cdots\ f_{i_{|\mathcal{I}|}}(X)],$$

where $|\mathcal{I}|$ is the number of the elements of $\mathcal{I}$, and the elements $i_1, i_2, \ldots, i_{|\mathcal{I}|} \in \mathcal{I}$ satisfy $1 \le i_1 < i_2 < \cdots < i_{|\mathcal{I}|} \le n$. According to this notation, for the matrix $X = [x_1\ \cdots\ x_n] \in \mathbb{R}^{d\times n}$ with the columns $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$, the collection of its columns corresponding to the indexes of $\mathcal{I}$ is denoted as

$$[x_j]_{j\in\mathcal{I}} = [x_{i_1}\ x_{i_2}\ \cdots\ x_{i_{|\mathcal{I}|}}].$$

The $[x_j]_{j\in\mathcal{I}}$-space is the $(d\times|\mathcal{I}|)$-matrix subspace of $\mathbb{R}^{d\times n}$ with the coordinate $[x_j]_{j\in\mathcal{I}}$ spanned by the basis $e_{dk}e_{nj}^\top \in \mathbb{R}^{d\times n}$ for $k = 1, 2, \ldots, d$ and $j \in \mathcal{I}$.

Let $\mathcal{P}_{\mathcal{I}} : \mathrm{pow}(\mathbb{R}^{d\times n}) \to \mathrm{pow}(\mathbb{R}^{d\times|\mathcal{I}|})$ be the projection of a set onto the $[x_j]_{j\in\mathcal{I}}$-space, namely, for a set $\mathcal{A} \subset \mathbb{R}^{d\times n}$

$$\mathcal{P}_{\mathcal{I}}(\mathcal{A}) = \{Y \in \mathbb{R}^{d\times|\mathcal{I}|} : \exists X \in \mathcal{A} \text{ s.t. } Y = [x_j]_{j\in\mathcal{I}}\}, \quad (6)$$

where $\mathrm{pow}(\cdot)$ is the power set of a set. For a matrix $X \in \mathbb{R}^{d\times n}$ and a set $\mathcal{A} \subset \mathbb{R}^{d\times n}$, their distance is defined as

$$\mathrm{dist}(X, \mathcal{A}) = \inf_{Y\in\mathcal{A}} \|X - Y\|. \quad (7)$$

For two sets $\mathcal{A} \subset \mathbb{R}^{d\times n}$ and $\mathcal{B} \subset \mathbb{R}^{d\times n}$, their Hausdorff distance is defined as

$$\text{H-dist}(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{X\in\mathcal{A}} \mathrm{dist}(X, \mathcal{B}), \sup_{Y\in\mathcal{B}} \mathrm{dist}(Y, \mathcal{A})\right\}.$$

B. Some concepts of graph theory

Consider a graph $G = (\mathcal{V}, \mathcal{E})$ with a vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and an edge set $\mathcal{E}$. The elements $i \in \mathcal{V}$ and $\{i, j\} \in \mathcal{E}$ are called a *vertex* and an *edge*, respectively. We assume that G is undirected and time-invariant.

For a vertex subset $\mathcal{C} \subset \mathcal{V}$, let $\mathcal{E}|_{\mathcal{C}} \subset \mathcal{E}$ be the edge subset such that both vertexes of each edge in $\mathcal{E}|_{\mathcal{C}}$ are contained by $\mathcal{C}$, namely, $\mathcal{E}|_{\mathcal{C}} = \{\{i, j\} \in \mathcal{E} : i, j \in \mathcal{C}\}$. Then, the graph $G|_{\mathcal{C}} = (\mathcal{C}, \mathcal{E}|_{\mathcal{C}})$ is said to be *induced* by $\mathcal{C}$, or an *induced subgraph* of $\mathcal{C}$. A vertex subset $\mathcal{C} \subset \mathcal{V}$ is said to be a *clique* in G if the induced subgraph of $\mathcal{C}$ is complete [31]. Clique $\mathcal{C}$ is said to be *maximal* if there is no other clique $\bar{\mathcal{C}}$ in G such that $\mathcal{C} \subset \bar{\mathcal{C}}$. Let $\text{M-clq}(G) \subset \mathrm{pow}(\mathcal{V})$ be the set of all the maximal cliques in G, and $\text{M-clq}_i(G) = \{\mathcal{C} \in \text{M-clq}(G) : i \in \mathcal{C}\}$ be the set of all the maximal cliques to which vertex i belongs. For clique $\mathcal{C}$, the number $|\mathcal{C}|$ of the vertexes is called the *order* of $\mathcal{C}$.

Example 1: Consider the graph $G = (\mathcal{V}, \mathcal{E})$ depicted in Fig. 3. This graph consists of three maximal cliques of orders $2, 3, 4$, and the set of them is given as

$$\text{M-clq}(G) = \{\{2, 4\}, \{1, 2, 3\}, \{4, 5, 6, 7\}\}.$$

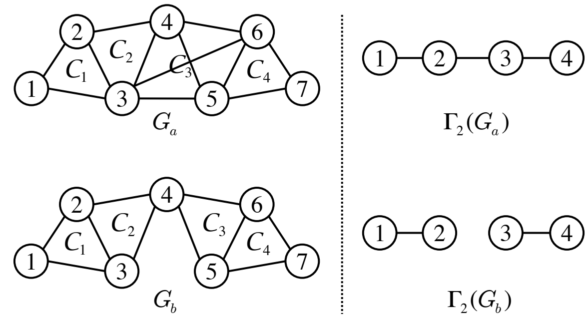

Fig. 4. Examples of 2-interconnection graphs of maximal cliques.

The set of the maximal cliques to which vertex $i \in \mathcal{V}$ belongs is given as follows:

$$\begin{aligned}
&\text{M-clq}_1(G) = \text{M-clq}_3(G) = \{\{1, 2, 3\}\},\\
&\text{M-clq}_2(G) = \{\{2, 4\}, \{1, 2, 3\}\},\\
&\text{M-clq}_4(G) = \{\{2, 4\}, \{4, 5, 6, 7\}\},\\
&\text{M-clq}_5(G) = \text{M-clq}_6(G) = \text{M-clq}_7(G) = \{\{4, 5, 6, 7\}\}.
\end{aligned}$$

■

Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{|\text{M-clq}(G)|} \in \text{M-clq}(G)$ be the maximal cliques in G. The *r-intersection graph* $\Gamma_r(G) = (\breve{\mathcal{V}}, \breve{\mathcal{E}}_r)$ of the maximal cliques in G is a graph whose vertexes correspond to the maximal cliques in G, with an edge between two vertexes whenever the corresponding two maximal cliques in G have at least r vertexes in common [32], [33]. Its vertex and edge sets are given by $\breve{\mathcal{V}} = \{1, 2, \ldots, |\text{M-clq}G|\}$ and the following, respectively:

$$\breve{\mathcal{E}}_r = \{\{\ell, m\}, \ell, m \in \breve{\mathcal{V}} : |\mathcal{C}_\ell \cap \mathcal{C}_m| \ge r, \ell \ne m\}.$$

Example 2: Consider the graphs in Fig. 4. Graph G_a consists of four maximal cliques: $\mathcal{C}_1 = \{1, 2, 3\}$, $\mathcal{C}_2 = \{2, 3, 4\}$, $\mathcal{C}_3 = \{3, 4, 5, 6\}, \mathcal{C}_4 = \{5, 6, 7\}$, which derive the 2-intersection graph $\Gamma_2(G_a)$ of the maximal cliques as a line graph. In contrast, as for graph G_b, the 2-intersection graph $\Gamma_2(G_b)$ of the maximal cliques is not connected due to the lack of the connection between vertexes 3 and 5 in G_b. ■

C. Discontinuous dynamics

Consider a differential equation

$$\dot{X}(t) = F(X(t)) \quad (8)$$

of a matrix variable $X(t) \in \mathbb{R}^{d\times n}$ with a matrix-valued function $F : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ measurable and essentially locally bounded, but not necessarily continuous. A matrix-valued function $X(t) \in \mathbb{R}^{d\times n}$ is called a *Filippov solution*, or simply *solution*, of (8) if $X(t)$ is an absolutely continuous function satisfying the differential inclusion

$$\dot{X}(t) \in \mathcal{K}[F](X(t)).$$

Here, $\mathcal{K}[F] : \mathbb{R}^{d\times n} \to \mathrm{pow}(\mathbb{R}^{d\times n})$ is the set-valued map defined by

$$\mathcal{K}[F](X) = \overline{\mathrm{co}}\left\{Y \in \mathbb{R}^{d\times n} : \exists X_k \in \mathbb{R}^{d\times n}\backslash\mathcal{S},\ k = 1, 2, \ldots \right.$$
$$\left. \text{s.t.} \lim_{k\to\infty} X_k = X,\ \lim_{k\to\infty} F(X_k) = Y\right\}$$

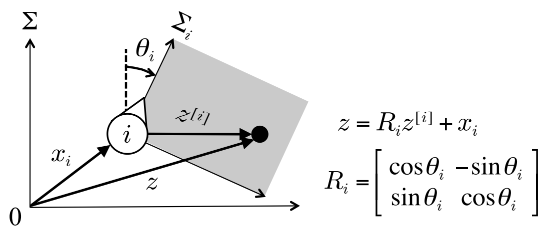

Fig. 5. Relation between the global and local frames through (11) in $\mathbb{R}^2$.

with a set $\mathcal{S} \subset \mathbb{R}^{d\times n}$ of measure zero, where $\overline{\mathrm{co}}(\cdot)$ is the closure of the convex hull of a set. Let $\mathcal{Z}(F) \subset \mathbb{R}^{d\times n}$ be the zero set of the function $F(X)$, defined by

$$\mathcal{Z}(F) = \{X \in \mathbb{R}^{d\times n} : 0 \in \mathcal{K}[F](X)\}, \tag{9}$$

which is the *equilibrium set* of system (8). We say that a closed, non-empty set $\mathcal{A} \subset \mathcal{Z}(F)$ is a *locally attractive equilibrium set* of (8) if there exists an open set $\mathcal{U} \supset \mathcal{A}$ such that

$$\lim_{t\to\infty} \mathrm{dist}(X(t), \mathcal{A}) = 0 \quad \forall X(0) \in \mathcal{U}. \tag{10}$$

Additionally, if $\mathcal{U} = \mathbb{R}^{d\times n}$ and $\mathcal{A} = \mathcal{Z}(F)$, we say that $\mathcal{A}$ is the *globally attractive equilibrium set* of (8).

III. Problem Formulation

A. Target system

Consider a group of n mobile agents in a d-dimensional space, where d is a positive integer describing the dimension of working space (e.g., $d = 2$, 3, but not limited to them). Let $\mathcal{V} = \{1, 2, \cdots, n\}$ be the set of the agent indexes. Each agent can obtain information from others over a network whose connectivity is described by graph $G = (\mathcal{V}, \mathcal{E})$ with the vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. Agents i, j can exchange information over the network if and only if $\{i, j\} \in \mathcal{E}$. The set of the neighbors of agent i is expressed as $\mathcal{N}_i = \{j \in \mathcal{V} : \{i, j\} \in \mathcal{E}\}$.

There are two types of frames to describe the positions of the agents: the *global frame* Σ and *local frames* $\Sigma_i(t)$ ($i \in \mathcal{V}$). The global frame Σ is fixed and common among all agents while the local frames $\Sigma_i(t)$ are time-varying and may be different from each other. Let $x_i(t) \in \mathbb{R}^d$ be the position of agent $i \in \mathcal{V}$ in Σ, corresponding to the origin of $\Sigma_i(t)$. For a point $z \in \mathbb{R}^d$ in Σ, let $z^{[i]} \in \mathbb{R}^d$ denote its expression in $\Sigma_i(t)$ according to the following relation:

$$z = R_i z^{[i]} + x_i(t), \tag{11}$$

where $R_i \in SO(d)$ is a transformation matrix associated with the rotation of $\Sigma_i(t)$ from Σ. Assume that R_i is constant and unknown to anyone including agent i itself. For $d = 2$, for example, R_i is determined by the angle $\theta_i \in [0, 2\pi)$ between the first axes of Σ and $\Sigma_i(t)$ as shown in Fig. 5.

Assume that the velocity of agent i can be directly controlled by the control input $u_i(t) \in \mathbb{R}^d$ in $\Sigma_i(t)$. Then, the dynamics of agent $i \in \mathcal{V}$ is described in Σ as

$$\dot{x}_i(t) = R_i u_i(t) \tag{12}$$

from (11). (See Appendix A for the deviation.) As for sensing, we assume that agent i can measure the relative positions $x_j^{[i]}(t) \in \mathbb{R}^d$ of the neighbors $j \in \mathcal{N}_i$ in $\Sigma_i(t)$, expressed as

$$x_j^{[i]}(t) = R_i^\top (x_j(t) - x_i(t)) \tag{13}$$

by replacing z with $x_j(t)$ in (11). Then, the control input $u_i(t)$ should be of the form

$$u_i(t) = f_i([x_j^{[i]}(t)]_{j\in\mathcal{N}_i}) \tag{14}$$

with a function $f_i : \mathbb{R}^{d\times|\mathcal{N}_i|} \to \mathbb{R}^d$. The control input of the form (14) is said to be *relative and distributed* over G.

