TROPICAL GEOMETRIC COMPACTIFICATION OF MODULI, I - M_g CASE -

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To the memory of Kentaro Nagao

Abstract. We compactify the classical moduli variety of compact Riemann surfaces by attaching moduli of (metrized) *graphs* as boundary. The compactifications do *not* admit the structure of varieties and patch together to form a big connected moduli space in which $\sqcup_g M_g$ is open dense.

The metrized graphs, which are often studied as "tropical curves", are obtained as Gromov-Hausdorff collapse by fixing diameters of the hyperbolic metrics of the Riemann surfaces. This phenomenon can be also seen as an archemidean analogue of the tropicalization of Berkovich analytification of M_g [ACP].

Contents

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1. Introduction

Let us recall that the moduli space of smooth projective curves admits a "canonical" modular compactification constructed in Deligne-Mumford [DM] first as an algebraic stack $\overline{\mathcal{M}_g}^{\rm DM}$. [1] Later on, the moduli stack was proved to have a coarse projective variety which is normal and of dimension $3g-3$ [KM, Especially, III], [Gie], [Mum2].

The boundary of the compactification still parametrizes geometric objects which are certain nodal curves called "stable curves" characterized by the GIT stability ([Gie], [Mum2]) or by the K-stability ([Od2, 4.1], also cf. [Mum2], [Od1], [LW, §7]). Hence the GIT construction ([Mum1]) applies ([Gie], [Mum2]) while it also fits to more general moduli existence conjecture for K-(semi)stable polarized varieties ("K-moduli" cf., [Od4]).

In this paper, we introduce a pair of new compactifications of M_g which are *no longer varieties* but compact Hausdorff toplogical spaces. In the first compactification which we denote as $\overline{M_g}^{\rm GH}$, the boundaries parametrize the Gromov-Hausdorff limits of compact Riemann surfaces with rescaled Poincaré (i.e., Kähler-Einstein) metrics with diameter 1, which we identify as certain graphs (Theorem 2.4). Hence we would like to call the compactification $\overline{M_g}^{\rm GH}$ *Gromov-Hausdorff compactification.*

In the second compactifications of M_g, we further encode some non-negative integer weights on the vertices of the limit graphs. We call the metrized graphs with such weights, *weighted metrized graphs.* The class of our limits graph is very close to what has been studied as "(stable) tropical curves" in the literatures (e.g., [BMV, Cap, MZ, CHMR]). Our point is that we can construct a refined compactification of M_g than $\overline{M_g}^{\rm GH}$ by encoding the weights. The obtained compactifications will be called "*tropical geometric compactifications*". We chose the term because the boundaries coincides with the moduli spaces of such tropical curves, which are also studied in the literatures (e.g., [BMV, Cap, MZ, CHMR] again), while we also avoided the term "*tropical compactification*" already used by J. Tevelev whose context is very different, namely, the problem of compactifying subvarieties of a torus in a toric variety (cf., [Tev]).

Let us explain the backgrounds by discussing a broader picture for moduli spaces of more general varieties. There are two major backgrounds for this work, which we recall now:

[1] Here we put the superscript "DM", often omitted in the literaturs, to clearly distinguish from the compactifications we introduce in this paper.

(i) The current extensive approach to the Strominger-Yau-Zaslow mirror symmetry conjecture ([SYZ]). Indeed, conjectures of Gross-Wilson [GW, §6], Todorov, Kontsevich-Soibelman ([KS]) (cf., e.g., the survey on the Gross-Siebert program[Gross]) speculates certain families of Calabi-Yau varieties with its Ricci-flat Kähler metrics collapse to integral affine manifolds with singularities in the Gromov-Hausdorff sense, which are recently often regarded as some tropical version of Calabi-Yau varieties.

(ii) The algebraicity of *non-collapsed* Gromov-Hausdorff limits of Kähler- Einstein manifolds ([DS]), its applications to moduli of Fano varieties ([Spo, OSS, Od4]), later followed by ([SSY, LWX, Od5]).

There is a similarity between the above two i.e. (i) and (ii) as the first i.e. (i) is in particular showing that the collapsed Gromov-Hausdorff limits of Kähler-Einstein manifolds are "tropical *algebraic*" objects while the second (ii) is showing that the non-collapsed limits of Kähler-Einstein (Fano) manifolds are *algebro-geometric* objects i.e., varieties.

For moduli spaces of Fano manifolds, which we discussed in (cf., [DS, OSS, Od4], [SSY, LWX, Od5]), the two kinds of the compactifications

(α) the Gromov-Hausdorff metric compactification of the moduli space of Kähler-Einstein manifolds with the rescaled Kähler-Einstein metrics with fixed diameters (our $\overline{M_g}^{\rm GH}$ and $\overline{M_g}^{\rm T}$ to be introduced in this paper are on this side) which is closer to the spirit of (i) and

(β) algebro-geometric compactified moduli of K-stable varieties, e.g. $\overline{M_g}^{\rm DM}$ as in (ii)

essentially coincide because of the non-collapsing of the metrics. However they "look" completely different in the non-Fano case due to collapse of the Kähler-Einstein metrics as we show in the present series of papers. Indeed, the author believes that the Gromov-Hausdorff compactification while fixing the *volume* (rather than the diameter), if it exists in an appropriate sense, should be closer in spirit to (β). Nevertheless, as we observe in the case of M_g in this paper, we believe that the two series of compactifications (α) and (β) must be deeply connected in general.

In the present paper, first we start with the classification of all the possible Gromov-Hausdorff limits of the compact Riemann surfaces with Kähler-Einstein metrics of diameters 1. Then using the classification, we construct the compactifications and proceed to analyze their structures.

Our connection between classical algebro-geometric compactifications and tropical moduli spaces can be seen as an archimedean analogue of the *tropicalization* (*skeleton*) of non-archimedean analytification of the moduli varieties which is recently studied in [ACP]. We discuss this analogy towards the end of the subsection 2.2.

Another interesting point of our compactifications $\overline{M_g}^{\mathrm{GH}}$, is that they naturally patch together to form a big (infinite dimensional) *conneted* moduli space in which M_g are open subsets for *all* g. We will call it *infinite join* and denotes it as $\overline{M_\infty}^{\mathrm{GH}}$.

It would be interesting to pursue this line of research for moduli varieties of other polarized varieties. For instance, the author conjectures that the moduli schemes of smooth canonical models, again with the rescaled Kähler-Einstein metrics of diameters 1, are also precompact for Gromov-Hausdorff distance and the corresponding collapses will be dual intersection complexes of KSBA semi-log-canonical models in certain generalized sense. Such speculation is inspired by the recent Kollár-Shepherd-Barron-Alexeev (KSBA) compactification (cf., e.g., the survey [Kol]) and the observation that it is a moduli scheme of K-stable varieties ([Od1, Od3], also [BG]).

Throughout this article, we work over the complex number field $\mathbb{C}$ unless otherwise stated.

Notes added, part 1. Two years after our original preprint of this paper, Boucksom-Jonsson [BJ, §2] generalized the Morgan-Shalen compactification [MS] which can be also further generalized to orbifolds in [Od6, Appendix]. It may be convenient to mention here that the compactification applied to M_g are *different* from our compactification. More precisely, although it can be set-theoritically identified with our $\overline{M_g}^{wT}$ but has *different* topology. See [Od6, Theorem 3.7] for the details.

Also, after that, we had other further developments with Yoshiki Oshima for the case of A_g and moduli of K-trivial varieties case (cf., [OO]). In *loc.cit*, we put a focus on the moduli of K3 surfaces, after the works of [GW], [KS], [GTZ16].

Acknowledgments. The first version of this paper appeared in June, 2014 (arXiv:1406.7772) and this is a revised exposition of the *former half*, i.e. the M_g case, of the original preprint. The companion paper

[Od6] is a revision of the *latter half*, i.e. the A_g part of arXiv:1406.7772, together which included later developments.

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This paper and its companion paper [Od6] are dedicated to fifteen years memory of *Kentaro Nagao*. Looking back, I can never stop deeply thanking Nagao-san for all the inspirations from the beginning and the warm friendliness. I hope he would be delighted again.

