Doctoral thesis

New viable theories of modified gravity: Minimal Theories and Quasidilaton

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Abstract

In the last few decades, several alternative models of gravity have been developed in the hope to produce large-scale deviations from general relativity (GR). The principal aim of this program is to uncover what physics could be at works behind the present-day acceleration of the Universe, other than a fine-tuned cosmological constant. Among other motivations to explore new models of gravity are also the other puzzles of cosmology, such as dark matter, as well as the incompleteness of our description of gravity in the UV. However, satisfying the constraints from current observations while guaranteeing full theoretical consistency is still a challenging task for several such constructions; both do indeed put strong restrictions on the theory space of gravitational theories. Two notable examples are the instabilities that plague some of the cosmological realizations of massive gravity theories and its extensions, or the recent restrictive bound on the speed of tensor modes from the multi-messenger observation of a neutron star binary merger.

In this thesis we show that there exist new classes of alternative theories of gravity that are observationally and theoretically viable, and produce interesting phenomenology. In particular, we focus first on minimally modified gravity (MMG) theories, which propagate only the two tensor modes by means of violations of the Lorentz symmetry in the gravitational sector. In this context, we present a class of theories constructed on the basis of the existence of an Einstein frame, in which the gravitational Lagrangian is equivalent to GR. As observational constraints, we consider in particular the bound on the speed of tensor modes, as well as on the variation of the gravitational constant. We find that there subsists a wide class of interesting possibilities to modify GR. In addition to this construction, we review other theories that fall into the MMG class.

As a second example of a new alternative theory of gravity, we construct and study the minimal theory of quasidilaton massive gravity (MQD). This theory is motivated by some difficulties to find viable Friedmann-Lemaître-Robertson-Walker cosmologies in the context of quasidilaton massive gravity theories, but can also be effectively understood as an extension of a specific MMG theory, the minimal theory of massive gravity, by rendering dynamical part of the fiducial metric structure. We show that MQD is viable for a wide region of its parameter space, that it will be efficiently constrained by future cosmological surveys, and can sustain interesting phenomenology, in particular produce weak gravity while propagating the same number of degrees of freedom as usual scalar-tensor theories.

In order to motivate the models we present, this thesis also includes two compact review chapters, which cover respectively the standard model of gravity and cosmology, and several alternatives to GR together with current and prospective constraints. In light of the large number of future observational efforts to constrain the cosmological dynamics as well as the behavior of gravity on these scales, model-building efforts come by as crucial tools to be able to interpret the future data most efficiently.

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Contents

Chapter 1

Introduction

General relativity (GR), as proposed by Einstein more than a century ago, is one of the most successful constructions of theoretical physics. What started only as a peculiar[1] unified view of space, time, and gravity, has now become a cornerstone of our understanding of Nature, and its predictions have led to various technological and epistemic advances[2]. As recently as in the last five years, the theory has provided humanity with new tools: thanks to the effort of large scientific collaborations such as the LIGO (the Laser Interferometer Gravitational-wave Observatory) and Virgo collaborations, we are now able to detect gravitational waves from compact binary inspirals, mergers, and ringdowns [2]; in parallel, recent interferometric observations have also let us obtain the first image of a black hole's photosphere [3]. These recent achievements are only the last in a series of confirmations of GR as a solid theory.

It is intriguing that, in spite of this success, there are in fact very good reasons to think that GR should be modified. One may distinguish two main physical clues in this direction: first, as far as is known today, the microscopic world follows the rules of quantum mechanics, and second, the largest, macroscopic scales harbor several unexplained phenomena. In the first case, the non-renormalizability of GR [4] indicates that while it is a valid approximate model at our scales—as confirmed by all the experimental successes until today—when shorter distances (or higher energies) are considered, it should be replaced by a more exhaustive theory. Our failure to date to find and confirm this encompassing theory may be referred as the *quantum gravity problem*, which has motivated a wide range of ambitious models (see, for example, [5, 6, 7, 8, 9]). It is however not the quantum gravity problem, but instead the unexplained phenomena at large scales (the aforementioned second reason to go beyond GR) that will be studied in this thesis. In a nutshell, both the understanding coming with GR as well as technological advances have opened the doors to the investigation of scales that range from galactic ($\mathcal{O}(10^5)\,\mathrm{ly}$) to cosmological ($\mathcal{O}(10^{10})\,\mathrm{ly}$) (not to mention precision science on solar-system scales). Applying our understanding of gravity on these scales has then led to the surprising indirect discovery of two yet misunderstood constituents of our Universe: *dark matter* and *dark energy*. Dark matter is misunderstood because it is optically invisible, and years after its indirect detection [10, 11] there remains the possibility that we will never observe it other than through gravitational signatures, or even that it is not matter *at all*, but instead that we do not understand some dynamics at these scales, hence a modification of the gravitational laws. Next, dark energy is formally speaking the constituent that drives the current acceleration of the expansion of the Universe (meaning on even larger scales than dark matter). It was detected some 20 years ago [12, 13], and it is still a veritable theoretical puzzle due to its small yet non-zero value [14] while experimentally its properties are not yet too strongly constrained. Several particular examples of theories and cosmological scenarios, such as brane-world scenarios (see e.g. [15]), extra fields (see e.g. [16, 17, 18]),... have since then effectively proved that modifications of the gravitational theory were especially apt to play a role in the *dark energy puzzle* as well.

Motivated by the fundamental question *"Can there be a consistent alternative to GR?"*, in this thesis we will explore the route of modified theories of gravity at the largest scales —commonly called *infra-red* (IR) modifications of gravity[3]—in particular following the clues of the dark energy puzzle. Focusing on the dark energy puzzle is a common approach, based both on simplicity, but also on the large separation of scales between the different aforementioned mysteries. Nevertheless, as of today, it is not possible to

[1] There were several critical views at the time, see for instance [1].

[2] Notably, we can now discuss understanding how gravitational waves will tell us about the origins of our Universe, all while driving guided by the GPS of a smartphone. Thanks Einstein!

[3] A common metonymy (which will be used from time to time in this thesis) is the use of the term "modified gravity" alone, which should be understood as modification to GR, or modification at some given scales...

exclude cosmological-scale physics eventually having an impact on our understanding of dark matter, at smaller scales, or even quantum gravity, the ultimate *ultra-violet* problem. It is while keeping this possibility in mind that we choose *not* to discuss in details either dark matter or high-energy physics.

Aside from the possibility to find a solution to the dark energy puzzle, there are several other technical reasons to consider alternatives to GR. On the one hand, it is important to push and refine the boundaries of our present theoretical knowledge of gravity: this includes for example exploring our "neighborhood" in the space of gravitational theories. One may also entertain the idea that GR may have been singled out (as a theory) for our Universe for some fundamental reason, and by exploring other possible theories one could potentially find answers to the question *"What makes GR special?"*, and hence deepen our understanding of GR itself. Maintaining a healthy skeptical view (here, of GR) may be the best route to serendipitous discoveries, as has been countless times across the history of science. On the other hand, it is especially interesting to study modifications *now*, due to the wealth of data that is projected to appear thanks to ambitious experiments in the next decades. The most interesting experiments for the purposes of this thesis are within cosmology; future lensing and spectroscopic galaxy surveys [19, 20, 21, 22, 23, 24], on a volume yet unseen, will allow to refine our statistical mapping of dark matter, and probe both the growth of structures and the background evolution of the Universe. In parallel to this, ever more numerous gravitational wave detections [25, 26], as well as novel frequency windows [27, 28, 29, 30, 31], will contribute to set other stringent constraints on alternative theories of gravity. In light of these future observations, the coming years look very promising in terms of research in gravity.

In the last two decades, several archetypes of modified theories of gravity have attracted more attention. *Scalar-tensor* theories[4], in which a scalar field couples non-minimally to gravity, have recently been explored more thoroughly, and a family of theoretically healthy theories has been delimited among higher-order theories [32]. Due to the high precision of tests at astrophysical scales [33], it is now commonly admitted that a screening mechanism (see e.g. [34]) should be incorporated into most (if not all) phenomenologically interesting scalar-tensor theories, in order to suppress extra forces within the regime of these tests. Several screening mechanisms have therefore been considered. Higher-order theories will for example tend to implement the Vainshtein screening mechanism [35]; other possibilities, for instance the chameleon mechanism [36, 37], have been proposed. Interestingly, notwithstanding the presence of screenings, many of these theories have already been constrained thanks to a measurement of the speed of the gravitational waves [38, 39][5]. Still, surviving scalar-tensor theories and more generally scalar fields may represent reasonable candidates of dark energy. Note that one may also consider other extra fields (in kind and number), for example vector fields, instead of just one scalar field (see e.g. [42]).

Intimately related to higher-order scalar-tensor theories, another class of theories has seen a renewal of interest in the same two decades: *massive gravity* theories and their extensions [43]. These were first revived in a form violating Lorentz symmetry [44, 45, 46], and a Lorentz-invariant theory was found a few years later [47]. Massive gravity theories endow the graviton with a mass, hence modifying the dynamics of gravity and, potentially, the force that is experienced by matter fields: one expects for example a Yukawa-type suppression of the gravitational interaction. Another particularity of several massive gravity theories is to provide a way to accelerate the Universe merely due to the vacuum expectation value of the graviton potential. These two features render the proposal of a massive gravity theory a priori very appealing. It was however found that cosmology within Lorentz-invariant massive gravity requires to go beyond the standard homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker cosmology [48], and that its extensions are seldom "cosmology-complete"[6] (i.e. can sustain a realistic cosmology) due to instabilities in the extra modes (see e.g. [49]). Other properties shared by most massive gravity theories include the use of the Vainshtein screening mechanism [35], and the presence of a high-energy cutoff scale parametrically lower than the Planck scale [44].

The technical difficulties of massive gravity and its extensions, as well as the recent strong hints that the speed of gravity should closely match the one in GR (and hence the possible stringent bounds of scalar-tensor theories), motivate us to *revisit the archetypes of modified gravity, and explore new avenues.* More particular questions we will try to address are, for instance: *"How well do we understand the boundaries of consistent modified gravity?"* and *"Can we develop new consistent (cosmology-complete) theories that have non-degenerate phenomenology, and that are testable in a foreseeable future?"* In particular, we explore wider applications of violations of Lorentz invariance on cosmological scales, and revisit an extension of massive gravity from this perspective. We will see that, in addition to satisfying the current bounds

[4] See [18] for a recent review.

[5] See [40] for some scrutiny regarding the scales. Conservatively, the measurements should therefore be seen as a hint rather than a pure constraint on the theory. See also [41] for additional theoretical constraints.

[6] Term coined here in analogy with *Pac-man completeness* common in computer sciences: a useful programming language should allow a full implementation of the pac-man game.

and including promising cosmological scenarios, there are other benefits to the theories we present, in particular the possibility of interesting phenomenology such as weakened gravity for cosmological matter perturbations.

More holistically, we will also argue in favor of *minimalism* as a guiding principle for constructing theories, which becomes particularly useful once violations of Lorentz invariance are allowed for. Minimal theories [50] modify GR while keeping the minimum number of degrees of freedom. This is not only practical from the point of view of simplifying computations, but it also ensures that no potentially unstable or unphysical modes are added. In the case of minimally modified gravity theories with two degrees of freedom, one may expect that screening mechanisms are not needed. In the case of massive gravity theories, the often unstable extra modes of the graviton can be removed by a minimization procedure [51].

Detailed outline of the thesis

General structure The thesis has been written with the intent of (i) giving the reader a clear introduction of the premises and the objectives of the field of modified gravity, and (ii) explaining why the models proposed are interesting. We would like to introduce the reader to the reasons to modify GR, how to modify GR, and what are some current limits of the alternatives to GR. We therefore start, in chapter 2, with a comprehensive review of GR, cosmology, and several tools of the trade, such as the Hamiltonian analysis or the study of cosmological perturbations. In chapter 3, we present several standard modified theories of gravity, as well as some limitations of these theories. These two chapters will have hopefully convinced the reader of the necessity to understand alternative theories further. In chapter 4 we then present a class of new constructions called the *minimally modified* theories of gravity. I have contributed to this class of theories through the works [50, 52]. Finally, in chapter 5 is presented another novel modified theory of gravity, the *minimal theory of quasidilaton massive gravity*, another original contribution to the field realized during this thesis [53, 54, 55]. Chapter 6 presents the conclusions. The rest of this section provides a more detailed summary of the arguments of the thesis.

Chapter 2 This is the first of the two review chapters. We start by a description of the basic elements that lead to GR, and explain in which sense it is a unique construction. The construction of the gravitational action of GR relies fundamentally on the equivalence principles (section 2.1.2), on the choice of semi-Riemannian geometry and the use of tensors (section 2.1.3), and on covariance and local Lorentz invariance (presented in section 2.1.4). Lovelock's theorem, introduced in section 2.1.5 allows then to single out the Einstein-Hilbert Lagrangian and the Einstein equations. We note in section 2.1.6 that the matter couplings are another *sine qua non* of a gravitational theory; we then have all the ingredients to define the action and equations of GR in 2.1.7. The end of the first part of the chapter, section 2.1.8, is devoted to the discussion of Hamiltonian structure of GR, together with an exposition of the Dirac analysis technique; we include also a preliminary remark on the field content of gravitational theories in section 2.1.9.

The second part of chapter 2, section 2.2, discusses the study of cosmology. It is the puzzles of cosmology, in particular the dark energy puzzle, that form the most important motivation for this thesis. The presentation of cosmology is customary, and we divide the presentation into two further parts, one concerning the homogeneous and isotropic Universe, and another devoted to the perturbative approach. In section 2.2.1, we present FLRW (homogeneous and isotropic) cosmology, as well as the basic terminology. We also present a rough picture of cosmological evolution, the different eras, and the importance of the diverse matter content of the Universe. Shortly put, we present what is (roughly) known about the Universe. This allows to set the basis for section 2.2.2, in which we present what is (definitely) *not* known in cosmology, that is the dark sector of the Universe: dark matter and dark energy. We then introduce the standard model of cosmology, and subsequently point out that this dark sector could be explained not only by additional matter content, but also by a modification of our description of gravity. Finally, we conclude the section by mentioning other possible future incognita, i.e. tensions within the standard model of cosmology. At this point we have introduced the dark Universe from the point of view of background cosmology, but it is not clear how to tell apart and constrain different models. For this, one indeed needs to go beyond FLRW cosmology, and we therefore move on to a short introduction of cosmological perturbation theory in section 2.2.3. We then present which observables are interesting for studying alternative theories of gravity, in section 2.2.4. Among these are, notably, weak gravitational lensing and the redshift-space distortion spectrum. We finally identify the common impact of modified gravitational dynamics on these observables.

Chapter 3 This is the second review chapter of this thesis. Motivated by our lack of knowledge of the dark sector of the Universe, in section 3.1 we explore some possibilities to go beyond the Lovelock theorem. In particular, we review the addition of scalar fields in section 3.1.1, which is one of the most commonly used forms of modification of gravity. After discussing possible symmetry breakings in section 3.1.2, we also review massive gravity, another notable example of modification to GR, in section 3.1.3. Massive gravity is promising as a simple modification of gravity on large scales, but has proved difficult to implement and has only recently been understood enough for cosmological applications. In fact, the need to build consistent/stable cosmologies has motivated several extensions of the original formulation. The second part of the chapter, section 3.2, is devoted to the bounds on large-scale modifications of gravity. We first discuss some current constraints from cosmological observables in section 3.2.1. We then emphasize the importance of planned future surveys for the characterization of the dark sector. Other regimes are also discussed, such as the weak-field regime relevant for e.g. the solar system in section 3.2.2, strong field regimes (e.g. black holes) in section 3.2.3, and the radiation regime in section 3.2.4. Gravitational radiation has become recently accessible, and has already provided several bounds on alternative theories of gravity. As a whole this chapter sets the context (and state of the art) for the original works in this thesis.

Chapter 4 We review and investigate the possibility to obtain new interesting alternative theories of gravity from Lorentz violations. The theories we present only propagate non-linearly two degrees of freedom, as does GR, and are hence called *minimally modified gravity theories (MMGs)*. In section 4.1 we discuss the motivations to study this class of theories specifically. Both consistency and simplicity motivate us to follow the idea of the least number of propagating degrees of freedom. In section 4.2 we review the general construction proposed in [56, 57]. The following sections 4.3 and 4.4 are then based on an original work realized during this thesis. In section 4.3 we segregate two types of MMGs, in the hope that this will help systematize the study of MMGs. In section 4.4, we propose a new class of theories built using specifically the property of one of the two types. The class of theories that we proposed allows for interesting modifications of background cosmology that are dependent on the matter energy momentum tensor. This modification is consistent with observations and can potentially answer the mystery posed by the current tension in the measurements of the Hubble rate. In section 4.5, we focus instead on a review of the *minimal theory of massive gravity (MTMG)* [51], which is an interesting example of the second type of theory. This theory sustains weak gravity on the scales of cosmological matter perturbations, and is hence especially interesting in view of future surveys. Finally we summarize the chapter in section 4.6.

Chapter 5 We present the original *minimal theory of quasidilaton massive gravity (MQD)*. We start in section 5.1 by a presentation of the previous quasidilaton theories, including an introduction to the motivations behind the addition of a quasidilaton field, and a comprehensive review of the extensions proposed to cure the problems of the original formulation. This allows to introduce MQD in section 5.2. This section and the following are based on the work realized during this thesis. We start by presenting the action of MQD in section 5.2.2. Since some properties of the theory can be better appreciated from the Hamiltonian perspective, we discuss the total Hamiltonian and the nature of the constraints in section 5.2.3. We then move on to the phenomenology of the theory with background cosmology in section 5.3, details on the attractor in section 5.3.1, and a detailed study of perturbations in section 5.3.2, including the phenomenology in the sub-horizon limit and within the quasi-static approximation. We conclude the section with a summary of the specificities of MQD in section 5.4, notably the possibility of weak gravity at linear perturbation scales.

Chapter 6 Finally, we present our conclusions, summarize the results of the work realized during thesis—in particular the phenomenological interest of the theories we introduced—address the problematics presented in this introduction, and discuss possible directions for future study.

Appendices This thesis also includes appendices containing useful tools and expressions that would have charged excessively the main text. In appendix A, we give for reference some tools useful for the study of cosmological perturbations. In appendix B, we give a few extra expressions for MTMG. In appendix C, we describe the Hamiltonian analysis of the shift-symmetric cubic Horndeski theory (this appendix was mostly reproduced from [54]). Finally, appendix D contains further details on the Hamiltonian analysis and the analysis of cosmological perturbation of MQD.

Declaration

The original part of this thesis is based, for the chapter on minimally modified gravity (chapter 4), on publications [50] published in *JCAP* in collaboration with Katsuki Aoki, Antonio De Felice, Chunshan Lin, and Shinji Mukohyama, and, for the minimal theory of quasidilaton massive gravity (chapter 5), on the publications [53, 54, 55] all published in *Physical Review D* in collaboration with Antonio De Felice and Shinji Mukohyama.

The original results presented within this thesis are:

1. The construction of a class of non-trivial modifications of gravity following from the existence of an Einstein frame. These theories propagate no extra degree of freedom, can pass present bounds such as on the speed of gravitational waves, and yet modify both the background (via a dynamical dark-energy equation of state) and the short scale gravitational constant (which becomes time-dependent). It is expected that the procedure presented can be generalized in several interesting directions.

2. The construction of the to-date most promising candidate among quasidilaton theories of massive gravity. The theory also generalizes MTMG so that viable cosmology with both a self-accelerating background and interesting phenomenology can be expected. Phenomenology with a cosmological matter fluid has been worked out.

We append to this thesis (in the version submitted towards the completion of the doctoral degree) two additional publications, [58] published in *Physical Review D* in collaboration with Antonio De Felice, Shinji Mukohyama, and Yota Watanabe, and [52] published in *Physical Review D* in collaboration with Antonio De Felice, François Larrouturou and Shinji Mukohyama. Their content is not directly included in this thesis. Indeed, although the two publications also explore alternative models of gravity, the results they propose do not concern directly the construction of minimal theories. In these two works we have shown that:

1. The proof-of-concept that a viable cosmology can be obtained within the chameleonic extension of bimetric theory [59]. We both explore the background evolution from radiation-dominated initial conditions, we study the stability of cosmological perturbations, and we show that the generalized Higuchi bound [49, 60] can be satisfied at all times.

2. MTMG allows for a range of essential GR-like spherical symmetric exact solutions, in particular the Schwarzschild solution with or without cosmological background. This has not yet been accomplished within other standard models of massive gravity [61].

These are two examples of necessary steps in order to test the validity of alternative theories of modified gravity.

Chapter 2

Theories of gravity

2.1 General relativity, behind the scenes

In this chapter, we will discuss some fundamental building blocks of general relativity. Simply speaking, it is by knowing more about these building blocks that one will garner a clear idea on how to think *beyond general relativity*. As will be seen in the next chapter, these foundational elements may seem at first glance difficult to evade, since generic modifications run into several problems, thus making the construction of alternative theories of gravity a challenge. Fortunately this challenge was successfully approached by several scientists and research groups, in rather inventive ways.

The content reviewed in this section can be found in several textbooks; we therefore chose to retain the concise style of a cursory reminder rather than delving into an in-depth exposition. We invite interested readers unfamiliar with these subjects to consider the textbooks [62, 63, 64, 65].

2.1.1 Notation

- Unless explicitly stated we will be working mostly in 4 space-time dimensions, with a $(-,+,+,+)$ signature of the metric.
- We work in natural units, where the speed of light and the reduced Planck constant are unity, $c = \hbar = 1$.
- Abstract spacetime indices follow the simple notation in which the 4-dimensional are written as Greek letters such as $\mu, \nu, \rho, \ldots$, whereas 3-dimensional indices have Latin letters such as $i, j, k, \ldots$ When we introduce vielbeins, we use capital curly Latin letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ for the local 4-dimensional indices, whereas 3-dimensional indices are written with capital Latin letters such as $I, J, K, \ldots$ Finally, internal indices are written using Latin letters $a, b, c, \ldots$ unless some confusion can arise, in which case we explicit which set of indices is in use.
- Symmetrization and anti-symmetrization of indices are respectively denoted $A_{(ij)} \equiv \frac{1}{2}\left(A_{ij} + A_{ji}\right)$ and $A_{[ij]} \equiv \frac{1}{2}\left(A_{ij} - A_{ji}\right)$.
- We use the Einstein convention for summed indices. Traces of tensors and matrices are commonly written by omitting the indices, or using square brackets and explicit indices whenever there is a chance of confusion.
- The symbols for metrics and derivatives are the following. For four-dimensional metrics we use the letters g and f, with ∇ the covariant derivative for the metric g. For spatial slices and three dimensional metrics we use the symbols γ, Γ, $\tilde{\gamma}$, and the covariant derivatives of γ and Γ are $\mathcal{D}$ and D, respectively. When necessary, we write the Christoffel symbol also using the symbol Γ.
- To prevent any confusions, we use the symbol H for the Hubble rate, and the symbol $\mathfrak{H}$ for Hamiltonians.

2.1.2 Equivalence principles

Einstein formulated his ideas about space-time by considering the *equivalence principle*, which ultimately helped giving form to GR. The rough idea of the equivalence principle is that our Universe exhibits broad *equivalence classes* of experimental setups for which the results are the same: for example, as was already captured by the famous experiments by Galileo, two objects of diverse composition but with the same mass will fall in the same way in the Earth's gravitational field.

Although nowadays several experiments allow for considerable constraints, clear-cut limits of these equivalence classes are still a subject of research. Different theories of gravity predict different equivalence classes, therefore a careful understanding of these classes, in particular how encompassing they are, is essential to understand gravity. In fact, after the advent of GR, with the refinement of experimental research on the foundations of gravitational theories started by Dicke [66], several versions of the equivalence principle were distinguished. Most notably, on has, from the least restrictive to the most restrictive: the *weak equivalence principle*, the *Einstein equivalence principle*, and the *strong equivalence principle*. Here we detail these principles, reproducing lines of [65].

The weak equivalence principle (WEP) is based on the idea that gravity should apply in the same way to all test bodies[1]. In details, as expressed by Will [65]:

> *(WEP)* "[I]f an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition."

The existence of this equivalence class is fundamental when choosing how gravitational fields couple to matter fields. Then, the Einstein equivalence principle (EEP) gives a broader equivalence class by focusing not only on trajectories but also on any type of non-gravitational experiment. Again as in [65]:

> *(EEP)* "(i) WEP is valid, (ii) the outcome of any local nongravitational test experiment is independent of the velocity of the (freely falling) apparatus, and (iii) the outcome of any local nongravitational test experiment is independent of where and when in the universe it is performed."

The existence of this equivalence class is fundamental as it can be related to the theory being a metric theory. Clearly, Lorentz invariance for the matter sector is also closely related to the EEP: locally, one can find coordinates in which the effects of gravity for the trajectory of point particles can be neglected, implying that special relativity should be recovered. Finally, the strong equivalence principle (SEP), gives an even broader equivalence class by broadening the class of test bodies and considering gravitational experiments. We reproduce again the statement from [65]:

> *(SEP)* "(i) WEP is valid for self-gravitating bodies as well as for test bodies, (ii) the outcome of any local test experiment is independent of the velocity of the (freely falling) apparatus, and (iii) the outcome of any local test experiment is independent of where and when in the universe it is performed."

This equivalence class is commonly related to GR alone, and the nonexistence of further fields in the theory. This related idea, that a gravitational theory should be a theory for the metric alone, is one of the hypothesis of Lovelock's theorem [67, 68], which we will state later in the text.

2.1.3 Basic elements

As mentioned above, the equivalence principles translate the idea that a theory of gravity should reduce locally to the ideas of special relativity, and should therefore be a theory of the space-time metric [65]. In this subsection, we would like to make the notions of metric and space-time more precise. One of the fundamental insights that the advent of general relativity has offered is that gravitational theories can be seen as *dynamical theories of geometry*, and in particular of the geometry of space-time. We therefore introduce here, rather canonically, a short list of tools that allow to describe this geometry.

It is generally assumed that space-time is well described by a semi-Riemannian geometry (in principle it would be possible to generalize this structure, for example to metric-affine geometries [69, 70], but for simplicity we will not do so). In such a description, one can measure (infinitesimal) distances using a symmetric, invertible, positive semi-definite metric tensor $g_{\mu\nu}$ via

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{2.1}$$

[1] Here in the sense of [65]: *"[a] body that has negligible self-interaction energy (as estimated using Newtonian theory) and that is small enough in size so that its coupling to inhomogeneities in the external fields can be ignored."* See the same reference for a detailed definition of a "local experiment".

where ds^2 is an infinitesimal space-time interval, dx^μ is the dual basis of the tangent space at location x^μ of the manifold, which can be interpreted as the small space-time displacements around x, with Greek letters as indices.

On the tangent space to space-time points, contravariant vectors with upper-indexed components, e.g. v^μ, are then defined as components of the vector space generated by the vector basis $\{\partial_\mu\}$ of the tangent space. Covariant vectors, e.g. w_μ, can be defined as duals to the contravariant vectors, with the basis $\{dx^\mu\}$. Tensors of type (l, m), with l upper indices and m lower indices can be defined within the tensor product of copies of both tangent space and its dual. Finally it is easy to understand the metric, a $(0, 2)$-tensor, as well as its inverse $g^{\mu\nu}$, a $(2, 0)$-tensor, as the maps between covariant and contravariant indices. Tensors have a simple transformation property under coordinate transformations, or rather diffeomorphisms $x \to x'(x)$,

$$T'^{\mu_1\ldots\mu_l}_{\nu_1\ldots\nu_m}(x') = T^{\alpha_1\ldots\alpha_l}_{\beta_1\ldots\beta_m}(x)\frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\mu_l}}{\partial x^{\alpha_l}}\frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\beta_m}}{\partial x'^{\nu_m}}\,. \tag{2.2}$$

In order to go further in the understanding of the surrounding space-time (to a given point), we need to define higher derivatives, especially derivatives of the tensors. The covariant derivative is defined using a connection $\Gamma^\gamma_{\mu\nu}$, for example acting on a vector as

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho\,, \tag{2.3}$$

which is easily generalized to tensors with all kinds of indices. Under metric compatibility with the covariant derivative $\nabla_\lambda g_{\mu\nu} = 0$ as well as under symmetricity in the two lower indices, the connection can be written as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\,, \tag{2.4}$$

defining what is called the Christoffel symbol. As we mentioned, these hypotheses are strictly-speaking not necessary, and using a connection independently of the metric leads to metric-affine theories of gravity[2], see for example [69, 70].

One may then define curvature tensors, notably the Riemann tensor, and its traces the Ricci tensor and the Ricci scalar

$$R^\rho{}_{\sigma\mu\nu} \equiv \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\,, \quad R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}\,, \quad R \equiv g^{\mu\nu}R_{\mu\nu}\,. \tag{2.5}$$

The Einstein tensor will also come in handy, we therefore define it as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\,. \tag{2.6}$$

Among others, it satisfies the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$.

2.1.4 Symmetries

Symmetries are central in Einstein's work. First, the Einstein equations, and later the Einstein-Hilbert action, are postulated in a way that leaves the physics completely independent of the coordinates used to describe it. This symmetry is coined *general covariance*, or *diffeomorphism invariance.* Although it was later recognized that general covariance can be implemented in many physical constructions, it played nevertheless an important role in the development of GR. In practice, general covariance is implemented by using tensors, with all indices contracted, to build the Lagrangian of the theory. This, in turn, leads to covariant equations of motion.

In fact, in GR invariance under diffeomorphisms rather translates a redundancy in our description of the physics. It is a local symmetry, and thus should be rather called *gauge invariance.* One may equivalently do calculations in any gauge (i.e. choice of coordinates), as long as the physically measured variables are carefully defined: covariance will ensure that they will be gauge-independent. In several cases, for example in context of cosmological perturbations, it is customary to form gauge-invariant quantities ab initio, which are much more practical when one needs to relate them to observables (the gauge-invariant variables for cosmological perturbations will be defined in section 2.2.3).