The control objective is to achieve (1) for any $i \in \mathcal{V}$ with some $(R(t), \tau(t)) \in \mathrm{SE}(d)$, where $x_{*i} \in \mathbb{R}^d$ ($i \in \mathcal{V}$) is the position of agent i in a desired formation. This objective can be rewritten as

$$\lim_{t\to\infty} \mathrm{dist}(X(t), \mathcal{T}_{X_*}) = 0 \tag{15}$$

with the collective positions $X(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)] \in \mathbb{R}^{d\times n}$ and the *target set* of $X(t)$ defined by

$$\mathcal{T}_{X_*} = \{X \in \mathbb{R}^{d\times n} \colon \exists (R, \tau) \in \mathrm{SE}(d) \text{ s.t. } X = RX_* + \tau \mathbf{1}_n^\top\}, \tag{16}$$

where $X_* = [x_{*1}\ x_{*2}\ \cdots\ x_{*n}] \in \mathbb{R}^{d\times n}$ represents the collective desired position.

B. Gradient-flow approach

For the control objective (15), we employ the gradient-flow approach, which is effective in designing cooperative controllers. Let $v : \mathbb{R}^{d\times n} \to \mathbb{R}_+$ be a function evaluating the achievement of a given task with the minimum value, zero. When $x_i(t)$ is governed by the gradient-flow of $v(X)$ as

$$\dot{x}_i(t) = -\frac{\partial v}{\partial x_i}(X(t)) \tag{17}$$

for all $i \in \mathcal{V}$, $v(X(t))$ is monotonically non-increasing, and $X(t)$ locally converges to the zero set $v^{-1}(0)$, as shown in Section VII. From $R_i \in \mathrm{SO}(d)$, the gradient-flow (17) of $v(X)$ is obtained in (12) by the control input

$$u_i(t) = -R_i^\top \frac{\partial v}{\partial x_i}(X(t)). \tag{18}$$

Then, to achieve (15), we just have to find a function $v(X)$ satisfying

$$v^{-1}(0) = \mathcal{T}_{X_*}. \tag{19}$$

We call a function $v(X)$ satisfying (19) an *indicator* to $\mathcal{T}_{X_*}$, which enables the agents to know whether the desired formation is achieved or not.

To design a relative and distributed control input (14) via the gradient-based control input (18), let us define two classes of functions. First, a continuous function $v(X)$ is said to have a *relative gradient* if it is differentiable almost everywhere and for every $i \in \mathcal{V}$ there exists a function $\bar{f}_i : \mathbb{R}^{d\times(n-1)} \backslash \bar{\mathcal{S}}_i \to \mathbb{R}^d$ with a set $\bar{\mathcal{S}}_i$ of measure zero such that

$$\frac{\partial v}{\partial x_i}(X) = -R_i \bar{f}_i([R_i^\top (x_j - x_i)]_{j\in\mathcal{V}\backslash\{i\}}) \tag{20}$$

holds for any matrix $R_i \in \mathrm{SO}(d)$. Let $\mathcal{F}_r$ be the set of all the continuous functions having relative gradients. From (13) and (20), the gradient-based control input (18) with $v(X) \in \mathcal{F}_r$ depends only on relative positions $x_j^{[i]}(t)$, but does not on R_i. Second, a continuous function $v(X)$ is said to have a *distributed gradient* over G if it is differentiable almost everywhere and for every $i \in \mathcal{V}$ there exists a function $\tilde{f}_i : \mathbb{R}^d \times \mathbb{R}^{d\times|\mathcal{N}_i|}\backslash\tilde{\mathcal{S}}_i \to \mathbb{R}^d$ with a set $\tilde{\mathcal{S}}_i$ of measure zero satisfying

$$\frac{\partial v}{\partial x_i}(X) = -\tilde{f}_i(x_i, [x_j]_{j\in\mathcal{N}_i}). \tag{21}$$

Let $\mathcal{F}_d(G)$ be the set of all the continuous functions having distributed gradients over G. The gradient-based control input (18) is relative and distributed over G for all $i \in \mathcal{V}$ if and only if $v(X) \in \mathcal{F}_r \cap \mathcal{F}_d(G)$.

C. Problem setting

Our goal is to design an indicator $v(X)$ to $\mathcal{T}_{X_*}$ belonging to $\mathcal{F}_r \cap \mathcal{F}_d(G)$. However, its existence depends on the topology of G. Namely, if the edges in graph G are insufficient, we probably cannot find any indicators. Taking this situation into account, we consider the following optimization problem so as to attain (19) as close as possible:

$$\min_{v(X)\in\mathcal{F}_r\cap\mathcal{F}_d(G)\cap\mathcal{F}_0(X_*)} \text{H-dist}(v^{-1}(0), \mathcal{T}_{X_*}), \tag{22}$$

where $\mathcal{F}_0(X_*)$ is the set of all the functions $v : \mathbb{R}^{d\times n} \to \mathbb{R}_+$ taking the minimum 0 at X_*. A solution $v(X)$ of (22) is said to be a *best approximate indicator* to $\mathcal{T}_{X_*}$ in $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$.

Now, the main problem in this paper is given as follows.

Problem 1: For graph G and the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), derive a best approximate indicator $v(X)$ to $\mathcal{T}_{X_*}$ in $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$. Moreover, from its gradient, design the control input $u_i(t)$ in the local frame $\Sigma_i(t)$ relative and distributed over G. ■

As stated above, it depends on the topology of G whether there exists an indicator. The next problem in this paper is to characterize graph topologies from this viewpoint.

Problem 2: For the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), find topologies of graph G such that there exists an indicator $v(X)$ to $\mathcal{T}_{X_*}$ in $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$. ■

Remark 1: Problem 2 requires the existence of an indicator, which satisfies (19). Nevertheless, to solve Problem 2, we just have to check whether one best approximate indicator is an indicator or not. If a best approximate indicator is an indicator, there exists an indicator; otherwise, (22) guarantees that there is no indicator over G. ■

Finally, we investigate local convergence of $X(t)$ to $v^{-1}(0)$ when using the gradient-flow approach. This is because for non-smooth $v(X)$, even local convergence is not guaranteed. Moreover, we consider finding a condition to achieve local or global convergence to $\mathcal{T}_{X_*}$.

Problem 3: For $v(X)$ and $u_i(t)$ designed in Problem 1, confirm that $v^{-1}(0)$ is a locally attractive equilibrium set of the system (12) with the control input $u_i(t)$. Moreover, find a topology of graph G under which $\mathcal{T}_{X_*}$ is a locally or the globally attractive equilibrium set. ■

IV. Solution to Problem 1

A. Preliminaries for pattern matching

In this subsection, first, a conventional approach to pattern matching is introduced. Next, to apply this approach to the multi-agent coordination problem, three relevant results are newly derived.

Now, we consider a rigid body placed in a d-dimensional space. Let $y_1, y_2, \ldots, y_m \in \mathbb{R}^d$ be a sequence of m points on the rigid body in a frame Σ_y, and $z_1, z_2, \ldots, z_m \in \mathbb{R}^d$ be the corresponding sequence in a different frame Σ_z. For every $i \in \{1, 2, \ldots, m\}$, the correspondence between these points is given as

$$y_i = Rz_i + \tau \tag{23}$$

with some rotation matrix $R \in \mathrm{SO}(d)$ and translation vector $\tau \in \mathbb{R}^d$. Pattern matching is a problem to find $(R, \tau) \in \mathrm{SE}(d)$ minimizing the sum of the square errors $\sum_{i=1}^m \|y_i - (Rz_i + \tau)\|^2$ in terms of (23), which is formulated as

$$\min_{(R,\tau)\in\mathrm{SE}(d)} \|Y - (RZ + \tau\mathbf{1}_m^\top)\|^2 \tag{24}$$

with $Y = [y_1\ y_2\ \cdots\ y_m] \in \mathbb{R}^{d\times m}$ and $Z = [z_1\ z_2\ \cdots\ z_m] \in \mathbb{R}^{d\times m}$.

To solve (24), the singular value decomposition (SVD) is used as

$$(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top)(Y - \mathrm{ave}(Y)\mathbf{1}_m^\top)^\top = USV^\top, \tag{25}$$

where $U, V \in \mathbb{R}^{d\times d}$ are orthogonal matrices and $S = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_d)$ are the diagonal matrix whose entries $\sigma_1, \sigma_2, \ldots, \sigma_d$ ($\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_d \geq 0$) are the singular values of the matrix in the left-hand side of (25). Then, the solution to (24) is given as

$$R = V\mathrm{diag}(\overbrace{1, \ldots, 1}^{d-1}, \det(UV))U^\top \tag{26}$$

$$\tau = \mathrm{ave}(Y - RZ). \tag{27}$$

Note that matrices U, V in (25) are not necessarily uniquely determined, and R satisfying (26) is not either. Let $\mathcal{R}_m : \mathbb{R}^{d\times m} \times \mathbb{R}^{d\times m} \to \mathrm{pow}(\mathrm{SO}(d))$ be the set-valued function of Y and Z consisting of all the matrices $R \in \mathrm{SO}(d)$ given by (26) with any orthogonal matrices $U, V \in \mathbb{R}^{d\times d}$ satisfying (25). Then, the conventional result for the pattern matching is obtained as follows.

Lemma 1: *[23] The solution to (24) is given by $R \in \mathcal{R}_m(Y, Z)$ and τ in (27).* ■

We can regard (24) as a function of Y, defined by $f : \mathbb{R}^{d\times m} \to \mathbb{R}_+$ as

$$f(Y) = \min_{(R,\tau)\in\mathrm{SE}(d)} \|Y - (RZ + \tau\mathbf{1}_m^\top)\|^2. \tag{28}$$

Since $f(Y)$ is the min function of the continuous function parameterized by R and τ, $f(Y)$ is continuous. Three properties of $f(Y)$ are derived as follows.

First, $f(Y)$ is differentiable almost everywhere as follows.

Lemma 2: *Assume that $Z \in \mathcal{M}_{dm}$. The function $f(Y)$ is differentiable almost everywhere, whose gradient is given as*

$$\frac{\partial f}{\partial Y}(Y) = 2(Y - \mathrm{ave}(Y)\mathbf{1}_m^\top - R(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top)) \tag{29}$$

with $R \in \mathcal{R}_m(Y, Z)$.

Proof: See Appendix B. ■

Next, the gradient of $f(Y)$ is rotationally and translationally invariant (i.e. $\mathrm{SE}(d)$-invariant [34]) with respect to Y as follows.

Lemma 3: Assume that $Z \in \mathcal{M}_{dm}$. *For any* $(\hat{R}, \hat{\tau}) \in \mathrm{SE}(d)$*, the gradient of* $f(Y)$ *almost everywhere satisfies*

$$\frac{\partial f}{\partial Y}(\hat{R}Y + \hat{\tau}\mathbf{1}_m^\top) = \hat{R}\frac{\partial f}{\partial Y}(Y). \tag{30}$$

Proof: See Appendix C. ■

Finally, the zero set of $f(Y)$ and that of the gradient are equivalent as follows.

Lemma 4: Assume that $Z \in \mathcal{M}_{dm}$. *The following holds:*

$$f^{-1}(0) = \mathcal{Z}\left(\frac{\partial f}{\partial Y}\right). \tag{31}$$

Proof: See Appendix D. ■

B. Design of a best approximate indicator

To solve Problem 1, we first find a class of functions $v(X)$ belonging to $\mathcal{F}_d(G)$. For this purpose, cliques play a crucial role. Actually, the authors' work [35] gives a specific characteristic of a function belonging to $\mathcal{F}_d(G)$ as follows.

Lemma 5: [35] For graph G*, a continuous and differentiable almost everywhere function* $v : \mathbb{R}^{d\times n} \to \mathbb{R}_+$ *belongs to* $\mathcal{F}_d(G)$ *if and only if* $v(X)$ *can be decomposed as*

$$v(X) = \sum_{\mathcal{C}\in \text{M-clq}(G)} v_{\mathcal{C}}([x_j]_{j\in\mathcal{C}}), \tag{32}$$

where $v_{\mathcal{C}} : \mathbb{R}^{d\times|\mathcal{C}|} \to \mathbb{R}_+$ *is continuous, differentiable almost everywhere and depends on* $[x_j]_{j\in\mathcal{C}}$ *for each clique* $\mathcal{C} \in \text{M-clq}(G)$. ■

From Lemma 5, we can design just a function $v_{\mathcal{C}}([x_j]_{j\in\mathcal{C}})$, but cannot change the structure of the sum in (32). For achieving (19), the function

$$v_{\mathcal{C}}([x_j]_{j\in\mathcal{C}}) = \frac{\alpha_{\mathcal{C}}}{2}(\text{dist}([x_j]_{j\in\mathcal{C}}, \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})))^2 \tag{33}$$

with a gain $\alpha_{\mathcal{C}} > 0$ is the most appropriate, which evaluates the discrepancy between X and $\mathcal{T}_{X_*}$ through the projections on the $[x_j]_{j\in\mathcal{C}}$-space. Actually, the following result is obtained, which is the solution to the first part of Problem 1.