2. Gromov-Hausdorff compactification of M_g

2.1. **Precompactness.** For each compact Riemann surface of genus $g(\geq 2)$, we put *rescale* of the Kähler-Einstein metric with the *diameter* 1. [2] Recall that the Kähler-Einstein metric is nothing but the famous Poincaré metric in this case. The first point we should clarify is the precompactness of M_g with the associated Gromov-Hausdorff distance (for its definition we refer to e.g. [BBI, Chapter 7]) on it. We denote the Gromov-Hausdorff distance as $d_{\rm GH}$. Recall that the precompactness of a subset of the space of all compact metric spaces means its closure with respect to the Gromov-Hausdorff topology is compact. During the process of degenerations i.e., going to boundary of M_g, the curvature tends to $-\infty$, so we can *not* apply the Gromov's precompactness theorem [Grom] in our situation. Instead we can apply the following theorem of Shioya [Shi] and the Gauss-Bonnet theorem to prove it.

Theorem 2.1 ([Shi, Theorem 1.1])**.** *For two fixed positive real numbers $D > 0$ and $c > 0$, consider the set $S(D, c)$ of closed 2-dimensional Riemannian manifolds (R, d) with*

(i) *the diameter* $\mathrm{diam}(d) < D$
(ii) *and the total absolute curvature $\int_R |K_{(R,d)}|\mathrm{vol}(R) < c$ where $K_{(R,d)}$ and $\mathrm{vol}(R)$ denotes the Gaussian curvature and the volume form with respect to the metric d.*

Then the set $S(D, c)$ is precompact with respect to the associated Gromov-Hausdorff distance.

[2]Readers will find later that this specific constant 1 does not have any specific meaning as we only meant to fix it, so we can rather set it to be any fixed positive constant.

By applying the above theorem, we get the following desired pre-compactness.

Corollary 2.2. (M_g, d_{GH}) *is precompact.*

First proof. It directly follows from the Shioya's theorem above (2.1) since our total absolute curvature is constant due to the Gauss-Bonnet theorem. □

We include another proof of Corollary 2.2 in the next section, in which we also classify all the Gromov-Hausdorff limits.

2.2. **Gromov-Hausdorff collapse of Riemann surfaces.** Before stating a theorem, we precisely fix some graph theoretic terminology we use in this paper.

Definition 2.3. In the present paper, a *metrized (finite) graph* means a finite connected non-directed graph with finite positive lengths attached to all edges. It is not necessarily simple, i.e., loops and several edges with the same ends are allowed. A *contraction* of a finite graph is a graph which can be obtained from the original graph by contracting some of its edges.

The main result of this section is the following theorem, which implies the precompactness of M_g and also classify all the possible Gromov-Hausdorff limits of compact hyperbolic surfaces while fixing their diameters.

Theorem 2.4. *Let* $\{R_i\}_{i\in\mathbb{Z}_{>0}}$ *be an arbitrary sequence of compact Riemann surfaces of fixed genus* $g \geq 2$. *Suppose* $\{(R_i, \frac{d_{\mathrm{KE}}}{\mathrm{diam(R_i)}})\}_i$ *converges in the Gromov-Hausdorff sense. Here* d_{KE} *denotes the Poincaré metric* [3] *on each* R_i *and its diameter is* $\mathrm{diam}(R_i)$.

Then the limit is the metric space associated to either

(i) *a metrized graph of diameter* 1 *or*
(ii) *a compact Riemann surface of genus* g.

Assume furthermore that the sequence R_i *converges to* $[R_\infty] \in \overline{M}_g^{\mathrm{DM}}$ *(which can be always be achieved by passing to a subsequence since* $\overline{M}_g^{\mathrm{DM}}$ *is compact). Then if* $[R_\infty] \in M_g$ *we are in case* (ii) *and* R_i *converges in the Gromov-Hausdorff sense to the metric space underlying* R_∞*; if, on the other hand,* $[R_\infty] \notin M_g$ *then we are in case* (i) *and the* R_i *converges to the metric space underlying a metrized graph whose underlying graph is a contraction of the dual graph of* R_∞.

[3] i.e., the hyperbolic metric which is also a Kähler-Einstein metric, hence the notation

Conversely, any metrized graph with diameter 1 *whose underlying graph is a contraction of some (possibly* 0*) edges of the dual graph of a stable curve of genus g, can occur in this way (*i*).*

Proof. We fix a reference compact Riemann surface S and regard the Teichmuller space T_g as the set of marked compact Riemann surfaces $[\phi\colon S \xrightarrow{\simeq} R]$ where we only care of the isotopy type of ϕ.

First we briefly recall the basic of the pair-of-pants decomposition of S, which we abbreviate as pants decomposition from now on for short, and later we will explain how to apply it.

$$S = \bigcup_{0\le a\le g-2} P_a$$

with the associated simple closed boundary geodesics $s_1, \cdots, s_{3g-3}$. Then in turn it naturally induces the corresponding pants decompositions of R

$$R = \bigcup_{0\le a\le g-2} P_a(R)$$

for all elements $[\phi\colon S \xrightarrow{\simeq} R]$ of T_g since we can take simple closed boundary geodesics in the corresponding homology classes. The associated simple closed boundary geodesics $\{s_j(R)\}_j$ of R gives the (real analytic) Fenchel-Nielsen coordinates on it

$$(l_1, \cdots, l_{3g-3}; \theta_1, \cdots, \theta_{3g-3})\colon T_g \cong \mathbb{R}_{>0}^{3g-3} \times (\mathbb{R}/2\pi\mathbb{Z})^{3g-3},$$

where l_j is the length of s_j and θ_j is corresponding twist parameters (cf., [IT]). Then the following well-known theorem is due to L. Bers.

Fact 2.5 ([Bers, Theorem 2 for the type $(g, 0)$ case])**.** *Fix a positive integer $g \ge 2$. Then there is a uniform constant C_g such that for an arbitrary compact hyperbolic Riemann surface R, there is a pant decomposition whose corresponding lengths l_j of any dividing simple closed geodesic satisfy $l_j < C_g$.*

We now argue as follows. Suppose we are given a sequence $\{R_i\}_{i\in\mathbb{Z}_{>0}}$ of compact Riemann surfaces of the fixed genus $g \ge 2$, as in the statement of Theorem 2.4. We replace it by its certain subsequence, after several steps as follows. Firstly, due to the compactness of the Deligne-Mumford compactification $\overline{M_g}^{\rm DM}$, we can replace the sequence $\{R_i\}$ by subsequence, if necessary, to ensure the existence of a limit of $[R_i]$ inside $\overline{M_g}^{\rm DM}$. By applying the Bers' theorem 2.5, for each i, we have a pants decomposition satisfying the assertion of Theorem 2.5, i.e., all the lengths of the corresponding simple closed geodesics are less than a uniform constant C_g. For each R_i, we fix such a pants decomposition from now on. On the other hand, note that for each pants decomposition there is a corresponding graph whose vertices are (pair of) pants while edges are common geodesics is 3-regular with $2g-2$ vertices. We call this graph the combinatorial type of the pant decomposition. See for instance [Ham, around Definition 1.5] for the details. The number of edges of such a graph is $3g-3$ so obviously there is only a finite possibilities for such graphs. Hence, there is only a finite possibilities of combinatorial type of pants decomposition. Therefore, by passing to an appropriate subsequence of $\{R_i\}$ again, if necessary, we can and do assume the combinatorial type of the pants decompositions we took, which satisfies the condition $l_j < C_g$ of Fact 2.5, stays fixed. By the upper bound of l_j, by further passing to an appropriate subsequence of $\{R_i\}$ again, if necessary, we can and do assume moreover that $\lim_{i\to\infty} l_j(R_i) = L_j$ for some constants $L_j \in [0, C_g] \subset [0,\infty)$ for each j.

The simple geodesics $s_j(R_i)$ of R_i with $L_j = 0$ are representing the vanishing cycles, i.e., all the cycles that shrink to nodal singularities of the corresponding limit in the Deligne-Mumford compactification $\overline{M_g}^{\mathrm{DM}}$. We make the following claim, although the author believes this has been known to or expected by the experts.

Claim 2.6. *There is an index j with $L_j = 0$, if and only if the diameter of the non-rescaled hyperbolic metrics (i.e., with constant Gaussian curvature -1) tends to $+\infty$. This is also equivalent to that the limit of the sequence $[R_i]$ does not belong to M_g.*

Otherwise, passing to a subsequence, the Gromov-Hausdorff limit R_∞ of $\{R_i\}_i$ exists as a compact Riemann surface of the same genus g.

proof of Claim 2.6. If all the L_j are non-zero, then the compactness of

$$\{(l_1, \cdots, l_{3g-3}; \theta_1, \cdots, \theta_{3g-3}) \mid L_i - \epsilon \le l_i \le C_g \text{ for } 1 \le \forall i \le 3g-3\} \subset T_g$$

for small enough positive real number ϵ straightforwardly implies that the corresponding points $[R_i] \in T_g$ converge inside T_g.