As a practical note, throughout the text, it will be often useful to rely on infinitesimal transformations to probe and understand important properties about given symmetries. In the context of covariance, infinitesimal diffeomorphism can be written as

$$x^\mu \to x^\mu + \xi^\mu(x)\,, \tag{2.7}$$

[2]This generalization can be interesting for alternative theories of gravity, see e.g. [71].

where x denotes the x^μ collectively, and where the ξ^μ are unspecified space-time functions. At linear order this transformation will hence act on the metric as

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) \,. \tag{2.8}$$

We now discuss a second important set of symmetries in GR. The equivalence principles is what accounts and enforces these symmetries, pertaining to the local structure of spacetime. In the formulation of the equivalence principles, the local notion of relativistic invariance is implied, i.e. reflecting a local Lorentz symmetry. In fact, it is possible to construct general relativity out of the local Poincaré symmetry, by gauging the translation subgroup of Poincaré. In order to include interactions with fermions, it becomes convenient to gauge the Lorentz group as well [72, 73, 74, 75, 76].

2.1.5 Lovelock's theorem

The study of invariants under coordinate transformations provides a beautiful guide to construct theories of gravity. Under the assumption of diffeomorphism invariance, and with a few more hypotheses essentially related to the strong equivalence principle, the number of Lagrangian densities that can be constructed out of the metric and its first and second order derivatives are limited. This is nicely expressed in the results derived by Lovelock [77], which limit the gravitational Lagrangian to the so-called Lovelock invariants given by:

$$\delta^{\mu_1 \dots \mu_{2d}}_{[\nu_1 \dots \nu_{2d}]} R_{\mu_1 \mu_2}{}^{\nu_1 \nu_2} \dots R_{\mu_{2d-1} \mu_{2d}}{}^{\nu_{2d-1} \nu_{2d}} \,, \tag{2.9}$$

where d is limited by the dimensionality of the space considered. In four dimensions, The Lovelock invariants are: i) the Ricci scalar R, ii) the Pontryagin invariant P, as well as iii) the Gauss-Bonnet invariant W. To this one may add the cosmological constant Λ. In the case of the Pontryagin and the Gauss-Bonnet invariants, they are purely boundary terms, and therefore do not contribute to the field equations [77].

On the basis of the study of the aforementioned invariants, and focusing on the gravitational field equations, Lovelock derived a powerful theorem [67, 68], which can be stated as follows:

> *Theorem (Lovelock)*—Assuming 4-dimensional space-time, and under the requirements of diffeomorphism invariance and exclusive use of the metric field, the most general second order equations of motion derivable from a local action and satisfying symmetricity and divergence-freeness, are the Einstein field equations in vacuum, i.e.
>
> $$G_{\mu\nu} + g_{\mu\nu}\Lambda = 0 \,,$$
>
> with $G_{\mu\nu}$ the Einstein tensor (defined in (2.6)) and Λ a constant.

The Lovelock theorem is one of the main guides to constructing theories beyond general relativity, since one will need to break at least one of its assumptions to bypass the Einstein equations in vacuum. This allows in particular to classify different modifications to the gravitational theory.

2.1.6 Minimal coupling

Inspired by the equivalence principles, the last assumption that we need to make to reach a complete gravitational theory—once we have trusted Lovelock for the vacuum Lagrangian—is about the way matter couples with the metric. In the case where only the metric is a gravitational field, the *minimal coupling prescription*

$$\eta_{\mu\nu} \to g_{\mu\nu} \,, \qquad \partial \to \nabla \,, \tag{2.10}$$

allows to correctly recover Lorentz invariance locally, starting from a flat space-time matter Lagrangian, and couples to matter in a non-derivative way (see e.g. [78]). Note that in the case of spinors, the situation is more involved (but can be treated in the vielbein formalism), and we will therefore only consider bosonic fields from now on for simplicity.

2.1.7 General Relativity

Now we have all the elements to write the action for general relativity, i.e. an action for the metric and the non-gravitational fields, which is diffeomorphism invariant. The Einstein-Hilbert action coupled with matter is

$$S_{\rm EH} + S_{\rm mat} = \frac{M_{\rm P}^2}{2} \int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_{\rm mat}[g, \psi], \tag{2.11}$$

which leads, as anticipated partly by the Lovelock theorem

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu}\,, \tag{2.12}$$

where the gravitational constant G_N is here related to the Planck mass by $M_{\rm P} = 1/\sqrt{8\pi G_N}$, and the energy momentum tensor of the matter fields $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = \frac{1}{2\sqrt{g}}\frac{\delta S_{\rm mat}}{\delta g^{\mu\nu}}\,. \tag{2.13}$$

The equations (2.12) are called the Einstein field equations. The conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ can be also obtained on-shell from the contracted Bianchi identity.

2.1.8 Constraint structure of GR

As Dirac showed [79, 63], the redundancies present in gauge theories can be best understood and studied through the Hamiltonian formalism. However, in order to define an Hamiltonian, one should also choose a direction of time: technically, one chooses a specific space-time foliation in 3-dimensional spatial hypersurfaces, i.e. with a time-like normal vector.

$3+1$ decomposition

In practice, one may perform a foliation through the Arnowitt-Deser-Misner (ADM) decomposition of the metric; one writes

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{2.14}$$

which defines the lapse function N, the shift vector N^i, and the spatial metric γ_{ij}, whose inverse is then denoted γ^{ij}. A foliation can then be defined via the normal vector u^μ, through

$$N = -u_0\,, \quad N^i = -\frac{u^i}{N}\,, \tag{2.15}$$

Working in this decomposition, the spatial metric can be pulled-back into a projection operator,

$$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu\,, \quad \gamma_{ij} = g_{ij}\,, \quad \gamma^{ik}\gamma_{kj} = \delta^i_j\,. \tag{2.16}$$

Indices on the spatial hypersurfaces, $i, j, k, \ldots$, can be raised an lowered thanks to the spatial metric. Note that in principle the ADM decomposition is general and needs no *a priori*, but it can be also reached starting with a scalar field (say ϕ) with time-like gradient whose surfaces of constant value define the foliation. The normal vector to the hypersurfaces can then be defined in each point via

$$u_\mu = \frac{\partial_\mu\phi}{\sqrt{-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}}\,, \tag{2.17}$$

which is then loosely equivalent to the decomposition (2.14). Going back to the ADM decomposition, it is easy to construct a 3-dimensional Riemann curvature tensor with the induced metric γ_{ij}, but this object will of course not include all the information of the full four dimensional curvature. To have a complete picture of space-time, one needs to define the *extrinsic curvature*

$$K_{ij} = \frac{1}{2N}\left(\dot\gamma_{ij} + 2\mathcal{D}_{(i}N_{j)}\right)\,, \tag{2.18}$$

where we have also used $\mathcal{D}_i$, the covariant derivative compatible with γ_{ij}, and the over-dot denotes a time derivative. The extrinsic curvature occupies a privileged place in the subsequent Hamiltonian analysis, as it carries time-derivatives of the spatial metric, the field of interest. Finally, the decomposition of the GR Lagrangian density is given by

$$\mathcal{L} = N\sqrt{\gamma}\left(R[\gamma] + K^{ij}K_{ij} - K^2\right) + \text{boundary term}\,. \tag{2.19}$$

More details on the 3+1 decomposition, in particular pertaining to the boundary term which will not be relevant within this thesis, can be found in the standard textbooks, e.g. [64, 80].

Hamiltonian of GR

The Hamiltonian is now obtained by Legendre transformation. Being fully agnostic about which ADM elements are dynamical, we have in principle the canonical variables N, N^i, and γ_{ij}, collectively named q_A. The first step is to compute the conjugated variables, which we respectively name π_N, π_i, and π^{ij}, collectively named p^A, and are given by

$$p_A = \frac{\delta \mathcal{L}}{\delta \dot{q}^A}. \tag{2.20}$$

where the dot denotes a time derivative. In the context of field theory it is customary to use a field as canonical variable, but this should really be understood as having a canonical variable at each point of space. The GR Hamiltonian can then be written

$$\mathfrak{H} \equiv \int d^3x \left(\sum_A p_A \dot{q}^A - \mathcal{L} \right) = \int d^3x \left(-N\mathcal{R}_0 - N^i \mathcal{R}_i \right),$$

where

$$\begin{aligned}
\mathcal{R}_0 &= \frac{M_\mathrm{P}^2}{2}\sqrt{\gamma}\, R[\gamma] - \frac{2}{M_\mathrm{P}^2}\frac{1}{\sqrt{\gamma}}\left(\gamma_{il}\gamma_{jk} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} + \mathcal{R}_{0,\mathrm{mat}}\,, \\
\mathcal{R}_i &= 2\sqrt{\gamma}\gamma_{ik}\mathcal{D}_j\left(\frac{\pi^{kj}}{\sqrt{\gamma}}\right) + \mathcal{R}_{i,\mathrm{mat}}\,.
\end{aligned}$$

In fact it turns out that both $\pi_N = \pi_i = 0$, hinting that N and N_i are not dynamical variables.

Dirac analysis

We now turn to a brief summary of the Dirac analysis [79, 63] which we then apply to GR. The Dirac analysis can make the constraints (for example gauge symmetries) in a system apparent; it relies on a procedure that allows to exhaust the constraints that apply on the phase space and to ultimately understand the correct number of degrees of freedom in a field theory. The analysis relies on the Poisson brackets, which we define here (in three dimensional space) as

$$\{f(x), g(y)\} = \int d^3z \sum_A \left(\frac{\delta f(x)}{\delta q^A(z)}\frac{\delta g(y)}{\delta p_A(z)} - \frac{\delta f(x)}{\delta p_A(z)}\frac{\delta g(y)}{\delta q^A(z)} \right). \tag{2.21}$$

With this definition, the Poisson brackets give the usual result

$$\{q^A(x), p_B(y)\} = \delta^A_B \delta^{(3)}(x-y)\,, \tag{2.22}$$

where $\delta^{(3)}$ is a Dirac δ function. For a given function of the canonical coordinates and momenta, the Poisson bracket with the full Hamiltonian allows to compute its time derivative,

$$\frac{d}{dt} f[q^A, p_A, t] = \{f, \mathfrak{H}\} + \frac{\partial f}{\partial t}\,. \tag{2.23}$$

In order to compute Poisson brackets with functions involving derivatives of the coordinates or momenta it is convenient to define corresponding smeared functionals, as in

$$f \to f[\phi] \equiv \int d^3x\, \phi(x) f(x)\,, \tag{2.24}$$

with $\phi(x)$ a well-behaved function falling to zero at infinity. This allows to conveniently study the result of Poisson brackets that would otherwise contain derivatives of Dirac δ functions. The result of the Poisson bracket between to smeared functionals $A_1[\phi_1]$ and $A_2[\phi_2]$ is chosen to be the kernel $A_3(x,y)$ defined by

$$\{A_1[\phi_1], A_2[\phi_2]\} \approx \iint d^3x\, d^3y \phi_1(x) A_3(x,y) \phi_2(y), \tag{2.25}$$

up to factors of the 3-dimensional measure $\sqrt{\gamma}$, in order to obtain only space-time scalars.

Finally we express the notion of *constraint surface*, which is the restriction of the phase space generated by the set of all constraints in the Hamiltonian, for example $\Phi(q^A, p_A, t) = 0$, where $\mathfrak{H} \ni$

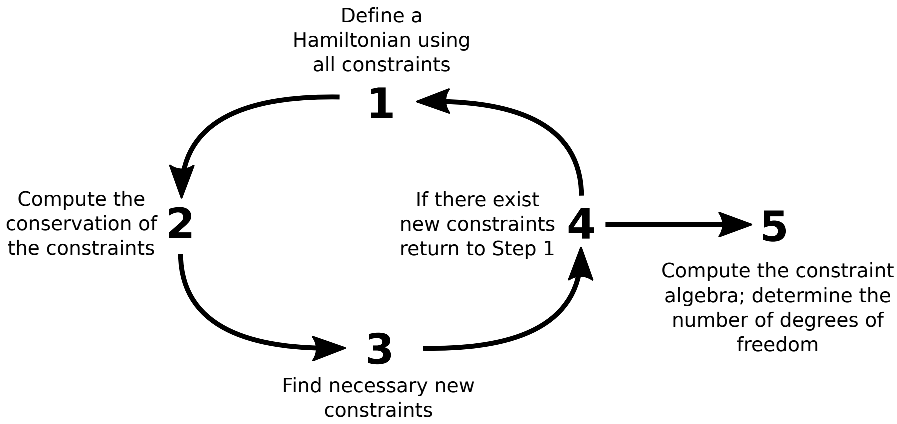

Figure 2.1: Operational scheme of the Dirac analysis of the Hamiltonian.

$\int d^3x\lambda\Phi(q^A, p_A, t)$ (the constraints are enforced via the equations of motion of auxiliary fields, the Lagrange multipliers λ). With the constraint surface, one can define the notion of weak equality, to be understood as *equality when restricted to the phase space surface*. This weak equality is denoted given two functions f and g of the phase space variables,

$$f \approx g\,. \tag{2.26}$$

Now that we have defined the basic tools, we can describe the Dirac procedure, as summarized in Fig. 2.1. This procedure goes as follows:

Step 1	Define a Hamiltonian with all available constraints.
Step 2	Compute the Poisson bracket of each constraint with the current Hamiltonian, which yields the conservation in time of the constraint.
Step 3	If any result of the previous step does not vanish weakly (on the constraint surface), and cannot be solved in terms of Lagrange multipliers, they should be added to the set of constraints.
Step 4	Repeat from Step 1, unless no new constraints have been generated at the previous step.
Step 5	Once the previous steps are finished, one may analyze the final constraint algebra (see below).

In the final analysis, one may distinguish two types of constraints: those that commute with all other constraints on the constraint surface, which are called *first-class*, and the rest, which are called *second-class*. When there are several non-commuting constraints, some scrutiny may be needed before judging whether there really is no first-class constraint. A foolproof method is to compute the determinant of the matrix formed by all Poisson brackets between the constraints. If this determinant vanishes, it means that there exists at least one first-class linear combination of constraints. One should therefore find and exclude these first class constraints to then count the number of remaining constraints.

Once the constraints are categorized between first-class and second-class, the counting of the degrees of freedom is straightforward: each first-class constraint removes 2 phase space degrees of freedom and each second-class constraint removes 1 instead. In fact, first class constraints represent gauge invariances [63], which can be seen as equivalence classes in the phase space. On the other hand second class constraints can be visualized as constraints restricting the dynamics on a lower dimensional hypersurface in the phase space.

The Hamiltonian analysis we described above is useful to determine degrees of freedom non-linearly. We will first apply it to GR in this chapter, but also discuss its application to several modified models of gravity in chapter 3. The analysis was especially important in the obtention of novel theories, as explained in chapters 4 and 5.

Hamiltonian analysis of GR

As we will see GR has a particular Hamiltonian structure: it is purely made of constraints. In the case of GR one can should first add $\pi_N = \pi_i = 0$ as constraints to the starting Hamiltonian. We obtain the primary Hamiltonian (this time in vacuum, for simplicity)

$$\mathfrak{H}^{(1)} \equiv \int d^3x \left(-N\mathcal{R}_0 - N^i\mathcal{R}_i + \xi^i\pi_i + \xi_N\pi_N\right), \tag{2.27}$$

where ξ^i and ξ_N are Lagrange multipliers. The constraint algebra is simple, the only Poisson bracket, $\{\pi_i, \pi_N\}$, yielding identically 0. The conservation of these constraints turns out to yield exactly $\mathcal{R}_0$ and $\mathcal{R}_i$, which we now should add including Lagrange multipliers Λ_N and Λ^i to the structure to form the secondary Hamiltonian

$$\mathfrak{H}^{(2)} \equiv \int d^3x \left(-(N+\Lambda_N)\mathcal{R}_0 - (N^i+\Lambda^i)\mathcal{R}_i + \xi^i\pi_i + \xi_N\pi_N\right). \tag{2.28}$$

From now, we call $\mathcal{R}_0$ and $\mathcal{R}_i$ the Hamiltonian and the momentum constraints. If one now computes the algebra of constraints, one finds commutators that vanish only on the constraint surface. The values of these commutators are

$$\begin{aligned}
\{\mathcal{R}_0[\phi], \mathcal{R}_i[f^i]\} &= -\int d^3x\sqrt{\gamma}\,\phi\mathcal{D}_i\big(\mathcal{R}_0 f^i\big) \approx 0\,,\\
\{\mathcal{R}_0[\phi_2], \mathcal{R}_0[\phi_2]\} &= \int d^3x\mathcal{R}_i\left(\phi_1\mathcal{D}^i\phi_2 - \phi_2\mathcal{D}^i\phi_1\right) \approx 0\,,\\
\{\mathcal{R}_i[f^i], \mathcal{R}_j[g^j]\} &= \int d^3x\mathcal{R}_i\left(g^j\mathcal{D}_j f^i - f^j\mathcal{D}_j g^i\right) \approx 0\,.
\end{aligned} \tag{2.29}$$

Now, it is trivial to see that these constraints are indeed conserved in time. The loop procedure is now finished, and one may move on to the inspection of the constraint algebra. In fact we already know that all constraints are first-class, as the Poisson brackets presented previously show. Therefore there are

$$12\ (\gamma_{(ij)}) + 6\ (N^i) + 2\ (N) - 2\times 1\ (\mathcal{R}_0) - 2\times 3\ (\mathcal{R}_i) - 2\times 3\ (\pi_i) - 2\times 1\ (\pi_N) = 4 \tag{2.30}$$

phase space degrees of freedom, i.e. two physical degrees of freedom. These are the two gravitational wave polarizations.

2.1.9 Remark on the field content of gravity

The question as to whether gravity can be correctly represented by a theory without dynamical metric but with a lower spin field can be answered negatively (refer to e.g. Feynman's lectures [81]). One may then ask whether it is still possible to include other fields in addition to the metric in the gravitational sector. In fact, this turns out to be difficult (although nowadays this difficulty is commonly bypassed through screening mechanisms as we will see in the next chapter), for the following reason: a new field which would couple to matter would add its own interaction to the game, thus affecting, for example, the forces within the solar system. Since the current measurements of gravitational interactions and effects within the solar system are very precise (see section 3.2.2 and [65]), a sizable new force is easily excluded, unless the field is very massive. To summarize, based on a direct interpretation of the observations on solar system scales, it is likely that the gravitational theory only contains the metric as a fundamental field. We will however see in the next chapter that this need not necessarily be the case.

2.2 Cosmology

In this section we first review the very basic elements of the Friedmann-Lemaître-Robertson-Walker cosmology. This is useful to present the basic definitions that will then be used in the latter chapters. The discussion of the background cosmology also let us introduce the idea of both dark matter and dark energy. We argue that these puzzles are one of the main reasons to try to modify gravity.

After background cosmology has been reviewed, we focus on the perturbed universe, motivated by the insufficiency of background cosmology to resolve between possible modified theories of gravity. We again discuss the basic elements of cosmological perturbation theory and move on to discuss how to probe the inhomogeneous Universe.

As in the previous section, the results of this section are found in classical textbooks e.g. [82, 83, 84], to which we refer the reader interested in more details.

2.2.1 The cosmological background

On super-cluster scales (from about 100 Mpc), observations can be interpreted as the Universe being roughly homogeneous and isotropic. This observation, is known as *cosmological principle*. The most general metric satisfying the cosmological principle, homogeneity and isotropy, can be written, using polar spatial coordinates

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) , \tag{2.31}$$

where $d\Omega^2$ is a shorthand notation for the line element on $\mathbb{S}_2$,

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 , \tag{2.32}$$

and where the constant k can take three values

$$k = \begin{cases} +1 & \text{closed Universe,} \\ 0 & \text{flat Universe,} \\ -1 & \text{open Universe.} \end{cases} \tag{2.33}$$

This metric is commonly called Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The function $a(t)$, the *scale factor*, relates spatial distances at two given times, and hence its increase/decrease can be understood as the expansion/contraction of the Universe. Note also that using time reparametrization symmetry one commonly sets $N(t) = 1$. However, since in the next chapters we will be dealing with cases without this symmetry, we will leave it from time to time in GR, and always in theories beyond GR[3]. Finally, since observations strongly favor the $k = 0$ case [85], it is not abnormal to consider only that case, for simplicity and agreement with observations (this is also motivated by inflationary models, see the discussion after equation (2.49)), giving the *flat FLRW metric* (here given in spatially Cartesian coordinates)

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j . \tag{2.34}$$

With the FLRW metric, it is customary to obtain time derivatives of $a(t)$ once the Christoffel and the curvature tensors are considered, therefore we define

$$H \equiv \frac{\dot{a}}{aN} , \tag{2.35}$$

the *Hubble rate* of expansion, with the over-dot denoting a derivative with respect to t. We will see that this quantity is rather important. The detailed expressions for the metric quantities are then left in the appendices. The existence of a time-dependent scale factor has some interesting consequences, in particular the existence of a cosmological *redshift* of radiation defined as

$$1 + z \equiv \frac{a(t_{\rm obs})}{a(t_{\rm em})} = \frac{\lambda_{\rm obs}}{\lambda_{\rm em}} , \tag{2.36}$$

where obs stands for "observation" and em stands for "emission". Redshift is commonly used to estimate and discuss distances and time in cosmology. A redshift $z = 0$ corresponds to today and *here*, while higher redshifts correspond to increasingly distant objects (and the light we receive from these objects was emitted in the past). Other common time-variables include the conformal time $dt = ad\eta$ or the e-folding number $N_e = \ln(a/a_i)$, with a_i a given initial scale factor.

The cosmological principle also restricts the form of the source term in the Einstein equations to

$$T_{\mu\nu} = (\rho(t) + P(t))\, u_\mu u_\nu + P(t) g_{\mu\nu} , \tag{2.37}$$

where u_μ is the vector normal to spatial hypersurfaces. In fact, in the coordinates of the FLRW space-time, one has $u^\mu = \delta^\mu_0$. It turns out that that the functions ρ and P can be interpreted as the density and pressure of a comoving perfect fluid that fills the Universe. This is the way matter fields will appear on cosmological scales.

Once plugged into the Einstein field equations, the FLRW metric and the fluid energy-momentum tensor yield the Friedmann equations

$$3H^2 = \frac{1}{M_{\rm P}^2}\rho - \frac{k}{a^2} + \Lambda , \tag{2.38}$$

$$2\dot{H} = -\frac{1}{M_{\rm P}^2}(\rho + P) , \tag{2.39}$$

[3] Leaving $N(t)$ is also useful to keep in general as an accounting device for time derivatives.

whereas the behavior of matter is directed by a conservation equation

$$\dot{\rho} + 3H\left(\rho + P\right) = 0\,. \tag{2.40}$$

In fact, only two of these three equations are independent, which can be understood by the Bianchi identities conditioning the Einstein tensor.

We would like to understand a little bit better what physics enters into the fluid energy-momentum tensor. One may define the *equation of state*,

$$w = \frac{P}{\rho}\,, \tag{2.41}$$

which turns out to be expressed simply for a range of different types of matter. First of all, the cosmological constant can be interpreted as a fluid with $w = -1$. We will detail the behavior of the Universe in its presence later. Then, one finds, after considering the microphysics, that non-relativistic matter will behave as pressureless dust with $w = 0$, whereas radiation has $w = \frac{1}{3}$. Finally one may consider the spatial curvature as behaving with $w = -\frac{1}{3}$. Indeed the conservation equation (for a constant w) yields

$$\rho \propto a^{-3(w+1)}\,. \tag{2.42}$$

This last equation is important in recognizing that given an initial amount of each kind of fluid, for, say, an expanding Universe, there may be different eras in which one or the other fluid dominates. In our Universe, one commonly admits that at early enough times (yet after inflation) the content of the Universe was given by (admitting here a non-zero k)

$$\Omega_{\rm r} \gg \Omega_{\rm d} \gg |\Omega_k| \gg \Omega_\Lambda\,, \qquad \text{at some } t \ll t_0\,, \tag{2.43}$$

where t_0 denotes the current time, and with the *density parameters*

$$\Omega_\alpha = \frac{\rho_\alpha}{3M_{\rm P}^2 H^2}\,, \tag{2.44}$$

where the subscript α reads Λ for the cosmological constant, k for the spatial curvature, d for dust, r for radiation. With such initial conditions one starts in a radiation-dominated Universe, and then subsequently reaches a dust-dominated era, whereas the Universe "ends" dominated by the contribution of the cosmological constant. See Fig. 2.2 for an example these dominance eras. The Universe in which

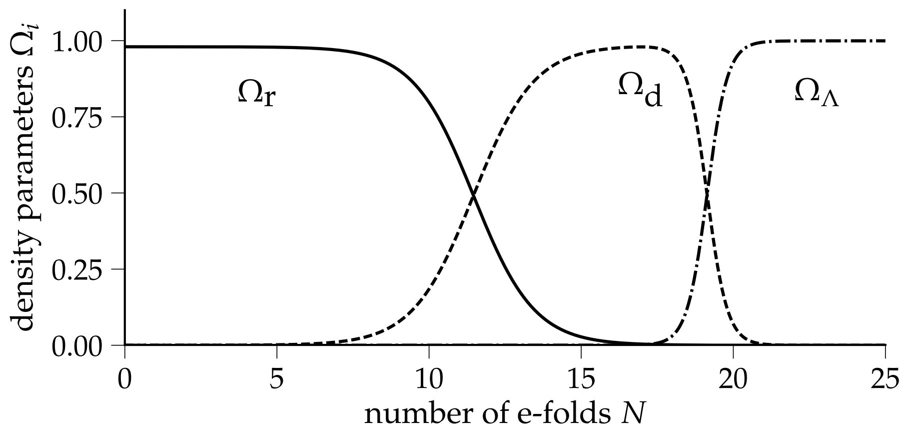

Figure 2.2: Example evolution of the density parameters Ω_α (the plots are indicative and should not be interpreted as the exact values), showing the succession of radiation-dominated, matter-dominated, and cosmological constant-dominated eras (here computed with $k = 0$).

we currently live is close to the moment of equality between the density of dust, $\Omega_{\rm d}$ and of what seems to be the cosmological constant Ω_Λ. More precisely, using the subscript 0 to denote present-day quantities (i.e. evaluated at t_0), we have

$$\Omega_{\Lambda 0} \approx 0.7\,, \quad \Omega_{\rm d} \approx 0.3\,, \quad \Omega_{\rm r} \approx 10^{-3}\,, \quad \Omega_k \leq 1 \cdot 10^{-2}\,. \tag{2.45}$$

We will come back as to how these are estimated.

As far as we understand our past history, the Universe has always expanded[4], and will do so in the future [12, 13]. Let us detail for example what lies ahead of us in the future of the Universe, if, as predicted by the standard model of cosmology, the cosmological constant Λ dominates. One may quickly see that the Friedmann equation (2.38), for the cosmological constant only, is solved by

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3M_{\rm P}^2}} \qquad \to \qquad a = a_0 e^{\sqrt{\frac{\Lambda}{3M_{\rm P}^2}}\, t} \,. \tag{2.46}$$

It is in fact possible to relate the FLRW metric with this scale factor to the *de Sitter space-time*, defined as the embedding of an hyperboloid in 5 dimensional Minkowski space,

$$-X_0^2 + X_1^2 + \ldots + X_4^2 = H^{-2} \,, \tag{2.47}$$

where H is a constant, which can be turned into

$$ds^2 = -dt^2 + e^{2Ht}\delta_{ij}dx^i dx^j \,, \tag{2.48}$$

with H given by (2.46), by the *flat chart*

$$X^0 + X^1 = \frac{1}{H}e^{Ht} \,, \qquad X^j = e^{Ht}x^j \,. \tag{2.49}$$

This is the reason the term de Sitter space is often employed to describe the late-time acceleration by a cosmological constant (the concept is of course also important during inflation, see below). The exponential expansion has several properties, the main one being to basically quickly dilute all matter content. Of course locally gravitationally bound neighborhoods may subsist for a (long) while. Another property is its ability to make the comoving horizon aH expand which is very relevant for perturbations with a given mode k. These two properties, because they can solve both the *flatness problem* and the *horizon problem* are at the origin of the proposal, called *inflation*, of a phase of accelerated expansion before the standard big-bang scenario unfolds. This inflationary scenario is widely accepted today. However inflation cannot be implemented by the cosmological constant Λ, simply because it is constant and therefore its value, which is small (we will come back to this), was relatively even more negligible in the dense early Universe. This is why inflation generally calls for at least one new ingredient: the inflaton. The end of inflation is generally understood as a decay of the inflaton field into standard-model fields, during a period called *reheating*. It is interesting to see the inflaton as the gauge boson of spontaneously broken time diffeomorphisms: this has led to the study of (single-field) inflation as an effective field theory [86].

The epoch from reheating until when stars and galaxies form, once one goes through the microphysics, is quite complex. We can and will not give the details but a short summary of what is called the *hot big bang scenario*. Several tests of gravity rely on some of the microphysics (e.g. the big-bang nucleosynthesis). A simplified chronology of the Universe is given in Table 2.1. Although the details of this chronology can be very complex, it mostly relies on considering the rates of interactions between the dominant matter species, while taking into account the decrease in temperature of the expanding Universe, as well as the expansion rate itself. For example, some interactions can be efficient only above given temperatures, such as the ionization of an atom, while other interactions may become inefficient simply because they are slower than the cosmic expansion itself.

The first part of this history takes place during a radiation dominated Universe, approximatively until the physics behind the cosmological microwave radiation (CMB) (see below) becomes relevant. In a radiation dominated universe, one should understand how many particles are relativistic and therefore contribute to radiation. The number of relativistic species is accounted by the quantity g_*, defined through

$$g_*(T) = \sum_{i\in\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i\in\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 \,, \tag{2.50}$$

where g_i is the degeneracy of the species, and which yields the radiation density

$$\rho_r(T) = g_*(T)\left(\frac{\pi}{30}\right)T^4 \,. \tag{2.51}$$

This, via the Friedmann equations, yields the cosmic expansion rate, H.

[4]Bouncing scenarios are not (yet?) within the standard model.

very early	Inflation or other scenarios
early	Reheating
$z \sim 10^{15}$	Electro-weak phase transition
$z \sim 10^{12}$	Quantum-chromodynamical phase transition
$z \sim 10^{10}$	Neutrino decoupling
$z \sim 10^{9}$	Electron-positron annihilation
$z \sim 4 \cdot 10^{8}$	Nucleosynthesis
$z \sim 3 \cdot 10^{3}$	Matter-radiation equality
$z \sim 10^{3}$	Recombination
$z \sim 30 - 40$	First stars form
$z \sim 0.3$	Dark energy-matter equality

Table 2.1: Rough cosmological chronology.