Theorem 1: For graph G *and the target set* $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ *in (16), a best approximate indicator to* $\mathcal{T}_{X_*}$ *in* $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$ *is given as*

$$v(X) = \sum_{\mathcal{C}\in \text{M-clq}(G)} \frac{\alpha_{\mathcal{C}}}{2}(\text{dist}([x_j]_{j\in\mathcal{C}}, \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})))^2 \tag{34}$$

for any positive numbers $\alpha_{\mathcal{C}}$ *(*$\mathcal{C} \in \text{M-clq}(G)$*).*

Proof: See Subsection IV-C. ■

To consider the second part of Problem 1, we calculate the gradient of (33). From (6), the projection of $\mathcal{T}_{X_*}$ in (16) onto the $[x_j]_{j\in\mathcal{C}}$-space is described as follows:

$$\mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*}) = \{Y \in \mathbb{R}^{d\times|\mathcal{C}|} : \exists(R,\tau) \in \mathrm{SE}(d) \ \text{s.t.}\ Y = R[x_{*j}]_{j\in\mathcal{C}} + \tau\mathbf{1}_{|\mathcal{C}|}^\top\}. \tag{35}$$

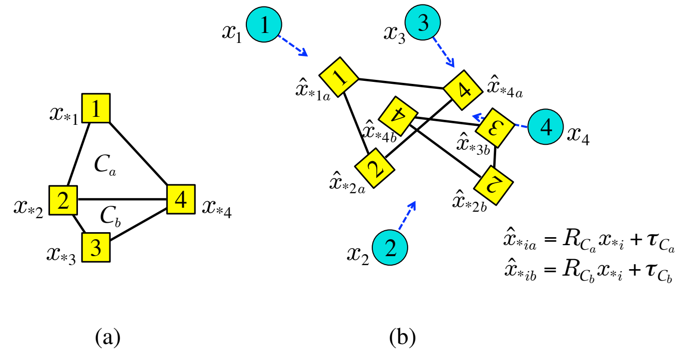

Fig. 6. Operation of the proposed method: (a) desired positions, (b) desired positions with rotation and translation over each clique (squares) and agent positions (circles).

From (7) and (35), (33) is reduced to

$$\begin{aligned} v_{\mathcal{C}}([x_j]_{j\in\mathcal{C}}) &= \frac{\alpha_{\mathcal{C}}}{2}\inf_{Y\in\mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})}\|[x_j]_{j\in\mathcal{C}} - Y\|^2 \\ &= \frac{\alpha_{\mathcal{C}}}{2}\min_{(R,\tau)\in\mathrm{SE}(d)}\|[x_j]_{j\in\mathcal{C}} - (R[x_{*j}]_{j\in\mathcal{C}} + \tau\mathbf{1}_{|\mathcal{C}|}^\top)\|^2. \end{aligned} \tag{36}$$

This is nothing but the optimization problem (24) for

$$Y = [x_j]_{j\in\mathcal{C}},\ Z = [x_{*j}]_{j\in\mathcal{C}},\ m = |\mathcal{C}| \tag{37}$$

and can be solved by the technique of pattern matching as Lemma 1. Moreover, by applying (37) in (29) of Lemma 2, we can derive the gradient of (36). Then, the solution to the second part of Problem 1 is obtained as follows.

Theorem 2: The gradient-based control input (18) with $v(X)$ *in (34) is achieved as*

$$u_i(t) = \sum_{\mathcal{C}\in\text{M-clq}_i(G)} \alpha_{\mathcal{C}}\{\text{ave}([x_j^{[i]}(t)]_{j\in\mathcal{C}}) + R_{\mathcal{C}}(t)(x_{*i} - \text{ave}([x_{*j}]_{j\in\mathcal{C}}))\} \tag{38}$$

for any $R_{\mathcal{C}}(t) \in \mathcal{R}_{|\mathcal{C}|}([x_j^{[i]}(t)]_{j\in\mathcal{C}}, [x_{*j}]_{j\in\mathcal{C}})$ *and any positive numbers* $\alpha_{\mathcal{C}}$. *The control input (38) is relative and distributed over* G.

Proof: See Subsection IV-C. ■

The proposed controller (38) works to achieve the desired formation via *distributed pattern matching*. This is explained in Fig. 6 as follows. Consider the desired positions x_{*i} and the network connections in (a). The corresponding graph consists of two maximal cliques $\mathcal{C}_a$ and $\mathcal{C}_b$ of order 3, forming triangles. Then, the matched desired positions $\hat{x}_{*ik}(t) = R_{\mathcal{C}_k}(t)x_{*i} + \tau_{\mathcal{C}_k}(t)$ $(k = a, b)$, illustrated by the two triangles in (b), are calculated as (38) over each clique $\mathcal{C}_k$ in a distributed way. Then, the agent positions $x_i(t)$ are controlled toward an intermediate point between $\hat{x}_{*ik}(t)$ for cliques $\mathcal{C}_k$ to which agent i belongs. Although the matched desired positions $\hat{x}_{*ia}(t)$ and $\hat{x}_{*ib}(t)$ are separated, they will gradually gather as continuously updated according to (38).

Remark 2: Distance-based formation control, e.g. [17], utilizes objective functions consisting of distance-errors between neighbors such as

$$v(X) = \sum_{\{i,j\}\in\mathcal{E}} \alpha_{ij}(\|x_i - x_j\|^2 - d_{ij}^2)^2 \tag{39}$$

for positive numbers α_{ij}, where $d_{ij} = \|x_{*i} - x_{*j}\|$. Although this function depends only on the distances, its gradient, generating control input, depends not only on the distances but also on the directions of neighbors. Therefore, the information required to the distance-based formation control is the same as the proposed controller (38). ■

C. Proofs of Theorems 1 and 2

Proof of Theorem 1: As a preliminary, we first consider a relaxed problem of (22) as

$$\min_{v(X)\in\mathcal{F}_d(G)\cap\mathcal{F}_0(\mathcal{T}_{X_*})} \text{H-dist}(v^{-1}(0), \mathcal{T}_{X_*}), \tag{40}$$

where $\mathcal{F}_0(\mathcal{T}_{X_*})$ is the set of all the continuous functions $v : \mathbb{R}^{d\times n} \to \mathbb{R}_+$ taking the minimum 0 on $\mathcal{T}_{X_*}$. From $X_* \in \mathcal{T}_{X_*}$,

$$\mathcal{F}_0(\mathcal{T}_{X_*}) \subset \mathcal{F}_0(X_*) \tag{41}$$

holds. Moreover, the following relation is obtained.

Lemma 6: The following inclusion holds:

$$\mathcal{F}_r \cap \mathcal{F}_0(X_*) \subset \mathcal{F}_0(\mathcal{T}_{X_*}). \tag{42}$$

Proof: See Appendix E. ■

Now, (i) we derive a solution of the relaxed problem (40), and (ii) show that this solution belongs to the feasible set of the original problem (22). From (42), the feasible region of (40) contains that of (22). Thus, the solution of (40) is also that of (22).

(i) The following lemma comes from the authors' previous work, which does not limit the set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ to (16).

Lemma 7: [35] For graph G and a closed, non-empty set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$, a solution to (40) is given by (34) for any $\alpha_{\mathcal{C}} > 0$ ($\mathcal{C} \in$ M-clq(G)) if (33) is differentiable almost everywhere for any $\mathcal{C} \in$ M-clq(G). ■

For the target set (16), Lemma 2 guarantees that (33) is differentiable almost everywhere. Thus, from Lemma 7, $v(X)$ in (34) is a solution to (40).

(ii) Consider $v(X)$ in (34). By applying (37) to (30) in Lemma 3 with $\hat{R} = R_i^\top$ and $\hat{\tau} = -R_i^\top x_i$,

$$\begin{aligned} R_i^\top \frac{\partial v_{\mathcal{C}}}{\partial x_i}([x_j]_{j\in\mathcal{C}}) &= \frac{\partial v_{\mathcal{C}}}{\partial x_i}(R_i^\top [x_j]_{j\in\mathcal{C}} - R_i^\top x_i \mathbf{1}_{|\mathcal{C}|}^\top) \\ &= \frac{\partial v_{\mathcal{C}}}{\partial x_i}([R_i^\top (x_j - x_i)]_{j\in\mathcal{C}}) \end{aligned} \tag{43}$$

is obtained. Thus, the gradient of $v_{\mathcal{C}}([x_j]_{j\in\mathcal{C}})$ is of the form (20). Hence, $v_{\mathcal{C}}([x_j]_{j\in\mathcal{C}}) \in \mathcal{F}_r$ holds for any $\mathcal{C} \in$ M-clq(G), and thus $v(X) \in \mathcal{F}_r$ is obtained from (34). This and Lemma 7 guarantee $v(X) \in \mathcal{F}_d(G) \cap \mathcal{F}_0(\mathcal{T}_{X_*}) \cap \mathcal{F}_r$, which leads to $v(X) \in \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*) \cap \mathcal{F}_r$ from (41). Thus, $v(X)$ in (34) belongs to the feasible region of (22). ■

Proof of Theorem 2: From Lemma 2, for clique $\mathcal{C} \in$ M-clq(G), if $i \in \mathcal{C}$,

$$\begin{aligned} \frac{\partial v_{\mathcal{C}}}{\partial x_i}([x_j]_{j\in\mathcal{C}}) = \alpha_{\mathcal{C}}\{&x_i - \text{ave}([x_j]_{j\in\mathcal{C}}) \\ &- R_{\mathcal{C}}(x_{*i} - \text{ave}([x_{*j}]_{j\in\mathcal{C}}))\} \end{aligned} \tag{44}$$

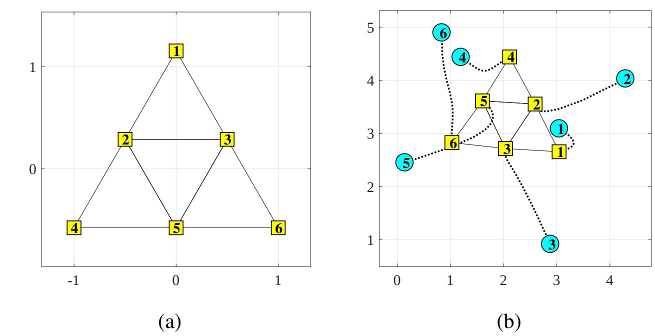

Fig. 7. Simulation result in $\mathbb{R}^2$: (a) desired positions x_{*i}, (b) agent positions $x_i(t)$ at $t = 0$ (circles) and at $t = 40$ (squares).

holds with $R_{\mathcal{C}} \in \mathcal{R}_{|\mathcal{C}|}([x_j]_{j\in\mathcal{C}}, [x_{*j}]_{j\in\mathcal{C}})$; otherwise, the partial derivative is zero. From (13), (43), and (44),

$$\begin{aligned} -R_i^\top \frac{\partial v_{\mathcal{C}}}{\partial x_i}([x_j]_{j\in\mathcal{C}}) &= -\frac{\partial v_{\mathcal{C}}}{\partial x_i}([R_i^\top (x_j - x_i)]_{j\in\mathcal{C}}) \\ &= -\alpha_{\mathcal{C}}\{R_i^\top (x_i - x_i) - \text{ave}([R_i^\top (x_j - x_i)]_{j\in\mathcal{C}}) \\ &\qquad - R_{\mathcal{C}}(x_{*i} - \text{ave}([x_{*j}]_{j\in\mathcal{C}}))\} \\ &= \alpha_{\mathcal{C}}\{\text{ave}([R_i^\top (x_j - x_i)]_{j\in\mathcal{C}}) + R_{\mathcal{C}}(x_{*i} - \text{ave}([x_{*j}]_{j\in\mathcal{C}}))\} \\ &= \alpha_{\mathcal{C}}\{\text{ave}([x_j^{[i]}]_{j\in\mathcal{C}}) + R_{\mathcal{C}}(x_{*i} - \text{ave}([x_{*j}]_{j\in\mathcal{C}}))\} \end{aligned} \tag{45}$$

with $R_{\mathcal{C}} \in \mathcal{R}_{|\mathcal{C}|}([R_i^\top (x_j - x_i)]_{j\in\mathcal{C}}, [x_{*j}]_{j\in\mathcal{C}})$ is obtained. From (32), the gradient-based control input (18) is reduced to

$$u_i(t) = -R_i^\top \frac{\partial v}{\partial x_i}(X(t)) = -\sum_{\mathcal{C}\in\text{M-clq}_i(G)} R_i^\top \frac{\partial v_{\mathcal{C}}}{\partial x_i}([x_j(t)]_{j\in\mathcal{C}}),$$

which leads to (38) with (45). ■

V. Numerical Examples

In this section, the effectiveness of the proposed method is demonstrated by simulations in both $\mathbb{R}^2$ and $\mathbb{R}^3$. In all simulations, the proposed controller (38) is employed with the gain $\alpha_{\mathcal{C}} = 0.25$.