Now, we denote the space of all compact metric spaces with the Gromov-Hausdorff topology as CMet. Here, we recall the following standard fact well-known to experts.

Fact 2.7 (Gromov-Hausdorff continuity on M_g)**.** *If we consider the map $\Phi\colon M_g \to \mathrm{CMet}$, sending $[R]$ to the underlying topological surface with the Poincaré metric. Also define $\Phi_1\colon M_g \to \mathrm{CMet}$ by sending $[R]$ to the underlying topological surface with the rescaled Poincaré metric with the diameter 1. Then these Φ and Φ_1 are both continuous with respect to the complex analytic topology on M_g.*

This is fairly standard but we write the arguments for convenience. Obviously, the continuity of Φ_1 follows from that of Φ because the diameters of the hyperbolic metrics vary continuously due to the continuity of Φ. In turn, the continuity of Φ follows, for instance, from the interpretation of the family as a family of the quotients of the upper half plane by continuously deforming Fuchsian subgroup of $PSL(2,\mathbb{R})$. (The isomorphic class of the Fuchsian group is not changed, as it is the isomorphic class of the fundamental group of genus g compact Riemann surface.) Or it also follows from the implicit function theorem applied to the constancy of the Gaussian curvature. Hence, in particular, the diameters of the (non-rescaled, original) Poincaré metrics of R_i are bounded and the Gromov-Hausdorff limit of R_i with the rescaled Poincaré metric is still a compact Riemann surface of genus g.

On the other hand, if $L_j = 0$ for at least one index j, then the famous collar theorem [Ke] applies and directly shows that for each i there is a cylinder (called "collar") inside R_i, including the closed geodesic l_j, whose diameter tends to $+\infty$. We end the proof of the Claim 2.6. □

From now on, we assume these equivalent conditions are satisfied i.e., $[R_\infty] \notin M_g$. Otherwise, the subsequence converges to a compact Riemann surface (i.e., "does not degenerate"), which corresponds to the case (ii) of Theorem 2.4. This is again because of the continuity of the surfaces with the rescaled Poincaré metrics parametrized by M_g with respect to the Gromov-Hausdorff topology.

Let us denote the diameter of the Poincaré (hyperbolic) metric $d_{\rm KE}$ of R_i as d_i. Then recall that what we are analysing is the metric behaviour of $(R_i, \frac{d_{\rm KE}}{d_i})$ and we wish to determine its Gromov-Hausdorff limit. For that, we analyze the behaviour of the pant $(P_a(R_i), \frac{d_{\rm KE}}{d_i})$ in this proof. We denote the three boundary geodesics of the pants as $s_b(P_a)(b = 1, 2, 3)$, or $s_b(R_i; P_a)(b = 1, 2, 3)$ for precision, which may partially be identified in the Riemann surface R_i i.e., e.g. $s_1(R_i; P_a) = s_2(R_i; P_a)$ can be possible. From now on, whenever the context is clear, we sometimes omit R_i and simply denote the pants of R_i as P_a, not $P_a(R_i)$ and its boundary geodesics $s_b(P_a)(b = 1, 2, 3)$ rather than $s_b(R_i; P_a)(b = 1, 2, 3)$.

Let us recall a standard fact in the Teichmuller theorey (cf., [IT, Chapter 3, §1.5, §2]) which claims that the pant $P_a(R_i)$ can be cut and separated into two isometric hyperbolic hexagons $Q_a(R_i)$ and $Q'_a(R_i)$ canonically by geodesics which connect different boundary geodesics of the pant $P_a(R_i)$. Let us also recall from [IT, Chapter 3, §1.5, §2] that the interior part of the hyperbolic hexagons $Q_a(R_i)$, with its hyperbolic metric, can be regarded as an open subset of a unit disc with the hyperbolic metric $d_{\rm KE}$, in a unique way up to the isometry group of the disk i.e., $PGL(2,\mathbb{R})$. We denote the center of the unit disc as p.

Let us call the 3 boundaries of the hexagon which were originally part of the boundaries of the pant P_a as *boundary geodesics*. In any case, the important invariants are the lengths of the 3 boundary geodesics which are half of the lengths of the boundary geodesics $s_b(R_i;P_a)(b=1,2,3)$ of the original pant P_a. Indeed, it is a well-known fact that biholomorphic type of Q_a (so also for P_a) is determined by the lengths of the three boundary geodesics (cf., e.g., [IT]). We now study the Gromov-Hausdorff limit of the hyperbolic hexagon Q_a while fixing diameters. Then, recall from the Claim 2.6, it follows that $d_{\rm KE}(p,s_b(R_i;P_a))\to+\infty$ for $i\to\infty$ if and only if the corresponding boundary geodesic $s_b(R_i;P_a)$ shrinks i.e., $\text{length}(s_b(R_i;P_a))\to 0$ for $i\to\infty$.

To each P_a, we associate a tree Γ_a, just as a combinatorial graph, with

- the vertex set $V(\Gamma_a):=\{v_a\}\sqcup\{w_b\mid s_b(R_i;P_a)\text{ shrinks}\}$ and
- the edge set $E(\Gamma_a):=\{\overline{v_aw_b}\mid s_b(R_i;P_a)\text{ shrinks}\}$.

Denote the diameter of the hyperbolic hexagon $Q_a(R_i)$ with respect to Poincaré metric as $d_i(a)$. (Recall that the diameter of whole R_i is d_i.) We analyze the asymptotic behaviour of $(R_i,\frac{d_{\rm KE}}{d_i})$ by further "decomposing" into that of $Q_a(R_i)$ as above.

First we fix a constant $0<\epsilon\ll 1$ so that the sequence of the half pant $\{Q_a(R_i)\}_i$ satisfies that the disk $D(p,(1-\epsilon))$ with center p and radius $(1-\epsilon)$ contains all non-shrinking boundary geodesics of $Q_a(R_i)$. Then thinking of the distance between each point in $(Q_a(R_i)\cap D(p,(1-\epsilon))$ and p, we straightforwardly obtain that the diameter of $\{(Q_a(R_i)\cap D(p,(1-\epsilon)),d_{\rm KE})\}_i$ is bounded above by C_ϵ. On the other hand, the diameters of the collar neighborhoods of shrinking boundary geodesics tends to $+\infty$ by the collar theorem [Ke]. Hence, we have that

Claim 2.8 (Diverging hyperbolic hexagon)**.** *$d_i(a)\to\infty$ for $i\to\infty$ if and only if there is an index b with* $\text{length}(s_b(R_i;P_a))\to 0$ *for $i\to\infty$.*

Claim 2.9 (Limit of hyperbolic hexagon, I)**.** *If we consider the sequence $(Q_a(R_i),\frac{d_{\rm KE}}{d_i(a)})$ for $i=1,2,\cdots$, it has the Gromov-Hausdorff limit as a metrized tree Γ_a in the case when* $\text{length}(s_b(R_i;P_a))\to 0$ *for some b when $i\to\infty$. Otherwise its Gromov-Hausdorff limit is still some hyperbolic hexagon.*

The last sentence of Claim 2.9 holds because, for any b, $\text{length}(s_b(R_i;P_a))$ converges to a positive real number from our assumption when $i\to+\infty$ and $d_i(a)$ are bounded above, converging to a positive real number as well. Thus, the Gromov-Hausdorff limit of $(Q_a(R_i), d_{\mathrm{KE}})$ can be taken simply as the Hausdorff limit inside the unit disk which implies the desired claim.

Next, we compare the diameters of each hyperbolic hexagon $Q_a(R_i)$ and the whole Riemann surface R_i.

Claim 2.10 (Diameters comparison)**.** *(*i*) For any i there is at least one $Q_a(R_i)$ (or equivalently, its index a) such that*

$$d_i \leq 12(g-1)d_i(a). \tag{1}$$

*(*ii*) Suppose that an index a satisfies that $d_i(a) \to \infty$ when $i \to \infty$. Then, for any a and large enough i, we have*

$$\frac{d_i(a)}{2} \leq d_i. \tag{2}$$

Proof of Claim 2.10. The second assertion (ii) easily follows from the definition. Indeed, it can be proven as follows. First we can assume $d_i(a)$ is the length of a geodesic $\gamma\colon [0,1] \to R_i$ connecting two points $\gamma(0), \gamma(1)$ in the union of the boundary geodesics. Then its midpoint $\gamma(\frac{1}{2})$ and one of the endpoints, say $\gamma(1)$, of the geodesic has the same distance in whole R_i i.e., after gluing the boundary geodesics. Hence (ii) follows.