At first, reheating leaves the Universe with a quark-gluon plasma, in which hadrons cannot be formed until the quantum-chromodynamical (QCD) phase transition. When the energy is low enough, baryons and antibaryons annihilate, as does also eventually happen with electrons and positrons. These annihilations leads in turn to the reheating of the photon bath. Nucleosynthesis is largely dependent on the ratio of neutrons to protons in the Universe, which is approximately set once neutrinos decouple and the freeze-out ratio is reached. At lower temperatures, once deuterium is not dissociated efficiently by energetic photons anymore, nuclear fusion reactions start leading to the *nucleosynthesis* of light atoms such as helium or lithium.

The times of matter-radiation equality roughly correspond to the epoch at which the CMB radiation we observe today was emitted. In fact, matter-radiation equality happens shortly before recombination, the epoch in which electrons are gradually captured and stabilized into (mostly) hydrogen atoms. Once the Universe is mostly neutral, and photodissociation is low enough, it becomes transparent leading to propagation of light, which will become CMB light after redshifting. The primordial plasma can therefore be "observed" via the CMB. All sorts of interesting early-time physics can be observed in the CMB, notably the primordial perturbation spectrum. One of its other most interesting features are the baryon acoustic oscillations (BAO). These oscillations were due to the interplay between gravitation and radiation pressure in the early-time plasma. Importantly they left an imprint both on the CMB and on the matter perturbations, leading to the possibility to estimate distances using these oscillation as a *standard ruler*.

2.2.2 The dark Universe

Until now we have brushed under the carpet one of the biggest teachings of cosmology: *we don't understand what exactly composes most of our Universe.* Shortly put, we don't know what makes up most of the dust component of the cosmological fluid, *dark matter*, and we are not sure of what exactly contributes to the late-time era in addition to or instead of Ω_Λ, i.e. *dark energy.* Of course, from cosmological observations, we know a lot of things about how they behave on averaged large scales (and will learn more in the near future), but a definite picture of the microphysics is completely missing. What we have now are a variety of scenarios that can roughly fit all the observations we have until today.

The understanding of the dark sector of the Universe is one of the largest tasks of cosmology, today, and in the future. In the next sections we will detail how some future tests will try to tackle this issue. For now, let us summarize the common lore about the dark Universe. We first give an introduction about dark matter. We then give a small summary about dark energy. Finally, we point out that as far as is known, alternative theories of gravity may be a viable option to tackle the dark Universe puzzles.

Dark matter

Dark matter has been recognized as a mystery for longer than dark energy. Several probes point at its existence, at different scales. Historically three main observations have been decisive

- Galactic scales: missing mass in galaxies [11] (most of them, see e.g. [87]).
- Extra-galactic scales: missing mass in the clusters of galaxies [10].
- Cosmological scales: the shape and evolution of the matter power spectrum, structure formation [88], and consistency with other cosmological probes [85].

From cosmological observations alone, one may estimate that baryonic matter represents about two-fifths of the current matter content of the Universe, with the rest being mostly dark matter. There are several plausible models for dark matter, but the standard one for now remains the so-called *cold dark-matter* model. Although it is possible that dark matter is entirely decoupled from standard model fields except from the gravitational interaction, some features such as its relic density may also indicate a possible interaction with the standard model, e.g. via the weak interaction. Direct searches have therefore been implemented [89]. Besides new fundamental fields, there are also other candidates for dark matter, such as primordial black holes (see e.g. [90] and references therein for a review).

Dark energy

In the previous section we have seen that a cosmological constant could lead to an accelerated expansion of the Universe. In fact, this acceleration has been observed for the first time not so long ago, and independently by two groups [12, 13]. However, although the cosmological constant seems to fit consistently the observations, there are two reasons to be careful when trying to explain this acceleration.

The first reason is simply that for now, the precision of observations is not enough (see section 3.2.1) to determine if the late-time acceleration is due to an effective cosmic fluid with equation of state $w = 1$ (the cosmological constant) or something else with $w \neq 1$.

The second reason is deeper, and involves considering what we know from the quantum world. All quantum fields are subject to fluctuation even in their ground state, leading to what is called *vacuum energy*, as far as the Hamiltonian is concerned. Although technically amounting to a tremendous energy once a volume integration is performed, for most applications of quantum field theory, this vacuum energy is however completely irrelevant. This of course not always the case, as for example the Casimir effect can be understood as a manifestation of these vacuum fluctuations. However, by far, the vacuum energy should appear once gravitational effects are considered. Since all contributions to the energy-momentum-tensor gravitate, the standard lore posits that this vacuum energy should interact with the gravitational field just like a cosmological constant (different possibilities are investigated, see for example the recent [91]). Still, there is a doubt as to whether the vacuum energy can be the cosmological constant that we observe: according to quantum field theory (see [14, 92]), the value of the quantum field's vacuum energy should not only be much larger (of order $\mathcal{O}(1)\, M_{\mathrm{P}}^2$ assuming a cutoff at the Planck scale) than the observed value of the cosmological constant, of order $\mathcal{O}(10^{-60})\, M_{\mathrm{P}}^2$—this could be admittedly be canceled by a bare cosmological constant that would come on top of the vacuum energy—but the value of the possible bare counter-terms are also largely unstable to vacuum corrections, and hence have to be fine-tuned again for each effective field theory cutoff one is working with. One arguable view of the problem is to separate the original problem into two problems: assume that a property of gravity in the UV will set to zero the contribution from the vacuum energy (this is called the old cosmological constant problem), in which case one simply needs to find a *technically natural*—read well-behaved under quantum corrections, e.g. thanks to an approximate symmetry—source for the accelerated expansion of the Universe (this is called the new cosmological constant problem). In this sense, models of dark energy tend to only solve the new cosmological constant problem. On the other hand, it still possible that we live in a world with a fine-tuning, e.g. due to anthropic selection [93][5].

Due to mainly these two reasons, it is tempting to be conservative, and not conclude prematurely that we understand what is accelerating our Universe. It is therefore common to describe the cosmic fluid, or mechanism, that causes the acceleration as *dark energy.* Taken as a fluid (be it effectively), we know that it amounts for about 7 parts in 10 of the total density content in the Universe, and that its equation of state is close to $w \sim -1$. Experimentally, understanding dark energy is one of the main motivations for more precise observations in cosmology, in particular future large-field surveys of the late-time Universe

[5]Despite all arguments against it, this explanation is at least a good last resort option, in lack of another one.

[19, 20, 21, 22, 23, 24]. Theoretically, it has also led to the development of alternatives to the cosmological constant (see for example the book [94] or the reviews [95, 96]), most notably models involving scalar fields. It was also possible to build an effective field theory involving several (yet not all) of these models, see [97, 98].

Finally let us mention recent conjectures emanating from progress in the understanding of string theory, most notably the de Sitter swampland conjecture [99]. If correct, it would provide a further motivation (which should not be conclusive, since it relies on string theory models and we don't know whether string theory is realized in our Universe) to consider alternatives to the cosmological constant, which would produce a pure late-time de Sitter Universe and would be disfavored by the conjectures; see e.g. [100] for the application of the swampland conjectures to dark energy models.

Although there is a doubt as to whether it is accelerating our Universe of not, it remains that the cosmological constant for now is the most "economical" way to satisfy the cosmological observations, and is therefore the scenario of reference in the standard model of cosmology. Since we have discussed previously dark matter, and now dark energy, we have all the ingredients to name the standard model of cosmology. Its name, ΛCDM, stands for the cosmological constant Λ and Cold Dark Matter.

Alternative theories of gravity

Several mysteries appear in cosmology as soon as large distances are considered: from galactic scales up the elusive dark matter, and at larger, cosmic scales, the observed accelerated expansion. As we have seen, it is possible to solve these mysteries by including new species of matter fields yet to be observed. However there is another interesting possibility: since gravity sets the very stage where every astrophysical phenomenon takes place, one could try to solve the problem not by adding new matter fields, but by changing the stage itself… One could try to find a new theory of gravity, *whose laws would be different from GR on larger scales.*

This alternative scenario is that of *modified theories of gravity*, and is not without advantages. It is potentially an economic and universal solution, and it may provide the basis for a fundamental change in our comprehension of space-time. Indeed, such a paradigmatic change could be expected from the current lack of a fully workable UV completion for general relativity. Of course, that is not to say that certain models of dark matter or dark energy wouldn't also be game-changers.

In the previous section, we have seen (in particular from Lovelock's theorem in section 2.1.5) that general relativity relies on several pillars. These are well established, but it is possible to replace them. As a simple example, one may take the route of adding a new field that would couple non minimally to matter (or gravity, in a conformally transformed frame). Several models are in this category, e.g. $f(R)$ gravity, *Horndeski* gravity, and so forth, all grouped under the denomination scalar-tensor gravity. By extension one may find theories such as vector-tensor, multi-metric theories, etc. However modifications to GR can take more diverse forms and more speculative scenarios: for example, diffeomorphism- or Lorentz-breaking models are just as interesting. In the next chapter we will present more in details this wide variety, but for the remainder of the present chapter we will remain agnostic with what enters in the theory.

Still, finding viable alternatives to GR is not trivial and is still an ongoing enterprise. Even considering dark matter and dark energy separately, it is not trivial to explore all possibilities, and within to find convincing solutions in this variety. In this text, we make the choice to focus only on the *infrared (IR) modifications of gravity* that may have a chance to solve the dark energy puzzle. By doing so, we will completely neglect a vast field dedicated to solve dark matter without introducing a new species of matter, most notably theories which aim at a MOND-like behavior [101, 102].

However a point is clear: in order to have a better idea of the dark sector, one needs to go beyond background cosmology. Indeed, as far as background (i.e. FLRW) cosmology is considered, dark energy, and dark matter, can be well modelized by a fluids with given properties. By this alone we will however not be able to distinguish, say, minimally coupled dark energy from alternative theories of gravity. It is therefore crucial to make the next step and consider at least the perturbative regime atop the background cosmology. We will discuss the inhomogeneous Universe in the next subsection.

Tensions

We finish this subsection about the dark Universe with a discussion which is not necessarily directly related to the dark fluids such as dark matter or dark energy, but that is nonetheless mysterious and will maybe lead us to revisit our models of cosmology. It turns out that assuming the standard model of cosmology, there is an increasing tension in several, in particular two, measurements: the measure of the current Hubble rate H_0, and the measure of σ_8 (we will define it in the next subsection). Let us focus on the first one. The tension (reported for example in [103]) arises from two independent measurements of H_0: on the one hand the measurements using standard candles such as type 1a supernovae in the Cepheids [103], on the other the measurement using the CMB data, or even BAO data [104]. It is therefore easy to describe this problem as a tension between early (CMB time) and late ($z \sim 0$) data, as for example other late-time probes such as lensing [105, 106] tend to agree with the supernova data. It is clear that if this tension subsists, as it seems to do until now, novel scenarios will be needed. In particular it is interesting to see that early-time modifications of gravity may be able to tackle this issue [107, 50].

2.2.3 Cosmological perturbation theory

It is clear that the cosmological principle doesn't hold on gradually more resolved scales. This simply means that the solutions of the Friedmann equations should be considered as the background solutions for the more complicated sub-Hubble(-radius) physics (see below for a definition). This also means that *on a range of scales* the subhorizon physics are well approximated by linear perturbation theory. Although not all-encompassing, both background and linear perturbation physics tell us a lot about our Universe, and in particular about the dark universe. As far as the results we present in this manuscript go (to give qualitative constraints on alternative theories of gravity), these two will be sufficient. One should however note that a lot of effort in theoretical cosmology goes into resolving smaller scales, which contain a treasure of increasingly complicated but also interesting physics, and are essentially needed to give more accurate constraints on gravity. In this subsection, we will focus on the theory of perturbations, which we will employ in the later chapters. We will leave to the next section the discussion of what can be learned from studying perturbative regimes.

Cosmological perturbation theory consists in studying the physics of general perturbations of the metric and the matter fields while considering the background geometry to be a solution of the Friedmann equations. For more detailed definitions, see e.g. [108, 109].

We first define the perturbations of the metric. Using the ADM decomposition (2.14), and restricting ourselves to the flat FLRW case we decompose each variable as

$$N = \bar{N}(1+\phi)\,, \quad N_i = \delta N_i\,, \quad \gamma_{ij} = a^2\delta_{ij} + \delta\gamma_{ij}\,, \tag{2.52}$$

where $\delta\gamma_{ij} \ll a$ and $\phi, \delta N_i \ll 1$, the shift being being pure perturbation since it is zero on the FLRW background. One may further decompose

$$\delta N_i = \bar{N}a(\partial_i\beta + \beta_i)\,, \quad \delta\gamma_{ij} = a^2\left[2\delta_{ij}\psi + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\Delta\right)e + \partial_{(i}e_{j)} + h_{ij}\right]\,, \tag{2.53}$$

where spatial indices can be raised an lowered with the Kronecker δ_{ij} (and hence for practical purposes the contra- or covariance become irrelevant), and where $\partial^i e_i = \partial^i\beta_i = \partial^i h_{ij} = 0$, $h^i_i = 0$, and $\Delta = \partial_i\partial^i$. After replacement we drop the bar from $\bar{N}$ for simplicity of notation.

This decomposition is not innocent. Indeed, from the rotational invariance of the background it follows [110] that at linear level (in the equations of motion) there is no mixing between the set of scalar modes

$$\{\phi, \psi, \beta, e\}, \tag{2.54}$$

the set of (divergenceless, or transverse) vector modes

$$\{\beta_i, e_i\}\,, \tag{2.55}$$

and the (transverse and traceless) tensor modes

$$\{h_{ij}\}\,; \tag{2.56}$$

this is called the scalar-vector-tensor (SVT) decomposition. The statement remains true when the matter modes are included (to the scalar set). These are defined simply as

$$\rho = \bar{\rho} + \delta\rho\,, \tag{2.57}$$

where, just as for the lapse, the bar on $\bar{\rho}$ will be dropped after replacement. For a convenient way to introduce the density perturbations in a Lagrangian treatment, see appendix A.3.1.

A second decomposition can be made, this time following from the spatial homogeneity of the background. One may decompose each perturbation into (again, linearly) non-interacting spatial Fourier modes as

$$\tilde{A}(t,\mathbf{k}) = \int \frac{d^3x}{(2\pi)^3} A(t,\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \,. \tag{2.58}$$

where A stands for any of the perturbations. In practice, this can be as simple as replacing spatial derivatives $\partial_i \to ik_i$ into the perturbed expressions.

The decomposition we have presented is far from unique, and even more, considering gauge invariance, there is a redundancy in our choice: some intermediate results will be dependent on which coordinates have been chosen. Due to gauge invariance it is not easy to understand whether a given perturbation is real, or is just an artifact of the coordinates. In such a situation it is important to rely on the so-called *gauge-invariant perturbations*. It easy to define such combinations out of the transformation of the perturbations under gauge transformations (see appendix A.1). We define here a few such variables, the Bardeen variables [110]

$$\Phi \equiv \phi + a\left(H\beta + \frac{\dot{\beta}}{N}\right) - \frac{1}{2N}\left(2a^2H\dot{e} + a^2\frac{\ddot{e}}{N} - a^2\dot{e}\frac{\dot{N}}{N^2}\right) \,, \tag{2.59}$$

$$\Psi \equiv \psi + \frac{k^2 e}{6} + Ha\left(\beta - \frac{a\dot{e}}{2N}\right) \,. \tag{2.60}$$

We also define the gauge-invariant density contrast

$$\delta \equiv \frac{\delta\rho_m}{\rho_m} + 3\,\zeta \,, \tag{2.61}$$

As mentioned, in addition to the use of gauge invariant variables, one may actually enforce some gauge in order to make calculations simpler. A wide variety of gauges can be taken, but maybe the most usual ones are

- the *Newtonian gauge*:
$$\beta = 0 \,, \qquad e = 0 \,, \qquad \beta_i = 0 \,, \tag{2.62}$$
given as in [84]. In this gauge the quantity ψ, on short scales, roughly corresponds to the Newtonian potential. This is however not a fully fixed gauge; for a complete gauge fixing one may fix $e_i = 0$ instead of $\beta_i = 0$.

- the *flat gauge*:
$$\psi = 0 \,, \qquad e = 0 \,, \qquad e_i = 0 \,. \tag{2.63}$$

- the *synchronous gauge*:
$$\phi = 0 \,, \qquad \beta = 0 \,, \qquad \beta_i = 0 \,, \tag{2.64}$$
often used for numerical calculations, since only the spatial hypersurfaces are perturbed, and one may keep a simple interpretation of time. This is however not a fully fixed gauge [84] and some extra condition may be needed.

In general, whether by working without gauge-fixing, or by working with alternative theories with e.g. extra fields, several variables will be non-dynamical and hence redundant. On of the main tasks within perturbation theory will generally be to integrate out these redundant degrees of freedom.

At the end of the day, even though the gauge freedom or integrating out variables may simplify the calculations, it is not always enough to establish a clear picture of the physics, in particular as several powers of the three-momentum come into play. Therefore one often relies on additional (physically relevant) limits. We will discuss here the *sub-* and *super-Hubble* limits.

The *sub-Hubble* limit, also called the *sub-horizon* limit, in reference to the comoving Hubble horizon $1/(aH)$, is simply the limit

$$k \gg aH \,. \tag{2.65}$$

This limit is interesting for considering perturbations when, or shortly after, they are created in the early Universe. It also simply is more representative of the behavior at certain observable scales, which by definition are smaller or at least comparable to the Hubble radius.

On the other end, one finds the *super-Hubble* limit, also called the *super-horizon limit*

$$k \ll aH\,. \tag{2.66}$$

This limit is useful in particular for the study of the fate of primordial perturbations. Indeed, these modes tend to exit the comoving Horizon during inflation, as the Universe expands in an accelerated way and the Hubble rate remains approximately constant. They then eventually re-enter later, once inflation has ended and the Hubble rate starts evolving again. In fact, we are at a very good time for cosmology: since it seems that we are entering a new de Sitter-like era, the super-Hubble modes will remain so in the future.

Note that it is of no use to know the precise value of a perturbation field at a given place, since all perturbations were sourced as by a random field. One should consider instead the statistical properties of the field, such as *correlation functions*

$$\langle \delta(\vec{x})\delta(\vec{x}')\rangle\ , \tag{2.67}$$

or, in momentum space, *power spectra*, e.g. the matter power spectrum $P_\delta(k)$ defined as

$$\langle \delta_{\vec{k}}\delta_{\vec{k}'}\rangle = \delta^{(3)}(\vec{k}+\vec{k}')P_\delta(k)\,. \tag{2.68}$$

Finally, one should bear with non-linear growth below given scales. The treatment of it was not relevant to this work, and hence we do not delve into it. It will be sufficient to know that due to the initial normalization of matter perturbations, typically the non-linearities become dominant below scales $\sim 8h^{-1}$Mpc with $h = H_0/(100\,\mathrm{km\,s^{-1}\,Mpc^{-1}})$. It is therefore common to normalize the observables by the variance of the matter density contrast within a sphere of $\sim 8h^{-1}$Mpc. This quantity is noted σ_8.

2.2.4 Phenomenology of perturbations

The dark universe puzzles arise at the largest scales, and at late times. However, as long as only the background cosmology is concerned, it is not possible to distinguish a modification of the laws of gravity from new matter species appearing as perfect fluids in the Friedmann equations (2.38) to (2.40). It is therefore important to consider at least the perturbative regime of cosmology.

Perturbations can be witnessed within the two cosmological fluids that we can presently confidently observe: baryonic matter and radiation[6]. Perturbations are also admitted within the dark sector(s), but these cannot be directly seen: we observe them only via gravitational effects. It is understood that all these perturbations were seeded quantum mechanically in the early Universe [111, 112], for example during inflation, and that their subsequent linear and non-linear evolution has led to the high inhomogeneities at observed short scales.

Perturbations in the CMB

An early imprint of the non-homogeneous and non-isotropic Universe can already be found on the CMB. Down to about the 10^{-4} level, and once some distortions (e.g. due to the motion of the Earth with respect to a comoving frame) are taken into account, the CMB is isotropic. However, the finer details reveal temperature fluctuations. At present, these have been best mapped by the Planck mission [85]. The spectrum is measured on a broad range of wavelengths, limited by our resolution on the short scales, and by cosmic variance (*we only have one Universe*) for the largest scales.

Although effects of alternative theories of gravity were not necessarily yet imprinted on the initial CMB surface, they could however generically be found on the distortions induced on the CMB light in their travel through the matter and then dark-energy dominated universe, filled with forming structures. One calls these new perturbations the *secondary anisotropies*, in order to distinguish them from the *primary anisotropies* present on the CMB initial surface (see for example [82]). There are two main secondary anisotropies relevant to studying modified gravity [113, 114, 115, 116, 117]: the integrated Sachs-Wolfe-effect (ISW) [118, 119, 120] on large scales, as well as CMB weak lensing [121, 122, 123, 124], on shorter scales. On top of this the baryon acoustic oscillations also play an important role as a standard ruler and will be useful for our discussion of matter perturbations.

For now, let us simply define how the ISW and lensing anisotropies can be related to the sum of the gauge invariant potentials Φ and Ψ, defined in (2.59) and (2.60) (later in the text, we will show how these potentials are a probe of modifications of GR). First, the ISW anisotropy is given by (see e.g. [84])

$$\Theta(\hat{n}) = \int_{\eta_*}^{\eta_0} d\eta \frac{\partial}{\partial\eta}\left(\Phi + \Psi - \hat{n}^i\bar{\Psi}_i - \hat{n}^i\hat{n}^j h_{ij}\right)\ , \tag{2.69}$$

[6] Although it is expected that within a foreseeable future we may be able to also observe primordial gravitational waves.

where $\Theta \equiv \frac{\delta T}{T}$ is the temperature perturbation, η_* is the conformal time of emission, $\hat{n}$ is the direction of observation and $\bar{\Psi}_i$ is defined as in [84]. In the case of adiabatic initial conditions it will be detectable for scales $\ell \lesssim 10$. CMB lensing, on the other hand, shows up in the convergence angular power spectrum [125, 126] by

$$C_\ell^{\kappa\kappa} = 8\pi^2 \frac{(\ell+1)^2}{\ell} \int_0^{\chi_*} d\chi\, \chi \left(\frac{\chi_* - \chi}{\chi_* \chi}\right) P_{\Psi+\Phi}\left(k = \frac{\ell}{\chi}, \chi\right) , \tag{2.70}$$

where we have used comoving distances χ, and $P_{\Psi+\Phi}$ is the the power spectrum of the potential $\Psi + \Phi$. We will explain how to relate the characteristics of the potential $\Psi + \Phi$ to modified gravitational dynamics along with the discussion of matter perturbations.

Let us also mention the baryon acoustic oscillations (BAO) in the context of the CMB. The BAO arises from the tight coupling of baryons and photons (through Thompson scattering), allowing for acoustic oscillations, which left an imprint both on the CMB and on the distribution of matter. The BAO spectrum is therefore important, as it can be found both on the CMB and on the matter power spectrum, and can hence be used as a standard ruler (see e.g. [83] for more details).

Perturbations of the matter density

After briefly discussing the perturbations in the CMB, we turn to the matter perturbations. Indeed, after the matter-radiation equality, the inhomogeneities in the matter sector (including dark matter), have dominated the growth of inhomogeneities. Matter perturbations originally seeded by primordial perturbations have gradually clustered more and more, up to *non-linear* regimes of evolution which have seen the birth of galaxies and stars. However on large enough scales *linear* perturbations are still a valid tool to understand how matter perturbations evolve. On scales below 100 Mpc matter takes a very characteristic filamentary structure (see figure 2.3).

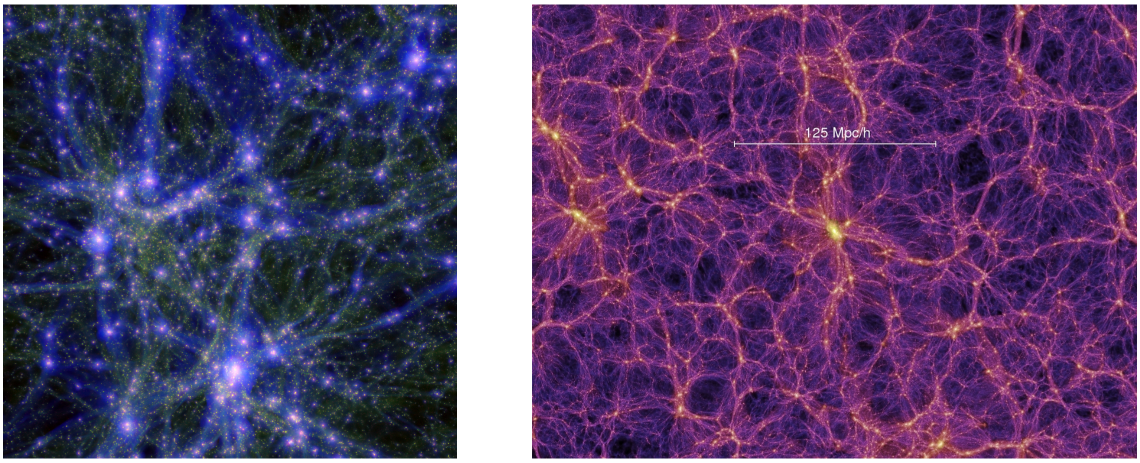

Figure 2.3: Simulations of the large scale structure. *Left*: MareNostrum simulation (composite of dark matter, gas, and temperature map) [127]. *Right:* Millennium Simulation Project (dark matter map) [128].

Here we follow [82]. In GR, whenever the linear regime is a good approximation, one may find the evolution equation

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_N \rho \delta = 0 \, , \tag{2.71}$$

where $\delta\sigma$ stands for a perturbations field, δ the density contrast as defined in (2.61), and G is the gravitational constant. Assuming a spatially-dependent initial density field, one may separate variables as

$$\delta(t, \vec{x}) = D(t)\epsilon(\vec{x}) \, . \tag{2.72}$$

written in terms of the scale factor, the purely temporal equation becomes

$$\frac{d^2 D}{da^2} + \left(\frac{1}{H}\frac{dH}{da} + \frac{3}{a}\right)\frac{dD}{da} - \frac{4\pi G}{H^2}\frac{\rho}{a^2} D = 0 \, , \tag{2.73}$$

which gives a decaying mode D_- and a growing mode D_+. Considering the continuity equation in Lagrangian coordinates one may characterize the divergence of the proper velocity perturbations by

$$\frac{1}{aH}\vec{\nabla}\cdot\vec{u} = -f\delta\,, \tag{2.74}$$

where f is the growth index defined as

$$f \equiv \frac{d\ln D}{d\ln a}\,, \tag{2.75}$$

and which therefore follows the equation [129]

$$\frac{df}{d\ln a} + f^2 + 2\left(1 + \frac{d\ln H}{d\ln a}\right) f - \frac{4\pi G\rho}{H^2} = 0\,. \tag{2.76}$$

One of the main issues relative to the matter power spectrum, is that it can only be inferred. Indeed, a large portion of matter being, as far as we understand, optically dark, we can only rely either on fractional and biased information from the observation of visible tracers (for example, galaxies), or on indirect information from the observation of gravitational lensing. Here we want to discuss two main methods which can constrain models of modified gravity: weak lensing [130], and redshift space distortions[7].

Starting with weak lensing, we can again use equation (2.70) [117], taking instead of $(0, \eta_*)$ an integration along the new line of sight. Again, the matter perturbations encountered by the light along its travel will impact through the potential $\Psi + \Phi$. We will explain further down how this will be impacted by modifications to the gravitational theory.

We now move on to our second method to study the matter power spectrum: redshift space distortions. The idea, here, is to use the distribution of galaxies to understand properties of the matter power spectrum. Two points are noteworthy. First, there is an intrinsic *bias*[8] when identifying properties of the total matter distribution by using the visible subset only. A naive approach would simply match both, but a proper pessimistic approach should introduce parameters to take into account the possible discrepancy between the distribution of dark matter and baryonic matter. Second point, we simply cannot measure directly the radial distance to galaxies. All we can do is use spectroscopic surveys to measure their redshift. This is why we use the term redshift-space.

Let us however define properly the *redshift space*: it is the distribution of galaxies projected by taking into account only the cosmological redshift of a fiducial cosmology (for example obtained by another type of observations). What remains is, of course, only a distorted notion of distance; other effects, in particular peculiar velocities, will also produce a sizable redshift. One therefore talks of *redshift space distortions*. We denote, in the vein of [117], the redshift-space distance vector by $\vec{s}$, while the one in real space is $\vec{r}$. The relation between the density fields is therefore (for more details see e.g. [82, 117])

$$\delta_s = \delta_r\left(1 + \frac{f(z)}{b(z)}\mu^2\right), \tag{2.77}$$

where $f(z)$ is the growth function, $b(z)$ is the *galaxy bias*, and where μ is the cosine between momentum of the matter perturbation and the line of sight. One can then relate the galaxy spectrum and matter power spectrum in respectively redshift and real spaces by

$$P_g^s(k,\mu,z) = \left(b(z) + f(z)\mu^2\right)^2 P_m^r(k,z)\,, \tag{2.78}$$

This equation tells us that two main observables in the RSD spectrum will be $f\sigma_8$, as well as $b\sigma_8$.

As we will see, in order to probe alternative models of gravity, it also makes special sense to cross-correlate RSD measurements with weak lensing probes, as one probes how gravity couples with matter while the other probes how gravity couples with light. Other tools that allow to make better sense of RSD data are standard ruler measurements such as BAO measurements.

Of course non-linear, i.e. shorter, scales need a more careful treatment. We won't go too much further into the complex treatment that is necessary to fully carry out RSD studies, but we leave the reader to reviews (e.g. [131]). An important approach in non-linear studies are calibrations using n-body simulations.

Recently, galaxy surveys have scaled in volume, and several future galaxy surveys are in preparation. We will not discuss more closely the status of observations in this chapter. It is in the next chapter (section 3.2) that we will detail the constraints that can be put on dark energy and modified gravity from current and future studies. Here we only give some pointers as to why the studies of lensing, redshift space, etc. are important to understand alternative theories of gravity.

[7] Other methods, such as galaxy clustering and 21cm-line cosmology, may have a word to say too.

[8] This is the commonly used technical term.