First, consider the multi-agent system consisting of six agents in $\mathbb{R}^2$. In Fig. 7(a), the desired positions $x_{*i} \in \mathbb{R}^2$ ($i \in \mathcal{V}$) and the topology of G are described by the squares and the edges, respectively. This graph consists of four maximal cliques of order 3. The simulation result is depicted in Fig. 7(b), where the circles and squares represent the initial ($t = 0$) and final ($t = 40$) positions of the agents, respectively, and the dotted curves denote the trajectories. It is observed that the desired formation in Fig. 7(a) is almost achieved at $t = 40$ with some rotation and translation. Figs. 8(a)∼(f) show the snapshots of this simulation. The circles represent the current positions $x_i(t)$, and each triangle composed of the squares denotes the matched desired positions $\hat{x}_{*ik}(t) = R_{\mathcal{C}_k}(t)x_{*i} + \tau_{\mathcal{C}_k}(t)$ for each clique $\mathcal{C}_k$, derived from the proposed controller (38). It is observed that at first the matched desired positions are separated according to the cliques, they gradually gather to form the desired formation as Fig. 8(f).

Next, consider seven agents in $\mathbb{R}^3$. In Fig. 9(a), the desired positions $x_{*i} \in \mathbb{R}^3$ ($i \in \mathcal{V}$) and the topology of G are described by the squares and the edges, respectively. This graph

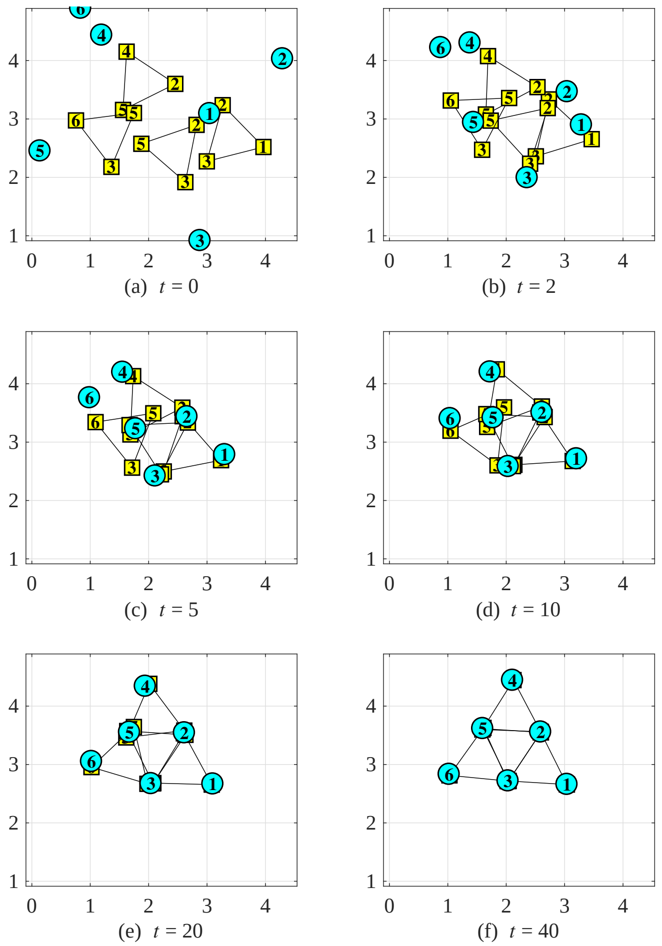

Fig. 8. Snapshots of the simulation: agent positions $x_i(t)$ (circles) and the matched desired positions $\hat{x}_{*ik}(t)$ of each clique $\mathcal{C}_k$ (squares).

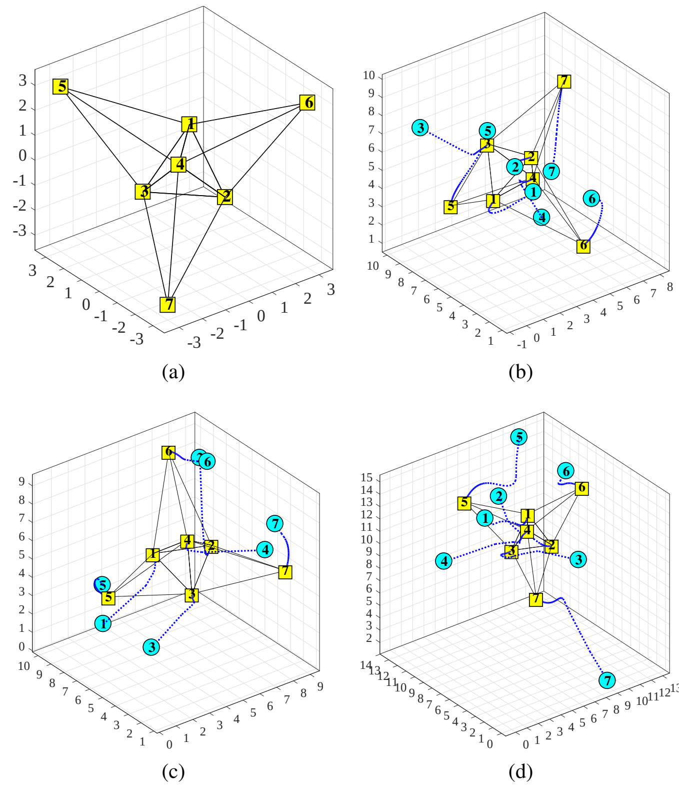

Fig. 9. Simulation results in $\mathbb{R}^3$ from different initial positions: (a) desired positions x_{*i}, (b)∼(d) agent positions $x_i(t)$ at $t = 0$ (circles) and at $t = 40$ (squares).

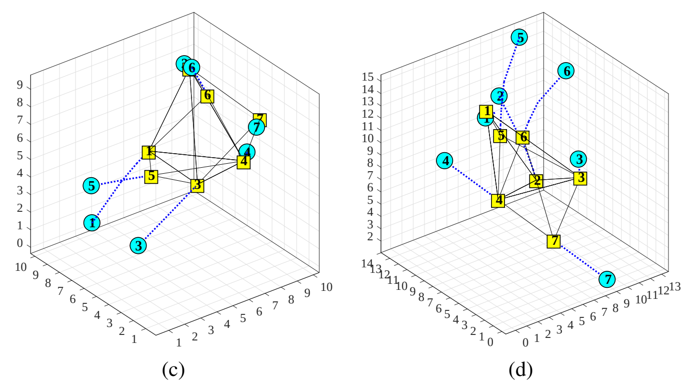

Fig. 10. Simulation results with the existing method from the initial positions of (c), (d) in Fig. 9: agent positions $x_i(t)$ at $t = 0$ (circles) and at $t = 40$ (squares).

consists of four maximal cliques of order 4. The simulation results from different initial positions are depicted in Figs. 9(b)∼(d), where the circles and squares represent the initial ($t = 0$) and final ($t = 40$) positions, and the dotted curves denote the trajectories. It is seen that the desired positions in Fig. 9(a) are finally achieved from any initial positions with different rotations and translations, and that the reflections of the formations do not occur due to the distributed pattern matching.

We compare the proposed method to the existing method, using the distance-based controller with (39). Simulation results with the existing method from the initial positions in Fig. 9(c), (d) are shown in Fig. 10(c), (d), respectively. It is observed that the final agent positions in Fig. 10(c), (d) are different from the desired positions in Fig. 9(a) with any rotations and translations, which is caused by reflections in parts of the formation.

These simulation results show the effectiveness of the proposed method in both $\mathbb{R}^2$ and $\mathbb{R}^3$. Note that a collision avoidance operation can be combined by adding repulsive potential functions to $v(X)$.

VI. Solution to Problem 2

In this section, we find a graph condition to achieve an indicator to $\mathcal{T}_{X_*}$ belonging to $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$. To this end, a new concept of graph connectivity is introduced as follows. For graph G and matrix $X \in \mathbb{R}^{d\times n}$, a pair (G, X) is called a *framework*. For a given graph G and matrix $X_* \in \mathbb{R}^{d\times n}$, framework (G, X_*) is said to be *clique-rigid* if the following holds with framework (G, X) for each matrix $X = [x_1\ x_2\ \cdots\ x_n] \in \mathbb{R}^{d\times n}$:

$$[x_j]_{j\in\mathcal{C}} \in \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})\ \forall \mathcal{C} \in \text{M-clq}(G) \ \Rightarrow\ X \in \mathcal{T}_{X_*}. \quad (46)$$

A framework (G, X_*) is clique-rigid if it is the only framework that can be constructed from the set of the frame-

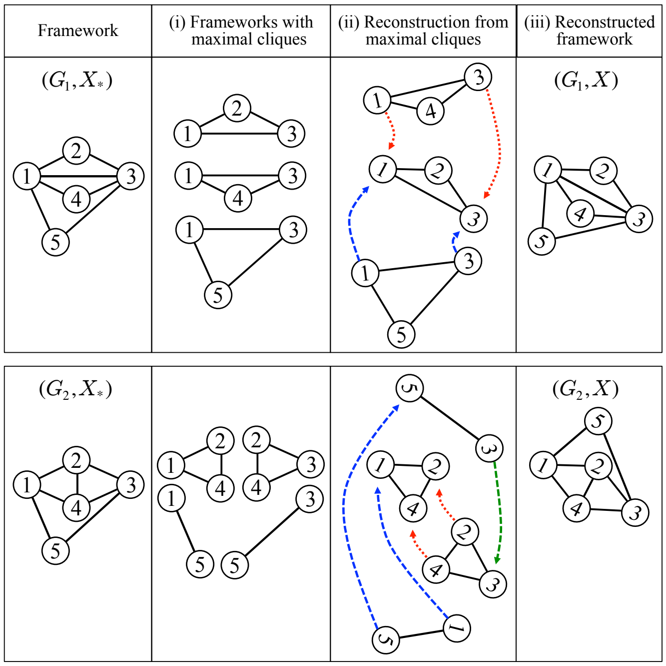

Fig. 11. Meaning of clique-rigidity of frameworks.

works $(G|_{\mathcal{C}}, [x_{*j}]_{j\in\mathcal{C}})$ induced by the maximal cliques $\mathcal{C} \in$ M-clq(G).

Example 3: In Fig. 11, framework (G_1, X_*) is clique-rigid as verified through the following procedure: (i) List all frameworks induced by the maximal cliques in G_1. (ii) Reconstruct a framework from the maximal cliques with some rotations and translations so as to match the vertex indexes. This operation corresponds to the assumption part of (46). (iii) A reconstructed framework (G_1, X) is given. Since it is equivalent to the original framework (G_1, X_*) with some rotation and translation, the conclusion part of (46) is satisfied. For any reconstructions, (46) is satisfied, and thus (G_1, X_*) is clique-rigid. On the other hand, (G_2, X_*) is not clique-rigid because there exists a reconstructed framework (G_2, X) different from (G_2, X_*) with any rotations and translations. ■

The following theorem is given as a solution to Problem 2.

Theorem 3: For graph G and the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), there exists an indicator $v : \mathbb{R}^{d\times n} \to \mathbb{R}_+$ to $\mathcal{T}_{X_*}$ in $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$ if and only if (G, X_*) is clique-rigid. One of such indicators is given by (34).

Proof: First, to show the sufficiency, assume that (G, X_*) is clique-rigid. Consider $v(X)$ in (34), belonging to $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$ from Theorem 1. Moreover, $v(X)$ is an indicator to $\mathcal{T}_{X_*}$ since (19) holds as follows:

$$\begin{aligned} v(X) = 0 &\Leftrightarrow \mathrm{dist}([x_j]_{j\in\mathcal{C}}, \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})) = 0 \ \forall \mathcal{C} \in \text{M-clq}(G) \\ &\Leftrightarrow [x_j]_{j\in\mathcal{C}} \in \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*}) \ \forall \mathcal{C} \in \text{M-clq}(G) \\ &\Leftrightarrow X \in \mathcal{T}_{X_*}, \end{aligned} \tag{47}$$

where the first relation is from (34), the second one is from (7) and the closedness of $\mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})$ from (35), and the third one is from (35) and the clique-rigidity defined by (46).

Next, to show the necessity, assume that (G, X_*) is not clique-rigid. From (46), there exists a matrix $\tilde{X} = [\tilde{x}_1 \ \tilde{x}_2 \ \cdots \ \tilde{x}_n] \in \mathbb{R}^{d\times n}$ such that $[\tilde{x}_j]_{j\in\mathcal{C}} \in \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*}) \ \forall \mathcal{C} \in$ M-clq(G) and $\tilde{X} \notin \mathcal{T}_{X_*}$. Then, $v(\tilde{X}) = 0$ holds for $v(X)$ in (34) from (47). This fact and $\tilde{X} \notin \mathcal{T}_{X_*}$ lead to $v^{-1}(0) \neq \mathcal{T}_{X_*}$. From Theorem 1, $v(X)$ is a solution of (22), which yields

$$\text{H-dist}(\tilde{v}^{-1}(0), \mathcal{T}_{X_*}) \geq \text{H-dist}(v^{-1}(0), \mathcal{T}_{X_*}) > 0$$

for any function $\tilde{v} : \mathbb{R}^{d\times n} \to \mathbb{R}_+$ belonging to $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$, where the last inequality follows from $v^{-1}(0) \neq \mathcal{T}_{X_*}$. Therefore, $\tilde{v}^{-1}(0) \neq \mathcal{T}_{X_*}$ holds; thus $\tilde{v}(X)$ is not an indicator to $\mathcal{T}_{X_*}$. As a result, no function in $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$ can be an indicator. ■

For designing clique-rigid frameworks, the concept of the intersection graph is important. Actually, the following theorem gives a sufficient condition of (G, X_*) to be clique-rigid associated with intersection graphs.