Our first assertion (i) is proved as follows. Take a shortest geodesic $\delta\colon [0,1] \to R_i$ connecting two points in R_i with $\mathrm{length}(\delta) = \mathrm{diam}(R_i)$. An elementary observation shows that the maximum number of the connected components of $\mathrm{Im}(\delta) \cap Q_a(R_i)$ is at most 3 so that we have

$$\mathrm{length}(\mathrm{Im}(\delta) \cap Q_a(R_i)) \leq 3d_i(a) \tag{3}$$

$$\mathrm{length}(\mathrm{Im}(\delta) \cap Q'_a(R_i)) \leq 3d_i(a). \tag{4}$$

Indeed, if we write

$$I_a := \{t \in [0,1] \mid \delta(t) \in Q_a\} = [\alpha_1, \alpha_2] \sqcup \cdots \sqcup [\alpha_{2m-1}, \alpha_{2m}],$$

with $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \alpha_{2m}$, then note that $\delta(\alpha_2)$ and $\delta(\alpha_{2m-1})$ are connected by a geodesic of length at most $d_i(a)$, by the definition of $d_i(a)$. Since δ is taken to be a shortest geodesic, $\sum_{1\leq k<m} \mathrm{length}(\delta([\alpha_{2k-1}, \alpha_{2k}])) \leq d_i(a)$ which gives our desired estimate (3), and also (4) similarly. Hence, by summing up, we obtain $d_i \leq 6\sum_a d_i(a)$. Since $\#\{a\} = 2(g-1)$, we obtain the desired inequality (2). ☐

From the Claims 2.8 and 2.10 (i),(ii) we have that $d_i \to \infty$ if and only if there is some P_a with $d_i(a) \to \infty$. Also it follows from the Claim 2.10, if P_a satisfies that for some b $\mathrm{length}(s_b(R_i; P_a)) \to 0$ for $i \to \infty$, by further passing to a subsequence we can assume that R_i satisfies that

$$\frac{d_i(a)}{2} \leq d_i \leq 12(g-1)d_i(a),$$

for a fixed a, say $a = 1$. On the other hand,

$$\frac{d_i(a)}{2} \leq d_i$$

holds for any a. Hence, combining Claim 2.9 and Claim 2.10, we have that

Claim 2.11 (Limit of hyperbolic hexagon, II)**.** *Under our assumption that* $[R_\infty] \notin M_g$*, if we consider the sequence* $(Q_a(R_i), \frac{d_{\rm KE}}{d_i})$ *for* $i \to \infty$*, it converges in the Gromov-Hausdorff sense to either a metrized tree* Γ *or a point.*

The convergence to the point occurs exactly when $\dfrac{d_i}{d_i(a)} \to +\infty$ for $i \to +\infty$. From the above claim 2.11, it follows that the global Gromov-Hausdorff limit of $(R_i, \frac{d_{\rm KE}}{d_i})$ is a metrized graph which is obtained by gluing all Γ_a at w_b's whose corresponding boundary geodesics s_j are the same in the whole Riemann surface R_i. The resulting graph is either the dual graph of the corresponding stable curve R_∞ or a graph obtained from the dual graph after contracting several edges to points. (We simply call such procedure a *contraction* of a graph in this paper.)

Now let us move on to the proof of the converse direction (the last paragraph of the statements of Theorem 2.4). That is, starting from an arbitrary finite metrized graph Γ of diameter 1 which satisfies the assumption of the last paragraph of Theorem 2.4, we wish to prove there is a sequence of compact Riemann surfaces $R_i (i = 1, 2, \cdots)$ of genus g such that Γ is the Gromov-Hausdorff limit of $(R_i, \frac{d_{\rm KE}}{d_i})$ i.e., the rescaled Poincaré metrics of diameter 1.

We fix an arbitrary stable curve R whose dual graph contracts to the underlying graph of Γ. Such R exists due to our assumption on Γ. Then take a smooth point in each of the irreducible components of R and denote them by p_i. Here the index i corresponds to each irreducible component. We take a semi-universal deformation of R as $\{R_{\vec{t}}\}_{\vec{t} \in U}$ with an open neighborhood $U' \subset \mathbb{C}^{3g-3}$ of $\vec{0}$, satisfying $R_{\vec{0}} = R$ and take $p_{i,\vec{t}}$ of $R_{\vec{t}}$ with $p_{i,\vec{0}} = p_i$ which is continuous with respect to $\vec{t}$. From here, we use a smaller open neighborhood of $\vec{0}$ denoted by $U \subset U'$ with $\bar{U} \subset U'$. Note that there is a discriminant locus $D \subset U$ such that $\vec{t} \notin D$ if and only if $R_{\vec{t}}$ is smooth. We fix a uniform pants decomposition of $R_{\vec{t}}$ so that the nodes x_k of R are shrunk dividing geodesics s_k of the decomposition. For each node x_k of R connecting the irreducible components including p_i and p_j, there is a corresponding shortest geodesic $\gamma_{k,\vec{t}}$ connecting $p_{i,\vec{t}}$ and $p_{j,\vec{t}}$ if $R_{\vec{t}}$ is smooth which intersects with s_k.

Recall that there is a standard submersive holomorphic map $\phi = \{\phi_k\}_k \colon U \to \prod_k \mathrm{Kur}(x_k)$, where $\mathrm{Kur}(x_k)$ stands for the Kuranishi space underlying a semi-universal deformation of the node singularity x_k, and ϕ_k is induced by restricting the deformation of R to a neighborhood of each node x_k. In this case, $\mathrm{Kur}(x_k)$ can be regarded as an open neighborhood of 0 in $\mathbb{C}$ and the discriminant locus D is the divisor $\cup_k \phi_k^{-1}(0)$. For the proof of the fact that ϕ is submersive, i.e., its differential $d\phi$ is surjective, see [DM, Proposition 1.5]. Since the distance of p_i, p_j for $i \neq j$ in R with respect to the hyperbolic metric is $+\infty$ (i.e., not defined as a real number), for a sequence $\{\vec{t_m}\}_{m=1,2,\cdots} \subset U \setminus D$,

$$\mathrm{length}(\gamma_{k,\vec{t_m}}; R_{\vec{t_m}}) \to +\infty$$

for $m \to \infty$ if and only if $\phi_k(\vec{t_m}) \to \vec{0}$.

On the side of Γ, for each node x_k of R, also an edge γ_k of Γ corresponds, which may be possibly contracted to a point. If it is contracted, we regard it as an edge of length 0.

From the above discussions with the surjectivity of ϕ, for large enough positive integers $m \gg 1$, there is $\vec{t_m} \in U \setminus D$

$$\mathrm{length}(\gamma_{k,\vec{t_m}}; R_{\vec{t_m}}) = m \cdot \mathrm{length}(\gamma_k; \Gamma) \text{ if } \gamma_k \text{ is not contracted in } \Gamma \tag{5}$$

$$\mathrm{length}(\gamma_{k,\vec{t_m}}; R_{\vec{t_m}}) = \sqrt{m} \text{ if } \gamma_k \text{ is contracted in } \Gamma. \tag{6}$$

Then, the above taken sequence of smooth compact Riemann surfaces $\{R_{\vec{t_m}}\}_m$ with the rescaled Poincaré metric converges to a metrized graph and from (5) and (6), the limit metrized graph coincides with Γ. We complete the proof of the last paragraph of Theorem 2.4. □

Remark 2.12. A while after the first version of this paper, we essentially gave another (logically independent) more moduli-theoritic proof of Theorem 2.4 in the sequel [Od6] by using [Wol]. Precisely speaking, Theorem 2.4 follows from [Od6, §3.2.1, Theorem 3.7 and its proof] which depends on [Wol].

Remark 2.13. In the simpler case of $g = 1$, i.e., elliptic curves case, we also have a similar phenomenon as discussed in the introduction of [GW]. It can be regarded as the easiest prototypical example of the sequel paper [Od6] on the moduli spaces of principally polarized abelian varieties and also well-known to the experts of the Strominger-Yau-Zaslow mirror symmetry conjetures. Thus we give only brief description as an introduction to our sequels [Od6], [OO].