Impact of modified gravity

Alternative theories of gravity directly impact the treatment of perturbations by complexifying it, even at linear level: there may be new fields (several models introduce extra degrees of freedom), or for example the integration of the non-dynamical degrees of freedom may be complicated. In these cases one may need to rely on further approximations, such as the *quasi-static approximation*. In what follows we give some details of this approximation in the case in which there is an extra scalar perturbation variable which may mix with the matter density contrast. Still the modifications of (2.82) and following equations can be found in a wider variety of alternative theories of gravity.

The quasi-static approximation is a refinement of the subhorizon limit, and is especially relevant in the case of models of dynamical dark energy and alternative theories of gravity. In such a case, one needs this approximation to obtain the effective gravitational potential induced by the presence of the dark energy field. The idea is that the perturbations of the dark energy field may be decomposed into a slow varying part (with time-variation of order H), and a fast varying part, which will then be neglected inside the sound horizon, i.e.

$$k \ll \frac{c_s}{aH} \,, \tag{2.79}$$

where c_s is the sound speed for the dark energy field perturbations [132]. It is therefore interesting to take this limit whenever $c_s \sim \mathcal{O}(1)$, as we do in the case of the model studied in chapter 5. The quasi-static approximation was used already in [133], which studied a model of quintessence for which the dark energy perturbations $\delta\sigma$ satisfy an equation of the type

$$\ddot{\delta\sigma} + 3H\dot{\delta\sigma} + \left(\frac{k^2}{a^2} + m_\sigma^2\right)\delta\sigma = \text{background source,} \tag{2.80}$$

where we do not specify the source terms other than that they arise from background quantities and hence vary at the "slow" rate H. It is clear that the fast varying part is typically sourceless within the Hubble radius, and will hence decay, so that for small enough k one can take

$$\frac{\ddot{\delta\sigma}}{N^2} \simeq H\,\frac{\dot{\delta\sigma}}{N} \simeq H^2\,\delta\sigma \ll \frac{k^2}{a^2}\,\delta\sigma \,. \tag{2.81}$$

One can then compare between time derivatives of the perturbation and the spatial derivatives in the other equations, which lead to an equation for the density contrast alone, for example,

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}\rho\delta = 0 \,. \tag{2.82}$$

The coefficients of the different terms may be different from standard GR. New coefficients such as G_{eff} can even be scale dependent. In fact, modifications of GR do not only affect the matter power-spectrum. Notably, the Poisson equations that usually characterize the Bardeen potentials Φ and Ψ (defined in (2.59) and (2.60)) may be modified even within the sound horizon as (in Fourier space, and neglecting the anisotropic stress)

$$-\frac{k^2}{a^2}\,\Psi = 4\pi\,G_\Psi\,\rho_m\,\delta \,, \tag{2.83}$$

$$-\frac{k^2}{a^2}\,\Phi = 4\pi\,G_\Phi\,\rho_m\,\delta \,, \tag{2.84}$$

hence typically the ratio Ψ/Φ, which is 1 in GR, can be modified. One therefore defines the *gravitational slip*

$$\eta \equiv \frac{\Psi}{\Phi} \,, \tag{2.85}$$

which will be in general different from 1. Taken together, G_{eff} and η are often taken as standard parametrization of alternatives theories of gravity (usually $G_{\text{eff}} = G_\Phi$), and some parameters already produce some interesting results[9], although improvement must still be done since the background is still taken as ΛCDM. We will detail these parametrizations in section 3.2.

The modifications we discussed mean for example that after relating $P_{\Psi+\Phi}$, which we encountered above in the discussion of weak lensing (equation (2.70)), to the matter power spectrum P_δ via the Poisson equations, one will be able to relate modifications of gravity to some observables. Another example is the growth index (2.75), which will also be affected by the modification in (2.82), and which will be ultimately tested by RSD data. Other aspects of modifications of GR may be tested by studying RSD data, for example screening mechanisms (e.g. [134]), which are detailed in the next chapter.

[9] e.g. to reduce the tension in H [107].

Chapter 3

Beyond general relativity

3.1 Theories beyond Lovelock's theorem

In the previous chapter, we have seen that the standard model of cosmology fails to elucidate the nature of dark energy (and dark matter). It is therefore necessary to investigate the basic assumptions behind the standard model; it is in particular possible that the acceleration of the expansion of the Universe receives contributions from modifications to general relativity at large scales. The aim of this chapter is to understand how to theoretically construct these alternatives to GR, and how these can be constrained by observations.

Lovelock's theorem [67, 68] (see section 2.1.5) identifies several assumptions behind general relativity, and hence is effectively a guide to construct new theories. However, these different hypotheses, as well as the equivalence principles (see section 2.1.2), also protect general relativity from a range of complications that could arise in generic extensions. For example, the requirement of having second order equations of motion avoids the Ostrogradsky instability [135, 136, 137]. Another example is that the Einstein equivalence principle requires Lorentz invariance [65], which in turn may be associated to a restricted notion of causation[1]. One should therefore tread carefully.

As we will show, it will be generally possible to evade these difficulties in different interesting ways. In the next few sections, we will concisely review different modifications of GR at large scales[2] that aim to tackle the dark energy puzzle. We will in particular discuss two ways to go beyond Lovelock's theorem: adding extra fields, and relaxing symmetries.

3.1.1 Adding fields

Adding fields to the gravitational sector is arguably the simplest way to build a different theory of gravity. Several types of field can be added to the theory, starting by scalar fields [18, 17, 16], vector fields [42, 140], and up to spin-2 fields [141]. In particular, there has been a renewed interest towards scalar-tensor (as well as vector-tensor) theories which could evade the presence of an Ostrogradsky-type ghost while having higher order derivatives. In this short account, we will focus on scalar-tensor theories. We discuss first Galileon theories, then Horndeski and its further generalizations, the theories beyond Horndeski, and finally DHOST theories. Finally we also describe another class of scalar-tensor theories, $f(R)$ theories (they can accommodate both IR and UV modifications), which do not have higher order derivatives but emerge as non-minimally coupled theories.

Scalar-tensor gravity

The idea of writing a novel theory of gravity involving a scalar field dates back to Jordan in the 1950s [142], who wrote a theory

$$S_{\rm Jordan} = \frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-g}\phi^\gamma\left(R - \frac{\omega}{\phi^2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) + S_{\rm matter}[g,\phi,\psi]\,, \tag{3.1}$$

where ψ are the matter fields. In general, even after field redefinitions, it is not possible to render the theory equivalent to GR plus a new minimally coupled scalar field. The simplest example example of

[1] For a discussion as to whether the Lorentz structure can be emergent see for example [138], and for a discussion of causality within the context of superluminalities see [139].

[2] For extensions of GR in the UV, see for example [17, 7], or the recent [9].

such theory is the Brans-Dicke theory [143] in which one choses $\gamma = 1$ in the *Jordan frame*—the frame in which matter fields are minimally coupled. This gives the action

$$S_{\text{Brans-Dicke}} = \frac{M_{\text{P}}^2}{2} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) + S_{\text{matter}}[g, \psi] \, . \tag{3.2}$$

Higher orders

With the advent of higher dimensional cosmological scenarios (in particular the DGP model [144]), it was understood that an (effective) scalar-tensor theory of gravity could include non-trivial derivative self-interactions, i.e. higher-derivative terms in the action. However, as crystallized by the long-standing result by Ostrogradsky [135, 136, 137], generic higher-order Lagrangians incur the risk of exciting ghost-like degrees of freedom.

The only way to circumvent the result by Ostrogradsky is by using trivial or non-trivial degeneracies of the kinetic matrix. A degenerate Lagrangian is one for which the kinetic matrix is degenerate. One can define the kinetic matrix even for a higher order theory simply by the adding auxiliary variables, making the Lagrangian only first-derivative dependent [145].

Galileon theories

Galileon theories are an early example of degenerate scalar tensor theory: although their actions include higher-order derivatives, they retain second order equations of motion, and thus evade the Ostrogradsky ghost. They have been built as a generalization of the effective theory for DGP [146], organized as a field theory on Minkowski background. In fact, the Galileon higher-order terms may be seen as the Wess-Zumino terms for the Goldstone-bosons of the spontaneously broken Galileon shift symmetry [147]. This symmetry reads

$$\phi \to \phi + v_\mu x^\mu + c \, . \tag{3.3}$$

which leads to the general Lagrangian

$$\begin{aligned} S_{\text{Gal}} = \int d^4x \Big\{ & c_1 \phi + c_2 X - c_3 X \Box \phi + c_4 X \left[(\Box \phi)^2 - \phi_{\mu\nu} \phi^{\mu\nu} \right] \\ & - \frac{c_5 X}{3} \left[(\Box \phi)^3 - 3 (\Box \phi) \, \phi_{\mu\nu} \phi^{\mu\nu} + 2 \phi_{\mu\nu} \phi^{\nu\lambda} \phi_\lambda{}^\mu \right] \Big\} \, , \end{aligned} \tag{3.4}$$

where here we consider only flat space derivatives, i.e. $\phi_{\mu\nu} \equiv \partial_\mu \partial_\nu \phi$, and $X \equiv \partial^\mu \phi \partial_\mu \phi / 2$. Galileon terms are not only important per se, but can be found as limits of other theories e.g. in the decoupling limit of Lorentz-invariant massive gravity [148].

Horndeski theory

Galileon theory can be seen as one of the first major steps that eventually led to the rediscovery of the Horndeski Lagrangian. Several generalizations were still needed, however. First of all, the theory had to be written on curved space (which led to breaking the original assumption of symmetry under Galilean shifts (3.3)). This was not trivial, as higher derivatives of the metric tensor potentially appearing in the equations had to be avoided [149]. Then, the theory was generalized to include a set of free functions, since the Galilean symmetry was broken anyway; that theory was called *generalized Galileon theory* [150]

$$S_{\text{gGal}} = \int d^4x \sqrt{-g} \sum_{i=2}^{5} \mathcal{L}_i + S_{\text{mat}}[g, \psi] \, , \tag{3.5}$$

with

$$\mathcal{L}_2 = G_2(X, \phi) \, , \tag{3.6}$$

$$\mathcal{L}_3 = G_3(X, \phi) \Box \phi \, , \tag{3.7}$$

$$\mathcal{L}_4 = G_4(X, \phi) R + G_{4,X} \left[(\Box \phi)^2 - \phi^{\mu\nu} \phi_{\mu\nu} \right] \, , \tag{3.8}$$

$$\mathcal{L}_5 = G_5(X, \phi) G^{\mu\nu} \phi_{\mu\nu} + \frac{G_{5,X}}{6} \left[(\Box \phi)^3 - 3 (\Box \phi) \, \phi^{\mu\nu} \phi_{\mu\nu} + 2 \phi_{\mu\nu} \phi^{\nu\lambda} \phi^\mu{}_\lambda \right] \, , \tag{3.9}$$

and where the G_i are free functions of X and ϕ (hence explicitly breaking Galilean shifts), and only covariant derivatives are considered, $\phi_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi$, $X \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. This theory is also commonly called *Horndeski theory*, for the simple reason that it was in fact discovered (and largely forgotten) well before by Horndeski [151]. Here we have chosen to present it from the point of view of Galileon theories, as these are in fact important in the context of other alternative theories of gravity.

Beyond Horndeski

Thinking back to section 2.1.6, one may have noticed that we had not really discussed matter couplings in scalar-tensor theories; yet, these play a crucial role. The freedom to chose a non-minimal matter coupling may in fact be seen as a hint that one may find novel theories beyond the class defined in (3.5). Some interesting matter couplings are the following: the *conformal* matter coupling, where matter couples minimally with

$$\tilde{g}_{\mu\nu} \equiv A(\phi) g_{\mu\nu} \,, \tag{3.10}$$

and the purely *disformal* matter coupling [152]

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} + B(\phi)\partial_\mu\phi\partial_\nu\phi \,. \tag{3.11}$$

Note that this coupling can be found in brane cosmology for a moving brane [153, 154, 155]. Both can be extended (and merged) into [156]

$$\tilde{g}_{\mu\nu} \equiv A(\phi, X) g_{\mu\nu} + B(\phi, X)\partial_\mu\phi\partial_\nu\phi \,. \tag{3.12}$$

One possible way to construct theories beyond Horndeski is to use these extended disformal transformations [157]. This allowed to construct healthy theories, that however had more than quadratic equations of motion. The teaching is therefore that there is a new class of healthy theories. It was noted that this class of theories relatable to Horndeski is not encompassing, since other examples were found. One common example is the GLPV theory [158]. Later, all quadratic *degenerate higher-order scalar-tensor theories* (DHOST) were found [145], and this was later extended to cubic theories [159]. DHOST theories where also found to be closed under disformal transformations (see [160] and references therein). For a review of higher order scalar tensor theories see for example references [32, 18].

Hamiltonian analysis with higher-order derivatives

In the chapter 5 we will perform a Hamiltonian analysis involving, as part of the full theory, the quadratic and cubic Horndeski theories $\mathcal{L}_2 + \mathcal{L}_3$. We therefore find it adequate to make some comments regarding the analysis of higher-derivative theories. Since higher-order scalar-tensor theories exhibit second time derivatives of the scalar function, one needs to either extend the Hamiltonian formalism presented in section 2.1.8, or find a way to eliminate these higher order derivatives; we will pursue this second avenue. There are two main approaches that have been used along with the development of higher-order scalar theories.

The first approach to avoid the higher time derivatives consists in using the gauge-freedom of the theory to fix the unitary gauge for the scalar field ϕ, in which the scalar field is assumed to be a given monotonic function of time, i.e. $\phi \sim f(t)$, or even $\phi \sim t$. Within the unitary gauge the scalar field is no longer a dynamical degree of freedom, and therefore higher-order time derivatives do not pose a problem. This approach was used in [161, 158, 162]. In the unitary gauge, after ADM decomposing the metric, the GLPV Lagrangian (a subclass of beyond Horndeski Lagrangians) is (see for example [161])

$$\mathcal{L}_2 = A_2(t, N) \,, \tag{3.13}$$
$$\mathcal{L}_3 = A_3(t, N) K \,, \tag{3.14}$$
$$\mathcal{L}_4 = A_4(t, N)\left(K^2 - K^{ij}K_{ij}\right) + B_4(t, N) R^{(3)} \,, \tag{3.15}$$
$$\mathcal{L}_5 = A_5(t, N)\left(K^3 - 3KK^{ij}K_{ij} + 2K^i{}_j K^j{}_k K^k{}_i\right) + B_5(t, N) K^{ij} G_{ij} \,. \tag{3.16}$$

One can then proceed to the usual steps. It is noteworthy that the class of theories that are degenerate in the unitary gauge is in fact larger than the DHOST class see [163, 164].

The second option to avoid the higher time derivatives is the use of *auxiliary fields*. This second option does not require to choose the unitary gauge, and hence is practical, as we will see, for theories that rely on Stückelberg fields to recover covariance. This method was used for example in [163]. There, one replaces in the action all occurrences of $\partial_\mu\phi$ by the four-vector A_μ, while ensuring the equivalence of the theory via the extra constraint

$$S_{\text{auxiliary}} = \int d^4x \lambda^\mu \left(\partial_\mu\phi - A_\mu\right) , \tag{3.17}$$

where λ^μ is a Lagrange multiplier. Now, at least for the Lagrangians that are at most quadratic in derivatives, it is possible to remove all second time derivatives. For restricted classes of higher-order theories, the use of A_μ is not the only option; for the cubic Horndeski it is for example possible to introduce, instead of a four-vector, two (in fact in principle even a single) scalar fields

$$S_{\text{auxiliary}} = \int d^4x \left[\chi\left(\mathfrak{X} - X\right) + \theta\left(S - \Box\phi\right)\right] , \tag{3.18}$$

where χ and θ are Lagrange multipliers, whereas $\mathfrak{X}$ and S are the auxiliary fields. This approach will be used in chapter 5, and we present in Appendix C the Hamiltonian analysis of $\mathcal{L}_2 + \mathcal{L}_3$.

$f(R)$ theories

Another direction for adding more complexity to the scalar-tensor theories, aside considering higher-order derivatives, is to add a potential or/and non-trivial interactions with matter. This is well encapsulated by a popular scalar-tensor theory named *$f(R)$-theory* (here we follow [17]), whose action reads

$$S_{f(R)} = \frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-g}f(R) + S_{\rm mat}[g,\psi] , \tag{3.19}$$

where f is a general function. One may be puzzled as to why this theory enters the scalar-tensor theory, as there is no explicit scalar in the construction. One may answer this question by the conformal transformation

$$\tilde{g}_{\mu\nu} \equiv \frac{\partial f}{\partial R}\, g_{\mu\nu} , \tag{3.20}$$

where in general $F \equiv \frac{\partial f}{\partial R}$ is assumed to be positive, and one can then define further

$$\phi = \frac{\sqrt{3/2}}{\kappa}\ln F , \tag{3.21}$$

which can be inverted for given $f(R)$. This field redefinition yields the action

$$S_{f(R)} = \int d^4x\sqrt{-\tilde{g}}\left(\frac{M_{\rm P}^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) + S_{\rm mat}[F^{-1}(\phi)\tilde{g}_{\mu\nu},\psi] , \tag{3.22}$$

with the scalar field potential

$$V(\phi) = \frac{M_{\rm P}^2}{2}\frac{FR - f}{F^2} . \tag{3.23}$$

By choosing both the potential and the coupling to matter appropriately, interesting regimes can be reached.

Screenings

An appealing point of theories involving either potentials or higher-derivative interactions, is the possibility to make their predictions consistent with astrophysical tests, while allowing for modifications compared to general relativity at large scales. On short scales, one crucial point is to get rid of the *fifth-force* effect by which scalars which couple with matter fields in the Einstein frame will intervene to modify the first post-Newtonian parameters (see section 3.2.2).

The mechanisms that remove the fifth forces are called *screening mechanisms*. Here we present two of the main ways to suppress the fifth forces. One is to make the scalar field acquire a mass, at least in astrophysical context, so that it can be integrated out and doesn't affect the dynamics at first approximation. One of such screening mechanisms is the *chameleon screening*. Theories such as $f(R)$ can sustain a chameleon mechanism. The other way, is to make use of the higher derivative terms and let them trivialize the profile of the scalar field. One has then a *Vainshtein screening* [35]. Examples of theories that have such a mechanism are the higher-order derivative theories developed above, notably Galileons and Horndeski theory [151, 150], notably cubic or above.

The idea behind the chameleon screening [36, 37] is to make use of the interplay between the non-minimal coupling of matter to the scalar field (in the Einstein frame) and the potential for the field. Here, for a qualitative discussion[3], we assume that this coupling is of conformal form, i.e.

$$S_{\rm m} = S_{\rm m}[A^2(\phi) g_{\mu\nu}, \psi_{\rm mat}]\,, \tag{3.24}$$

with $A(\phi) = e^{\beta\phi/M_{\rm P}}$, β being a constant. In the context of cosmology for example, the equations for the scalar field are then

$$\ddot{\phi} + 3H\dot{\phi} = -V'_{\rm eff}(\phi, t)\,, \qquad V'_{\rm eff} \equiv \frac{\beta}{M_{\rm P}} A^4 (\rho - 3P) + V'(\phi)\,, \tag{3.25}$$

where $V(\phi)$ is the Einstein frame potential of the field and a prime denotes a partial derivative with respect to ϕ (note that the screening can also be seen in the Jordan frame, though the interpretation may be different). From this equation, several regimes can be found: during radiation domination, the contribution of matter fields being negligible, the field will tend to harmlessly roll the potential; during matter domination the field will couple to matter, but perturbations will acquire an effective mass in proportion to the matter density; with the decrease of the matter density towards dark energy domination the chameleon field will acquire a stronger influence, and may be important during late-time dynamics. This evolution is summarized in figure 3.1. The equation for the field for, say, a spherical

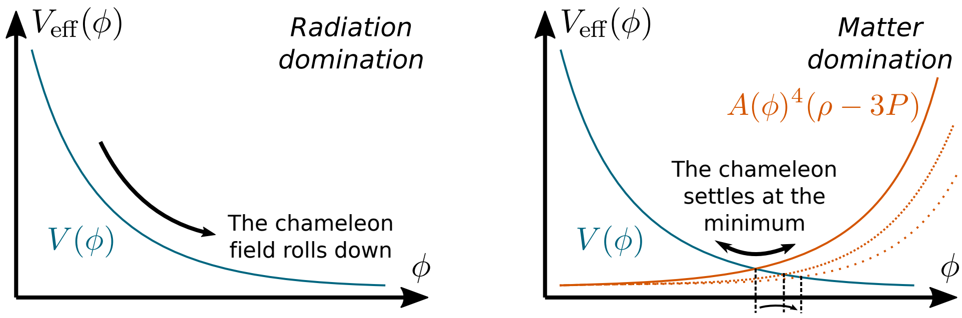

Figure 3.1: Cartoon of the effective potential for the chameleon field, during radiation domination (*left*) and during matter domination (*right*). $V_{\rm eff}$ is given by the sum of the contributions of V (*blue*) and the trace of the energy momentum tensor of matter (*orange*). During radiation domination there is only V, and hence the field rolls down the potential. During matter-(and Λ-)domination the potential has a minimum.

body

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = V'_{\rm eff}(\phi, r) \tag{3.26}$$

requires a more careful treatment (see [37]). However, in the case of a homogeneous body, one may easily find that the field settles to a given value (and a high effective mass) within the body, and that it will relax to its cosmological value outside the body. The body is screened since perturbations of the field near the body have a large mass, and can be integrated out. Finally, note that in $f(R)$ theories, a chameleon mechanism can be found if the coupling $F^{-1}(\phi)$ in (3.22) is $F^{-1}(\phi) \sim e^{\beta 2\phi/M_{\rm P}}$ [17].

To present qualitatively the case of the Vainshtein screening, we follow closely the simplified discussion from [18]. We admit here only perturbations about a slowly varying cosmological background, and choose to neglect second time derivatives. The reader may find a more complete treatment in e.g. [165]. We also focus solely on the subset of Horndeski (3.5) with $c_T^2 = 1$, i.e. containing $G_2(X,\phi)$, $G_3(X,\phi)$, and $G_4(\phi)$ only[4]. This gives the following Lagrangian for linear perturbations, in Newtonian gauge and under the quasi-static approximation (see sections 2.2.3 and 2.2.4),

$$\mathcal{L} = a\left\{M\left(-\Psi\partial^2\Psi + 2\Psi\partial^2\Phi\right) - \frac{\eta}{2}(\partial\varphi)^2 - 6\left[\left(M\xi + \frac{\mu\bar{X}}{\Lambda^3}\right)\Phi - 2M\xi\Psi\right]\partial^2\varphi\right\} + \frac{\mu}{a\Lambda^3}\mathcal{L}_3^{\rm Gal} - a^3\Phi\delta\rho\,, \tag{3.27}$$

[3]Indeed, it is possible to consider a coupling which doesn't couple universally (for example with dark matter). In such a case, the weak equivalence principle may be violated in some way.

[4]This is both because $c_T^2 = 1$ seems to be preferred by observations (see section 3.2.4.), and because this subset is the most relevant to the discussion of chapter 5.

where we have used $\bar{X} = \frac{1}{2}\dot{\bar{\phi}}^2$ and the perturbation

$$\phi(x) = \bar{\phi}(t) + \varphi(x)\,, \tag{3.28}$$

and defined $M^2 \equiv G_4(\bar{\phi})$,

$$\mathcal{L}_3^{\mathrm{Gal}} \equiv -\frac{1}{2}(\partial\varphi)^2\partial^2\varphi\,, \tag{3.29}$$

and

$$M\xi \simeq G_{4,\phi}\,, \qquad \mu \simeq -\Lambda^3 G_{3,X}\,. \tag{3.30}$$

Λ is the typical energy scale at which higher-order terms become relevant. One then studies the equations on a (quasi-)static spherical configuration $\varphi = \varphi(r)$, $\Phi = \Phi(r)$, $\Psi = \Psi(r)$, which read

$$\frac{\eta}{2}x - \left(\xi + \frac{\mu\bar{X}}{\Lambda^3 M}\right) y + 2\xi z + \mu x^2 = 0\,, \tag{3.31}$$

$$y = A + \left(\frac{\mu\bar{X}}{\Lambda^3 M} - \xi\right) x\,, \tag{3.32}$$

$$z = A + \left(\frac{\mu\bar{X}}{\Lambda^3 M} + \xi\right) x\,, \tag{3.33}$$

after defining

$$x \equiv \frac{1}{\Lambda^3}\frac{\varphi'}{r}\,, \qquad y \equiv \frac{M}{\Lambda^3}\frac{\Phi'}{r}\,, \qquad z \equiv \frac{M}{\Lambda^3}\frac{\Psi'}{r}\,, \qquad A \equiv \frac{1}{M\Lambda^3}\frac{\mathcal{M}}{8\pi r^3}\,, \qquad \mathcal{M} \simeq 4\pi\int_0^r \delta\rho(r')r'^2 dr'\,, \tag{3.34}$$

and taking $a = 1$. In the limit $A \gg 1$, i.e. deep inside the Vainshtein radius

$$r_{\mathrm{V}} \equiv \left(\frac{\mathcal{M}}{8\pi M_{\mathrm{P}}\Lambda^3}\right)^{\frac{1}{3}}\,, \tag{3.35}$$

the equation for x then becomes

$$\left[\frac{\eta}{2} - \left(\xi + \frac{\mu\bar{X}}{M\Lambda^3}\right)^2\right] x + \mu x^2 = \left(\frac{\mu\bar{X}}{M\Lambda^3} - \xi\right) A \quad \to \quad x \sim A^{1/2} \ll A \sim y \sim z\,. \tag{3.36}$$

Within the Vainshtein radius, the GR behavior is recovered as $\Psi = \Phi = -\frac{G_N\mathcal{M}}{r}$, on the other hand, when $A \sim 1$ the radial profile of the field, x, will contribute to the potentials. Considering the mass of the Sun, the Vainshtein radius encompasses much more than the solar system, indeed $r_{\mathrm{V},\odot} \sim \mathcal{O}(10^3)$ ly. Even within the Vainshtein radius, the cosmological evolution may impact with a slowly time-varying G_N. The fate of the Vainshtein screening in theories beyond Horndeski has been studied e.g. in [166] for GLPV theories. It is found for example that if the screening is efficient as much as in the spherical case one then expects (within the subset of GLPV $G_5 = F_5 = 0$ for which the Vainshtein is stable and exhibits the correct Newtonian behavior at short scales[5])

$$G_N = G_{\mathrm{cosm}} = \frac{1}{16\pi}\left(G_4 - 4XG_{4,X} - 4X^2G_{4,XX} - 10X^2F_4 - 4X^3F_{4,X}\right)^{-1}\,, \tag{3.37}$$

evaluated on the (time-dependent) cosmological background, where G_N is the gravitational constant as measured in a screened environment, and G_{cosm} is the gravitational constant for the background Friedmann equation, as measured at early times when dark energy is irrelevant.

Phenomenology within the quasi-static limit

As mentioned in section 2.2.4, on large enough scales (yet within the sound horizon of the new gravitational sector), modified gravity modifies the dynamics of matter perturbations and of the gravitational potentials. The modification can be encapsulated by effective gravitational constants G_{eff}, G_Φ, and G_Ψ

[5] For more details see [166] and references therein.

(see equations (2.82)-(2.84)). In fact, generically, one has $G_{\rm eff} = G_\Phi$. The expression for these, within Horndeski theories (we chose these for simplicity) for example, is [167]

$$G_{\rm eff} = \frac{1}{16\pi} \frac{c_T^2}{Q_T} \left\{ 1 + \frac{2Q_T \left[c_T^2 (1+\alpha_B) - (1+\alpha_M) \right]^2}{Q_s c_s^2 c_T^2 (1+\alpha_B)^2} \right\} , \tag{3.38}$$

$$G_\Phi = \frac{1}{16\pi} \frac{1}{Q_T} \left\{ 1 + \frac{2Q_T \alpha_B \left[c_T^2 (1+\alpha_B) - (1+\alpha_M) \right]}{Q_s c_s^2 (1+\alpha_B)^2} \right\} , \tag{3.39}$$

where $Q_s > 0$ is the scalar no-ghost condition, c_s the scalar sound speed, α_B and α_M some functions evaluated on the background (we do not specify here the previous quantities), $Q_T > 0$ the tensor no-ghost condition, which is given by [168]

$$Q_T = G_4 - 2XG_{4,X} + XG_{5,\phi} - H\dot{\phi}XG_{5,X} , \tag{3.40}$$

and c_T the tensor speed given by

$$c_T^2 = \frac{1}{Q_T} \left(G_4 - XG_{5,\phi} - X\ddot{\phi}G_{5,X} \right) , \tag{3.41}$$

where the quantities are evaluated on the background, e.g. $X = \dot{\phi}^2/2$. In fact, due to the recent measurement of the gravitational wave speed, theories with $c_T = 1$ have become more interesting (see section 3.2.4). To have $c_T^2 = 1$ on every background, one requires [168]

$$2G_{4,X} - 2G_{5,\phi} + \left(H\dot{\phi} - \ddot{\phi} \right) G_{5,X} = 0 . \tag{3.42}$$

This indicates that this subclass, barring fine-tunings[6], should have no G_5 dependence and G_4 should not depend on X. More details can be found in the review [168].

3.1.2 Symmetry breaking

In the previous section we have studied some types of gravity theories that can be built by adding extra fields to the gravitational action. In this section we want to study another way to bypass the Lovelock hypotheses: reducing the set of invariances of the action. In GR, the action has both a gauge invariance and local symmetries which are enforced on the gravitational action through the equivalence principles. Space-time diffeomorphisms are gauge-invariances; indeed, GR can be recovered as a gauging of local translations. As we will see, it is possible to recover this redundancy by the introduction of extra fields. On the other hand, the Lorentz invariance in the gravitational sector is related to the implicit choice of a Lorentzian geometry for space-time. Breaking Lorentz invariance may therefore in principle lead to different notions of geometry. Note that Lorentz invariance is best emphasized by the introduction of a tetrad (or vielbein), which renders GR explicitly invariant under Lorentz boosts on the flat space-time indices, which is called *local Lorentz invariance.*

There are two main ways to "break" a symmetry. The first case is called *spontaneous symmetry breaking* and denotes the case in which a field takes an non-trivial expectation value, and hence provides a preferred background for perturbative expansions. In such a case, the symmetry is understood as a symmetry of the underlying theory, but not necessarily of the solutions. In the case in which there doesn't exist symmetric solutions in the theory, the idea of "spontaneity" of the breaking disappears, but one retains the terminology anyway. This is the case if, say, one considers matter fields on curved background: they will break some symmetries anyway. In the case of spontaneous symmetry breaking, several powerful results associate the symmetry breaking pattern to the emergence, within effective low-energy theories, of degrees of freedom called *Nambu-Goldstone bosons* [169, 170, 171] (at least when global symmetries are concerned). These results can be extended, with some scrutiny, to the case of spacetime symmetries (see e.g. [172]). The other way to break a symmetry is sometimes called an *explicit symmetry breaking*, in the sense that the underlying action does not possess the symmetry, at most an approximate version of it.