Theorem 4: For graph $G = (\mathcal{V}, \mathcal{E})$ and matrix $X_* \in \mathbb{R}^{d\times n}$, framework (G, X_*) is clique-rigid if the d-intersection graph $\Gamma_d(G) = (\check{\mathcal{V}}, \check{\mathcal{E}}_d)$ of the maximal cliques in G is connected and

$$[x_{*j}]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} \in \mathcal{M}_{d|\mathcal{C}_\ell\cap\mathcal{C}_m|} \ \forall \{\ell, m\} \in \check{\mathcal{E}}_d. \tag{48}$$

Proof: Assume that $\Gamma_d(G) = (\check{\mathcal{V}}, \check{\mathcal{E}}_d)$ is connected. Consider a matrix $X \in \mathbb{R}^{d\times n}$ satisfying the assumption in (46). Then, for each maximal clique $\mathcal{C}_\ell \in$ M-clq(G), $[x_j]_{j\in\mathcal{C}_\ell} \in \mathcal{P}_{\mathcal{C}_\ell}(\mathcal{T}_{X_*})$ holds, which indicates

$$[x_j]_{j\in\mathcal{C}_\ell} = R_\ell [x_{*j}]_{j\in\mathcal{C}_\ell} + \tau_\ell \mathbf{1}_{|\mathcal{C}_\ell|}^\top \tag{49}$$

with some $(R_\ell, \tau_\ell) \in \mathrm{SE}(d)$ from (35). Consider two maximal cliques $\mathcal{C}_\ell$ and $\mathcal{C}_m$, namely $\ell, m \in \check{\mathcal{V}}$, satisfying $\{\ell, m\} \in \check{\mathcal{E}}_d$, and then $|\mathcal{C}_\ell \cap \mathcal{C}_m| \geq d$ holds from the definition of the interconnection graph. From (3) and (49),

$$\begin{aligned} [x_j]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} M_{|\mathcal{C}_\ell\cap\mathcal{C}_m|} &= R_\ell [x_{*j}]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} M_{|\mathcal{C}_\ell\cap\mathcal{C}_m|} \\ &= R_m [x_{*j}]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} M_{|\mathcal{C}_\ell\cap\mathcal{C}_m|} \end{aligned} \tag{50}$$

is obtained. From (5), (48), and $|\mathcal{C}_\ell \cap \mathcal{C}_m| \geq d$,

$$\begin{aligned} \mathrm{rank}([x_{*j}]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} M_{|\mathcal{C}_\ell\cap\mathcal{C}_m|}) &= \min\{d, |\mathcal{C}_\ell \cap \mathcal{C}_m| - 1\} \\ &\geq d - 1 \end{aligned} \tag{51}$$

holds. From (50), (51), and the fact that $R_\ell, R_m \in \mathrm{SO}(d)$, we obtain $R_\ell = R_m$. Then, from (49),

$$[x_j]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} - R_\ell [x_{*j}]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} = \tau_\ell \mathbf{1}_{|\mathcal{C}_\ell\cap\mathcal{C}_m|}^\top = \tau_m \mathbf{1}_{|\mathcal{C}_\ell\cap\mathcal{C}_m|}^\top$$

is obtained, and averaging row-wise the matrix leads to

$$\tau_\ell = \tau_m = \frac{1}{|\mathcal{C}_\ell \cap \mathcal{C}_m|} \mathrm{ave}([x_j]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m} - R_\ell [x_{*j}]_{j\in\mathcal{C}_\ell\cap\mathcal{C}_m}).$$

Since $\Gamma_d(G)$ is connected, (R_ℓ, τ_ℓ) for all the maximal cliques $\mathcal{C}_\ell \in$ M-clq(G) agree with each other in the same way. Since all the $(R_\ell, \tau_\ell) \in \mathrm{SE}(d)$ are equal for $\ell \in \check{\mathcal{V}}$, define $(\tilde{R}, \tilde{\tau}) \in \mathrm{SE}(d)$ as those common elements. Since each vertex is contained by a maximal clique, from (49), $X = \tilde{R} X_* + \tilde{\tau} \mathbf{1}_n^\top$ is obtained. Thus, $X \in \mathcal{T}_{X_*}$ holds from (16) and the conclusion part of (46) is achieved. Therefore, (G, X_*) is clique-rigid. ■

We compare the clique-rigidity with conventional connectivities, such as global rigidity and rigidity, defined as follows [36]: First, framework (G, X_*) is said to be *globally rigid* if

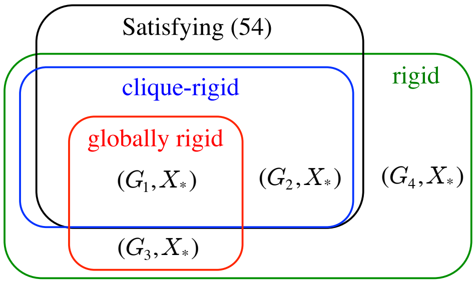

Fig. 12. Relations between clique-rigid, globally rigid, and rigid frameworks.

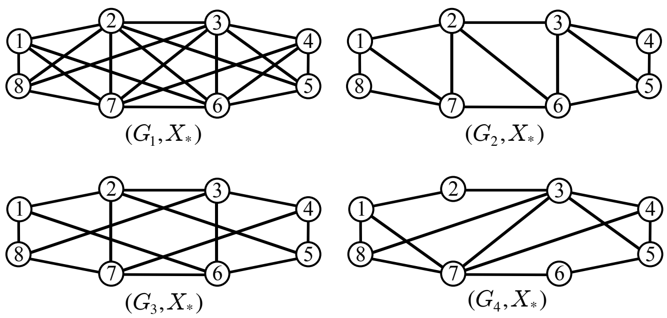

Fig. 13. Examples of frameworks corresponding to Fig. 12.

the following holds for framework (G, X) with each matrix $X \in \mathbb{R}^{d\times n}$:

$$\begin{aligned} &\|x_i - x_j\| = \|x_{*i} - x_{*j}\| \ \forall \{i,j\} \in \mathcal{E} \\ &\quad \Rightarrow \|x_i - x_j\| = \|x_{*i} - x_{*j}\| \ \forall i,j \in \mathcal{V}. \end{aligned} \tag{52}$$

Next, framework (G, X_*) is said to be *rigid* if the following holds for framework $(G, \Phi(s))$ with each continuous matrix-valued function $\Phi = [\phi_1 \ \phi_2 \ \cdots \ \phi_n] : [0, 1] \to \mathbb{R}^{d\times n}$ satisfying $\Phi(0) = X_*$:

$$\begin{aligned} &\|\phi_i(s) - \phi_j(s)\| = \|x_{*i} - x_{*j}\| \ \forall \{i,j\} \in \mathcal{E}, s \in [0,1] \\ &\quad \Rightarrow \|\phi_i(s) - \phi_j(s)\| = \|x_{*i} - x_{*j}\| \ \forall i,j \in \mathcal{V}, s \in [0,1]. \end{aligned} \tag{53}$$

Then, the following two theorems are given.

Theorem 5: For graph G and matrix $X_* \in \mathbb{R}^{d\times n}$, framework (G, X_*) is clique-rigid if (G, X_*) is globally rigid and there exists a clique $\mathcal{C}$ satisfying

$$|\mathcal{C}| \geq d + 1, \quad [x_{*j}]_{j\in\mathcal{C}} \in \mathcal{M}_{d|\mathcal{C}|}. \tag{54}$$

Proof: See Appendix F. ■

Theorem 6: For graph G and matrix $X_* \in \mathbb{R}^{d\times n}$ if framework (G, X_*) is clique-rigid, (G, X_*) is rigid.

Proof: See Appendix G. ■

These concepts of rigidity relate to each other as shown in Fig. 12, which indicates the following: (i) The clique-rigidity is stronger than the rigidity. (ii) If there is a clique satisfying (54), the clique-rigidity is weaker than the global rigidity. Fig. 13 shows examples of frameworks in $\mathbb{R}^2$, corresponding to frameworks $(G_1, X_*) \sim (G_4, X_*)$ in Fig. 12. As shown in Fig. 13, clique-rigid framework (G_2, X_*) in $\mathbb{R}^2$ consists of triangles (cliques of order 3); globally rigid framework (G_3, X_*) consists of quadrangles (non-cliques) some of whose edges connect distant vertexes.

Remark 3: In the setting of this paper, the formation control is not necessarily achievable over just a connected graph because the control input is restricted to a relative one as (14). If the agents could exchange the direct information on (R_i, τ_i), they just had to reach a consensus of (R_i, τ_i) over a connected graph. However, $(R_i, \tau_i) \in \mathrm{SE}(d)$ cannot be expressed through the relative states $x_j^{[i]}(t)$, defined as (13). Actually, Theorem 3 shows that the clique-rigidity is necessary to achieve the formation control via relative states, and the condition of graph topologies cannot be relaxed. ■

VII. Solution to Problem 3

In this section, we investigate the convergence properties of the system (12) with the control input (38). This control input is derived from the gradient-based one (18) with $v(X)$ in (34), which is not differentiable everywhere. To deal with such non-smooth dynamics, the following lemma is available, derived from the non-smooth version of LaSalle's invariance theorem [37], [38], [39].

Lemma 8: Let $v : \mathbb{R}^{d\times n} \to \mathbb{R}_+$ be a locally Lipschitz, regular function and $X(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)] \in \mathbb{R}^{d\times n}$ be a Filippov solution of (17). Then, for a compact positively invariant set $\Omega \subset \mathbb{R}^{d\times n}$ of $X(t)$, from any initial state $X(0) \in \Omega$, $v(X(t))$ is non-increasing with respect to t, and (10) holds for $\mathcal{A} = \mathcal{Z}(\partial v/\partial X) \cap \Omega$ and $\mathcal{U} = \Omega$.

Proof: See Appendix H. ■

To apply Lemma 8 to $v(X)$ in (34), we have to verify the existence of a compact positively invariant set Ω. Since this $v(X)$ is not radially bounded, its level set is not compact. Instead, we construct Ω by using the fact that $X(t)$ is bounded as shown in the following lemma.

Lemma 9: For graph G and the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), let $X(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)] \in \mathbb{R}^{d\times n}$ be the solution of the system (12) with the control input (38). Then, (i) the average of $x_i(t)$ for $i \in \mathcal{V}$ belonging to each connected component of G is constant and (ii) $\|X(t)\|$ is bounded.

Proof: See Appendix I. ■

From Lemmas 8 and 9, we can derive the convergence properties of the target system as follows.

Theorem 7: For graph G, the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), and the function $v(X)$ in (34), (i) $\mathcal{Z}(\partial v/\partial X)$ is the globally attractive equilibrium set, and (ii) $v^{-1}(0)$ is a locally attractive equilibrium set of the system (12) with the control input (38).

Proof of (i): Let $X(t)$ be the solution of of (12) with (38) starting from $X(0) = X_0 \in \mathbb{R}^{d\times n}$, and $\Omega(X_0) \subset \mathbb{R}^{d\times n}$ be a compact set containing $X(t)$ for all $t \geq 0$. Such $\Omega(X_0)$ exists due to the boundedness of $\|X(t)\|$ from Lemma 9 (ii). From Theorem 2, the control input (38) is reduced to (18) with $v(X)$ in (34), with which (12) leads to (17). Therefore, from Lemma 8, (10) holds for $\mathcal{A} = \mathcal{Z}(\partial v/\partial X) \cap \Omega(X_0)$ and $\mathcal{U} = \Omega(X_0)$. Since this is the case for any $X_0 \in \mathbb{R}^{d\times n}$, (10) holds for $\mathcal{A} = \mathcal{Z}(\partial v/\partial X)$ and $\mathcal{U} = \mathbb{R}^{d\times n}$. Thus, $\mathcal{Z}(\partial v/\partial X)$ is the globally attractive equilibrium set.