Suppose there is a sequence of elliptic curve $\{\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau_i)\}_i$ where τ_i belongs to the standard fundamental domain W of the upper half plane $\mathbb{H}$ modulo the modular group $\mathrm{SL}(2,\mathbb{Z})$, that is

$$W := \{\tau \in \mathbb{H} \mid |\mathrm{Re}(\tau)| \leq 1, |\tau| \geq 1\}.$$

If $\mathrm{Im}(\tau_i)$ does *not* diverge, then after passing to a subsequence, they converge in the Gromov-Hausdorff sense to an elliptic curve. If $\mathrm{Im}(\tau_i)$ diverges, then the Gromov-Hausdorff limit of a subsequence of $\left\{\left(R_i, \frac{d_{\mathrm{KE}}}{\mathrm{diam}(d_{\mathrm{KE}})}\right)\right\}_{i=1,2,\cdots}$ is $S^1(\frac{1}{2\pi})$, the circle of radius $\frac{1}{2\pi}$. On the other hand, for a family of elliptic curves over the punctured disk, the compactified Néron model after suitable base change is well-known to be n-gon with some $n \in \mathbb{Z}_{>0}$. Thus their dual graphs are topologically S^1, which is homeomorphic to the Gromov-Hausdorff limit discussed above.

Remark 2.14. For the case of curves with punctures (marked points), i.e., elements of $M_{g,n}$ with $n \geq 1$, as the natural hyperbolic metric has hyperbolic cusp singularities of *infinite* diameters around the punctures, we have not been able to make a suitable formulation to study Gromov-Hausdorff collapses.

Professor Y-G.Oh kindly pointed out to me that a different but similar kind of "graph-like thin" metrics also appear as "*(general) minimal area metric*" studied by Zwiebach and Wolf-Zwiebach (cf., e.g., [Z], [WZ]) for constructing closed string field theory. The metrics are expected to be isometric to *flat* semi-infinite cylinders around the punctures. The graph structure is regarded as a version of Feynman diagrams there.

Remark 2.15. Our Theorem 2.4 suggests that the conjectures of Gross-Wilson [GW, §6], Kontsevich-Soibelman [KS] and Gross-Siebert (cf., [Gross]) on the correspondence of Gromov-Hausdorff limit and dual complex of degenerating *Calabi-Yau manifolds* may well have an analogue in *negative* Ricci curvature Kähler-Einstein case, i.e., those projective manifolds with ample canonical classes.

Let us trace again the proof of our Theorem 2.4 to see some analogy with the tropicalization of the Berkovich analytification [ACP]. The one page arguments below does *not* contain any substantially concrete results and rather we mean to give a re-interpretation of our Theorem 2.4 and compare with [ACP]. In our theorem 2.4, starting with an arbitrary sequence of compact hyperbolic surfaces, we took a nice subsequence which converges to a stable curve in the Deligne-Mumford compactification and also converging in the Gromov-Hausdorff sense (while fixing the diameter). Let us call such sequence of compact hyperbolic surfaces of genus $g(\geq 2)$ "strongly convergent sequence". We denote the set of such strongly convergent sequences of compact hyperbolic Riemann surfaces as [4] $\mathcal{SM}_g$.

Definition 2.16. For the positive integer $g \geq 2$, let S_g be the set of the underlying metric spaces of the metrized graphs which appear as the Gromov-Hausdorff limits of sequences of compact Riemann surfaces of genus $g(\geq 2)$, and associate Gromov-Hausdorff distance structure on it.

Note that S_g is also compact by Theorem 2.4 and the simple fact that S_g is closed under the Gromov-Hausdorff convergence.

Then what we have constructed in the proof of Theorem 2.4 is the following two kinds of limiting maps

$$r\colon \mathcal{SM}_g \to \overline{M_g}^{\mathrm{DM}} \tag{7}$$

which maps $\{R_i\}$ to the limit (Deligne-Mumford) stable curve and

$$t\colon \mathcal{SM}_g \to S_g \tag{8}$$

which maps $\{R_i\}$ to the Gromov-Hausdorff limit. Furthermore, we proved that r and t are compatible in the sense that the underlying graph of $t(\{R_i\})$ is a contraction of the dual graph of the limit stable curve $r(\{R_i\})$.

On the other hand, in the recent paper [ACP] by Abramovich-Caporaso-Payne, the following is proved.

> *Fix an algebraically closed base field k with trivial valuation. If we consider the Berkovich analytification $\overline{M_g}^{an}$* [Berk1] *of the Deligne-Mumford compactification $\overline{M_g}$, then the deformation retraction to the Berkovich skeleton* [Berk2] *is the "tropicalization" map towards the moduli of tropical curves of genus g.*

Note that the Berkovich analytification parametrises stable curves over valuation fields which contains k (with trivial valuation) and it can be regarded as (a subspace of) this as an "algebro-geometric" analogue of the set of strongly convergent sequence of compact Riemann surfaces $\mathcal{SM}_g$. From this viewpoint, their tropicalization (deformation retract) is an analogue of our map t. The analogue of r in the Berkovich geometric setting [ACP] is the reduction map $\overline{M_g}^{an} \to \overline{M_g}^{\mathrm{DM}}$.

[4] Here, $\mathcal{S}$ stands for a sequence.

2.3. **The construction of $\overline{M_g}^{\rm GH}$.** We define our *Gromov-Hausdorff compactification* of the moduli space of curves, first set-theoretically as

$$\overline{M_g}^{\rm GH} := M_g \sqcup S_g.$$

Recall that we have defined S_g in Definition 2.16 as the moduli space of the underlying metric spaces of the metrized graphs which appear as the Gromov-Hausdorff limits of sequences of compact Riemann surfaces of genus $g(\geq 2)$. Then we put a topology on it, whose open basis consists of the following two kinds of subsets:

(i) open subsets of M_g (with respect to the complex analytic topology) and
(ii) the metrics balls with centers are in S_g.

What we mean by the metric ball, with its center $[G] \in S_g \subset \overline{M_g}^{\rm GH}$ (G is a metrized graph) and radius $r \in \mathbb{R}_{>0}$, is simply defined as

$$B([G], r) := \{[C] \in \overline{M_g}^{\rm GH} \mid d_{\rm GH}([C], [G]) < r\}.$$

The obtained topological space $\overline{M_g}^{\rm GH}$ is compact due to our Theorem 2.4. It also satisfies the Hausdorff separation axiom simply because the Gromov-Hausdorff limit as compact metric space is unique as general theory (cf., [BBI]).

The readers may wonder why we do not simply use the notion of the metric completion above. However, note that the complex conjugate $\iota \in {\rm Aut}(\mathbb{C}/\mathbb{R})$ reverses the natural orientation of the corresponding Riemann surface, which does not change it metric space strucuture. A subtle technical point here is that $\overline{M_g}^{\rm GH}$ is not exactly the metric completion with respect to the Gromov-Hausdorff topology, of the set of compact Riemann surfaces of genus g by regarding the Riemann surfaces just as metric spaces. That is because it would discard the complex structures and ignore the effect of ι above (cf., e.g., [Spo],[OSS]).

Recall that S_g is defined as the moduli spaces of the underlying metric spaces of our limit metrized graphs as in Theorem 2.4. For each finite (metrized) graph Γ, let us denote the number of 1- valent vertices by $v_1(\Gamma)$ and denote the first betti number of Γ by $b_1(\Gamma)$. Then, more specifically and concretely, S_g can be described as follows.

Proposition 2.17. *The metric spaces parametrized by S_g can be characterized by a purely topological condition that the underlying topological spaces of the metrized graphs satisfy $v_1(\Gamma) + b_1(\Gamma) \leq g$.*

Note there is a subtle distinction between the metrized graph and the underlying metric space, which is simply a 1-dimensional CW complex with a metric. The reason is that the underlying metric space does *not* see the 2-valent vertices. It is also not enough to consider metrized graphs without 2-valent vertices since a circle can not be obtained in that way.

proof of Proposition 2.17. From Theorem 2.4, we only need to specify the class of dual graphs of stable curves with genus g.

A stable curve C of genus g whose irreducible decomposition is $\cup_i C_i$ with dual graph Γ satisfies

$$g = \sum_i g(C_i^\nu) + b_1(\Gamma), \tag{9}$$

where ν denotes the normalization and b_1 denotes the first Betti number. From the stability condition, for each component C_i which corresponds to a 1-valent vertex of Γ, $g(C_i^\nu) \geq 1$. This is essentially the only numerical stability condition. Thus we have $g = \sum_i g(C_i^\nu) + b_1(\Gamma) \geq v_1(\Gamma) + b_1(\Gamma)$. Tracing back the above discussion, it is also easy to see that it is a sufficient condition as well. □

Remark 2.18. One remark, which the author hopes to be useful, is that in the above characterisation of metrized graphs which are parametrised in S_g, rather than putting the "diameter 1" condition, it may be easier to impose that "the sum of lengths of edges is 1" when we try to concretely describe the structure of our compactifications. Note that these two moduli spaces are naturally homeomorphic, simply by rescaling the metrics on our metrized graphs.