[6]Note on fine-tunings: another (here unrelated) example of *unwanted fine-tuning* would be to demand $G_{4,\phi} = -XG_{3,X}$, in a theory with $c_T^2 = 1$. In such a case generically one would expect $G_{4,\phi}$ to depend on X, and hence depart from the requirement mentioned above.

Effective field theories

The distinction between the two modes of symmetry breaking requires the knowledge of the fundamental or underlying theory, which is clearly not always the case, especially in the context of gravity. In this sense, a top-down effective field theory (EFT) approach (such as the coset construction [172, 173]) cannot be used without assuming symmetries in the UV. On the other hand, a bottom-up EFT approach may be used: one constructs, out of the geometrical tools needed to describe a given system (e.g. in cosmology, the ADM variables and the associated derivatives and curvatures, or simply different components of the metric), the most general action consistent with given symmetries (see e.g. [97, 98, 174]). In this bottom-up EFT approach, there will be still some underlying assumptions permeating the choice of which building blocks will be used.

In the context of modified gravity, several such bottom-up EFTs have been used. For the purposes of this thesis, the EFT of dark energy [97, 98] is particularly interesting in that it allows for the description of all models which propagate an extra dynamical scalar and possess time-dependent diffeomorphisms as a residual symmetry. The first few terms of EFT action are for example given by

$$S_{\mathrm{EFTofDE}} = \int d^4x\sqrt{-g}\left[\frac{m_0^2}{2}\left(1+\Omega(t)\right)R^{(4)} + \Lambda(t) - c(t)\delta g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K \ldots\right], \tag{3.43}$$

in which one uses building blocks compatible with a metric description of spacetime and the residual symmetries (here $SO(3)$), for example δg^{00} or δK (the perturbations around FLRW of the 00-component of the metric and of the extrinsic curvature). Note that we have eluded several relevant operators for the sake of presentation. Generalizations include for example higher derivatives [175]. General stability properties in the presence of matter have been studied in [176]. Scalar-tensor theories can be naturally related to a (gauge-) symmetry breaking. In fact, the presence of a time-like scalar field can be seen as a spontaneous breaking of time-diffeomorphisms. From the point of view of the theory with diffeomorphisms, one may chose a particular time coordinate to coincide with the scalar field, say choose slicings such that $\phi = f(t)$. This is called the *unitary gauge* of the theory. On the other hand, one may obtain a covariantized action by the Stückelberg trick which we present below. Note also that in this chapter we want eventually want to go beyond the scalar-tensor framework, which we discussed in the previous section.

Stückelberg trick

Gauge-symmetries can be recovered by using the so-called Stückelberg trick, which involves adding new fields to the theory. This trick is named after Stückelberg[7] who first used it [177, 178] to build a manifestly $U(1)$-gauge-invariant alternative to the Proca Lagrangian for the massive vector field A_μ,

$$\mathcal{L}_{\text{Stückelberg}} = -\partial_\mu A_\nu \partial^\mu A^\nu + m^2 A_\mu A^\mu + \partial_\mu B \partial^\mu B - m^2 B^2\,,$$

with help of a scalar field B transforming under $U(1)$ as

$$A_\mu \to A_\mu + \partial_\mu \alpha\,, \qquad B \to B + m\alpha\,.$$

Hence it is possible to write the theory with a $U(1)$ gauge symmetry, although the original theory (recovered by setting $B \to 0$) is not $U(1)$ invariant. Very interestingly for spacetime symmetries, this approach can be generalized to the recovery of diffeomorphism invariance (for a review see [179]). The simplest example is the EFT of dark energy (3.43), in which one may recover the broken time diffeomorphisms by sending $t \to \phi(x)$ and the 3+1 decomposition back from (2.17). In this case, the theory with broken time diffeomorphisms maps well to scalar tensor theories.

We will see that in fact one may recover spatial diffeomorphism invariance in more complicated setups, with even less residual symmetries. In the case the full set of diffeomorphisms is broken, one may recover them by introducing 4 Stückelberg fields (since four gauged translations have been broken). It would therefore be tempting to simply generalize the EFT of dark energy to these 4 fields, an approach studied in [180, 181]. However, it turns out that in general cases, and in particular at non-linear level, one of the fields has negative energy excitations [182].

[7] For Ernst Stückelberg's (1905-1984) lecture notes see http://cours-physique.org/.

3.1.3 Massive gravity

Following the idea of the Proca field—for which the $U(1)$ gauge symmetry is broken (*à bon entendeur*) by the mass term—it is possible to find novel theories of gravity by violating diffeomorphism invariance, the gauge redundancy in GR. One of these types of theory is massive gravity, graviton mass terms being prime examples of terms breaking general covariance. In what follows we will detail on these theories.

Massive gravity is interesting for several reasons, but most notably for its phenomenology at large scales. In fact, it is often regarded as the simplest example, in principle, of an infrared (IR) modification of gravity: again in principle, it is expected that the novel scale given by the graviton mass should help set an energy scale above which GR is recovered, and under which (for example at cosmological distances) gravity is modified. Massive gravity can be schematically written

$$S_{m\mathrm{G}} = S_{\mathrm{EH}}[g_{\mu\nu}, \partial_\rho g_{\mu\nu}] + S_m[g_{\mu\nu}] + S_{\mathrm{mat}}[g_{\mu\nu}, \Psi_a]\,, \tag{3.44}$$

where S_m is the mass term and has some polynomial structure in $g_{\mu\nu}$, and where the subscripts m are the new scale characterizing the graviton mass. The equations of motion will then also involve a novel contribution $\Lambda_{\mu\nu}$, as

$$G_{\mu\nu} + \Lambda_{\mu\nu} = \frac{1}{M_{\mathrm{P}}^2} T_{\mu\nu}\,. \tag{3.45}$$

The contribution $\Lambda_{\mu\nu}$ will not generally vanish even in very symmetric situations. In cosmology, where the curvature radius roughly depends on the average matter content, it may play an important role whenever the energy momentum tensor of matter is low, typically at late times with $\Lambda_{\mu\nu} \propto m^2 \sim H_0^2$ where m is the graviton mass parameter, and H_0 the current Hubble rate. The hope is that the graviton mass thus contributes to the acceleration of the Universe. We will see that *self-accelerating* models can be found, i.e. acceleration without a dark component.

Massive gravity can also affect gravity at large distances in other ways, e.g. by changing the gravitational interaction between bodies. The most direct modification is the appearance of a Yukawa-type modification to the Newtonian potential

$$\Phi_{m\mathrm{G}} = -G_N \frac{\mathcal{M}}{r} e^{-mr}\,, \tag{3.46}$$

with, again, m as the graviton mass parameter, and $\mathcal{M}$ the mass of the source. As will be seen in the latter chapters 4 and 5, other modifications, such as a scale-dependent effective gravitational constant $G_{\mathrm{eff}} = G_{\mathrm{eff}}(k)$ may be produced, in particular in the context of cosmological perturbations.

It turns out that the previously mentioned scale separation may not necessarily be so simple, especially in the most conservative versions of massive gravity, as cutoffs and strong coupling scales hint at possible new physics even at higher energy scales. In the latter chapters we will study theories for which a simpler scale separation is realized, but we chose to remain rather general in this chapter.

Fierz-Pauli theory

The first theory of massive gravity, consistent at linear order in perturbation theory around Minkowski ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), and Lorentz invariant, was given by Fierz and Pauli [183]. *Fierz-Pauli* theory reads

$$\mathcal{L}_{\mathrm{FP}} = \frac{M_{\mathrm{P}}^2}{8}\left(-\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - 2\partial_\mu h \partial_\nu h^{\mu\nu} + 2\partial_\mu h^{\mu\nu}\partial^\alpha h_{\alpha\nu} + \partial_\mu h \partial^\mu h\right) - \frac{m^2 M_{\mathrm{P}}^2}{8}\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)\,, \tag{3.47}$$

where the first term can be compacted in $-\frac{M_{\mathrm{P}}^2}{4} h^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta}$, $\mathcal{E}^{\alpha\beta}_{\mu\nu}$ being called the Lichnerowicz operator. The equations of motion for the field are roughly (i.e. after choosing an appropriate gauge and a proper variable redefinition)

$$\left(\Box - m^2\right) h_{\mu\nu} = 0\,, \tag{3.48}$$

those of a propagating massive field. The action also explicitly breaks linearized diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$. Interestingly the special structure of the mass term preserves Lorentz-invariance but also exorcises a ghost degree of freedom (which reappears in the non-linear theory), so that although four gauged symmetries are broken, only three new degrees of freedom appear, for a total of five—as expected for a massive spin-2 theory.

Just as for the $U(1)$ invariance broken by the Proca mass term, the diffeomorphism invariance broken by the mass term can fortunately be recovered via the the Stückelberg trick. This leads to a theory which has a novel version of general covariance implemented by a transformation of the Stückelbergs.

It is simple to implement the new scalar fields. In the linear theory of massive gravity, the idea is to perform a coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$, and promote the $\xi^\mu(x)$ to fields belonging to the theory, i.e. $\xi^\mu(x) \to -\chi^\mu(x)$. One can then reabsorb further coordinate transformations by corresponding transformations of the $\chi^\mu(x)$. This allows to see the disappearance of a ghost at linear level.

Fierz-Pauli theory may be consistent at linear level, but it was recognized early on that the limit of zero mass is in fact disconnected from GR, fact known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity [184, 185]. This discontinuity is a hint that higher orders need to be taken into account; shortly after the discontinuity was found, Vainshtein argued that the problem could be solved through the screening of the extra mode by the non-linear interactions [35]. Unfortunately, it was also argued by Boulware and Deser (BD) that generic non-linear completions of Fierz-Pauli theory would invite a ghost-like degree of freedom into the theory [182]. As we will see, it was not until several years later that *non-linear* ways to remove the BD ghost would be found, using two different approaches: Lorentz-violations, and non-trivial degeneracies of the Lagrangian.

Non-linear massive gravity?

Before approaching the non-linear theories of massive gravity, we note that it is also possible to define the Stückelberg trick non-linearly. In a non-linear theory, one would like to write a potential directly for the metric $g_{\mu\nu}$ not just for the perturbation $h_{\mu\nu}$. Since this is trivially impossible ($g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha$) without introducing derivatives of the metric, a novel structure which will couple with the metric has to be defined at some level. In fact, one may first break diffeomorphisms using a fixed but generic extra background structure $f_{\rho\nu}$, which is often called the *fiducial* metric. The interaction term can then roughly be given by traces of $g^{\mu\rho}f_{\rho\nu}$. To implement the Stückelberg trick, one may then write

$$f_{\mu\nu} = f_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \tag{3.49}$$

where f_{ab} can be understood as a metric over an internal space with indices $a, b, \ldots$, and ϕ^a are the Stückelberg fields. Since the ϕ^a are space-time scalars, general covariance is recovered. The frozen structure $f_{\mu\nu}$ hence can be understood as the pull-back of the internal space metric into real space. Since the interaction term only consist in traces, it is common as well [43] to consider a second version of the non-linear Stückelberg trick. In that approach one recovers a covariant theory by writing it in terms of traces of $g^{ac}f_{cb}$ where $g^{ab} \equiv g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$ is the push-forward of the 4-dimensional metric onto the internal space. In [54], for example, we have used this second approach.

It can prove useful, at times, to write the theory in a non-covariant fashion, just as in GR one may choose convenient coordinates to simplify some treatment. Starting from the covariantized theory, and using the new diffeomorphism, one may freely chose to work in the full space-time *unitary gauge*. In the non-linear case, this gauge is defined as

$$\phi^a(x) \to x^a\,, \tag{3.50}$$

which translates in the linearized case to setting all the χ^μ to zero.

Now that we have discussed the non-linear Stückelberg, we can draw the lines of a non-linear theory of massive gravity.

Lorentz violating massive gravity

The first avenue leading to non-linear massive gravity was opened using Lorentz violations. Most commonly one uses a Lorentz breaking which leaves 3-dimensional rotations, i.e. $SO(3)$ untouched after a choice of preferred time-direction, i.e. a preferred foliation. This is justified in the context of cosmology, since matter fields will spontaneously break boosts anyway (and which for example has led foremost to theories of a single Stückelberg field as seen in the previous section). For massive gravity, one still wants to include all four spacetime Stückelberg fields, but making a difference between the temporal one ϕ^0, which defines the preferred slicing, and the spatial ones ϕ^i. Out of these fields it is indeed possible to build the following objects

$$N = \sqrt{-g^{\mu\nu}\partial_\mu\phi^0\partial_\nu\phi^0}\,, \qquad n_\mu = N\partial_\mu\phi^0\,, \tag{3.51}$$

$$N^i = n^\mu\partial_\mu\phi^i\,, \tag{3.52}$$

$$Y^{ij} = g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j\,, \tag{3.53}$$

$$\gamma^{ij} = Y^{ij} + n^\mu n^\nu\partial_\mu\phi^i\partial_\nu\phi^j\,, \tag{3.54}$$

which are all 3-dimensional tensors (except the normal vector n^μ). It is then easy to construct $SO(3)$ invariant contractions out of them by using δ_{ij}. The most general mass term is

$$S_m = \frac{M_{\rm P}^2 m^2}{2} \int d^4x \sqrt{-g}\, V(N, N^i, Y^{ij})\,. \tag{3.55}$$

which also includes Lorentz-invariant theories (see below). To avoid the BD ghost as well as strong couplings one should avoid having functions depending only on the lapse N. Indeed it is through the lapse that the constraint that removes the BD ghost arises. When linearized this yields

$$S_m = \frac{M_{\rm P}^2}{8} \int d^4x \left(m_0^2 h_{00}^2 + 2m_1^2 h_{0i}^2 - m_2^2 h_{ij}^2 + m_3^2 h^2 - 2m_4^2 h_{00} h \right)\,. \tag{3.56}$$

The Lagrangian (3.56) allows discussing a classification in terms of residual symmetry groups. When breaking diffeomorphisms, in addition to keeping global linear subgroups such as $SO(3)$, one may in fact maintain more complex combinations. An example maintaining time-dependent spatial reparametrizations

$$x^i \to x^i + \xi^i(t) \tag{3.57}$$

will restrict, at linear order, $m_1 = 0$, i.e. there should be no quadratic contraction of the lapse. Theories that benefit from this symmetry will propagate less than 6 degrees of freedom non-linearly. However it turns out that this type of residual symmetry also leads to a strong coupling around Minkowski [43]. Other choices are possible, for example, asking that the theory is at most linear in the lapse gives (at linear level) $m_0 = 0$, and this ultimately allows to build a constraint that removes the Boulware-Deser ghost.

Supersolid Lagrangians and massive gravity

Finally we comment on the distinguishability of massive gravity from other theories of gravity with extra fields. Thanks to the presence of what one could call the *Stückelberg frame*, it is clear that the theories developed on the basis of a breaking of symmetries can be arguably conflated with particular multi-scalar-tensor theories. Although generic scalar-tensor theories may involve a wider array of symmetry (breaking) patterns and other couplings than the ones of massive gravity, it is true that a different perspective on massive gravity, e.g. as self-gravitating medium, can be taken [181]. It was found that the internal space construction of *supersolid* medium in fact coincides with Lorentz violating massive gravity. This is due to the symmetry breaking pattern on which general supersolids are constructed: it is the same as the one characterizing the Stückelberg sector for massive gravity, i.e.

$$SO(3) \quad \& \quad \phi^a \to \phi^a + cst. \tag{3.58}$$

Note that more classes of four-dimensional media (solids, superfluids, perfect fluids, and Λ-media) can be constructed with larger residual symmetry groups [180, 181].

Lorentz-invariant massive gravity

Drawing inspiration from Galileon theories (see section 3.1.1), it was realized (and achieved) by de Rham, Gabadadze, and Tolley (dRGT) that a particular structure could both solve the problems encountered at non-linear order (BD ghost, vDVZ discontinuity) while keeping Lorentz invariance.

To preserve Lorentz invariance in the unitary gauge, it is not enough anymore to contract terms with δ_{ij} as one may do in $SO(3)$ preserving massive gravity. Instead the full Minkowski matrix should be used. Also, it does not make sense anymore to write the theory from the standpoint of a foliation, since a Lorentz invariant theory should have no preferred time direction. Therefore, we start by constructing nontrivial contractions with the fiducial metric $f_{\mu\nu}$, introduced with the non-linear Stückelberg trick (see above), on the scheme of

$$\dots g^{\beta\gamma} f_{\gamma\delta} \dots\,. \tag{3.59}$$

This will be the unitary gauge form of the building blocks of Lorentz invariant theory (in particular Lorentz invariance will manifest whenever one choses $f_{\mu\nu} = \eta_{\mu\nu}$). Moving on towards the particular structure of dRGT theory, we first introduce here the notation in terms of the fundamental matrix square-root $\mathcal{K}^\alpha{}_\beta$, defined by

$$\mathcal{K}^\alpha{}_\gamma \mathcal{K}^\gamma{}_\beta = g^{\alpha\gamma} f_{\gamma\beta}\,. \tag{3.60}$$

Note that dRGT theory may be written in different ways (including the convenient vielbein form which we introduce in chapter 4). Finally, the special structure that allows to write a non-linear, Lorentz invariant, and ghost free theory is reached by using exclusively symmetric elementary polynomials of $\mathcal{K}^{\alpha}{}_{\beta}$, the $e_i(\mathcal{K})$, $i = 0, \ldots, d$, where the polynomials are defined by

$$e_n(X) = \frac{1}{n!(d-n)!}\epsilon^{\mu_1\ldots\mu_n\lambda_{n+1}\ldots\lambda_d}\epsilon_{\nu_1\ldots\nu_n\lambda_{n+1}\ldots\lambda_d}X^{\nu_1}{}_{\mu_1}\ldots X^{\nu_n}{}_{\mu_n}\,, \tag{3.61}$$

with d the space-time dimension, and X any matrix, or through a recursive relation

$$e_n(X) = \frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1}\left[X^i\right]e_{n-i}(X)\,, \tag{3.62}$$

with $\left[X^i\right]$ the trace of the i-th power of X, and choosing a Minkowski fiducial metric, i.e. $f_{\mu\nu} = \eta_{\mu\nu}$. The graviton mass term is then written

$$S_m = \frac{M_{\rm P}^2 m^2}{2}\int d^4x\sqrt{-g}\sum_{i=0}^{4}c_i e_i(\mathcal{K})\,. \tag{3.63}$$

This graviton potential, taken together with the Einstein-Hilbert action, defines dRGT theory.

Hamiltonian analysis of nonlinear massive gravity

We now discuss the Hamiltonian analysis which will help understand how the BD ghost is removed from the theories described above. First, since a generic mass term breaks 4 gauged translation invariances, one may expect that the number of degrees of freedom for a generic massive gravity theory should be 6. In the Hamiltonian language, the breaking of diffeomorphism invariance leads to completely losing the Hamiltonian and momentum constraints that GR enjoys. On the other hand, based on Lorentz symmetry one expects that a healthy spin-2 theory should have only 5 degrees of freedom, hinting at the 6th mode entering generically as a ghost (the BD ghost). Fortunately, as we have seen, there are constructions that propagate less degrees of freedom and hence avoid this ghost. In this section, we will try to understand better how this counting arises.

The absence of the BD ghost translates in the Hamiltonian language as at least one novel constraint (in fact either two second-class constraints or a single first-class constraint) allowed by the specific structure of the action. The theories that have this special structure are the cases of Lorentz violating massive gravity, and ghost-free Lorentz invariant massive gravity.

In the case of Lorentz violating massive gravity, the lapse should enter linearly in the action. This implies at least one related constraint, reminiscent of the Hamiltonian constraint of GR. Schematically one has

$$\mathfrak{H}^{(\rm tot)}_{{\rm LV}m{\rm G}} = \int d^3x\left[-N\tilde{\mathcal{R}}_0 - \lambda\mathcal{C} + \ldots\right]. \tag{3.64}$$

where $\mathcal{C}$ represents an a secondary constraint automatically generated by $\dot{\tilde{\mathcal{R}}}_0 \approx 0$, λ is a Lagrange multiplier, and where the ellipsis denotes possible additional constraints, depending on the shape of the mass term. We do not discuss here an alternative class of Lorentz-violating theories which leads to an infinitely strong coupling around Minkowski space-time [43][8] (the one with $m_1 = 0$ at linear level).

For Lorentz-invariant massive gravity, the mass form of the mass term is more constrained than in the Lorentz-breaking theory. However the idea remains the same: find an appropriate form of the mass term that will leave the lapse as a Lagrange multiplier. We will follow [187] for this explanation, whereas the original non-linear proof of ghost-freeness was presented in [188, 189]. One may start by noticing that, by using a lapse-dependent redefinition of the shift variable $N^i \to n^i$, the matrix $\mathcal{K}^i{}_j$ may be rewritten as a combination

$$\mathcal{K}^i{}_j = \frac{1}{N}\mathcal{K}^i_{(-1)j}(n^i,\gamma_{ij}) + \mathcal{K}^i_{(0)j}(n^i,\gamma_{ij})\,, \tag{3.65}$$

where both $\mathcal{K}_{(-1)}$ and $\mathcal{K}_{(0)}$ do not depend on the lapse N, and where

$$\mathcal{K}^i_{(-1)j} = u^i v_j\,, \tag{3.66}$$

here without need of specifying the details of u^i, v_j, and $\mathcal{K}_{(0)}$. Due to this peculiar structure, one may build contractions that are linear in $\frac{1}{N}\mathcal{K}_{(-1)}$ by considering *only traced polynomials anti-symmetrized on*

[8]For another general discussion on the Hamiltonian analysis of massive gravity theories, see [186].

either of the indices (which are a limited number in given dimensions). We end up with the set presented previously, the symmetric polynomials $e_i(\mathcal{K})$ defined in (3.61). From here on we assume that the mass term is (3.63).

Now, having made sure that only symmetric polynomials are included in $\mathcal{L}_{\rm dRGT}$, we perform the Legendre transformation, yielding the primary Hamiltonian

$$\mathfrak{H}^{(1)}_{\rm dRGT} = \int d^3x \left[-N\mathcal{R}_0 - N^i(N, n^i, \gamma_{ij})\mathcal{R}_i + \frac{M_{\rm P}^2 m^2}{2}\Big(N\mathcal{H}_0(n^i, \gamma_{ij}) + M\mathcal{H}_1(n^i, \gamma_{ij})\Big) + \xi^i \pi_i \right], \tag{3.67}$$

where we didn't include a term $\xi_N \pi_N$ as we already know that N is simply a Lagrange multiplier. Indeed, an important information (in addition to N having a vanishing conjugated momentum) is that the equation for n^i can be inverted solely in terms of the spatial metric γ_{ij}, and will hence not add powers of the lapse. M denotes here the fiducial lapse function (defined in the same way as N but with the fiducial metric).

Considering systematically the conservation in time of the constraints, one finds a single additional constraint $\mathcal{C}$ (in addition to the one fixing the shift function, $\mathcal{P}_i$), leading ultimately to the secondary and total Hamiltonian

$$\mathfrak{H}^{(\rm tot)}_{\rm dRGT} = \int d^3x \left[-N\tilde{\mathcal{R}}_0 - \mathcal{H}_{\rm rest} + \lambda\mathcal{C} + \xi^i\pi_i + \zeta^i\mathcal{P}_i \right]. \tag{3.68}$$

where we have left all non-constraint parts in $\mathcal{H}_{\rm rest}$. We have therefore shown the broad lines of how the special structure of dRGT grants the propagation of 5 degrees of freedom instead of 6. The Boulware-Deser ghost has been exorcised again.

Regimes of massive gravity

It would be easy to (naively) assume that the phenomenology of dRGT massive gravity tends to GR at high energies, but it turns out this issue is not so simple, as exemplified already within Fierz-Pauli theory by the presence of the vDVZ discontinuity. In fact a more careful treatment can be done by considering which are the relevant operators for each degree of freedom in the theory. It is possible to render explicit a scalar- and a vector-graviton via a further decomposition of the Stückelberg. Looking for the canonical normalization of the different modes helps setting the scales right. Around a flat background, one has

$$\phi^a \to x^a - \frac{1}{M_{\rm P}}\left(\frac{1}{m}A^a + \frac{1}{m^2}\eta^{ab}\partial_b\pi\right). \tag{3.69}$$

Going from low energies-up, it has been shown (again, see [43] and references therein) that the first relevant non-linear interactions in dRGT theory arise at the scale $\Lambda_3 \equiv \left(M_{\rm P}m^2\right)^{1/3}$. It makes therefore sense to define a *decoupling* limit (a limit in which modes may only decouple but the number degrees of freedom remains unchanged)

$$M_{\rm P} \to \infty\,, \qquad m \to 0\,, \qquad \Lambda_3 \text{ fixed.} \tag{3.70}$$

In this decoupling limit, considering only the scalar- and tensor-graviton for simplicity, one finds after unmixing the interactions (reproduced from [43])

$$\mathcal{L}_{\Lambda_3} = -\frac{1}{4}\left[\tilde{h}^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}\tilde{h}_{\alpha\beta} + \sum_{n=2}^{5}\frac{b_n}{\Lambda_3^{3(n-2)}}\mathcal{L}_n^{\rm Gal}[\pi] - \frac{2b_5}{b_1\Lambda_3^6}\tilde{h}^{\mu\nu}X^{(3)}_{\mu\nu}\right], \tag{3.71}$$

where $\mathcal{E}^{\alpha\beta}_{\mu\nu}$ is the Lichnerowicz operator, the b_i are related to the constants c_i from (3.63), the $\mathcal{L}_n^{\rm Gal}$ correspond to the different parts of the action (3.4), and

$$X^{(3)}_{\mu\nu}[\Pi] \equiv \left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right)\eta_{\mu\nu} - 3\left([\Pi]^2\Pi_{\mu\nu} - 2[\Pi]\Pi^2_{\mu\nu} - 3[\Pi^2]\Pi_{\mu\nu} + 2\Pi^3_{\mu\nu}\right), \tag{3.72}$$

with $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$. Ultimately, the presence of $X^{(3)}_{\mu\nu}$ indicates that in order to be phenomenologically viable the theory necessitates $b_5 = 0$ [43]. However, once this choice is made, the emergence of a Galileon theory in the decoupling limit hints that dRGT massive gravity (as does Fierz-Pauli theory) enjoys the same Vainshtein screening as in the case of the Galileon, with a Vainshtein radius (see section 3.1.1)

$$r_V = \left(\frac{\mathcal{M}}{\Lambda_3^3 M_{\rm P}}\right)^{1/3}, \tag{3.73}$$

which, taking the mass of the Sun as a reference $\mathcal{M}$, and $m \sim H_0$ the current expansion rate, gives a radius parametrically larger than the solar system ($r_{\mathrm{V},\odot} \sim \mathcal{O}(10^3)\,\mathrm{ly}$). Note however, that besides $b_5 = 0$ other parameters have to be tuned to ensure stability [43].

In a generic theory of massive gravity non-linear interactions appear at lower energies $\Lambda_5 \equiv \left(M_{\mathrm{P}} m^4\right)^{1/5}$, while in Lorentz violating models the interactions may appear much higher e.g. $\Lambda_2 \equiv \sqrt{M_{\mathrm{P}} m}$ [44, 46], or even higher [51]. Considering the phenomenologically interesting $m \sim H_0$, one has for example $\Lambda_2^{-1} \sim \mathcal{O}(10^{-4})\,\mathrm{m}$, or $\Lambda_3^{-1} \sim \mathcal{O}(10^6)\,\mathrm{m}$. These scales are only indicative, especially in the case of the presence of a Vainshtein screening, since they should be redressed in a screened environment, thus running towards higher energies/shorter distances (see section 10.2 of [43]). Finally, it should be emphasized that the Vainshtein screening is more difficult to study in time-dependent or non-spherical cases. In these cases the conclusions may be modified.

Cosmology for dRGT massive gravity

In this chapter we have focused on models that can address or at least contribute to solve the dark energy puzzle. Scalar-tensor (and vector-tensor alike) are very diverse in this regard, but the difficulties in finding cosmologies are generally mild. On the other hand recovering cosmologies within models of massive gravity has been more complicated. Notable difficulties are (i) (non-)existence of FLRW solutions, (ii) (non-)existence of viable [9] background dynamics, (iii) linear and non-linear (in)stability of the solutions, (iv) (non-)viability of other phenomenology w.r.t. observations. In this part, we briefly review cosmologies within massive gravity theories.

Massive gravity, in its dRGT formulation, does not necessarily accommodate the FLRW ansatz 2.31. This fact arises for several reasons depending on the generality of the ansatz for the fiducial metric. In the case of the original ansatz with the Minkowski metric

$$ds_f^2 = -dt^2 + \delta_{ij} dx^i dx^j \,, \tag{3.74}$$

one finds that the divergence of the Einstein equations within the unitary gauge systematically [190] leads to

$$\dot{a} = 0 \,, \tag{3.75}$$

which precludes the existence of interesting cosmologies excepted a Minkowski universe. This condition can alternatively be obtained using the equation for the Stückelberg fields since (see for example [191])

$$\nabla^\mu \left(\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \right) = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \phi^a} \partial_\nu \phi^a \,. \tag{3.76}$$

Non-trivial open FLRW solutions can be found when choosing, for the fiducial metric, an open slicing of the Minkowski metric instead of the ansatz above [192]. However these solutions exhibit strong couplings or non-linear instabilities, as worked out in [193, 48]. More general fiducial metrics, still compatible with the symmetries of FLRW[10]

$$ds_f^2 = -M(t) dt^2 + \tilde{a}(t)^2 \delta_{ij} dx^i dx^j \,, \tag{3.77}$$

have been studied for example in [60] where it was found that a Higuchi-type ghost arises. Higuchi [49] originally derived a bound on the graviton mass from the study of spin-2 excitations on a de Sitter background. In fact, this bound can be generalized to more general cosmological backgrounds (see for example [194]). References [60, 195] then explored the bound in the context of bimetric theories and dRGT massive gravity on FLRW background. We do not reproduce here the explicit bound but give its approximate form

$$m_S^2 \gtrsim \mathcal{O}(1) H^2 \,, \tag{3.78}$$

where the subscript S stands for scalar, as this mode arises in the scalar sector (from the scalar part of the massive graviton).