Proof of (ii): Since $\mathrm{SO}(d)$ and $\mathbb{R}^d$ are analytic manifolds (i.e. manifolds with analytic transition maps), $\mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})$ in

(35) is an analytic manifold. Thus, the function $v_{\mathcal{C}}([x_j]_{j\in\mathcal{C}})$ in (33) is analytic on an open set around $v_{\mathcal{C}}^{-1}(0)$ [40], [41]. Therefore, $v(X)$ in (34) is analytic on an open set $\mathcal{O}$ around $\bigcap_{\mathcal{C}\in\text{M-clq}(G)} v_{\mathcal{C}}^{-1}(0) = v^{-1}(0)$. For analytic functions, Łojasiewicz's inequality [42], [43] is available, and there exists $\theta \in (0,1)$ such that

$$(v(X))^{\theta} \leq \left\| \frac{\partial v(X)}{\partial X} \right\| \quad \forall X \in \mathcal{O}. \tag{55}$$

Let $\mathcal{L}(v,c) = \{X \in \mathbb{R}^{d\times n} : v(X) \leq c\}$ be the level set of $v(X)$ for $c > 0$. Since $\mathcal{O}$ is open, $v^{-1}(0)$ is closed, and $\mathcal{L}(v,0) = v^{-1}(0) \subset \mathcal{O}$, $\mathcal{L}(v,c_0) \subset \mathcal{O}$ holds for a sufficiently small $c_0 > 0$. This inclusion and (55) yield

$$v^{-1}(0) \supset \mathcal{Z}\left(\frac{\partial v}{\partial X}\right) \cap \mathcal{O} \supset \mathcal{Z}\left(\frac{\partial v}{\partial X}\right) \cap \mathcal{L}(v,c_0),$$

which leads to

$$\text{dist}(X, v^{-1}(0)) \leq \text{dist}\left(X, \mathcal{Z}\left(\frac{\partial v}{\partial X}\right) \cap \mathcal{L}(v,c_0)\right). \tag{56}$$

Let $X(t)$ be the solution from $X(0)$ satisfying $v(X(0)) \leq c_0$. Then, $X(t) \in \mathcal{L}(v,c_0)$ holds for any $t \geq 0$ since $v(X(t))$ is non-increasing from Lemma 8. Therefore,

$$\text{dist}\left(X(t), \mathcal{Z}\left(\frac{\partial v}{\partial X}\right) \cap \mathcal{L}(v,c_0)\right) = \text{dist}\left(X(t), \mathcal{Z}\left(\frac{\partial v}{\partial X}\right)\right) \tag{57}$$

is achieved. From (i), (56), and (57), we obtain $\lim_{t\to\infty} \text{dist}(X(t), v^{-1}(0)) = 0$. Thus, (10) is satisfied for $\mathcal{A} = v^{-1}(0)$ and $\mathcal{U} = \text{int}(\mathcal{L}(v,c_0))$, where $\text{int}(\cdot)$ is the interior of a set. Therefore, $v^{-1}(0)$ is a locally attractive equilibrium set. ■

Now, we can guarantee the local convergence to the target set for clique-rigid frameworks as follows.

Theorem 8: For graph G and the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), if framework (G, X_*) is clique-rigid, $\mathcal{T}_{X_*}$ is a locally attractive equilibrium set of the system (12) with the control input (38).

Proof: For a clique-rigid framework (G, X_*), Theorem 3 guarantees that (34) is an indicator, satisfying (19). From (19) and Theorem 8 (ii), $\mathcal{T}_{X_*}$ is a locally attractive equilibrium set of the system (12) with the control input (38). ■

It is known that conventional methods for distance-based formation control do not guarantee global convergence to the desired formation even over the complete graph [17]. On the other hand, the proposed method achieves global convergence to $\mathcal{T}_{X_*}$ over the complete graph as follows.

Theorem 9: For graph G and the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), if G is complete, $\mathcal{T}_{X_*}$ is the globally attractive equilibrium set of the system (12) with the control input (38).

Proof: If G is complete, $\text{M-clq}(G) = \{\mathcal{V}\}$ holds. Then, $v(X)$ in (34) consists of the only one term (33) for $\mathcal{C} = \mathcal{V}$, and $v^{-1}(0) = \mathcal{Z}(\partial v/\partial X)$ holds from Lemma 4. Moreover, (46) always holds and (G, X_*) is clique-rigid for any X_*. Thus, Theorem 3 guarantees that $v(X)$ is an indicator, namely, (19) is achieved. Then, we obtain $\mathcal{T}_{X_*} = v^{-1}(0) = \mathcal{Z}(\partial v/\partial X)$, and thus Theorem 7 (i) guarantees that $\mathcal{T}_{X_*}$ is the globally attractive equilibrium set. ■

The necessary condition to realize the global convergence via relative and distributed gradient-based control input is the clique-rigidity as follows.

Theorem 10: For graph G and the target set $\mathcal{T}_{X_*} \subset \mathbb{R}^{d\times n}$ in (16), only if framework (G, X_*) is clique-rigid, there is a function $v : \mathbb{R}^{d\times n} \to \mathbb{R}$ in $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$ such that $\mathcal{T}_{X_*}$ is the globally attractive equilibrium set of the system (12) with the gradient-based control input (18).

Proof: Assume that framework (G, X_*) is not clique-rigid, and Theorem 3 guarantees that there is no indicator in $\mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$. Then, consider one function $v(X) \in \mathcal{F}_r \cap \mathcal{F}_d(G) \cap \mathcal{F}_0(X_*)$, and $v(X)$ does satisfy (19). Because $v(X) \in \mathcal{F}_r \cap \mathcal{F}_0(X_*)$, $v(X) \in \mathcal{F}_0(\mathcal{T}_{X_*})$ is obtained from Lemma 6, which implies that $v(X) = 0$ for any $X \in \mathcal{T}_{X_*}$. Therefore, $\mathcal{T}_{X_*} \subset v^{-1}(0)$ holds. From this inclusion and the fact that (19) does not hold, there exists a matrix $X_0 \in v^{-1}(0)\backslash\mathcal{T}_{X_*}$. Because $v^{-1}(0)$ is an equilibrium set, $X(t)$ with the initial state $X(0) = X_0$ stays at $X_0 \notin \mathcal{T}_{X_*}$ for all time and does not converge to $\mathcal{T}_{X_*}$. Therefore, $\mathcal{T}_{X_*}$ is not the globally attractive equilibrium set. ■

Theorems 9 and 10 give a sufficient and a necessary condition, respectively, for the global convergence to $\mathcal{T}_{X_*}$. Thus, there exist graph topologies ensuring the global convergence between clique-rigid frameworks and the complete graph.

VIII. Conclusion

This paper addressed a distributed formation control problem using relative positions and local bearings. The idea is to regard this problem as pattern matching of image data, which enabled us to propose a formation control method via distributed pattern matching. In this way, the proposed method successfully combined the techniques of the different fields: control theory (distributed control) and computer vision (pattern matching). Next, a strict network condition is derived to achieve the desired formation by introducing the clique-rigidity. A graph based on the Euclidean distance naturally derives clique-rigid frameworks rather than globally rigid ones. For example, the Delaunay graph always gives clique-rigid frameworks in the two-dimensional space, known as the Delaunay triangulation [44]. From this nature, the proposed method is more reliable in many practical situations. Finally, the simulations demonstrated the effectiveness of this method regardless of the dimension of the working space. Future work includes generalization of the concept of the relative states from (11) to extend the class of observable outputs. We should investigate what kind of relative states are suitable for controller design and practical application.

Appendix A
Deviation of (12)

From the assumption above (12), the velocity of agent i is controlled by $u_i(t)$ in $\Sigma_i(t)$, which means that

$$\lim_{\delta\to+0} \frac{x_i^{[i]}(t+\delta) - x_i^{[i]}(t)}{\delta} = u_i(t). \tag{58}$$

Note that $x_i^{[i]}(t) = 0$ is achieved by replacing z with $x_i(t)$ in (11). From this, by replacing z with $x_i(t+\delta)$ in (11) for $\delta > 0$ and dividing the resultant equation by δ,

$$\frac{x_i(t+\delta) - x_i(t)}{\delta} = R_i \frac{x_i^{[i]}(t+\delta) - x_i^{[i]}(t)}{\delta} \tag{59}$$

is obtained. Take the limits of δ to zero in (59), and (12) is achieved with (58). ■

Appendix B
Proof of Lemma 2

From (4), (29) is rewritten as

$$\frac{\partial f}{\partial Y}(Y) = 2(YM_m - RZM_m). \tag{60}$$

The rest of the proof is devoted to showing that (i) RZM_m is uniquely defined almost everywhere, and that (ii) (60) holds.

(i) If $m \geq d+1$, because the matrix in the left-hand side of (25) has full rank almost everywhere from $Z \in \mathcal{M}_{dm}$, the SVD in (25) is uniquely determined almost everywhere, and so is R in (26). Thus, only the case that $m \leq d$ has to be considered as the following lemma.

Lemma 10: For $Z \in \mathcal{M}_{dm}$ and $m \leq d$, R in (26) satisfies

$$RZM_m = V_1 U_1^\top ZM_m, \tag{61}$$

and RZM_m is unique almost everywhere in terms of $Y \in \mathbb{R}^{d\times m}$, where $U_1, V_1 \in \mathbb{R}^{d\times(m-1)}$ are the matrices such that the SVD in (25) is achieved with

$$S = \mathrm{diag}(S_1, \overbrace{0,\dots,0}^{d-m+1}),\ U = [U_1\ U_2],\ V = [V_1\ V_2]. \tag{62}$$

Proof: From (25) and (62),

$$\begin{aligned}((YM_m)(U_2^\top ZM_m)^\top)^\top &= U_2^\top ZM_m(YM_m)^\top = U_2^\top USV^\top \\ &= U_2^\top[U_1\ U_2]\mathrm{diag}(S_1,0,\dots,0)V^\top \\ &= [O\ E_{d-m+1}]\mathrm{diag}(S_1,0,\dots,0)V^\top \\ &= 0 \end{aligned} \tag{63}$$

holds, where $O \in \mathbb{R}^{(d-m+1)\times(m-1)}$ is the zero matrix. Because $YM_m \in \mathbb{R}^{d\times m}$ and $m \leq d$, $\mathrm{rank}(YM_m) = m-1$ holds almost everywhere. Then, the matrix $YM_m \in \mathbb{R}^{d\times m}$ has the kernel of one dimension spanned by $\mathbf{1}_m$ because $YM_m\mathbf{1}_m = 0$ holds from (3). Then, (63) yields

$$(U_2^\top ZM_m)^\top = \mathbf{1}_m c \tag{64}$$

for some $c \in \mathbb{R}^{1\times(d-m+1)}$. By multiplying $\mathbf{1}_m^\top/m$ for (64) from the left, $0 = c$ is obtained from (3), which leads to $U_2^\top ZM_m = 0$ from (64). From this, (26), and (62),

$$\begin{aligned} RZM_m &= (V_1U_1^\top + V_2\mathrm{diag}(\overbrace{1,\dots,1}^{d-m}\det(UV))U_2^\top)ZM_m \\ &= V_1U_1^\top ZM_m \end{aligned}$$

is obtained, which yields (61). Because U_1 and V_1 in the SVD (25) with the decomposition (62) are unique almost everywhere, so is RZM_m. ■

(ii) From Lemma 1, $f(Y)$ in (28) is reduced to

$$f(Y) = \|Y - \mathrm{ave}(Y)\mathbf{1}_m^\top - R(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top)\|^2 \tag{65}$$

with $R \in \mathcal{R}_m(Y,Z)$. Let

$$D = \mathrm{diag}(\overbrace{1,\dots,1}^{d-1}, \det(UV)). \tag{66}$$

From (4) and (26), (65) is reduced to

$$\begin{aligned} f(Y) &= \|YM_m - VDU^\top ZM_m\|^2 \\ &= \mathrm{tr}((YM_m)^\top YM_m) - 2\mathrm{tr}((YM_m)^\top VDU^\top ZM_m) \\ &\quad + \mathrm{tr}((ZM_m)^\top ZM_m). \end{aligned} \tag{67}$$

As shown below, the gradients of the terms in (67) are calculated as

$$\frac{\partial \mathrm{tr}((YM_m)^\top YM_m)}{\partial Y} = 2YM_m \tag{68}$$

$$\frac{\partial \mathrm{tr}((YM_m)^\top VDU^\top ZM_m)}{\partial Y} = RZM_m, \tag{69}$$

and (60) is achieved.

Since (68) is obvious, just (69) is shown. Each component of the left-hand side of (69) is calculated as

$$\begin{aligned}\frac{\partial \mathrm{tr}((YM_m)^\top VDU^\top ZM_m)}{\partial y_{ij}} &= \mathrm{tr}\left(M_m\frac{\partial Y^\top}{\partial y_{ij}}VDU^\top ZM_m\right) \\ &\quad + \mathrm{tr}\left((YM_m)^\top\frac{\partial VDU^\top}{\partial y_{ij}}ZM_m\right)\end{aligned} \tag{70}$$

for $Y = [y_{ij}]$, where the matrices U, V, and D depend on Y as (25) and (66). The first term of the right-hand side of (70) is reduced to

$$\begin{aligned}&\mathrm{tr}\left(M_m\frac{\partial Y^\top}{\partial y_{ij}}VDU^\top ZM_m\right) = \mathrm{tr}((e_{di}e_{mj}^\top)^\top RZM_m^2) \\ &= \mathrm{tr}(e_{di}^\top RZM_m e_{mj}) = e_{di}^\top RZM_m e_{nj}\end{aligned}$$

from (26) and (66). As shown below, the second term of the right-hand side of (70) is zero, and (69) is obtained.