3. Related moduli spaces and comparison

In this section, we further study $\overline{M_g}^{\mathrm{GH}}$ somewhat indirectly by comparing with other moduli spaces in literatures, and also construct some variants of $\overline{M_g}^{\mathrm{GH}}$ on the way, including what we call tropical geometric compactifications and denote by $\overline{M_g}^{\mathrm{T}}$.

3.1. **Comparison with tropical moduli spaces.**

Recently Brannetti-Melo-Viviani [BMV] constructed moduli space M_g^{tr} of the weighted metrized graphs, i.e., $(\Gamma, w \colon V(\Gamma) \to \mathbb{Z}_{\geq 0})$ of where Γ is a metrized graph such that

- 1-valent vertex v must have $w(v) \geq 2$,
- 2-valent vertex v must have $w(v) \geq 1$,
- $b_1(\Gamma) + \sum_{v \in V(\Gamma)} w(v) = g$,

with a natural topology (as well as some finer "stacky fan" structure) on it. Caporaso [Cap] introduced its log versions $M_{g,n}^{trop}$. See [BMV, Cap] for the details. The moduli space M_g^{tr} is similar to our boundary S_g

but there is an essential difference which is the presence of the weight function w above that morally encodes *genus* of each component of the limit Deligne-Mumford stable curves.

Similarly to what is done in [CV], [BMV], [Cap], the combinatorial type of the underlying graph of a metrized graph gives a stratification on S_g such that each strata is a finite quotient of a simplex. A basic property of our moduli space S_g is the following.

Proposition 3.1. *The function $S_g \ni [\Gamma] \mapsto v_1(\Gamma) + b_1(\Gamma)$ is a lower semicontinuous function on S_g with respect to the Gromov-Hausdorff topology which has been previously considered.*

Proof. The assersion follows easily from Theorem 2.4 combined with the precompactness (Corollary 2.2) but let us give a more straightforward combinatorial proof.

It is enough to see that if we contract one edge e, the function v_1+b_1 does not increase. If the edge e is a loop, then the process decreases b_1 by 1 and v_1 increases at most 1. If the edge e is not a loop, then the contraction does not change the homotopy type of the graph so that it keeps b_1 unchanged, and v_1 does not increase (it may decrease by 1 or 2).

□

Note that through the modular interpretations, there is a sequence of *canonical* closed embeddings

$$S_g \hookrightarrow S_{g+1} \hookrightarrow \cdots, \tag{10}$$

while other compactifications of moduli of curves and the moduli of *weighted* tropical curves by [BMV], [Cap], [CHMR] do *not* have this chain of canonical inclusions.

Inside the moduli space M_g^{tr} of (weighted) tropical curves in the sense of [BMV], let us consider the closed locus S_g^{wt} which parametrizes those with the diameter 1 ("wt" of S_g^{wt} stands for weights.)

Proposition 3.2. *We have natural morphisms as follows.*

$$\partial M_g^{tr} := M_g^{tr} \setminus \{\text{a point with weight } g\} \cong S_g^{wt} \times \mathbb{R}_{>0} \twoheadrightarrow S_g^{wt} \twoheadrightarrow S_g. \tag{11}$$

S_g has a finite stratification which satisfies that each strata is a finite group quotient of an open simplex and S_g is "purely" $(3g-4)$ dimensional for each $g(\geq 2)$ in the sense that, if we denote the union of $(3g-4)$-dimensional strata as $S_g^{oo} \subset S_g$, then it is an open dense subset. In addition, the last morphism of (11) *is a proper map such that each fiber over S_g^{oo} is finite.*

Proof. A tropical curve in the sense of [BMV] has finite non-zero diameter unless it is a point, so that we get the first isomorphism. Secondly, starting from a tropical curve which is not topologically a point, just by forgetting the weights and the 2-valent vertices, we get the underlying metric space of a metrized graph. It defines the last morphism $S_g^{wt} \twoheadrightarrow S_g$, which we denote as r. It follows straightforward from the topology on S_g^{wt} in [BMV] that this morphism is continuous and this is surjective by Proposition 2.17. From the compactness of S_g^{wt} and S_g, it follows automatically that the morphism is proper. Note that for any point p in S_g^{wt} which has 2-valent vertices, $r^{-1}(r(p))$ is non-finite. It is because that for each metric space X corresponding to a point in S_g, if it is underlying metric space of certain weighted tropical curve (weighted metrized graph) Γ parametrized in S_g^{wt}, once we know the locations of vertices in X, there are only finite choices of Γ which corresponds to the decomposition of $g - b_1(X)$ into non-negative integer weights attached to the vertices.

It is easy to see that S_g has a natural finite stratification by the homeomorphic class of the underlying graphs. Each strata can be seen as the moduli of metrized graphs with the same underlying graph, with the sum of the length of edges are 1 by rescaling the metrics. Hence it is homeomorphic to the quotient of an open simplex with respect to a linear action of a finite group, which is the automorphism group of each graph. Next we proceed to the proof of the fact that S_g is purely $3g-4$-dimensional as in the statement of Proposition 3.2. Indeed, for any given (underlying metric space of) a metrized graph Γ of the diameter 1 which satisfies $v_1(\Gamma) + b_1(\Gamma) < g$, by attaching small circles or short edges and rescaling, the corresponding point $[\Gamma] \in S_g$ can be easily perturbed to a point inside the strata with $v_1 + b_1 = g$. The strata can be easily checked to have dimension $3g-4$, as $3g-3$ is the number of edges inside Γ following elementary graph theory. This fact is also well known in the algebro-geometric field of study of the so-called Mumford curves. Thus, the union S_g^{oo} of $(3g-4)$-dimensional cells form open dense subset. For each $p = [\Gamma] \in S_g^{oo}$, the r-fiber $r^{-1}(r(p)) = \{p\}$ since for a point $[\Gamma']$ in the fiber, the vertices of the graph Γ' are nothing but the non-smooth points of $r(\Gamma') = r(\Gamma)$ as an underlying topological space and furthermore Γ' does not have any positive weights on the vertices because of the formula (9). We complete the proof of Proposition 3.2. □

3.2. **Construction of $\overline{M_g}^{\mathrm{T}}$.** It is possible to modify our construction of $\overline{M_g}^{\mathrm{GH}}$ to make more compatibility with the above "*weighted* tropical moduli spaces" of [BMV], [Cap], [CHMR]. That is, for a collapsing sequence of genus g compact Riemann surfaces as in Theorem 2.4, we can encode the information of the genera of the irreducible components of the limiting stable curves on the limiting graph. More precisely speaking, first we consider the set

$$\overline{M_g}^{\rm T} := M_g \sqcup S_g^{wt},$$

on which we put a topology as follows. A subset C of $\overline{M_g}^{\rm T}$ is *closed* if and only if

- $C \cap S_g^{wt}$ is closed in S_g^{wt} and
- any Gromov-Hausdorff collapsed graphs of compact Riemann surfaces which are in $C \cap M_g$, attached with the genera of components of the normalization of the limit stable curve in [DM] sense, which we suppose to exist, is actually in $C \cap S_g^{wt}$.

The compactness, the Hausdorff property of $\overline{M_g}^{\rm T}$, and the fact that M_g is open and dense inside $\overline{M_g}^{\rm T}$ all follow straightforwardly from our Theorem 2.4 and its proof. We would like to call this compactification $\overline{M_g}^{\rm T}$ of M_g as the *tropical geometric compactification* of M_g.

From the construction we have a natural continuous surjective map

$$\overline{M_g}^{\rm T} \twoheadrightarrow \overline{M_g}^{\rm GH},$$

which restricts to the identity map on the open subset M_g.

3.3. Finite join $\overline{M_{\le g}}^{\rm GH}$ and infinite join $\overline{M_\infty}^{\rm GH}$.

An interesting phenomenon is that, as the following definitions show, our Gromov-Hausdorff compactification $\overline{M_g}^{\rm GH}$ naturally patches together for different g thanks to the sequence of the canonical inclusions (10) of S_g.