Taken together, the previous results show how massive gravity does not accommodate FLRW solutions. After this realization, it was argued that one should instead consider anisotropic (see e.g. [196, 197], or inhomogeneous cosmology (see e.g. [190, 198, 199, 200]). Several considerations, such as the possible super-Hubble inhomogeneity of the Universe, and the presence of the Vainshtein screening at short scales, are arguments that may point in this direction [43]. In other words, the non existence of FLRW solutions doesn't preclude the existence of healthy and viable cosmologies within dRGT theory. However,

[9] In the sense of being roughly compatible with the expansion history of the Universe.

[10] Else one might see deviations from isotropy and homogeneity within the dynamics of cosmological perturbations.

the treatment of perturbations, especially structure growth, is rendered more difficult by background anisotropies or inhomogeneities (the usual mode decompositions cannot be used), and a comprehensive treatment has not been concluded yet.

Extensions of dRGT massive gravity

The difficulties of dRGT theory with FLRW cosmology have been one of the main motivations to explore cosmology within extensions. The overarching idea is that new dynamical modes can be added to for example de-trivialize expression (3.75). Maybe the most notable of the extensions is the framework of bimetric theories as implemented by Hassan and Rosen (HR) [141] (later extended to multi-metric theories). In this extension, one renders the fixed background metric dynamical by adding a second Einstein-Hilbert action

$$S_{\rm HR} = S_{\rm EH}[g] + S_{\rm EH}[f] + S_m[\mathcal{K}^\alpha{}_\beta] + S_{\rm mat}[g, \Psi_{\rm mat}], \tag{3.79}$$

in this case healthy FLRW solutions may be indeed found (while there exists also unhealthy regimes). Other extensions of dRGT massive gravity include a generalization of the coefficients as functions of the Stückelberg or the addition of scalar fields. Among these latter category, dubbed *extended massive gravity*, one finds mass-varying massive gravity [190, 201, 202, 203] and quasidilaton massive gravity. Chapter 5 will focus in particular on the quasidilaton theories.

Let us summarize the cosmology of these extensions:

- In generalized modified gravity [200] the mass parameters are allowed to depend on space-time, effectively allowing for stability of the gravitational degrees of freedom. Matter perturbations were not studied further.

- In bigravity theory (3.79), several situations are interesting: (i) it is possible to have a self-accelerating universe albeit degenerate with GR by exploiting a hierarchy between the two Planck scales[11] [204], (ii) it is possible to have a non-GR-degenerate cosmology with a self-acceleration given by $m \sim H_0$ at the cost of having a gradient instability [205] (see also [206]) in the scalar sector at early times (which may however potentially be cured through Vainshtein mechanisms [207]), (iii) rely on less studied exotic branches, which for example including bouncing cosmologies (for a discussion see [206]). Although in principle interesting [208], a fourth fine-tuned option with $m \ll H_0$ has been recently shown to be difficult to achieve [209], but can be cured by an extension [59, 58]. Finally, note that bimetric theories are especially interesting for the study of massive gravity, as the latter can be found as a decoupling limit of the former [43].

- Cosmology in quasidilaton massive gravity has been shown to be unstable [203], unless extra elements are included. See chapter 5 for more details.

- Mass-varying massive gravity may have viable cosmologies [201, 202, 203] but cosmological perturbations (within a very wide parameter space) in presence of matter still have to be investigated.

Finally, we discuss theories of Lorentz-violating massive gravity, well summarized as a subset of the self-gravitating media [181]. In these references the question of matter perturbations has not yet been analyzed. More recently, the subset of theories satisfying $c_T^2 = 1$ (see section 3.2.4) has been shown in [210] to have a phenomenology of matter perturbations degenerate with GR inside the sound horizon[12].

Two common (yet not generic) patterns arise: complications and difficulties emerge in the construction of a stable cosmology, or one has a tendency towards cosmologies degenerate with GR on the phenomenological level. In the case of mass-varying massive gravity further study is needed to approach the very large parameter space. Of all possibilities considered, a few known ones have a chance to produce interesting new phenomenology, as shown in [210]. All the previous models considered, the current lack of a fully elucidated alternative to ΛCDM cosmology, excepted few cases, is ultimately a motivation to consider minimal models as in chapter 4 or at least reductions of the number of degrees of freedom as in chapter 5.

[11] One may argue that introducing a scale higher than the Planck cutoff of the theory should be taken with a grain of salt. This is nevertheless an interesting and calculable limit of bigravity.

[12] In some specific cases it is possible to have a small sound horizon, in which case deviations from GR may be detectable.

3.2 Constraining modified gravity

In this section, we discuss several ways to test and constrain alternative theories of gravity, and hence test GR for modifications or inconsistencies. The simple idea is that one needs to apply the theory to given regimes (e.g. homogeneous and isotropic for cosmology, weak field regime,...), derive certain characteristic observables, and constrain these estimates with observations. The observables may often be statistical, as is the case for example in cosmology. In such a case, the confrontation of different theories will be made using appropriate statistical tools.

3.2.1 Cosmological constraints

Background observables

As discussed in section 2.2.2, dark energy and modified gravity models can (although not necessarily) already impact on the background evolution of the Universe in the form of modified equations of state $w_{\rm DE} \neq -1$, i.e. deviations may be measured w.r.t. the ΛCDM model already in background observables. One may also try to put generic constraints on these deviations from ΛCDM, which will in turn constrain all candidate theories.

One complication is the time-dependence of the equation of state in models of dark energy and modified gravity. Constraining a full free function of time with limited statistical data (in addition to the other ΛCDM parameters) is difficult, hence one needs to rely on a parametrization $w_{\rm DE}(a)$, unless one wants to constrain a very particular (and probably unlikely) solution of a particular model. A typical example of parametrization, in addition to the constant one, is (see for example [85])

$$w_{\rm DE} = w_0 + (1-a)\, w_a\,, \tag{3.80}$$

with $w_0, w_a = cst$. The bounds one obtains are largely dependent on the parametrization, and on the dataset used to constrain the model. The re-analysis (done in 2018) by the Planck collaboration [85] gives the following constraints for a constant parametrization

$$w_0 = -1.028 \pm 0.032\,, \tag{3.81}$$

and for parametrization (3.80)

$$w_0 = -0.961 \pm 0.077\,, \qquad w_a = -0.28^{+0.31}_{-0.27}\,, \tag{3.82}$$

in both cases for 68% confidence limits on the combined Planck, SNe, and BAO datasets. If one uses current BAO/RSD and WL data (as of 2019) instead of the combination of SNe and BAO data, one finds less stringent constraints

$$w_0 = -0.76 \pm 0.20\,, \qquad w_a = -0.72^{+0.62}_{-0.54}\,, \tag{3.83}$$

but which will neatly improve with future galaxy surveys. Other recent datasets give analogous constraints, see for example [211] for the inclusion of 3-year-data of the Dark Energy Survey (DES). Note again that tests presupposing a general parametrization are limited, and one needs to adjust this parametrization case by case for more model-specific constraints.

Other cosmological observables can be used to constrain modified background dynamics. Big-bang nucleosynthesis is an example which gives a powerful and early-time estimation of the gravitational constant and of the expansion rate [212, 213, 214] (for reviews mentioning this possibility see e.g. [215, 33, 65]). The idea is that the final cosmological abundance of given light elements (for example Helium or Lithium) is controlled, in a first approximation[13], by two rates: the weak interaction rate Γ_W and the Hubble rate H. One of the primary bottlenecks is the freeze-out of the weak interactions (see previous references for a more precise description). From the Friedmann equation one has roughly

$$H \sim \sqrt{G_{\rm cosm} N_* k_{\rm B}^2 T^2}\,, \tag{3.84}$$

with N_* the number of relativistic degrees of freedom, while the weak interaction rate scales as

$$\Gamma_W \sim G_{\rm F}^2 k_{\rm B}^5 T^5\,. \tag{3.85}$$

Therefore, there is a crossing temperature which is roughly dependent on the gravitational constant $G_{\rm cosm}$ or other factors affecting the expansion rate. Considering a varying gravitational constant (and a non-varying Fermi constant) gives a bound $|G_{\rm cosm} - G_N|/G_N \lesssim 10\%$, where G_N is the gravitational constant as measured within the solar-system.

[13] See the previous references for more precise discussions.

Perturbative observables

However, considering only background observables, the details of the new component—in particular whether one has a given model of dark energy or modified gravity—are mostly unintelligible; in order to differentiate models one has to study the inhomogeneous Universe (see sections 2.2.3 and 2.2.4 for an introduction within GR). One can roughly split between linear scales, for which linear perturbation theory works, and non-linear scales. On linear scales, modified gravity and dark energy are often parametrized on their impact on the Poisson equations (2.83) and (2.84), which we reproduce here:

$$-\frac{k^2}{a^2}\,\Psi = 4\pi\, G_\Psi\, \rho_m\, \delta\,, \tag{3.86}$$

$$-\frac{k^2}{a^2}\,\Phi = 4\pi\, G_\Phi\, \rho_m\, \delta\,, \tag{3.87}$$

where Ψ and Φ are the Bardeen gravitational potentials, and on the evolution of matter perturbation (2.82). One may then define the related variables μ and η by

$$G_\Phi = G_{\rm eff} \equiv \mu(a,k) G_{\rm cosm}\,, \qquad G_\Psi \equiv \eta(a,k)\mu G_{\rm cosm}\,, \tag{3.88}$$

where different notations exist for these variables, e.g. with Ψ and Φ interchanged, and where $G_{\rm cosm}$ is the gravitational constant for the cosmological background evolution. Another widely used quantity is $\Sigma = \mu(1+\eta)/2$, which controls the potential for lensing $\Psi + \Phi$. Note in particular, these quantities may be time- and scale-dependent, again the same issue as for the parametrization of $w_{\rm DE}$. Within available data analyses, typical parametrizations focus again on a simple Taylor expansion at linear order (see e.g. (56a) and (56b) of [85])

$$\mu(z) = 1 + E_{11}\Omega_{\rm DE}(z)\,, \qquad \eta(z) = 1 + E_{12}\Omega_{\rm DE}(z)\,, \tag{3.89}$$

with constants E_{11}, E_{12}. This parametrization will indeed cover only a subset of models and solutions, but is simple to implement. In addition to this parametrization, one often assumes that the background evolution is given by ΛCDM —another unfortunate but (until now) necessary non-trivial assumption. Current bounds, for example from [85], are

$$\mu_0 - 1 = -0.07^{+0.19}_{-0.32}\,, \qquad \eta_0 - 1 = -0.32^{+0.63}_{-0.89}\,, \tag{3.90}$$

where the subscript 0 here stands for the present-time value, and where the bounds are derived from Planck data (including CMB lensing) combined with BAO/RSD and WL. The presence of two different parameters shows the importance of combined future studies of RSD and weak lensing. In [85] one used RSD BOSS data [216, 217]; other recent datasets include e.g. FastSound [218], VIPERS [219], and WiggleZ [220]. For weak lensing DES data was used [221, 211]; another dataset is for example the KiDS [222, 223]. Note also that an alternative to the parametrizations (3.89), (3.80), etc. is to use the so-called *principal component analysis* (PCA) on a scale- and/or time-binning of the quantities of interest (see for example [224]). This is interesting as a way to remain model-independent; one can then balance the number of bins with the desired precision.

Several future observations will take the RSD and weak lensing data to the next level. For a comparison of some expected constraints see for example [19], which will reach errors of $\mathcal{O}(1\%)$ on parameters such as μ and Σ, of course given a particular parametrization (here affine in a). For ground-based based missions, the first on the line is DESI (survey 2019-2024) [225, 21], which is spectroscopic survey mission, with focus on the BAO and growth of large-scale structures. Other future interesting ground-based missions include the Square Kilometer Array (SKA) (2023-) [226, 20] which will perform among other radio weak-lensing observations; the Large Synoptic Survey Telescope (LSST) (survey 2022-) will also allow for weak lensing measurements [22]. On the side of space-based missions, up next is Euclid (launch 2020), which will perform an imaging and spectroscopic survey of a large number of galaxies. Euclid RSD expects constraints of about 20% on μ and Σ, while Euclid WL will be able to constrain Σ up to 10% [23]. Wide Field Infrared Survey Telescope (WFIRST) (launch 2025), if launched, will also help observe the dark Universe [227, 24] on infrared scales through lensing and wide-field surveys.

The discussion wouldn't be complete without mentioning the real tool to connect theories and observables on linear scales: the Einstein-Boltzmann solver codes. These codes (see for example CAMB [228], CLASS [229], or PyCosmo [230] to cite a few) solve the coupled equations for the evolution of the perturbations at linear level, including the full transfer function that allows to go from given initial conditions to the observables. In the context of modified gravity, these codes have to be modified to account for the new equations (and other subtleties, such as the choice of gauge, etc.). The parametrizations given above, e.g. (3.89) or (3.80) are meant to be implemented in these codes. However, the models of modified gravity will in general have more specific features. It is therefore of utmost importance to modify the codes in a way that accommodates models. Finally, one needs to solve multi-dimensional fitting problems, usually using *Markov Chain Monte-Carlo* (MCMC) techniques. All this has yet to be done for several models, in particular the ones exposed in this thesis.

Finally, note that modified gravity gravity can be distinguished from dark energy mainly from the existence of $\eta \neq 1$. Even including anisotropic stress producing dark energy models such as vector field models, the η parameter should deviate from unity only at horizon scales (see for example [231] and references therein).

Non-linear observables

Non-linear cosmology, even in the context of modified gravity, has been developed extensively; we primarily refer the reader to reviews [116, 232]. In this thesis, we will completely stick to linear observables and thus we give only a very brief account of the field. One important application for non-linear studies within modified gravity is the study of the screening mechanisms.

Growth becomes non-linear approximately on scales $\lesssim 8h^{-1}$Mpc, hence the linear treatment used above will break down. At the interface between linear and non-linear scales, one may use higher-order perturbation theory—an approach called standard perturbation theory (SPT) —to approximate the dynamics on increasingly small scales (for a review see [131], and for a recent work see [233]). In the case of modified gravity models one may apply similar techniques (for an early work [234], and for a more recent discussion [235]). The difference between real-space and redshift-space must also be taken into account [236]. However there is a limit to this endeavor, and one needs to move to N-body simulations to reach smaller scales. Initial conditions are set using the Zel'dovich approximation [237], or with higher order perturbation theory [238]; several codes may then be used for the evolution of the perturbations, e.g. [239]. In the case of modified gravity, N-body simulation codes are especially important because they allow to study short scales hence the *efficiency of screening mechanisms.* For a comparison and a more recent reference see e.g. [240, 241]. Finally, codes for computing observables quasi-nonlinear scales may then use N-body simulation data input to become more efficient (for example [242]).

3.2.2 Observations in the weak field regime

Aside from late-time cosmology, weak field regimes include astrophysical regimes. In particular, solar system tests are potentially interesting since they allow very tight constraints. We first review a time-tested parametrized framework, and then discuss briefly its limits.

As we have seen with cosmology, since there are multiple models of modified gravity, it is important to build generalized frameworks that try to encapsulate modifications of gravity in an agnostic way. This is an effort that cannot completely replace model-specific tests, since some models may have unique particularities, but should go hand in hand with them.

The most famous of such frameworks in the weak-field regime is the parametrized post-Newtonian (PPN) formalism [65] which considers an expansion for small $\frac{v}{c}$, where v is the characteristic velocity for a virialized system. Indeed for such systems

$$\frac{G_N \mathcal{M}}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,, \tag{3.91}$$

where $\mathcal{M}$ here denotes the typical mass of the system. It is sufficient to consider matter as a perfect fluid (not necessarily comoving, as we had assumed in cosmology), with

$$T_{\mu\nu} = \left[\rho\,(1+\Pi) + P\right] u_\mu u_\nu + P g_{\mu\nu} \tag{3.92}$$

where the rest-mass density ρ, the specific energy density $\Pi \sim \frac{v^2}{c^2}$, and the pressure $P \sim \rho \frac{v^2}{c^2}$, are enough to characterize the fluid up to one order in $\frac{v^2}{c^2}$ above the Newtonian level, i.e. at first*post-Newtonian* level. Out of the matter variables one may build relevant functionals, called *potentials*, and one may parametrize different theories of gravity by the coefficients for each of them. By *relevant*, it is understood that a range of further assumptions are made, to ensure for example that the metric becomes asymptotically flat, etc. Ultimately, this leads to a set of parameters that can be tested against observations, and constrained, and hence constrain the theories that predict these parameters.

Derivatives are assumed to scale as $\partial_0 \sim \frac{v}{c}\partial_i$. As far as the Einstein equations are concerned, hence the Ricci curvature tensor, to compute them at one order $\frac{v^2}{c^2}$ above the Newtonian limit one needs therefore only to expand the metric as

$$\begin{aligned} g_{00} &= -1 + h_{00}^{(0)} + h_{00}^{(1)} + \mathcal{O}(\epsilon^6) \\ g_{0i} &= -h_{0i}^{(1/2)} + \mathcal{O}(\epsilon^5)\,, \\ g_{ij} &= \ \delta_{ij} + h_{ij}^{(0)} + \mathcal{O}(\epsilon^4)\,, \end{aligned} \tag{3.93}$$

where the superscript denotes the order in $\frac{v^2}{c^2}$ above the Newtonian order, and $\epsilon = \frac{v}{c}$. At leading order, Newtonian physics is recovered, and there is only one relevant parameter (i.e. one potential), which fixes the gravitational constant. However, at higher order, the parameters one can build out of the lower-order potentials and fluid variables multiply quickly. One finds them in the *PPN metric* [65],

$$\begin{aligned} g_{00} =& -1 + 2U - 2\beta U^2 - 2\xi\Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\,\Phi_1 \\ &+ 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 \\ &+ 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - (\zeta_1 - 2\xi)\mathcal{A} + \mathcal{O}(\epsilon^6)\,, \\ g_{0i} =& -\frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)V_i - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)W_i + \mathcal{O}(\epsilon^5)\,, \\ g_{ij} =& \ (1 + 2\gamma U)\delta_{ij} + \mathcal{O}(\epsilon^4)\,, \end{aligned} \tag{3.94}$$

where the gravitational constant, which is in fact determined by the coefficient of $2U$ in g_{00}, has been set to 1, and where U, Φ_W, Φ_1, Φ_2, Φ_3, Φ_4, $\mathcal{A}$, V_i, and W_i are commonly used[14] potentials defined in [65]. All other quantities are the parameters of the parametrized expansion. In order to obtain the parameters for a given theory of gravity, one simply needs to compute the Einstein equations, as well as the equations for the other gravitational fields, order by order. In particular, the parameters for GR are $\gamma = \beta = 1$, with all other parameters 0. Working at order $\mathcal{O}(\epsilon^2)$ already allows for deviations from gravity, since $\gamma \neq 1$ may appear.

In the previous section about cosmology, we have discussed the high-momentum, *sub-Hubble limit*. Since the matter source in cosmology is also assumed to be a perfect fluid (usually comoving, in particular), it is in general possible to find a relation between the gravitational constant for dust perturbations, as well as the gravitational slip (2.85) into the parameters $G_{\rm N}$ and γ. We will use this in chapter 4.

The PPN parameters are well bounded by laboratory (e.g. torsion balance), solar-system (e.g. the motion of planets), and astrophysical (e.g. pulsar timings) experiments. We reproduce in table 3.1 the current bounds on the PPN parameters, as given in table 4 of [33]. It denotes for example bounds $\gamma - 1 \leq 2.3 \cdot 10^5$ derived from time delay for radio waves from the Cassini probe, or the bound $\beta - 1 \leq 8 \cdot 10^{-5}$ derived from a measurement of Mercury's perihelion shift. See the reference for more details on the different tests.

The presence of strong constraints on modifications of gravity on astrophysical and in particular solar system scales, is a strong motivation to implement screening mechanisms (see section 3.1.1). Furthermore, while the PPN formalism is useful for some theories of modified gravity, it is not appropriate in screened regimes for which non-linearities become important, e.g. when the Vainshtein screening mechanism is effective. In this case dedicated studies are necessary. A notable simple example is the case of binary pulsars, in which the pulsation rate diminishes slowly due to the emission of gravitational waves (and potentially other types of radiation in modified gravity) by the system (for a descriptions of these systems see for example [243]). There have been several studies of these systems[15] with a Vainshtein screening [246, 247]. Numerical studies have also been important in exploring the screened regimes dimension; in the case of binary systems, see for example [248]. Eventually, going beyond pulsar binaries (see for example [249] for a study in the solar system), a full set astrophysical tests of modified gravity may then be designed to explore screenings mechanisms per se (see chapter 6 of [116] for a review). Note that in the case of the Vainshtein screening, new formalisms have recently been developed to push in this direction, such as the post-Vainshteinian formalism [250], or the use of effective field theory techniques [251].

[14]There may be more, since the metric (3.94) is obtained using gauge freedom. In a theory without the full gauge-freedom this is not necessarily possible, and more potentials may be needed.

[15]See also [244] for the impact of massive tensor modes only, and for example [245] for other screening mechanisms.

Parameter	Effect	Limit	Remarks
$\gamma - 1$	time delay	2.3×10^{-5}	Cassini tracking
	light deflection	2×10^{-4}	VLBI
$\beta - 1$	perihelion shift	8×10^{-5}	$J_{2\odot} = (2.2 \pm 0.1) \times 10^{-7}$
	Nordtvedt effect	2.3×10^{-4}	$\eta_{\rm N} = 4\beta - \gamma - 3$ assumed
ξ	spin precession	4×10^{-9}	millisecond pulsars
α_1	orbital polarization	10^{-4}	Lunar laser ranging
		7×10^{-5}	PSR J1738+0333
α_2	spin precession	2×10^{-9}	millisecond pulsars
α_3	pulsar acceleration	4×10^{-20}	pulsar $\dot{P}$ statistics
ζ_1	—	2×10^{-2}	combined PPN bounds
ζ_2	binary acceleration	4×10^{-5}	$\ddot{P}_{\rm p}$ for PSR 1913+16
ζ_3	Newton's 3rd law	10^{-8}	lunar acceleration
ζ_4	—	—	not independent (see Eq. (71) of [33])

Table 3.1: Table of the current limits on the PPN parameters. Reproduced and adjusted from [33].

3.2.3 Strong field regimes

Existence of black hole solutions

Another horizon for modified gravity is the existence of black hole solutions. Black holes, as very fundamental systems in astrophysics (whose photosphere we can now see [3]), *should* be described by any theory of gravity (or at least equivalent compact objects). On top of this, the black hole metrics are cornerstones of many analysis currently done in astrophysics, cosmology, and other fundamental inquiries. It is therefore natural that one of the first steps for alternative theories of gravity is to try and obtain black hole solutions. However, with the increased complexity of these theories, black hole solutions, have in general proved more difficult to obtain.

One direction of research involves looking first for Schwarzschild-like solutions, for example in the Schwarzschild coordinates

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2\,, \tag{3.95}$$

with $f(r) \equiv 1 - \frac{2G_N M}{r}$ with M the black hole mass[16]. One may also look for new black-hole solutions, as we will discuss. It would be out of place to offer a complete review on the efforts to recover black holes in different theories of gravity. We nevertheless give a short account specifically on scalar-tensor and massive gravity theories.

Black holes in scalar-tensor theories

Black hole in theories with extra fields immediately lead to the question of the celebrated no-hair theorem, which restricts the quantities characterizing a black hole to its mass, angular momentum, and electric charge (see e.g. [252]). If a no-hair result exists, this means that the field should have a trivial configuration around a black hole. In several early models of scalar-tensor theories, new no-hair theorems have been shown to exist, for example in $f(R)$ and related theories [253], or in Galileon theories [254]. However, a number of assumptions (in particular staticity of the extra fields) may be broken and it was shown that a range of hairy solutions can be found. Time dependent solutions may also be found. Some the modifications one can make are for example the non-minimal coupling of the fields, such as in [255, 256, 257], or within higher-order scalar-tensor theories in special cases [258, 259, 260], through non-trivial asymptotics as in e.g. [261, 262], or time-dependent scalar fields, e.g. in [259, 260, 263]. More references can be found in the reviews and classifications [264, 265, 18, 266].

[16]One defines the Schwarzschild radius as $r_{\rm s} \equiv 2GM$.

$f_{\mu\nu} \propto g_{\mu\nu}$	$f_{\mu\nu} \not\propto g_{\mu\nu}$	
	both diagonal	~~both diagonal~~
$\times$	Singularity at the horizon [268]	Infinite strong coupling. [61]

Table 3.2: Static and spherically symmetric, Schwarzschild-(anti-)de Sitter-like black holes solutions in Lorentz-invariant massive gravity. The mention "both diagonal" indicated that the metrics are both diagonal simultaneously, while "~~both diagonal~~" indicates the complementary case. There are trivially no solutions where the metrics are proportional to each other. The categorization was adapted from [61].

Black holes in massive gravity

Finding black holes in massive gravity theory, in particular dRGT theory, has been a recent challenge (for reviews see [267, 61], also for bimetric theories) The known astrophysically relevant black holes in massive gravity have no hair [267]. Under the assumption of sphericity and staticity, a few classes of solutions can be detailed, and all have a pathology: either a physical singularity or a strong coupling [61]. We present in table 3.2 an account for these spherical static solutions. The classification depends on the form and relation between physical and fiducial metric, which allows to systematize the analysis. First, one segregates between solutions in which the metrics are proportional to each other for simplicity we note here as the case i) as opposed to solutions ii) in which the metrics are not proportional. In the case ii) one may further differentiate ii.a) the bidiagonal case, and ii.b) its complement. To summarize, every different case i), ii.a), or ii.b), a static Schwarzschild-like solution either does not exist or has a given unhealthy property. This has led to believe that a different route needs to be taken to explore black holes in dRGT. The recent works by Rosen [269, 270] give a strong push to the idea [271] that black holes in dRGT can exist if a time dependence is introduced. A full solution (not only a near-horizon expansion) is yet to be shown. Numerical solutions are also one of the future hopes for the field. Finally, note that with Lorentz-violating massive gravity, in particular in the minimal theory of massive gravity (MTMG) described in this thesis, the situation is different, and Schwarzschild solutions can be found [52] (see also [272] for a study of black holes in other Lorentz violating massive gravity theories).

Relativistic stars

Of course, black holes are not the only strong gravity systems that get modified. Self-gravitating matter systems, in this case neutron stars in particular, are also impacted by modifications of gravity. Here again we will not offer a thorough review but simple point to certain references, especially in the case the theory relies on Vainshtein mechanism. As for other systems, the question of relativistic stars is largely dependent on which modification of gravity one considers. The case of higher-order scalar-tensor theories beyond Horndeski has been considered in [273, 274], then revisited in [275, 276] (see also references therein). The equations of state in the latter works are yet to be extended to the more realistic polytropic case. As for massive gravity, the solutions were explored in [277, 278][17]

Probing the strong-field regimes

The hope is that all these modifications may be observed through, for example, gravitational wave signals. The advent of gravitational wave astronomy is indeed very exciting as a future window into strong-field dynamics. For reviews on these prospects see [280, 281]. To use most efficiently the future observations one will need to classify and systematize the different solutions and/or theories. See for example [282] for an effective field theory take on the quasinormal modes. See also [283] for a classification of modified gravity theories with respect to their black hole solutions. Finally, obtaining gravitational wave templates in theories of modified gravity is an important future challenge. At present, the post-Newtonian perturbative analysis of compact object inspirals has studied in a subclass of scalar-tensor theories (see e.g. [284, 285, 286]). However, the studies are yet to be extended to higher orders, and full templates including strong-field phases of the coalescence of binary systems are still missing.

[17] See also [279] for bigravity.

3.2.4 Constraints from the propagation of GW

Gravitational wave astronomy is not only interesting as a probe of the strong-field regime, but has been able to put constraints on models of modified gravity from the mere fact that the gravitational waves have traveled long distances ($\mathcal{O}(10^1) \sim \mathcal{O}(10^3)$ Mpc).

The most notable constraint on modifications of gravity was given by a binary neutron star merger detected as the gravitational wave event GW170817 [39] and the gamma ray burst GRB170817A [38] (and references therein). This sets several constraints on alternative theories of gravity via the difference in timing of arrival of both signals (which was of a few seconds). The current constraint is $|1 - c_T/c_{\rm EM}| = \mathcal{O}(1) \times 10^{-15}$, at around 100 Hz. This bound is a strong motivation to consider only theories with $c_T = 1$. This assumption would constrain several scalar-tensor theories as explanations for the cosmic acceleration, in particular the Horndeski class (3.5) would be restricted to the subset of up to cubic theories in addition to $G_4(\phi)$ instead of $G_4(\phi, X)$ [287, 288, 289, 290, 291, 167, 168]. This, at least within phenomenologically interesting scalar-tensor theories, implies that one should expect $\mu > 1$ (defined in (3.88)), if the Vainshtein mechanism is efficient (see equations 3.37 to 3.40).

Another possibility lies into reaching dynamically $c_T = 1$ at present time [292, 293]. Scrutiny should however be put on this bound, due to the frequency window close to the cutoff of the theories [40]. In parallel, it was pointed out that $c_T = 1$ should be preferred due to the possible decay of gravitational waves into dark energy [41].

Another concurrent bound from gravitational waves is on the mass of the graviton. From event GW170104 [294], the graviton mass μ_T has been bounded to $|\mu_T| < 7.7 \times 10^{-23}$ eV. This is only marginally weaker than the bound from solar-system observations $|\mu_T| < \mathcal{O}(1) \times 10^{-24}$ eV [295]. Future space-based interferometers such as LISA [27] will provide an increasingly better bound, of order $|\mu_T| < \mathcal{O}(1) \times 10^{-26}$ eV [296]. Several other model dependent constraints have and will be investigated (for a review see [297], and for an interesting proposal [298]).