The rest of this proof is devoted to showing that the second term of the right-hand side of (70) is zero. This term is reduced to

$$\begin{aligned}&\mathrm{tr}\left((YM_m)^\top\frac{\partial VDU^\top}{\partial y_{ij}}ZM_m\right) \\ &= \mathrm{tr}\left(ZM_m(YM_m)^\top\frac{\partial VDU^\top}{\partial y_{ij}}\right) \\ &= \mathrm{tr}\left(USV^\top\left(\frac{\partial V}{\partial y_{ij}}DU^\top + V\frac{\partial D}{\partial y_{ij}}U^\top + VD\frac{\partial U^\top}{\partial y_{ij}}\right)\right) \\ &= \mathrm{tr}\left(V^\top\frac{\partial V}{\partial y_{ij}}DS\right) + \mathrm{tr}\left(S\frac{\partial D}{\partial y_{ij}}\right) + \mathrm{tr}\left(SD\frac{\partial U^\top}{\partial y_{ij}}U\right)\end{aligned} \tag{71}$$

from (4) and (25). The partial derivative of the constraint of the orthogonal matrix V, say $V^\top V = E_d$, is derived as follows:

$$\left(V^\top\frac{\partial V}{\partial y_{ij}}\right)^\top + V^\top\frac{\partial V}{\partial y_{ij}} = 0.$$

Hence, $V^\top(\partial V/\partial y_{ij})$ is skew-symmetric, and all the diagonal entries of this matrix are zero. Then, all the diagonal entries of the matrix $V^\top(\partial V/\partial y_{ij})DS$ are zero because D and S are diagonal. Hence,

$$\mathrm{tr}\left(V^\top \frac{\partial V}{\partial y_{ij}}\right) = 0 \tag{72}$$

$$\mathrm{tr}\left(V^\top \frac{\partial V}{\partial y_{ij}} DS\right) = 0 \tag{73}$$

are achieved. In the same way,

$$\mathrm{tr}\left(SD \frac{\partial U^\top}{\partial y_{ij}} U\right) = 0 \tag{74}$$

is obtained. From (72),

$$\frac{\partial \det(V)}{\partial y_{ij}} = \det(V)\mathrm{tr}\left(V^\top \frac{\partial V}{\partial y_{ij}}\right) = 0$$

is derived. Similarly, $\partial \det(U)/\partial y_{ij} = 0$ is obtained. Then,

$$\begin{aligned}\frac{\partial(\det(UV))}{\partial y_{ij}} &= \frac{\partial \det(U)}{\partial y_{ij}}\det(V) + \det(U)\frac{\partial \det(V)}{\partial y_{ij}} \\ &= 0\end{aligned} \tag{75}$$

is derived. From (66) and (75), $\partial D/\partial y_{ij} = 0$ is achieved. From this, (73), and (74), the right-hand side of (71) is zero. ■

Appendix C
Proof of Lemma 3

Let

$$\bar{Y} = \hat{R}Y + \hat{\tau}\mathbf{1}_m^\top \tag{76}$$

and consider a matrix $\bar{R} \in \mathcal{R}_m(\bar{Y}, Z)$. Then, from Lemma 1, there exist orthogonal matrices $\bar{U}, \bar{V} \in \mathbb{R}^{d\times d}$ satisfying

$$\bar{R} = \bar{V}\mathrm{diag}(1,\ldots,1,\det(\bar{U}\bar{V}))\bar{U}^\top \tag{77}$$

$$(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top)(\bar{Y} - \mathrm{ave}(\bar{Y})\mathbf{1}_m^\top)^\top = \bar{U}\bar{S}\bar{V}^\top \tag{78}$$

and $\bar{S} = \mathrm{diag}(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_d)$ for the singular values $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_d$ ($\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq \cdots \geq \bar{\sigma}_d \geq 0$) of the matrix in the left-hand side of (78). From (3), (4), (25), and (76),

$$\begin{aligned}(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top)(\bar{Y} - \mathrm{ave}(\bar{Y})\mathbf{1}_m^\top)^\top &= ZM_m(\bar{Y}M_m)^\top \\ = ZM_m(\hat{R}YM_m)^\top &= (ZM_m(YM_m)^\top)\hat{R}^\top \\ = (USV^\top)\hat{R}^\top &= US(\hat{R}V)^\top\end{aligned} \tag{79}$$

is derived.

Assume that $m \geq d+1$, and from (78) and (79), $\bar{S} = S$ holds, and

$$\bar{U} = U,\ \bar{V} = \hat{R}V \tag{80}$$

hold almost everywhere because of the uniqueness of the SVD. Replace the matrices as (80) in (77), and

$$\bar{R} = \hat{R}V\mathrm{diag}(1,\ldots,1,\det(U\hat{R}V))U^\top = \hat{R}R \tag{81}$$

is derived for $R \in \mathcal{R}_m(Y,Z)$ from (26) and $\det(\hat{R}) = 1$. From (3), (29), (76), and (81),

$$\begin{aligned}\frac{\partial f}{\partial Y}(\bar{Y}) &= 2(\bar{Y}M_m - \bar{R}ZM_m) = 2(\hat{R}YM_m - \hat{R}RZM_m) \\ &= \hat{R}\frac{\partial f}{\partial Y}(Y)\end{aligned} \tag{82}$$

is obtained, which yields (30).

Assume that $m \leq d$, and the SVD in (25) is achieved with the matrices in (62) for $U_1, V_1 \in \mathbb{R}^{d\times(m-1)}$. In the same way, the SVD in (78) is obtained with the matrices

$$\bar{S} = \mathrm{diag}(\bar{S}_1, \overbrace{0,\ldots,0}^{d-m+1}),\ \bar{U} = [\bar{U}_1\ \bar{U}_2],\ \bar{V} = [\bar{V}_1\ \bar{V}_2] \tag{83}$$

for $\bar{U}_1, \bar{V}_1 \in \mathbb{R}^{d\times(m-1)}$. From (78) and (79), $S_1 = \bar{S}_1$ holds, and $\bar{U}_1 = U_1$ and $\bar{V}_1 = \hat{R}V_1$ hold almost everywhere. Then, from Lemma 10, the equations

$$\bar{R}ZM_m = \bar{V}_1\bar{U}_1^\top ZM_m = \hat{R}V_1U_1^\top ZM_m = \hat{R}RZM_m$$

are obtained. Hence, (82) holds also in this case, and (30) is achieved. ■

Appendix D
Proof of Lemma 4

Consider a matrix $\tilde{Y} \in f^{-1}(0)$, then from (65), $\tilde{Y} = \mathrm{ave}(\tilde{Y})\mathbf{1}_m^\top + R(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top)$ is obtained with some $R \in \mathcal{R}_m(\tilde{Y}, Z)$. Replace this in the gradient of $f(Y)$ for $Y = \tilde{Y}$, and from Lemmas 2 and 3,

$$\begin{aligned}\frac{\partial f}{\partial Y}(\tilde{Y}) &= \frac{\partial f}{\partial Y}(\mathrm{ave}(\tilde{Y})\mathbf{1}_m^\top + R(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top)) \\ &= R\frac{\partial f}{\partial Y}(Z) \\ &= 2R(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top - \tilde{R}(Z - \mathrm{ave}(Z)\mathbf{1}_m^\top))\end{aligned}$$

holds almost everywhere with any $\tilde{R} \in \mathcal{R}_m(Z,Z)$, which yields $0 \in \mathcal{K}[\partial f/\partial Y](\tilde{Y})$ because $\tilde{R} = E_d$ can be assigned from $E_d \in \mathcal{R}_m(Z,Z)$. Thus, $\tilde{Y} \in \mathcal{Z}(\partial f/\partial Y)$ holds from (9). The opposite relation can be verified in the same way. ■

Appendix E
Proof of Lemma 6

Let $\partial v : \mathbb{R}^{d\times n} \to \mathrm{pow}(\mathbb{R}^{d\times n})$ be the generalized gradient of $v(X)$ [45], defined by

$$\partial v(X) = \mathcal{K}\left[\frac{\partial v}{\partial X}\right](X). \tag{84}$$

Let $\mathcal{D}_F v : \mathbb{R}^{d\times n} \to \mathrm{pow}(\mathbb{R})$ be the set-valued derivative of v with respect to F, defined by

$$\begin{aligned}\mathcal{D}_F v(X) = \{a \in \mathbb{R} : \exists W \in \mathcal{K}[F](X) \\ \text{s.t.}\ \mathrm{tr}(P^\top W) = a\ \ \forall P \in \partial v(X)\}.\end{aligned} \tag{85}$$

Then, the following holds to system (8).

Lemma 11: [37] For a measurable and essentially locally bounded matrix-valued function $F : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$, let $X(t) \in \mathbb{R}^{d\times n}$ be the solution of system (8). For a locally Lipschitz and regular function $v : \mathbb{R}^{d\times n} \to \mathbb{R}$, the following holds almost everywhere:

$$\frac{\mathrm{d}v(X(t))}{\mathrm{d}t} \in \mathcal{D}_F v(X(t)).$$

■

Consider a function $v(X) \in \mathcal{F}_r \cap \mathcal{F}_0(X_*)$. Then, $v(X)$ takes the minimum zero at $X = X_*$, which leads to

$$0 \in \partial v(X^*) = \mathcal{K}\left[-[\bar{f}_i([x_j - x_i]_{j\in\mathcal{V}\setminus\{i\}})]_{i\in\mathcal{V}}\right](X_*) \tag{86}$$

with some $\bar{f}_i(\cdot)$, where the equation is achieved from $v(X) \in \mathcal{F}_r$ and (84) by choosing $R_i = E_d$ in (20). Since $\mathrm{SE}(d)$ is a path-connected set, so is $\mathcal{T}_{X_*}$ from (16). Then, for any matrix $\tilde{X} \in \mathcal{T}_{X_*}$, there exists a continuous function $\Phi : [0,1] \to \mathcal{T}_{X_*}$ such that $\Phi(0) = X_*$ and $\Phi(1) = \tilde{X}$. Then, $(\bar{R}, \bar{\tau}) : [0,1] \to \mathrm{SE}(d)$ is defined as the function satisfying $\Phi(s) = \bar{R}(s)X_* + \bar{\tau}(s)\mathbf{1}_n^\top$. From $v(X) \in \mathcal{F}_r$ and (84), by choosing $R_i = \bar{R}(s)$ in (20), the following is derived:

$$\begin{aligned}
&\partial v(\Phi(s)) \\
&= \mathcal{K}\left[[-R_i \bar{f}_i(R_i^\top [x_j - x_i]_{j\in\mathcal{V}\setminus\{i\}})]_{i\in\mathcal{V}}\right](\bar{R}(s)X_* + \bar{\tau}(s)\mathbf{1}_n^\top) \\
&= \mathcal{K}\left[[-R_i \bar{f}_i(R_i^\top [(x_j - \bar{\tau}(s)) - (x_i - \bar{\tau}(s))]_{j\in\mathcal{V}\setminus\{i\}})]_{i\in\mathcal{V}}\right] \\
&\qquad (\bar{R}(s)X_*) \\
&= \mathcal{K}\left[[-R_i \bar{f}_i(R_i^\top \bar{R}(s)[x_j - x_i]_{j\in\mathcal{V}\setminus\{i\}})]_{i\in\mathcal{V}}\right](X_*) \\
&= \mathcal{K}\left[-\bar{R}(s)[\bar{f}_i([x_j - x_i]_{j\in\mathcal{V}\setminus\{i\}})]_{i\in\mathcal{V}}\right](X_*). \qquad (87)
\end{aligned}$$

Since $\bar{R}(s) \in \mathrm{SO}(d)$ is non-singular, from (86) and (87),

$$0 \in \partial v(\Phi(s)) \quad \forall s \in [0,1] \qquad (88)$$

is obtained. From Lemma 11, there exists a matrix $W \in \mathcal{K}[\mathrm{d}\Phi/\mathrm{d}s](\Phi(s))$ such that

$$\frac{\mathrm{d}v(\Phi(s))}{\mathrm{d}s} = \mathrm{tr}(P^\top W) \ \forall P \in \partial v(\Phi(s))$$

holds almost everywhere. Thus, $\mathrm{d}v(\Phi(s))/\mathrm{d}s$ is possible to be zero almost everywhere from (88). From this and $v(X) \in \mathcal{F}_0(X_*)$, namely $v(X_*) = v(\Phi(0)) = 0$, $v(\Phi(s)) = 0$ for all $s \in [0,1]$ is a possible solution. Moreover, this solution is unique, and $v(\tilde{X}) = v(\Phi(1)) = 0$ holds. Consequently, for all $\tilde{X} \in \mathcal{T}_{X_*}$, $v(\tilde{X}) = 0$ is satisfied. Thus, $v(X) \in \mathcal{F}_0(\mathcal{T}_{X_*})$ holds, and (42) is derived. ■

Appendix F
Proof of Theorem 5

Assume that (G, X_*) is globally rigid and that there exists a clique $\mathcal{C}$ satisfying (54). Consider a matrix $X \in \mathbb{R}^{d\times n}$ satisfying the assumption part of (46). Then, from (35), for each $\mathcal{C} \in \mathrm{M\text{-}clq}(G)$, there exists $(R_\mathcal{C}, \tau_\mathcal{C}) \in \mathrm{SE}(d)$ such that

$$[x_j]_{j\in\mathcal{C}} = R_\mathcal{C}[x_{*j}]_{j\in\mathcal{C}} + \tau_\mathcal{C}\mathbf{1}_{|\mathcal{C}|}^\top, \qquad (89)$$

which yields

$$\|x_i - x_j\| = \|x_{*i} - x_{*j}\| \quad \forall i,j \in \mathcal{C}. \qquad (90)$$