Definition 3.3. The *finite joins* of our Gromov-Hausdorff compactifications are defined inductively as topological spaces

$$\overline{M_{\le 0}}^{\rm GH} := \overline{M_0}^{\rm GH} = \{\text{ Riemann sphere } \mathbb{CP}^1\} \text{ (singleton)},$$

$$\overline{M_{\le 1}}^{\rm GH} := \overline{M_1}^{\rm GH} := M_1 \sqcup \{S^1(\frac{1}{2\pi})\}(= \overline{A_1}^{\rm T} \text{ in the next section })$$

(one point compactification)

and for $g \ge 2$ as

$$\overline{M_{\le g}}^{\rm GH} := (\overline{M_{\le (g-1)}}^{\rm GH} \cup \overline{M_g}^{\rm GH})/\sim,$$

where the equivalence relation $\sim$ is simply the identification of the closed subset $S_{g-1} \subset S_g$ and another closed subset $S_{g-1} \subset \overline{M_{\leq(g-1)}}^{\mathrm{GH}}$. From the definition, we have natural inclusion relations

$$\cdots \overline{M_{\leq(g-1)}}^{\mathrm{GH}} \subset \overline{M_{\leq g}}^{\mathrm{GH}} \cdots .$$

Then we set

$$\overline{M_\infty}^{\mathrm{GH}} := \varinjlim_g \overline{M_{\leq g}}^{\mathrm{GH}} = \cup_g \overline{M_{\leq g}}^{\mathrm{GH}},$$

and call it *infinite join* of our Gromov-Hausdorff compactifications.

The boundary of $\overline{M_\infty}^{\mathrm{GH}}$ by which we mean the natural subset $\cup_g(\partial\overline{M_g}^{\mathrm{GH}} = S_g)$, should be regarded as a tropical version of the space[5] "M_∞" introduced and studied recently by Ji-Jost [JJ].

Also note that $\overline{M_\infty}^{\mathrm{GH}}$ is connected and all our Gromov-Hausdorff compactification $\overline{M_g}^{\mathrm{GH}}$ is inside this infinite join.

3.4. **Comparison with the Outer spaces.** There is a classical theory of the *outer space* X_n by Culler-Vogtman [CV], which is an analogue of the Teichmuller space for metrized graphs. There, the analogous discrete group to the mapping class group is the outer automorphism group $\mathrm{Out}(F_n)$ of the free group F_n with rank n. From now on, we use g instead of their n to unify our notation.

Recall that the quotient $X_g/\mathrm{Out}(F_g)$ parametrizes graphs Γ with $b_1(\Gamma) = g$ with $v_1(\Gamma) = 0$.

We introduce another moduli space of graphs as a subset of S_g (with the induced topology) as

$$S_g^o := \{\Gamma \in S_g \mid v_1(\Gamma) + b_1(\Gamma) = g\} \subset S_g.$$

It is simply the complement of $S_{g-1} \subset S_g$ by the definition. The following proposition essentially goes back to [CMV].

Proposition 3.4. *There is a canonical cellular open embedding* $X_g/\mathrm{Out}(\mathrm{F_g}) \hookrightarrow S_g^o(\subset S^g)$. *The image of* $X_g/\mathrm{Out}(\mathrm{F_g})$ *is open dense in* S_g *(thus so is* S_g^o*).*

Proof. First of all, it follows from the lower semicontinuity of the first Betti number of metrized graphs $b_1(\Gamma)$ that $X_g/\mathrm{Out}(F_g)$ is an open subset of S_g^o. For each $\Gamma \in S_g^o$ with $v_1(\Gamma) + b_1(\Gamma) = g$ and $0 < \epsilon \ll 1$, we define graph(s) $\phi_\epsilon(\Gamma)$ as follows. For each leave vw where v is a 1-valent vertix, we put a small loop of length $\epsilon l(vw)$. Doing the same for all edges and rescale the metric on whole graph to make its diameter 1, we get a metrized graph which we denote as $\phi_\epsilon(\Gamma)$.

[5] They call it "universal moduli spaces"

This construction naturally defines a perturbation of elements of S_g^o to those of $X_g/\mathrm{Out}(F_g)$. The fact that all of these are unions of relative interiors of the cells with respect to that CW complex structure follow straightforward from the definitions.

We also need to prove S_g^o is dense inside S_g. We provide an elementary proof for convenience. Let us analyze the neighborhood of $\Gamma \in S_{g-1} \subset S_g$. Starting from any such Γ with a point $p \in \Gamma$, we can similarly consider Γ's deformation $\psi_t(\Gamma) \in X_g/\mathrm{Out}(F_g)$ for $t > 0$, for example, as follows. Set $v_1(\Gamma) + b_1(\Gamma) = g - d$. Taking a point p, we define $\psi_t(\Gamma)$ as a union of Γ and a bouquet i.e., the union of d length t loops which passes through p. Thus in particular $X_g/\mathrm{Out}(F_g)$ is open and dense in S_g and hence so is S_g^o as well. □

Notes added, part 2. We end this section with the following notes added, about the relation with [LL] which was kindly taught by its author L.Lang in June of 2015. I appreciate him for informing it.

Remark 3.5. L. Lang defined "tropical convergence" of compact Riemann surfaces to metrized graphs as the convergence of the ratios of the lengths of shrinking geodesics, which represent vanishing cycles, in his [LL, Definition 1.1]. As also written in [LL, v2, §1.3], that notion of convergence is *not* equivalent to ours, i.e. Gromov-Hausdorff convergence of hyperbolic metrics. See more details on the original paper [LL]. The author also gives more detailed arguments in [Od6, §3].

4. Investigating topology

We would like to make the first step of investigation of the topology of our compactifications and their boundaries.

First, we recall the fact that the moduli space of smooth projective curves has vanishing higher homology groups, proved by J. Harer [Har]. His proof shows the existence of a deformation retract via the cell complex structure of the Teichmuller space (the so called "arc complex").

Theorem 4.1 ([Har, Theorem 4.1])**.** *For $g \geq 2$ and $i > 4g-5$, we have*

$$H_i(M_g; \mathbb{Q}) = 0 \text{ and } H^i(M_g; \mathbb{Q}) = 0.$$

So combined with the Poincaré-Lefschetz duality for orbifold, we get that for $i \leq 2g-2$

$$H_{\mathrm{c}}^i(M_g; \mathbb{Q}) = 0 \text{ and } H_i^{\mathrm{BM}}(M_g; \mathbb{Q}) = 0,$$

where H_{c}^i denotes the cohomology group with compact supports and H_i^{BM} denotes the Borel-Moore homology group.

The above theorem 4.1 has the following consequence.

Corollary 4.2. *For $i < 2g - 2$, we have*

$$H^i(\overline{M_g}^{\mathrm{T}};\mathbb{Q}) = H^i(S_g;\mathbb{Q}),$$
$$H_i(\overline{M_g}^{\mathrm{T}};\mathbb{Q}) = H_i(S_g;\mathbb{Q}).$$

Proof. It follows simply from the exact sequences of compactly supported cohomology groups or the Borel-Moore homology groups. □

Thus the study of homology and cohomology of our Gromov-Hausdorff compactification is reduced to that of the boundary for a specific range of degrees. Motivated by it, let us study the topology [6] of our boundary S_g. First, we sketch the following cases of small g.

Example 4.3. S_1 is just a point which stands for the circle of length 1. S_2 is a two 2-simplices (triangles) patched together along one of their edges for each. In one side of the 2-simplex, the inner points parametrize a union of two circles and a segment connecting them. The other side of the 2-simplex, the inner points parametrize a union of circle with a segment connecting two points in the circle. We refer to the picture below, where the parametrized metrized graphs are pictured around each stratum.

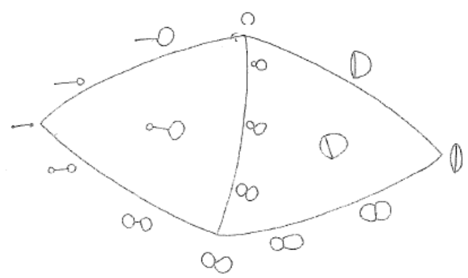

FIGURE 1. The boundary S_2 of $\overline{M_2}^T$

Note that obviously S_1 and S_2 are both contractable.

Since the open dense locus S_g^o of S_g is a rational classifying space of $\mathrm{Out}(F_g)$ as known to [CV], it has in general highly nontrivial topology. Indeed its cohomology is those of $\mathrm{Out}(F_g)$ (cf., e.g., [EVHS] for non-vanishing cohomology for $g = 5$ case), we expect interesting topological structure on S_g for large g.

[6]A while after the appearance of the first version of this paper as arXiv:1406.7772, Chan-Galatius-Payne [CGP] appears which systematically studies the topology of the moduli of weighted metrized graphs *with $n(> 0)$-marked points* i.e. the "log version" of S_g^{wT}.

We define

$$S_\infty := \varinjlim S_g = \cup_g S_g,$$

the injective limit with respect to the canonical embeddings $S_{g-1} \hookrightarrow S_g \hookrightarrow S_{g+1} \cdots$ (cf., (10)). After a kind suggestion of the referee, the author learnt that our S_∞ can be informally (but not logically) seen as a tropical analogue of the infinite union of the classical moduli spaces studied in [Cod, JJ].