Finally note that gravitational waves will test several other aspects of gravity. To cite one, LISA-type interferometers will be sensitive to the different polarizations of gravitational waves [299], whereas this is only partially the case current detectors.

Chapter 4

Minimally modified gravity

4.1 Philosophy of minimal theories

As was exposed in part 2.1.5, it is clear that to build novel theories one must relax some of the assumptions of Lovelock's theorem. A common conception is that breaking any of these assumptions is generally equivalent to adding degrees of freedom. Several examples speak for this line of thought: dRGT massive gravity propagates 5 degrees of freedom due to breaking general covariance, brane scenarios have naturally emergent scalar fields (e.g. [300, 15]), higher-dimensional compactifications [301], higher derivative theories [17], spontaneously-broken continuous symmetries [97, 98], ... all of these generically involve new modes in their effective low-energy theory. From this perspective alone, it is no wonder that scalar-tensor theories are seen as an archetypal modifications of gravity.

Yet as we will show explicitly in this chapter, breaking Lorentz invariance (LI) allows for theories which do not necessarily propagate new gravitational fields. Intuitively relaxing LI allows to treat temporal and spatial components and derivatives independently. Hence it is possible to choose appropriate terms, or to set constraints, that will render the extra modes non-dynamical. In fact, this is similar to what has been done in dRGT theory to remove the BD ghost; however, in that case, by restricting firmly the structure of the potential term, Lorentz invariance still constrains the massive graviton to propagate five degrees of freedom as a massive spin-2 field should. Therefore in dRGT it was not possible to remove more than the BD ghost except on given strongly coupled backgrounds. On the contrary, in the context of Lorentz-violating massive gravity, relaxing LI allows to consider subgroups of unbroken diffeomorphisms [45, 46], and hence potentially less degrees of freedom [181]. As we will show, in addition to considering residual symmetries, it is also possible to consider constrained Lagrangians. In the Hamiltonian formalism this corresponds to second-class constraints. It should be noted that we will only consider cases in which $SO(3)$ invariance is retained. In addition to allowing for a simple implementation of cosmology, this also allows to think easily in terms of the scalar-vector-tensor (SVT) decomposition.

Since relaxing LI allows for more freedom in constructing theories, it is interesting to ponder whether there are practical guiding principles that could orient the model building efforts. Here, we discuss two lines of thought, one centered on *consistency* and another based on the idea of *simplicity*. First, consistency comes as an essential criterion in physics. In our case, we would for example demand the (UV-)*stability* over "reasonable" backgrounds, as well as the possibility to sustain a "realistic" cosmology, *cosmology-completeness*. These two criteria are completely heuristic, and should be eventually replaced by confrontation against empiric observations and in-depth studies of stability. Nevertheless, they are useful since they allow to select subclasses of theories (e.g. $SO(3)$-invariant), which have a good chance to offer a good description of our world.

The second criterion, simplicity, can be seen as a continuation of Occam's ideas in the context of field theories. In the case of Lorentz invariant theory, one may easily conflate number of parameters with the number of fundamental fields, and hence the number of degrees of freedom. However, when symmetries are not so clear cut, theories with the same number of fields may propagate a different number of degrees of freedom, as is the case in massive gravity, especially classically. We propose to choose as a working principle to consider theories with the least number of degrees of freedom. Following on the naming of the *minimal theory of massive gravity* (MTMG) [51] and [56], this criterion can be named *minimalism*[1]. Interestingly, one can expect that this specific choice for the second criterion, minimalism, can in fact impact positively on the study of consistency, since less degrees of freedom may indicate less numerous

[1] This idea was first proposed in [302].

and therefore less stringent stability conditions. It will be also generally easier to satisfy observational constraints and to relieve the necessity of screening mechanisms.

In the next sections, we will therefore study some classes of modified gravity which fall under this idea of minimalism. This research program can be traced back to [51, 56]. In the next section, a first review section, we will summarize the analysis of [56]. The example introduced allows then to present with more perspective the work realized during this thesis [50] in sections 4.3 and 4.4. Finally, we close the chapter with another review section, presenting the model introduced in [51].

4.2 Unbroken symmetries

As a first simple example of MMGs, one may consider theories in which although Lorentz is violated, the full group of diffeomorphisms is only broken up to a given subgroup. Since we are interested in retaining $SO(3)$ invariance, it makes special sense to conserve spatial diffeomorphisms, i.e.

$$x^i \to x^i + \xi^i(t, x^j)\,. \tag{4.1}$$

as the subgroup of diffeomorphisms. This approach was considered in [56], in which general conditions for which a theory propagates only two degrees of freedom were derived. Another approach based directly on the Hamiltonian has been pursued in [57]. The class of theories considered is linear in the lapse function, and was written

$$S = \int dtd^3x\sqrt{\gamma}NF\left(K_{ij}, R_{ij}, \mathcal{D}_i, \gamma^{ij}, t\right)\,, \tag{4.2}$$

with all spatial indices contracted, and where all the quantities can be found defined in section 2.1.8. Note that the action does not depend on the shift, other than in K_{ij}, as this could lead to a violation of (4.1). Mixed derivatives are also not taken into account and would need a separate treatment. By considering theories that are linear in the lapse function, one may expect to obtain a degeneracy such that the new dynamical mode of the metric in fact is not dynamical at all.

In fact, in the Hamiltonian formalism, it is easy to see that there exists a primary constraint associated with the linearity in the lapse which is noted $\mathcal{C}$. Due to the residual symmetry (4.1), it is also possible to find three first-class constraints noted $\tilde{\mathcal{R}}_i$. It is however non-trivial to ensure that either the constraint $\mathcal{R}_0$ can be linearly combined with other second class constraints[2] to find a first-class constraint, or that there exists a tertiary second-class constraint due to consistency. Indeed, recall that one may start with twelve phase space degrees of freedom from the three-metric γ_{ij} and that the constraints $\tilde{\mathcal{R}}_i$ remove six of them. To reach four phase space degrees of freedom either there exists a first-class constraint, or two independent second-class constraints including $\mathcal{R}_0$.

The authors of [56] find that the sufficient and necessary condition is

$$\int d^3x\sqrt{\gamma}\left\{\mathcal{D}^j\left(\frac{1}{\sqrt{\gamma}}\frac{\delta\left\langle\sqrt{\gamma}F\alpha\right\rangle}{\delta R_{kl}(x)}\right)\mathcal{D}^i\left[\left(Q_{jl}\gamma_{ik} - \frac{1}{2}Q_{kl}\gamma_{ij} - \frac{1}{2}Q\gamma_{ik}\gamma_{jl}\right)\beta\right] - (\alpha \leftrightarrow \beta)\right\} \approx 0 \quad \forall\alpha\,,\,\beta\,, \tag{4.3}$$

where Q_{ij} is an auxiliary field introduced to replace K_{ij} in the Lagrangian; in fact, this condition is in general sufficient also in the more general class

$$S = \int dtd^3x\sqrt{\gamma}N\left[F\left(K_{ij}, R_{ij}, \mathcal{D}_i, \gamma^{ij}, t\right) + G\left(R_{ij}, \mathcal{D}_i, \gamma^{ij}, t\right)\right]\,. \tag{4.4}$$

This general construction has allowed to build new MMGs, for example[3] *square-root gravity* (SQGR), defined by

$$F = \sqrt{A(t)\left(K^{ij}K_{ij} - K^2\right) + B(t)}\sqrt{C(t)R + D(t)} + \Lambda(t)\,, \tag{4.5}$$

which has four first-class constraints. SQGR has been shown to be perturbatively equivalent to GR in vacuum [303] (see also [304]), at least up to the five-point functions. Using the coupling to matter, however, one may obtain theories which are not GR. One example of coupling is given in [305], which preserves the structure with four first-class constraints, which however do not correspond to four-dimensional

[2] For simplicity, we have not alluded here to the use of auxiliary fields to take care of the terms with overall more than two time-derivatives, due to F being a free function of K_{ij}. This leads to an extended phase space, with other second-class constraints than only $\mathcal{R}_0$.

[3] Another example is exponential gravity, see [56].

diffeomorphisms. Another possibility of matter coupling was proposed in [306], which relies on a gauge-fixing, thus reducing the first-class constraint related to the lapse into two second class constraints, thus allowing for the usual minimal coupling to matter.

Finally note that, as we will see later, spatial diffeomorphisms are not necessary to build MMGs (notably [51]).

4.3 Type-I and type-II MMGs

As we have seen, one may define a MMG theory as a theory that only carries the same gravitational degrees of freedom as general relativity. Considering the existence of large classes of MMGs, it is especially interesting to proceed to a categorization of the theories. It is interesting to further segregate *type-I gravity theories* from *type-II gravity theories*, on the basis of the existence of an Einstein frame [50]. *Type-I theories* have an Einstein frame, and therefore can be written, via field redefinitions, as the Einstein-Hilbert Lagrangian, plus a Lagrangian involving the kinetic terms of other fields, and plus a matter Lagrangian (in general non-minimally coupled). This is a rather common type of theories, e.g. many scalar-tensor theories (say, Brans-Dicke) fall into this category. On the other hand, two examples of *type-II gravity theories* are Hořava-Lifshitz gravity and MTMG.

This categorization is a concrete aid to the discussion of theories of gravity which can often be linked through change of variables, and as we will see it can also be an aid to the construction of MMGs. In Figure 4.1, we present a scheme of the theory space of MMG theories, organized on the basis of the constraint algebra of their Hamiltonian formulation. The low population of the map may be a hint that there are yet several theories or fundamenta to understand.

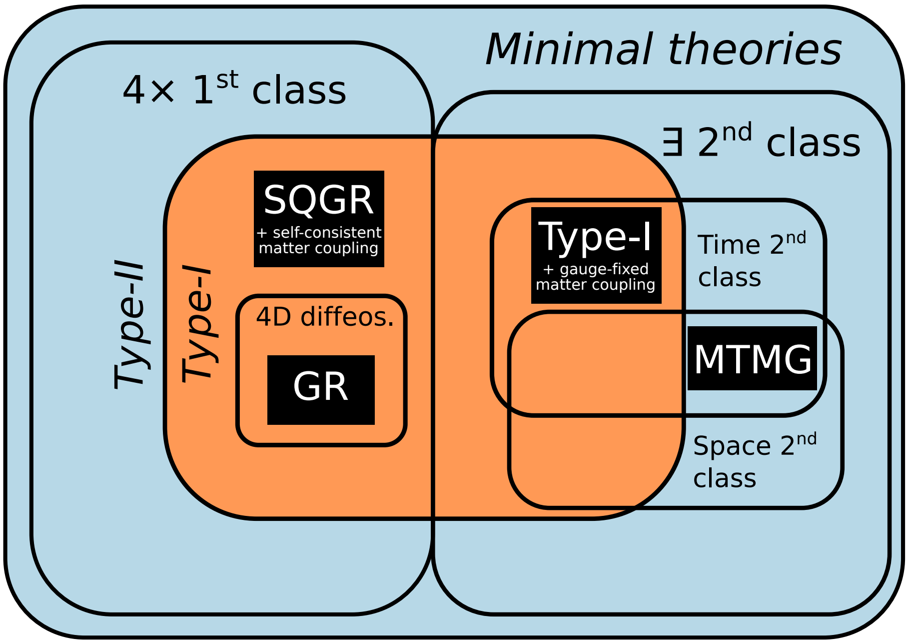

Figure 4.1: Scheme of the space of minimally modified theories of gravity (MMGs), here noted simply *minimal theories*. Type-I (in orange) and type-II (in blue) theories are disjoint sets. We further separate the case in which there are four first-class constraint, and the case in which second-class (in particular when time and/or space diffeomorphisms are downgraded to second-class) constraints exist. SQGR denotes the square root gravity of [56] with the self-consistent matter coupling [305]. Type-I theories with a gauge-fixed matter coupling include the theories constructed via canonical transformation of [306, 50], but do include theories such as SQGR when a gauge-fixed matter coupling is used. MTMG is the model of massive gravity [51].

In the following chapter we move on to consider MMG theories through two examples: a construction of a class of *type-I MMG theories* based on [306, 50], and the presentation of MTMG [51] as a *type-II MMG theory*. In particular, we will show that the phenomenology of MMGs can be of interest for late-time cosmology.

4.4 Type-I theories from canonical transformation

In this part we review our construction of a non-trivial type-I minimally modified gravity theory. The idea behind the construction is to use the Hamiltonian language, in which it is easy to understand the constraint structure, i.e. the structure of the diffeomorphism algebra, and in which we can use the tool of canonical transformations. This idea was first explored in [306], where however it was recognized that more generic canonical transformations would be needed to obtain non-GR phenomenology.

4.4.1 Construction

The idea is to start from the total GR gravitational Hamiltonian (the matter fields will be treated afterwards), i.e.

$$\mathfrak{H}_{\rm tot} = \int d^3x(\mathcal{N}\mathcal{H}_0[\Gamma,\Pi] + \mathcal{N}^i\mathcal{H}_i[\Gamma,\Pi] + \lambda\Pi_N + \lambda^i\Pi_i)\,, \tag{4.6}$$

with

$$\begin{aligned}\mathcal{H}_0 &:= \frac{2}{M^2\sqrt{\Gamma}}\left(\Gamma_{ik}\Gamma_{jl} - \frac{1}{2}\Gamma_{ij}\Gamma_{kl}\right)\Pi^{ij}\Pi^{kl} - \frac{M^2\sqrt{\Gamma}}{2}R[\Gamma]\,,\\ \mathcal{H}_i &:= -2\sqrt{\Gamma}\Gamma_{ij}D_k\left(\frac{\Pi^{jk}}{\sqrt{\Gamma}}\right)\,,\end{aligned}$$

where we rename for later convenience, compared with chapter 2, the lapse into $\mathcal{N}$, the shift into $\mathcal{N}^i$, and the spatial metric into Γ_{ij}, and D_i is the covariant derivative compatible with Γ_{ij}. Proceeding just as in (2.20) allows us to define the conjugated momenta, respectively $\Pi_{\mathcal{N}}$, Π_i, and Π_{ij}. M is the UV mass scale of this purely gravitational theory. Note that the lapse and shift are considered as variables as well.

From any Hamiltonian it is possible to define an equivalent Hamiltonian by a canonical transformation of the variables. A general canonical transformation of the second type (see for example [307]) can be defined through a generating functional

$$\mathcal{F} = -\int d^3x F(P^A, q_A, t)\,, \tag{4.7}$$

denoting collectively the canonical momenta and variables as P^A and Q_A, and where we have introduced the new set of canonical variables q_A, target of the canonical transformation. One may relate the two sets of variables and momenta by

$$\begin{aligned}Q^A &= -\frac{\delta\mathcal{F}}{\delta P_A}\,,\\ p_A &= -\frac{\delta\mathcal{F}}{\delta q^A}\,,\end{aligned}$$

where p_A are the momenta conjugated to the q^A. Finally the new Hamiltonian is written as

$$\bar{\mathfrak{H}}_{\rm tot} = \mathfrak{H}_{\rm tot} + \frac{\partial\mathcal{F}}{\partial t}\,.$$

The procedure is completed as it can be written in terms of the new variables

It is possible to apply this procedure to the gravitational theory in (4.6) just by setting $Q^A = (\mathcal{N}, \mathcal{N}^i, \Gamma_{ij})$. Although the construction should work in generality, here we restrict ourselves to the case

$$\mathcal{F} = -\int d^3x F(P^A, q_A, t)\,, \tag{4.8}$$

$$F = -\int d^3x(M^2\sqrt{\gamma}f(\tilde{\Pi}, \tilde{\mathcal{H}}) + N^i\Pi_i)\,, \tag{4.9}$$

where

$$\tilde{\Pi} = \frac{1}{M^2\sqrt{\gamma}}\Pi^{ij}\gamma_{ij}\,, \qquad \tilde{\mathcal{H}} = \frac{1}{M^2\sqrt{\gamma}}\Pi_{\mathcal{N}}N\,, \tag{4.10}$$

and f is an arbitrary function of $\tilde{\Pi}$ and $\tilde{\mathcal{H}}$. Note that this generating functional does not depend explicitly on time and so the Hamiltonian is formally left the same. Then, we obtain

$$\Gamma_{ij} = -\frac{\delta F}{\delta \Pi^{ij}} = f_{\tilde{\Pi}} \gamma_{ij} \,, \tag{4.11}$$

$$\mathcal{N} = -\frac{\delta F}{\delta \Pi_{\mathcal{N}}} = f_{\tilde{\mathcal{H}}} N \,, \tag{4.12}$$

$$\pi^{ij} = -\frac{\delta F}{\delta \gamma_{ij}} = f_{\tilde{\Pi}} \Pi^{ij} + \frac{M^2}{2}\sqrt{\gamma}\gamma^{ij}\left(f - f_{\tilde{\Pi}}\tilde{\Pi} - f_{\tilde{\mathcal{H}}}\tilde{\mathcal{H}}\right) \,, \tag{4.13}$$

$$\pi_N = -\frac{\delta F}{\delta N} = f_{\tilde{\mathcal{H}}} \Pi_{\mathcal{N}} \,, \tag{4.14}$$

For convenience (since we want to keep f general we prefer not to invert it) we define the auxiliary variables $\phi \approx \tilde{\Pi}$ and $\psi \approx \tilde{\mathcal{H}}$ by using the constraints

$$\mathcal{C} := \pi^{ij}\gamma_{ij} - \frac{M^2}{2}\sqrt{\gamma}(3f - f_\phi \phi) + \frac{3M^2}{2}\sqrt{\gamma} f_\psi \psi \approx 0 \,, \tag{4.15}$$

$$\mathcal{B} := \pi_N N - M^2 \sqrt{\gamma} f_\psi \psi \approx 0 \,. \tag{4.16}$$

The Hamiltonian after the canonical transformation becomes therefore

$$\mathfrak{H}_{\rm tot} = \int d^3x (N f_\psi \mathcal{H}_0 + N^i \mathcal{H}_i + \lambda \pi_N + \lambda^i \pi_i + \lambda_C \mathcal{C} + \lambda_D \mathcal{B} + \lambda_\phi \pi_\phi + \lambda_\psi \pi_\psi) \,, \tag{4.17}$$

with

$$\begin{aligned}
\mathcal{H}_0 = {} & \frac{2}{M^2 f_\phi^{3/2}\sqrt{\gamma}}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\left[\pi^{ij} - \frac{M^2}{2}\sqrt{\gamma}(f - f_\phi\phi - f_\psi\psi)\gamma^{ij}\right]\left[\pi^{kl} - \frac{M^2}{2}\sqrt{\gamma}(f - f_\phi\phi - f_\psi\psi)\gamma^{kl}\right] \\
& - \frac{M^2 f_\phi^{1/2}\sqrt{\gamma}}{2}\left[R[\gamma] - 2\mathcal{D}^2 \ln f_\phi - \frac{1}{2}\mathcal{D}_i \ln f_\phi \mathcal{D}^i \ln f_\phi\right] \,,
\end{aligned} \tag{4.18}$$

$$\mathcal{H}_i = -2\sqrt{\gamma}\gamma_{ij}\mathcal{D}_k\left(\frac{\pi^{jk}}{\sqrt{\gamma}}\right) + \mathcal{C}\partial_i \ln f_\phi + \mathcal{B}\partial_i \ln f_\psi - \pi_N N \partial_i \ln f_\psi \,, \tag{4.19}$$

and where $\mathcal{D}_i$ is the covariant derivative with respect to γ_{ij}.

Expressions (4.6) and (4.17) are strictly equivalent. However, it is still possible to define different gravity theories using them, in a straightforward way: by choosing a matter coupling. Although many properties of a theory of gravity is defined by the gravitational Lagrangian, even in GR, the matter coupling is still one of the foundations.

We chose to set the canonically transformed theory (4.17) as the Jordan frame of our theory, hence defining the original theory as the Einstein frame. Setting a theory as being the Jordan frame one translates into choosing a minimal coupling, i.e. after Legendre transformation,

$$\begin{aligned}
N\sqrt{\gamma}\mathcal{L} = M^2 f_1 N \sqrt{\gamma}\Bigg[& \frac{f_0'^{3/2}}{2}\left(\frac{1}{f_1^2}K^{ij}K_{ij} - \frac{1}{3f_1^2}K^2 + \frac{1}{f_0'}R(\gamma)\right) + \frac{K}{f_1}\left(f_0 - \frac{1}{3}f_0'\phi\right) \\
& - \frac{f_0''}{f_0'^{1/2}}\mathcal{D}^2\phi - \left(\frac{\mathcal{D}_i\phi}{f_0'^{3/4}}\right)^2\left(f_0' f_0''' - \frac{3}{4}f_0''^2\right) + \frac{1}{3}f_0'^{1/2}\phi^2\Bigg] + \sqrt{\gamma}\,\mathcal{L}_{\rm gf} + \sqrt{\gamma}\,\mathcal{L}_{\rm mat} \,,
\end{aligned} \tag{4.20}$$

where we have introduced K_{ij} as the extrinsic curvature for γ_{ij}, $K_{ij} := (\dot{\gamma}_{ij} - 2\mathcal{D}_{(i}N_{j)})/2N$. On the constraint surface, $\psi = 0$, allowing us to expand f as

$$f(\phi, \psi) = f_0(\phi) + f_1(\phi)\psi + \mathcal{O}(\psi^2) \,.$$

Notice that we have added a new piece yet to define, $\mathcal{L}_{\rm gf}$. This can be best understood by reflecting about diffeomorphisms.

In the Hamiltonian language, the introduction of the minimally coupled matter downgrades the Hamiltonian constraint, associated to time diffeomorphisms, from a first class into a second class constraint. This in principle is fatal to the theory, since as a result, a badly defined half degree of freedom would emerge. To prevent this, one can fix a gauge by the introduction of a gauge fixing term, which has the effect of splitting the Hamiltonian constraint into two second-class constraints, a primary and a secondary one. Now the primary would-be Hamiltonian constraint cannot be downgraded anymore.

Finally, one may chose a gauge, in our case,

$$\mathcal{L}_{\rm gf} = -\lambda^i_{\rm gf}\partial_i\phi \,. \tag{4.21}$$

this gauge allows to ultimately rewrite the Lagrangian of the new theory as

$$\mathcal{L} = M^2 f_1\left[\frac{f_0'^{3/2}}{2}\left(\frac{1}{f_1^2}K^{ij}K_{ij} - \frac{1}{3f_1^2}K^2 + \frac{1}{f_0'}R(\gamma)\right) + \frac{K}{f_1}\left(f_0 - \frac{1}{3}f_0'\phi\right) + \frac{1}{3}f_0'^{1/2}\phi^2\right] - \lambda^i_{\rm gf}\partial_i\phi + \mathcal{L}_{\rm mat}\,. \tag{4.22}$$

4.4.2 Cosmology

We first consider the background dynamics[4] with a flat FLRW ansatz (2.34), which yields the equations

$$\frac{M^2}{3}\phi^2 f_1\sqrt{f_0'} = \rho\,, \tag{4.23}$$

$$6H\sqrt{f_0'}\left(2f_0' - f_0''\phi\right) = -\phi\left(4f_0'f_1 + f_0''f_1\phi + 2f_0'f_1'\phi\right)\,, \tag{4.24}$$

$$\dot\rho + 3H(\rho+P) = 0\,, \tag{4.25}$$

where H is the Hubble expansion rate, and ρ and P derive from the total energy momentum tensor in the Jordan frame, including the cosmological constant and the contribution from matter fields, i.e.

$$\rho = M^2\Lambda + \rho_{\rm mat}\,, \qquad P = -M^2\Lambda + P_{\rm mat}\,. \tag{4.26}$$

For clarity, we define

$$f_{0m} \equiv 2f_0' - f_0''\phi\,, \tag{4.27}$$

$$d_f \equiv 4f_0'f_1 + f_0''f_1\phi + 2f_0'f_1'\phi\,; \tag{4.28}$$

this allows to combine (4.23) and (4.24) into an effective Friedmann equation

$$3M^2H^2 = \rho + \frac{M^2\phi^2}{12}\left(-4f_1\sqrt{f_0'} + \frac{1}{f_0'}\frac{d_f^2}{f_{0m}^2}\right)\,. \tag{4.29}$$

The gravitation constant for the background can be identified as $G_{\rm cosm} \equiv 1/(8\pi M^2)$. As we will see later on, this value does in general not coincide with the effective gravitational constant that drives the linear dynamics of the dark matter fluid fluctuations, $G_{\rm eff}$. Note also that modifications of gravity contribute to the Friedmann equation (4.29), and the form of the modification is controlled by functions f_1 and f_0, as well as by the density of matter fields through (4.23). One may understand this as a contribution to dark energy.

Moving on to the study of perturbations, it is first of all clear that tensor modes propagate with a modified speed, which we can relate to the speed of light (in this expression noted $c_{\rm EM}$) as

$$c_T^2 = c_{\rm EM}^2\frac{f_1^2}{f_0'}\,. \tag{4.30}$$

As we will discuss further, since modifications to the propagation of gravitational waves have been constrained, this will motivate us to look for a viable subclass of models. Once matter is added $f_0'^{3/2}/f_1$ becomes the no-ghost condition for the tensor modes. This is however already satisfied, since both free functions were already chosen as positive.

Since vector modes are trivially absent, we now focus on scalar modes. After integrating out the non-dynamical variables defined as in section 2.2.3, we obtain the quadratic order action for the density perturbations

$$N\sqrt{\gamma}\mathcal{L}_\delta = \mathcal{A}\,\dot\delta^2 - \mathcal{B}\,\delta^2\,, \tag{4.31}$$

From this Lagrangian, one may write the equation of motion of the density perturbations,

$$\ddot\delta + 2CH\dot\delta - 4\pi G_\delta\bar\rho\delta = 0\,, \tag{4.32}$$

[4] As already mentioned, we assume the gauge condition imposed by (4.21).

and we can express the Bardeen potentials as

$$-\frac{k^2}{a^2}\Psi = 4\pi G_\Psi \rho\delta + 4\pi \mathcal{G}_\Psi \rho\dot{\delta}\,, \tag{4.33}$$

$$-\frac{k^2}{a^2}\Phi = 4\pi G_\Phi \rho\delta + 4\pi \mathcal{G}_\Phi \rho\dot{\delta}\,. \tag{4.34}$$

where C, G_δ, G_Ψ, G_Φ, $\mathcal{G}_\Psi$, and $\mathcal{G}_\Phi$ are (gauge- and) scale-dependent and are given by

$$G_\delta = G_\Psi = G_\Phi \frac{\mathcal{K}_1}{\mathcal{K}_2} = G_{\rm cosm}\frac{1}{f_1\sqrt{f_0'}}\frac{\mathcal{K}_1}{\mathcal{K}_2^2}\,, \tag{4.35}$$

$$C = \frac{\mathcal{K}_3}{\mathcal{K}_1}\,, \tag{4.36}$$

$$\mathcal{G}_\Psi = G_{\rm cosm}\frac{a^2}{k^2}\frac{2c_T^2\phi(f_0'f_{1p} - 2f_0''f_1\phi)}{f_1 f_0' f_{0m}}\frac{1}{\mathcal{K}_1}\,, \quad \mathcal{G}_\Phi = G_{\rm cosm}\frac{a^2}{k^2}\frac{c_T^2 d_f\phi^2}{2f_1 f_0' f_{0m}}\frac{1}{\mathcal{K}_1}\,; \tag{4.37}$$

the functions $\mathcal{K}_i$ are defined by

$$\mathcal{K}_1 := 1 + \frac{a^2}{k^2}\frac{c_T^2\phi^2}{2}\,, \tag{4.38}$$

$$\mathcal{K}_2 := 1 + \frac{a^2}{k^2}\frac{\phi^2}{2}\,, \tag{4.39}$$

$$\mathcal{K}_3 := 1 + \frac{a^2}{k^2}\frac{3c_T^2\phi^2(f_{0m}f_1 + 2f_0'f_1'\phi)}{2d_f}\,, \tag{4.40}$$

with

$$f_{1p} := f_1 + 2f_1'\phi\,. \tag{4.41}$$

The decomposition we have just presented allows to straightforwardly take the subhorizon limit as in (2.79), since in that limit $\mathcal{K}_i \to 1$ and $\mathcal{G}_i \to 0$, and hence we obtain

$$G_\delta \to \frac{G_{\rm cosm}}{f_1\sqrt{f_0'}} \equiv G_{\rm eff}\,,\ C \to 1\,,\ \eta \to 1\,, \tag{4.42}$$

where we have included the slip parameter $\eta := \Psi/\Phi$. As a result, the only change caused by the canonical transformation for the evolution of the density perturbations in the sub-horizon limit is the change of the gravitational constant. The gravitational constant at short scales $G_{\rm eff}$ is generally time dependent since f_1 and f_0' are functions of ϕ. We may also compute the scalar no-ghost condition

$$Q = \lim_{k\to\infty}\frac{\mathcal{A}}{a^3} = \frac{1}{2}\frac{a^2}{k^2}\frac{\rho^2}{\rho + P}\,, \tag{4.43}$$

Therefore, there is no ghost instability as long as $\rho + P > 0$. This result is equivalent to the standard case in GR (the null energy condition). Along the same lines we find that matter waves, in the high-k limit, propagate with the squared-speed

$$c_s^2 = \lim_{k\to\infty}\frac{a^2}{k^2}\frac{\mathcal{B}}{\mathcal{A}} = c_{\rm mat}^2\,. \tag{4.44}$$

with $c_{\rm mat}$ defined as for example (A.20). This result extends to multiple matter fields, since all matter fields share the same minimal coupling in the Jordan frame.

4.4.3 Considering constraints

In this section, we take the results we have obtained in the previous section, and discuss them taking into account observations that can constrain the theory. We have considered two main constraints on:

1. the speed of propagation of gravitational waves,
2. the time-variation of the gravitational constant.

Starting with the speed of propagation, the recent multi-messenger observation of a binary neutron star merger has put a constraint ($\lesssim 10^{-15}$) on the relative difference between speed of light and speed of gravitational waves (see section 3.2.4). It is thus worthwhile to restrict ourselves to the case $c_T^2 = 1$ for all times, i.e. $f_1^2 = f_0'$, for this discussion. From the action (4.22) we see that the same will also hold on different backgrounds. Note however that the constraint was verified only at late-times, and it would therefore be technically possible to have $f_1^2 \neq f_0'$ at early times.