For each edge $\{i,j\} \in \mathcal{E}$, there exists a maximal clique $\mathcal{C}$ satisfying $i,j \in \mathcal{C}$, where $\mathcal{C}$ can depend on the edges. Hence, (90) leads to the assumption part of (52). Then, from the global rigidity, the conclusion part of (52) holds. Thus, there exists an orthogonal matrix $R \in \mathbb{R}^{d\times d}$, not necessarily belonging to $\mathrm{SO}(d)$, and a vector $\tau \in \mathbb{R}^d$ such that

$$X = RX_* + \tau\mathbf{1}_n^\top. \qquad (91)$$

Consider the clique $\mathcal{C}$ satisfying (54). From (3), (89), and (91), the following equations are obtained:

$$[x_j]_{j\in\mathcal{C}} M_{|\mathcal{C}|} = R[x_{*j}]_{j\in\mathcal{C}} M_{|\mathcal{C}|} = R_\mathcal{C}[x_{*j}]_{j\in\mathcal{C}} M_{|\mathcal{C}|},$$

which leads to $R = R_\mathcal{C}$ since $\mathrm{rank}([x_{*j}]_{j\in\mathcal{C}} M_{|\mathcal{C}|}) = d$ holds from (54). From $R_\mathcal{C} \in \mathrm{SO}(d)$, $R \in \mathrm{SO}(d)$ holds, and $X \in \mathcal{T}_{X_*}$ is achieved from (16) and (91). Thus, the conclusion part of (46) is obtained, and hence (G, X_*) is clique-rigid. ■

Appendix G
Proof of Theorem 6

Assume that (G, X_*) is clique-rigid. Let $\Phi = [\phi_1 \ \phi_2 \ \cdots \ \phi_n] : [0,1] \to \mathbb{R}^{d\times n}$ be a continuous function satisfying $\Phi(0) = X_*$ and the assumption part of (53). Consider a maximal clique $\mathcal{C} \in \mathrm{M\text{-}clq}(G)$, and $\{i,j\} \in \mathcal{E}$ holds for any $i,j \in \mathcal{C}$. Then, $\|\phi_i(s) - \phi_j(s)\| = \|x_{*i} - x_{*j}\|$ holds for any $i,j \in \mathcal{C}$ and $s \in [0,1]$, and $[\phi_j(0)]_{j\in\mathcal{C}} = [x_{*j}]_{j\in\mathcal{C}}$ holds. From these equations, there exists a continuous function $(R_\mathcal{C}, \tau_\mathcal{C}) : [0,1] \to \mathrm{SE}(d)$ such that $\phi_i(s) = R_\mathcal{C}(s)x_{*i} + \tau_\mathcal{C}(s)$ holds for any $i \in \mathcal{C}$ and $s \in [0,1]$, which is reduced to $[\phi_j(s)]_{j\in\mathcal{C}} \in \mathcal{P}_\mathcal{C}(\mathcal{T}_{X_*})$ from (35). This inclusion holds for any $\mathcal{C} \in \mathrm{M\text{-}clq}(G)$, and from the clique-rigidity (46), $\Phi(s) \in \mathcal{T}_{X_*}$ holds for any $s \in [0,1]$. Then, from (16), the conclusion part of (53) is achieved. ■

Appendix H
Proof of Lemma 8

Let $X(t) \in \mathbb{R}^{d\times n}$ be the solution of (17), equivalently the solution of (8) with

$$F(X) = -\frac{\partial v}{\partial X}(X). \qquad (92)$$

Assume that there exists an element $a > 0$ of $\mathcal{D}_F v(X)$. From (84), (85), and (92), there exists $W \in -\partial v(X)$ satisfying $\mathrm{tr}(P^\top W) = a > 0$ for each $P \in \partial v(X)$. This is however not the case because $\mathrm{tr}(P^\top W) = -\mathrm{tr}(W^\top W) \le 0$ for $P = -W \in \partial v(X)$. There is no positive element in $\mathcal{D}_F v(X)$, and

$$\mathcal{D}_F v(X) \subset (-\infty, 0] \qquad (93)$$

is achieved. From Lemma 11 and (93), $v(X(t))$ is non-increasing with respect to t. Then, the non-smooth version of LaSalle's invariance theorem [37] guarantees

$$\lim_{t\to\infty} \mathrm{dist}\left(X(t), \mathcal{Z}_s(\mathcal{D}_F v) \cap \Omega\right) = 0, \qquad (94)$$

where

$$\mathcal{Z}_s(\mathcal{D}_F v) = \{X \in \mathbb{R}^{d\times n} : 0 \in \mathcal{D}_F v(X)\}. \qquad (95)$$

As shown below,

$$\mathcal{Z}_s(\mathcal{D}_F v) = \mathcal{Z}\left(\frac{\partial v}{\partial X}\right) \qquad (96)$$

holds for (92). Equations (94) and (96) lead to (10) for $\mathcal{A} = \mathcal{Z}(\partial v/\partial X) \cap \Omega$ and $\mathcal{U} = \Omega$.

Equation (96) is shown. Consider a matrix $X \in \mathcal{Z}(\partial v/\partial X)$, and $0 \in \mathcal{K}[\partial v/\partial X](X)$ holds from (9). Then, $0 \in \mathcal{D}_F v(X)$ is achieved because $\mathrm{tr}(P^\top 0) = 0$ holds for any $P \in \partial v(X)$ from (85). Thus, from (95), $X \in \mathcal{Z}_s(\mathcal{D}_F v)$ is obtained. Conversely, consider a matrix $X \in \mathcal{Z}_s(\mathcal{D}_F v)$, then $0 \in \mathcal{D}_F v(X)$ holds from (95), and there exists a matrix $W \in -\partial v(X)$ satisfying $\mathrm{tr}(P^\top W) = 0$ for all $P \in \partial v(X)$ from (85). Hence, $0 \in \partial v(X)$ is satisfied because otherwise $\mathrm{tr}(P^\top W) < 0$ holds for $P = -W \in \partial v(X)$. Then, from (9) and (84), $X \in \mathcal{Z}(\partial v/\partial X)$ is achieved, which yields (96). ■

APPENDIX I
PROOF OF LEMMA 9

Proof of (i): Let q be the number of the connected components of graph G, and let $G_h = (\mathcal{V}_h, \mathcal{E}_h)$ for $h = 1, 2, \ldots, q$ be the subgraphs of G representing the connected components. For each h, from (13) and (38),

$$\begin{aligned}\sum_{i\in\mathcal{V}_h} u_i(t) &= \sum_{i\in\mathcal{V}_h} \sum_{\mathcal{C}\in\text{M-clq}_i(G_h)} \alpha_{\mathcal{C}}\{\text{ave}([x_j^{[i]}(t)]_{j\in\mathcal{C}}) \\ &\qquad + R_{\mathcal{C}}(t)(x_{*i} - \text{ave}([x_{*j}]_{j\in\mathcal{C}}))\} \\ &= \sum_{\mathcal{C}\in\text{M-clq}(G_h)} \alpha_{\mathcal{C}} \left\{ R_i^\top \sum_{i\in\mathcal{C}} (\text{ave}([x_j(t)]_{j\in\mathcal{C}}) - x_i(t)) \right. \\ &\qquad \left. + R_{\mathcal{C}}(t) \sum_{i\in\mathcal{C}} (x_{*i} - \text{ave}([x_{*j}]_{j\in\mathcal{C}})) \right\} \\ &= 0\end{aligned}$$

is obtained. Thus, from (12), the average of $\dot{x}_i(t)$ for $i \in \mathcal{V}_h$ is zero, and that of $x_i(t)$ is constant.

Proof of (ii): First, we show that $\|x_i(t) - x_j(t)\|$ is bounded for any edge $\{i, j\} \in \mathcal{E}$. From Theorem 2, $x_i(t)$ is governed by (17) with $v(X)$ in (34). For each $\{i, j\} \in \mathcal{E}$, there exists a clique $\mathcal{C}$ satisfying $i, j \in \mathcal{C}$. Then, the following is obtained:

$$\begin{aligned}\|x_i - x_j\| &= \|[x_\ell]_{\ell\in\mathcal{C}}(e_{|\mathcal{C}|i_{\mathcal{C}}} - e_{|\mathcal{C}|j_{\mathcal{C}}})\| \\ &= \|[x_\ell]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|}(e_{|\mathcal{C}|i_{\mathcal{C}}} - e_{|\mathcal{C}|j_{\mathcal{C}}})\| \\ &\le \sqrt{2}\|[x_\ell]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|}\|, \end{aligned} \tag{97}$$

where $i_{\mathcal{C}}$ is the order of i in $\mathcal{C} = \{\ell_1, \ell_2, \ldots, \ell_{i_{\mathcal{C}}} = i, \ldots, \ell_{|\mathcal{C}|}\}$ such that $\ell_1 < \ell_2 < \cdots < \ell_{i_{\mathcal{C}}} < \cdots < \ell_{|\mathcal{C}|}$. From Lemma 1 and the triangle inequality,

$$\begin{aligned}\text{dist}([x_\ell]_{\ell\in\mathcal{C}}, \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})) &= \|[x_\ell]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|} - R_{\mathcal{C}}[x_{*\ell}]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|}\| \\ &\ge \|[x_\ell]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|}\| - \|[x_{*\ell}]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|}\| \end{aligned} \tag{98}$$

holds with some $R_{\mathcal{C}} \in \mathcal{R}_{|\mathcal{C}|}([x_\ell]_{\ell\in\mathcal{C}}, [x_{*\ell}]_{\ell\in\mathcal{C}})$. From (97) and (98), for $v(X)$ in (34),

$$\begin{aligned}\|x_i - x_j\| &\le \sqrt{2}(\text{dist}([x_\ell]_{\ell\in\mathcal{C}}, \mathcal{P}_{\mathcal{C}}(\mathcal{T}_{X_*})) + \|[x_{*\ell}]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|}\|) \\ &\le \sqrt{2}\left(\sqrt{\frac{2}{\alpha_{\mathcal{C}}} v(X)} + \|[x_{*\ell}]_{\ell\in\mathcal{C}} M_{|\mathcal{C}|}\|\right) \end{aligned} \tag{99}$$

is obtained. Lemma 8 guarantees that $v(X(t))$ is non-increasing. From this and (99), $\|x_i(t) - x_j(t)\|$ is bounded.

Now, we show that $\|X(t)\|$ is bounded. Let T_h be a connected spanning tree of G_h, and let $\{i_{hk}, j_{hk}\}$ for $k \in \{1, 2, \ldots, |\mathcal{V}_h| - 1\}$ be the edges of T_h. Then, the incidence matrix of T_h is defined by $B_h \in \mathbb{R}^{|\mathcal{V}_h|\times(|\mathcal{V}_h|-1)}$ whose (ℓ, k) entry $(B_h)_{\ell k}$ is given as

$$(B_h)_{\ell k} = \begin{cases} 1 & \text{if } \ell = i_{hk} \\ -1 & \text{if } \ell = j_{hk} \\ 0 & \text{otherwise.} \end{cases}$$

From the property of the incidence matrices of connected trees, the following holds:

$$\text{rank}([B_h \;\; \mathbf{1}_{|\mathcal{V}_h|}]) = |\mathcal{V}_h|. \tag{100}$$

Consider one connected component G_h, and let $\chi_\ell \in \mathbb{R}^{|\mathcal{V}_h|}$ ($\ell \in \{1, 2, \ldots, d\}$) be the ℓ-th column of $[x_j]_{j\in\mathcal{V}_h}^\top$ as $[x_j]_{j\in\mathcal{V}_h}^\top = [\chi_1 \; \chi_2 \; \cdots \; \chi_d]$. The following expressions hold:

$$\begin{aligned}&\left\| \left[[x_{i_{hk}} - x_{j_{hk}}]_{k\in\{1,2,\ldots,|\mathcal{V}_h|-1\}} \;\; \sum_{i\in\mathcal{V}_h} x_i \right] \right\|^2 \\ &= \|[x_j]_{j\in\mathcal{V}_h}[B_h \;\; \mathbf{1}_{|\mathcal{V}_h|}]\|^2 \\ &= \sum_{\ell\in\{1,2,\ldots,d\}} \|[B_h \;\; \mathbf{1}_{|\mathcal{V}_h|}]^\top \chi_\ell\|^2 \\ &\ge \sum_{\ell\in\{1,2,\ldots,d\}} (\sigma_{\min}([B_h \;\; \mathbf{1}_{|\mathcal{V}_h|}]))^2 \|\chi_\ell\|^2 \\ &= (\sigma_{\min}([B_h \;\; \mathbf{1}_{|\mathcal{V}_h|}]))^2 \|[x_j]_{j\in\mathcal{V}_h}\|^2, \end{aligned} \tag{101}$$

where $\sigma_{\min}(\cdot)$ represents the minimum singular value of a matrix. From (100), $\sigma_{\min}([B_h \;\; \mathbf{1}_{|\mathcal{V}_h|}]) > 0$ is obtained. From this, (i), the boundedness of $\|x_i(t) - x_j(t)\|$, and (101), $\|[x_j(t)]_{j\in\mathcal{V}_h}\|^2$ is bounded. Then, $\|X(t)\|$ is bounded since $\|X(t)\|^2 = \sum_{h=1}^q \|[x_j(t)]_{j\in\mathcal{V}_h}\|^2$. ■

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