While we expect that each S_g has highly nontrivial topologies in general, we observe the following.

Theorem 4.4. *The topological space S_∞ is contractible. In particular, for any $k \geq 0$, $\varinjlim_g H_k(S_g; \mathbb{Q}) = 0$.*

Proof. Consider the cone of S_g, i.e., $CS_g := (S_g \times [0,1])/(S_g \times \{1\})$. It is enough to construct a series of continuous maps $\{\phi_g : CS_g \to S_\infty\}_{g \geq 2}$ which satisfies

(i) ϕ_g maps $(S_g \times \{1\})$ to a point as $\phi_g(S_g \times \{1\}) = \{$the unit interval $[0,1]$ (as a metrized graph)$\}$,
(ii) $\phi_g|_{S_g \times \{0\}} = \mathrm{id}|_{S_g}$,
(iii) and $\phi_{g+1}|_{CS_g} = \phi_g$.

Indeed, from the third condition, they glue together to form a continuous map

$$\phi_\infty : CS_\infty \to S_\infty$$

and this gives a deformation retract of S_∞ into a point of S_∞ which corresponds to the unit interval $[0,1]$ again as a metrized graph.

We construct the map ϕ_g by the following three steps.

Step 1 (Adding vertices)**.** First we construct $\phi_g|_{S_g \times [0,\frac{1}{3}]}$. For any $(\Gamma, t) \in S_g \times [0, \frac{1}{3}]$, suppose the set of vertices of Γ is $V(\Gamma) = \{p_1, \cdots, p_m\}$ and the set of edges is $E(\Gamma) = \{e_1, \cdots, e_n\}$. We define a new metrized graph $\psi_g(\Gamma, t)$ for $t \in (0, \frac{1}{3}]$ by setting the vertices set as $\{p_1, \cdots, p_m\} \sqcup \{p'_1, \cdots, p'_m\}$ and define the set of edges and their lengths as follows. The set of edges of $\psi_g(\Gamma, t)$ is $E(\Gamma) \sqcup \{\overline{p_i p'_i} \mid 1 \leq i \leq m\}$. We call an edge in $E(\Gamma) \subset E(\psi_g(\Gamma, t))$ as *old edge* in this proof, while the edges of the form $\overline{p_i p'_i}$ will be called *new edges*. We put their length $l(\overline{p_i p'_i}) = t$ while we keep the length of old edges as the same as Γ. Then we rescale the length of all edges of $\psi_g(\Gamma, t) (0 < t \leq \frac{1}{3})$ to make the diameter 1 and denote the obtained metrized graph as $\phi_g(\Gamma, t)$. Note that the image of $\phi_g|_{S_g \times [0, \frac{1}{3}]}$ is a priori *not* inside S_g. Indeed, while the metrized graphs parametrized in S_g are characterized by $v_1 + b_1$ by Proposition 2.17, we have that

$$v_1(\phi_g(\Gamma, t)) = \#V(\Gamma),$$

which is bigger than $v_1(\Gamma)$ if and only if Γ is not homeomorphic to the closed interval. This $\phi_g|_{S_g\times[0,\frac{1}{3}]}$ is continuous from the construction.

Step 2 (Contraction of old edges)**.** Our next step is the construction of $\phi_g|_{S_g\times[\frac{1}{3},\frac{2}{3}]}$. Roughly speaking, in this step of t increasing from $\frac{1}{3}$ to $\frac{2}{3}$, we gradually contract the old edges i.e., those which belong to $E(\Gamma)$. We make this rigorous as follows.

First, as in Step 1, we construct $\psi_g(\Gamma, t)$ for $t \in [\frac{1}{3}, \frac{2}{3}]$ by setting its vertices set and edges set as

$$\begin{aligned}
V(\psi_g(\Gamma, t)) &:= V(\phi_g(\Gamma, \frac{1}{3})) \\
&= \{v_1, \cdots, v_m\} \sqcup \{w_1, \cdots, w_m\} \text{ for } t \in \big[\frac{1}{3}, \frac{2}{3}\big), \\
V(\psi_g(\Gamma, t)) &:= \{v\} \sqcup \{w_1, \cdots, w_m\} \text{ for } t = \frac{2}{3}, \\
E(\psi_g(\Gamma, t)) &:= E(\phi_g(\Gamma, \frac{1}{3})) \\
&= E(\Gamma) \sqcup \{\overline{v_i w_i} \mid 1 \le i \le n\} \text{ for } t \in \big[\frac{1}{3}, \frac{2}{3}\big), \\
E(\psi_g(\Gamma, t)) &:= \{\overline{v w_i} \mid 1 \le i \le n\} \text{ for } t = \frac{2}{3}.
\end{aligned}$$

Then we put the metrics on the edges of $\phi_g(\Gamma, t)$ as follows. [7]

$$\begin{aligned}
\text{length}(\overline{v_i w_i}; \phi_g(\Gamma, t)) &:= \frac{1}{3}, \\
\text{length}(\overline{v_i v_j}; \phi_g(\Gamma, t)) &:= (2 - 3t)\text{length}(\overline{v_i v_j}; \Gamma).
\end{aligned}$$

The above construction of $\psi_g(\Gamma, t)$ realizes shrink of old edges in $\phi_g(\Gamma, \frac{1}{3})$. Then finally we define the metrized graph $\phi_g(\Gamma, t)$ as rescale of $\psi_g(\Gamma, t)$ with the diameter 1.

From the constrution, the continuity of $\psi_g|_{S_g\times[\frac{1}{3},\frac{2}{3}]}$ and $\phi_g|_{S_g\times[\frac{1}{3},\frac{2}{3}]}$ are obvious. The limit graph $\phi_g|_{t=\frac{2}{3}}$ is a metrized tree whose edges all share a common vertex so that its shape looks like "$*$". Precisely speaking, it is a metrized graph graphs whose

- vertices set is $\{v\} \sqcup \{w_i \mid 1 \le i \le m\}$ and
- edges set is $\{\overline{v w_i} \mid 1 \le i \le m\}$.

[7]The notation of the following is that the length of edge l in a graph G is denoted as length(l, G).

Let us call this type of tree "$*$-type" with $n(=\#E(\Gamma))$ leaves.

Step 3 (Deforming to the unit interval)**.** The final step is the construction of $\phi_g|_{S_g\times[\frac{2}{3},1]}$. The moduli space of $*$-type trees Γ (as we defined and discussed above in Step 2) with n leaves of diameter 1, with the Gromov-Hausdorff topology, is homeomorphic to the moduli space of those whose sum of lengths of edges is 1, simply by rescaling. And the latter is the simplex

$$\Delta_n := \{(x_1,\cdots,x_n) \mid 0\leq x_1\leq x_2\leq \cdots x_n\leq 1, \sum_{i=1}^{n} x_i = 1\}.$$

The contractability of the simplex above ensures, or we can directly see that there is a deformation retract of each $\Gamma\in\Delta_n$ to the interval $[0,1]$. This gives $\phi_g|_{S_g\times[\frac{2}{3},1]}$.

The desired properties (i), (ii), (iii) are all straightforward from the construction. We complete the proof of Theorem 4.4. To help understanding for the readers, we summarize our 3 Steps below as an example picture.

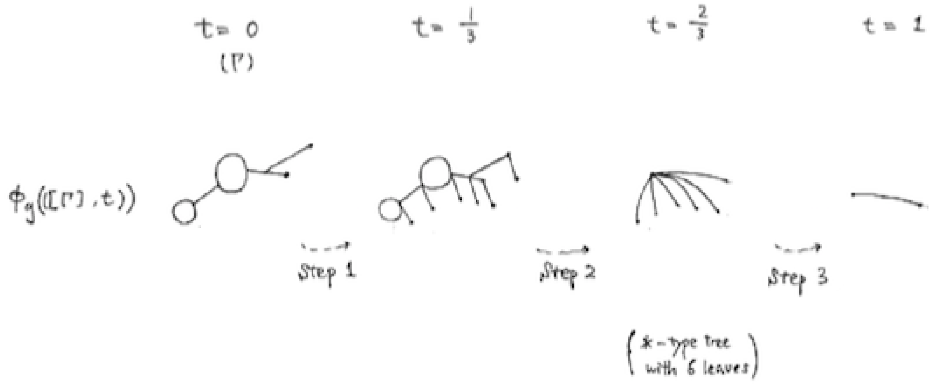

Figure 2. Picture proof of Theorem 4.4

□

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