On choosing $c_T = 1$, we may combine the equations (4.23)-(4.25) to find

$$\dot{\phi} + \frac{\rho + P}{4\rho}(3H\phi - \phi^2) = 0\,. \tag{4.45}$$

The Friedmann equation (4.29) is then

$$3M^2H^2 = \rho_{\rm mat} + \rho_{\rm DE}\,, \tag{4.46}$$

where

$$\rho_{\rm DE} := M^2\Lambda + \frac{M^2\phi^2}{3}\left[\frac{(2f_0' + \phi f_0'')^2}{(2f_0' - \phi f_0'')^2} - f_0'\right], \tag{4.47}$$

is the effective energy density of the dark energy. The effective pressure of the dark energy is found using equation (4.45) as

$$P_{\rm DE} := -P_{\rm mat} - 2M^2\dot{H} - \rho_{\rm DE} - \rho_{\rm m}\,. \tag{4.48}$$

In the case in which matter is a cold dust fluid, i.e. $P_{\rm mat} = 0$ and $c_{\rm mat} = 0$, the equation of state parameter of dark energy is

$$w_{\rm DE} := \frac{P_{\rm DE}}{\rho_{\rm DE}} \tag{4.49}$$

$$= -\frac{6\Lambda(4f_0'^2 - 5\phi^2 f_0''^2 + 4\phi f_0' f_0'' + 4\phi^2 f_0' f_0''') + \phi^3(8\phi f_0' f_0''^2 - \phi^2 f_0''^3 - 4f_0'^2 f_0'' - 8\phi f_0'^2 f_0''')}{(2f_0' - \phi f_0'')[3\Lambda(2f_0' - \phi f_0'')^2 - \phi^2\{4(f_0'-1)f_0'^2 - 4\phi f_0'(f_0'+1)f_0'' + \phi^2(f_0'-1)f_0''^2\}]}\,. \tag{4.50}$$

We need to assume $w_{\rm DE} < -1/3$, at least at low redshifts, in order to have an accelerating universe.

Considering perturbations, the sub-horizon $G_{\rm eff}$ is gauge independent and can be constrained with existing and future data (e.g. RSD data, see section 3.2.1). In our case, we have

$$8\pi M^2 G_{\rm eff} = \frac{1}{f_0'}\,. \tag{4.51}$$

whereas as mentioned $\eta := \Psi/\Phi = 1$, even away from the case $c_T^2 = 1$. Furthermore, since the gravity at short scales is dominated by $G_{\rm eff}$, we will set that $G_{\rm eff}(z = 0) = G_N$, where z is the redshift and G_N is the Newton gravitational constant.

We may now consider constraints coming from the time-variation of the gravitational constant. Indeed, one will have in general a time-dependent $G_{\rm eff}$, since ϕ depends on the density of the matter fields (including the cosmological constant). This is slightly different from the usual simple models of time-varying G_N [65], since in these models the gravitational constant on cosmological scales will tend to agree with the short scale one.

A stringent constraint comes from big bang nucleosynthesis (BBN) (see section 3.2.1). BBN is sensitive to the difference between the gravitational constant at cosmological scales at BBN, hence $G_{\rm cosm}$ and today's measured Newton constant, i.e. G_N, the difference being constrained to less than 10%. BBN therefore puts a constraint

$$|f_0'(z=0) - 1| \lesssim 0.1\,. \tag{4.52}$$

On the other hand, we conservatively require that general relativity be recovered at early times, hence

$$f_0'(z \gg 1) = 1\,. \tag{4.53}$$

Of course, a less stringent constraint could be extracted if we were to allow $c_T^2 \neq 1$ at early times. A cartoon of these constraints and possible scenarios is given in Figure 4.2.

In what follows we try to construct models that satisfy these constraints. We will first show that the accelerating expansion of the universe can be realized even in the case $\Lambda = 0$, which would however have important consequences on the future evolution of the Universe. We then present more conservative models with a non-zero Λ, and show that one may have interesting time evolutions for $G_{\rm eff}$ as well as $w_{\rm DE}$.

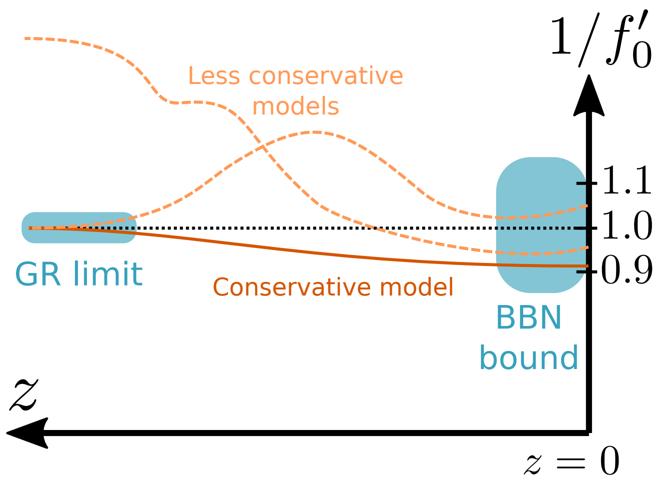

Figure 4.2: Cartoon of some constraints on the function f_0'. A conservative choice would respect the big bang nucleosynthesis (BBN) bound, in addition to having an early-time GR limit (see for example the model (4.57)). The BBN effectively translates into a bound on f_0' at present time $z = 0$.

Models without Λ

We are interested in first building a model that produces acceleration without cosmological constant. Setting $\Lambda = 0$ changes the asymptotic behavior of equation (4.23), which under $c_T^2 = 1$ is written

$$M^2 f_0' \phi^2 - 3\rho = 0\,. \tag{4.54}$$

Since ρ now vanishes at late-times, one will have either $f_0' \to 0$, or $\phi, H \to 0$, or both, from equation 4.24. The same argument in fact holds even without imposing $c_T = 1$, based on (4.23)-(4.25). A simple model that realizes GR at early times and a de Sitter expansion at late times is

$$f_0' = \frac{\phi_c^2 \phi^2 + \phi^4}{(3H_* - \phi)^4}\,, \tag{4.55}$$

where ϕ_c marks the transition epoch and H_* the Hubble rate at late times. Also, since

$$H = \frac{6H_*\phi_c^2 + 9H_*\phi^2 - \phi^3}{6\phi_c^2 + 9H_*\phi + 3\phi^2}\,, \tag{4.56}$$

one should ensure that there is no singularity of the Hubble rate at finite time in the past by choosing $8\phi_c^2 > 9H_*^2$, so that the denominator does not vanish.

As far as phenomenology is concerned, we find that the late-time $G_{\rm eff} > G_{\rm cosm}$, however, in the far future the same $G_{\rm eff} \to \infty$. At early times $w_{\rm DE} \to -1/2$.

Dark energy with $w_{\rm DE} \neq -1$ from Λ

An interesting example in the case $\Lambda \neq 0$ is

$$f_0' = \frac{(M_*/M)^2 + (\phi/\phi_c)^2}{1 + (\phi/\phi_c)^2}\,, \tag{4.57}$$

where M_* and ϕ_c are constants, which can satisfy both the BBN bound and enjoy an early-time GR limit, and hence is one of the *conservative* models depicted in figure 4.2. Again ϕ_c fixes the transition between two regimes: one has the limits $f_0' \to 1$ for $|\phi| \gg |\phi_c|$ and $f_0' \to M_*^2/M^2$ for $|\phi| \ll |\phi_c|$. This model can therefore be seen as two GR limits with different effective gravitational constants for $|\phi| \gg |\phi_c|$ and $|\phi| \ll |\phi_c|$.

Again, some scrutiny is needed: equation (4.54) gives a solution for ϕ,

$$\frac{\phi^2}{\phi_c^2} = \frac{1}{2}\left(\frac{3\rho}{M^2\phi_c^2} - \frac{M_*^2}{M^2} \pm \sqrt{\left(\frac{3\rho}{M^2\phi_c^2} - \frac{M_*^2}{M^2}\right)^2 + \frac{12\rho}{M^2\phi_c^2}}\right)\,, \tag{4.58}$$

hence, in the late-time limit $\rho \ll M^2\phi_c^2$, one finds two branches

$$\phi^2 \to 0 \ (+\,\text{branch}), \quad -M_*^2\phi_c^2/M^2 \ (-\,\text{branch})\,. \tag{4.59}$$

The minus branch is unphysical, and we therefore consider only the plus branch. It is then easy to show that this model reproduces the expected behavior, hence a GR limit at early times, and $G_{\text{eff,late}} = 1/(8\pi M_*^2)$. The BBN bound can be realized simply by choosing an appropriate M_*. In the presented model, one has still some freedom, and for example setting $\phi_c^2 \simeq \Lambda \simeq H_0^2$, will lead to the modification from GR to only appear in the present universe. We present in Fig. 4.3 the time evolutions of w_{DE} and G_{eff}/G_N for the cases $M_*^2/M^2 > 1$ and $M_*^2/M^2 < 1$.

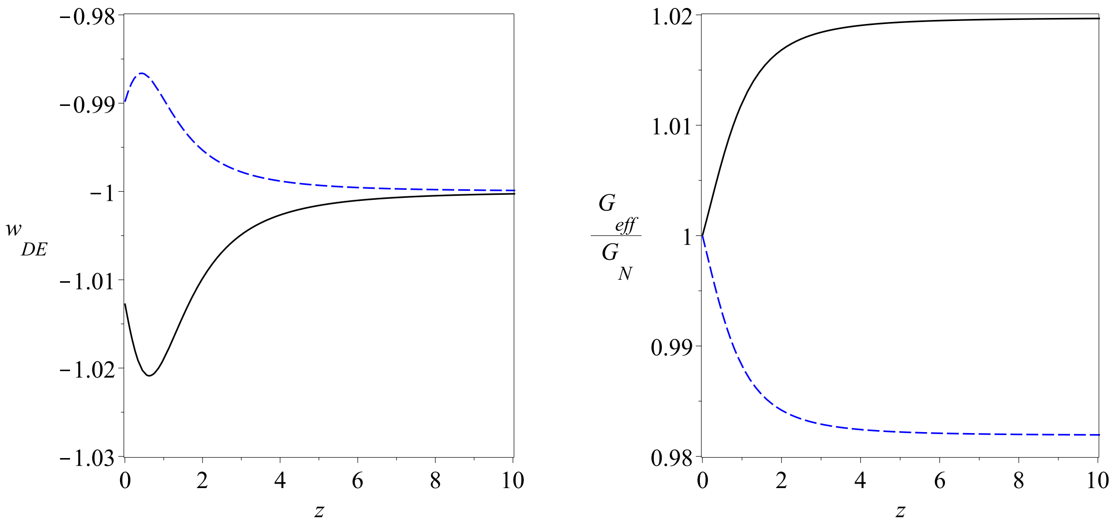

Figure 4.3: Time evolutions of the equation of state parameter and the effective gravitational constant for the model (4.57) with $M_*^2/M^2 = 1.1$ (black solid curves) and $M_*^2/M^2 = 0.9$ (blue dashed curves) and $\Lambda = \phi_c^2$. Here, we have also set $\Omega_{m0} = 0.3$. *Image reproduced from [50].*

The effective equation of state parameter w_{DE} is dynamical even though the acceleration is caused by a cosmological constant. At late times, $\lim_{\phi\to-\infty} w_{\text{DE}} = -1$. Hence, this model achieves early-times weak or strong gravity regimes for the dust-fluid fluctuations but still with $w_{\text{DE}} = -1$ at high redshifts.

Although a more detailed study would be needed, the possibility to modify the short scale gravitational constant may be interesting from the standpoint of the appearing tensions within the standard model of cosmology (see section (2.2.2)). Indeed if data sets point more and more towards the possibility of early-times modifications of gravity, for example as described in [107], then the type of models presented in this section will have a good chance to mend these tensions.

Reconstructed Dark Energy models

Finally, we present the possibility of reconstructing the $f_0(\phi)$ function from the background dynamics. The reconstruction program for dark energy was initiated several years ago, and has been reviewed for example in [133]. For a given (i.e. obtained from observations) evolution of the Hubble rate

$$H = H(a) = H_0\,E(a)\,, \tag{4.60}$$

where E is a given function of the scale factor which determines a particular dark energy dynamics, which satisfies the condition $E(a=1) = 1$, one may obtain at least numerically the function $f_0(\phi)$. One should consider the presence of radiation, matter and a cosmological constant, and hence the Friedmann equation can be written

$$3M^2H_0^2E^2 = \rho_m + \rho_r + \rho_{\text{DE}}\,, \tag{4.61}$$

where ρ_r and ρ_m are energy densities of radiation and matter, respectively, and ρ_{DE} is given by (4.47). One may show [50], that the system of background equations of motion may be written as a self-consistent system of ODEs giving the evolution of $\bar{\phi}$, i.e. ϕ after normalization.

After introducing the e-fold variable $\bar{N} := \ln(a)$, the system of differential equations is written as

$$\begin{aligned}
\partial_{\bar{N}}\bar{\phi} &= \frac{\bar{\phi}\,(3\Omega_{m0}\,a + 4\Omega_{r0})\left(\beta\bar{\phi} - 3\,E\right)}{4E\,(\lambda\,a^4 + 3\,\Omega_{m0}\,a + 3\Omega_{r0})}\,, &(4.62)\\
\partial_{\bar{N}}a &= a\,, &(4.63)\\
\bar{\phi}(\bar{N}=0) &= 1\,, &(4.64)\\
a(\bar{N}=0) &= 1\,, &(4.65)
\end{aligned}$$

where we have used the definition (2.44), and the subscript 0 stands for the present-day value. With the solution of these ODEs, we are able to evaluate

$$\frac{G_{\rm eff}}{G_N} = \frac{(\lambda + 3\,\Omega_{m0} + 3\,\Omega_{r0})\,\bar{\phi}^2 a^4}{\lambda\,a^4 + 3\,\Omega_{m0}\,a + 3\,\Omega_{r0}}\,. \tag{4.66}$$

The reconstructed models can be tested for empirical consistency, in principle, via the dynamics of the gravitational constant $G_{\rm eff}$ compared with the background $G_{\rm cosm}$ through several data sets (see section 3.2.1): RSD, BBN, etc.

4.5 Type-II theories

In the previous section we have presented Type-I theories. Type-II theories, by contraposition, are those theories that do not have an Einstein frame, i.e. they cannot be phrased as GR plus non-minimally coupled matter. As we will see, this allows for other interesting signatures that may be of interest in the context of late-time cosmology.

Although a direct construction that relies on the transformation between Einstein and Jordan frame was possible for type-I theories—one example of which we have presented in the previous subsection —by definition, such a direct construction will not exist for type-II theories. Instead, two ways to construct these theories can be described, along the lines of [56] and [51]. In the first method, as in [56], one constructs a general Lagrangian which respects a set of desired symmetries, including at least spatial rotations, $SO(3)$. Through a Hamiltonian analysis, one may then find the conditions under which the theory propagates two or less degrees of freedom. The non-existence of an Einstein frame should then be shown via the non-equivalence, in vacuum, with GR. This program has not yet been explored.

The second approach, employed in [51], depends on the knowledge of a non-minimal theory, which may or may not have an Einstein frame. One proceeds then to what we will call here a *minimization*, i.e. a self-consistent choice of new constraints that will reduce the number of degrees of freedom. The conditions for the existence, in general, of such a self-consistent choice have not yet been explored thoroughly, but the *minimal theory of massive gravity* (MTMG) [51], as well as the *minimal theory of quasidilaton massive gravity* (MQD) [53] (the subject of next chapter), are two examples of *minimizations*. In what follows, we review MTMG from the point of view of this minimization procedure. This section is a review section and does not directly include new results obtained during this thesis, aside some compact notations for the theory.

4.5.1 Constrained vielbein formalism

Before reviewing MTMG, we briefly discuss the vielbein formulation of massive gravity, in particular the constrained vielbein formalism. A partially constrained vielbein was also introduced in [308] to allow for a consistent non-minimal matter coupling within dRGT massive gravity.

Lorentz invariant massive gravity [47] can be expressed in an equivalent way using vielbeins (see for example [309]), which allows to bypass the square-root structure we presented previously. The vielbeins $e^{\mathcal{A}}{}_{\mu}$ are defined by

$$g_{\mu\nu} \equiv \eta_{\mathcal{AB}}\, e^{\mathcal{A}}{}_{\mu} e^{\mathcal{B}}{}_{\nu}\,, \tag{4.67}$$

where $\eta_{\mathcal{AB}}$ is the Minkowski metric, and indices $\mathcal{A}, \mathcal{B}, \ldots \in \{0, \ldots 3\}$ are local Lorentz indices. This description is redundant (the vielbeins have 6 more components w.r.t. the metric), which accounts for the possibility to perform a Lorentz transformation at each point in space-time. Once vielbeins are introduced, hence, local Lorentz invariance is rendered explicit by the possibility to transform the internal coordinates labeled by $\mathcal{A}, \mathcal{B}, \ldots$ as

$$e'^{\mathcal{A}}{}_{\mu} = \Lambda^{\mathcal{A}}{}_{\mathcal{B}} e^{\mathcal{B}}{}_{\mu}\,, \tag{4.68}$$

where $\Lambda^{\mathcal{A}}{}_{\mathcal{B}}$ is a Lorentz transformation with $\Lambda^{\mathcal{A}}{}_{\mathcal{A}'}\Lambda^{\mathcal{B}}{}_{\mathcal{B}'}\eta_{\mathcal{AB}} = \eta_{\mathcal{A}'\mathcal{B}'}$. On a side note, this can prove practical when coupling standard model fields to gravity. Returning to massive gravity, one may define a similar set of vielbeins for the fiducial metric as

$$f_{\mu\nu} \equiv \eta_{\mathcal{AB}} E^{\mathcal{A}}{}_{\mu} E^{\mathcal{B}}{}_{\nu} \,. \tag{4.69}$$

This decomposition makes the Lorentz invariance on the internal coordinates explicit. With this decomposition it is possible to write down the potential for the graviton as

$$S_m = \frac{M_{\rm P}^2 m^2}{2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mathcal{ABCD}} \left(c_0 \frac{1}{4!} E^{\mathcal{A}}{}_{\mu} E^{\mathcal{B}}{}_{\nu} E^{\mathcal{C}}{}_{\rho} E^{\mathcal{D}}{}_{\sigma} + c_1 \frac{1}{3!} E^{\mathcal{A}}{}_{\mu} E^{\mathcal{B}}{}_{\nu} E^{\mathcal{C}}{}_{\rho} e^{\mathcal{D}}{}_{\sigma} \right.$$
$$\left. + c_2 \frac{1}{2!2!} E^{\mathcal{A}}{}_{\mu} E^{\mathcal{B}}{}_{\nu} e^{\mathcal{C}}{}_{\rho} e^{\mathcal{D}}{}_{\sigma} + c_3 \frac{1}{3!} E^{\mathcal{A}}{}_{\mu} e^{\mathcal{B}}{}_{\nu} e^{\mathcal{C}}{}_{\rho} e^{\mathcal{D}}{}_{\sigma} + c_4 \frac{1}{4!} e^{\mathcal{A}}{}_{\mu} e^{\mathcal{B}}{}_{\nu} e^{\mathcal{C}}{}_{\rho} e^{\mathcal{D}}{}_{\sigma} \right) , \tag{4.70}$$

where the ϵ are the fully antisymmetric Levi-Civita symbols with $\epsilon_{0123} = 1$. This formulation is in fact entirely equivalent to dRGT massive gravity. Since the Einstein-Hilbert kinetic term is written down using exclusively one of the metrics, the Lorentz invariance is maintained, and one may transform independently both vielbeins. However the mass term is written down by combining both types of vielbeins, it is therefore not possible to perform an independent transformations anymore: only the simultaneous transformation

$$e'^{\mathcal{A}}{}_{\mu} = \Lambda^{\mathcal{A}}{}_{\mathcal{B}} e^{\mathcal{B}}{}_{\mu} \,, \qquad E'^{\mathcal{A}}{}_{\mu} = \Lambda^{\mathcal{A}}{}_{\mathcal{B}} E^{\mathcal{B}}{}_{\mu} \,, \tag{4.71}$$

is a symmetry of the action. Still one may use the Lorentz transformation to write one of the vielbeins, say, the physical one, into e.g. the ADM vielbein form

$$(e^{\mathcal{A}}{}_{\mu}) = \begin{pmatrix} N & \vec{0}_j \\ e^I{}_i N^i & e^I{}_j \end{pmatrix} . \tag{4.72}$$

The potential with the ADM vielbein formalism seems to be linear in the shift and lapse. It is however not so in the metric formalism, in which at least the shift appears non-linearly. However, in both cases the shift can be integrated out at the end of the day, and hence this is not a problem for the equivalence.

Another important point is that Lorentz invariance of the matter sector and the structure of the Einstein-Hilbert Lagrangian imply (see e.g. [310])

$$G_{\mu\nu} e_{\mathcal{B}}{}^{\mu} e_{\mathcal{A}}{}^{\nu} = G_{\mu\nu} e_{\mathcal{A}}{}^{\mu} e_{\mathcal{B}}{}^{\nu} \,, \quad \text{and} \quad T_{\mu\nu} e_{\mathcal{B}}{}^{\mu} e_{\mathcal{A}}{}^{\nu} = T_{\mu\nu} e_{\mathcal{A}}{}^{\mu} e_{\mathcal{B}}{}^{\nu} \,, \tag{4.73}$$

where we have use vielbein contractions to emphasize that one should now compute equations of motions for the vielbein instead of the metric. Therefore, in dRGT theory with unconstrained (dynamical) vielbeins, the equations of motion imply that the following condition must be satisfied

$$\frac{\delta S_m}{\delta e^{\mathcal{A}}{}_{\alpha}} e^{\mathcal{C}}{}_{\alpha} \eta_{\mathcal{CB}} = \frac{\delta S_m}{\delta e^{\mathcal{B}}{}_{\alpha}} e^{\mathcal{C}}{}_{\alpha} \eta_{\mathcal{CA}} \,, \tag{4.74}$$

and therefore, unless some parts of S_m vanish, one needs to have

$$\mathcal{Y}_{[\mathcal{AB}]} = 0 \,, \tag{4.75}$$

where $\mathcal{Y}_{\mathcal{A}}{}^{\mathcal{B}} \equiv (E^{-1})_{\mathcal{A}}{}^{\mu} e^{\mathcal{B}}{}_{\mu}$. This condition is interesting since it is sufficient for showing that there indeed exists a square root structure $(\sqrt{f^{-1}g})^{\alpha}{}_{\beta}$, once one goes back to the metric formulation. In the case of a minimal coupling (as in the original formulation of dRGT), instead of finding (4.75) through the equations of motion, one may equivalently set this condition from the start, resulting in the *constrained vielbein* formulation of dRGT gravity. On the other hand, the so-called *partially constrained vielbein* formalism [308], implements the following condition

$$Y_{[IJ]} = 0 \,, \tag{4.76}$$

where $Y_I{}^J \equiv (E^{-1})_I{}^i e^J{}_i$. This is a Lorentz breaking condition off-shell, since space- and time-like (space-time) indices are treated independently. However, since one may fix a boost to set the ADM form (4.72), and a residual three-rotation $O^I{}_J$, it is possible to find an equivalence

$$\exists\, O^I{}_J \in SO(3) \,| \qquad \mathcal{Y}_{[IJ]} = Y_{[IJ]} + O^L{}_{[I} \delta_{J]K} (E^{-1})_L{}^0 e^K{}_0 = Y_{[IJ]} \,. \tag{4.77}$$

Therefore, as long as (4.75) is valid on-shell, the partially constrained vielbein is also equivalent to dRGT.

If one chooses to put the physical vielbein in the ADM form (4.72), one can parametrize the fiducial vielbein as a corresponding (fixed) ADM form on which acts a general boost (which will eventually depend on the dynamical variables of the physical vielbein).

$$(E^{\mathcal{A}}{}_{\mu}) = \begin{pmatrix} M\gamma + M^k b_L E^L{}_k & b_L E^L{}_j \\ M b^I + E^L{}_i M^i \left(\delta^I_L + \frac{b^I b_L}{1+\gamma}\right) & E^L{}_j \left(\delta^I_L + \frac{b^I b_L}{1+\gamma}\right) \end{pmatrix}. \tag{4.78}$$

where the Lorentz boost is determined by b^I with $\gamma = \sqrt{1 + b^L b_L}$.

dRGT Hamiltonian in the vielbein formalism

It is instructive to understand the Hamiltonian analysis of dRGT theory from the "ADM-style" vielbein perspective (see [309]). We define the momentum for the three dimensional Hamiltonian as

$$\Pi_I{}^j \equiv \frac{\delta \mathcal{S}_{\mathrm{dRGT}}}{\delta e^I{}_j}\,. \tag{4.79}$$

We assume that the vielbein forms (4.72) and (4.78) have been replaced everywhere. From the absence of momentum for N and N^i and b^I one may already extract seven primary constraints $\pi_N \approx 0$, $\pi_i \approx 0$, $p_I \approx 0$. However, the conservation in time of $\pi_i \approx 0$ could be solved for the b^I [5] , and since these have no momenta they can simply be integrated out. The conservation in time of $\pi_N \approx 0$ remains and this yields the would-be Hamiltonian constraint $\tilde{\mathcal{R}}_0$. In fact there are also additional primary constraints: a particular combination of momenta that does not appear,

$$\mathcal{P}_{[IJ]} \equiv \Pi_{[I}{}^i \delta_{J]K} e^K{}_i\,. \tag{4.80}$$

The associated secondary constraint then enforces the symmetry condition (4.76). There is a unique tertiary constraint $\mathcal{C}$, found considering the conservation in time of $\tilde{\mathcal{R}}_0$, and which closes the procedure. On then finds the total Hamiltonian

$$\mathfrak{H}^{(\mathrm{tot})}_{\mathrm{dRGT}} = \int d^3x \left(-N\tilde{\mathcal{R}}_0 + \lambda\mathcal{C} + \mathcal{H}_{\mathrm{rest}} + \alpha^{IJ}\mathcal{P}_{[IJ]} + \beta^{IJ} Y_{[IJ]} + \xi\pi_N\right)\,, \tag{4.81}$$

in clear correspondence with (3.68), with all secondary constraints except π_N (which could be removed by considering the lapse as a Lagrange multiplier). Going back to our argument, we now would like to find, via a Lorentz transformation, new constraints that could reduce the number of degrees of freedom.

4.5.2 Precursor of MTMG

Considering Lorentz violations, it turns out to be especially interesting that the vielbein formulation renders the local Lorentz invariant structure explicit and practically dissociates it from diffeomorphism invariance. Indeed, it is possible to make use of the vielbein formalism to break Lorentz invariance in simple yet interesting ways. In particular, following the ideas presented in section 4.1 we are interested in breaking Lorentz in a way that can minimize the degrees of freedom.

Thinking back to the Hamiltonian formalism, massive gravity is characterized by the absence of the momentum constraints that would come from the linearity in the shift. The graviton potential term is indeed characterized by non-linearities in the shift, and therefore the theory ends up having more degrees of freedom than GR. It was found in [51] that one could render the potential term linear in the shifts (and hence add a set of new constraints) by an appropriate Lorentz violation. In this subsection we expose this particular way to break Lorentz.

The starting point is the result of the previous subsection, which states that by an appropriate boost one may choose the ADM form for one of the vielbeins, while keeping a residual $SO(3)$ invariance, and a boosted ADM form with boosts parameters b^I. It turns out that, indeed, it is possible to add constraints to the dRGT Hamiltonian simply by setting the boost parameters b^I to zero. This is a Lorentz-violating condition. In such a way, the conservation $\pi_i \approx 0$ cannot be solved for other variables without dynamics, and hence one should add new corresponding constraints. This choice is in particular compatible with

[5] Alternatively, one could have solved the conservation of the $p_I \approx 0$ for other variables, but we do not discuss this equivalent case.

the symmetry condition (4.76). In principle it can also be made compatible with the four-dimensional condition (4.75) if

$$\forall I\,,\quad \mathcal{Y}_{[0I]} = \frac{1}{M}\eta_{IJ}e^{J}{}_{i}\left(N^i - M^i\right) = 0 \quad \Rightarrow \quad M^i = N^i\,, \tag{4.82}$$

although we won't need this feature in the discussion that follows. Indeed, M^i doesn't appear in the action, and hence this condition is harmless unless the theory is formulated directly with four-dimensional metrics. The potential term is then given (we leave the explicit form to the following subsection) by

$$S_m = -\frac{M_{\rm P}^2 m^2}{2}\int d^4x\,(N\mathcal{H}_0 + M\mathcal{H}_1)\,. \tag{4.83}$$

This time directly recognizing N and N^i as Lagrange multipliers one can write a primary Hamiltonian

$$\mathfrak{H}^{(1)}_{\rm PMmG} = \int d^3x\left(-N\tilde{\mathcal{R}}_0 - N^i\mathcal{R}_i + \mathcal{H}_{\rm rest} + \alpha^{IJ}\mathcal{P}_{[IJ]}\right)\,, \tag{4.84}$$

where PMmG stands for *precursor minimal massive gravity*, since it will not be minimal and we will need more steps to minimize it (as seen further). $\mathcal{H}_{\rm rest}$ is the only part of the Hamiltonian that is not a constraint. Note that although we are using the same names as in (4.81), for example, the quantities are different and should be compared exclusively in terms of structure. The conservation in time of $\tilde{\mathcal{R}}_0$ and $\mathcal{R}_i$ is interesting since one has

$$\{\tilde{\mathcal{R}}_0, \mathcal{R}_i\} \not\approx 0 \qquad \text{and} \qquad \{\tilde{\mathcal{R}}_0, \mathcal{H}_{\rm rest}\} \not\approx 0\,,\ \{\mathcal{R}_i, \mathcal{H}_{\rm rest}\} \not\approx 0\,, \tag{4.85}$$

the first one due to the contribution from the graviton mass term to $\tilde{\mathcal{R}}_0$ (here in the metric formalism),

$$\{\mathcal{H}_0[\phi], \mathcal{R}_i[f^i]\} = -\int d^3x\phi\gamma_{k(i}\mathcal{D}_{j)}(f^k)\left\{\sqrt{\tilde{\gamma}}c_2\gamma^{li}\left(\mathfrak{K}^{-1}\right)^j{}_l + \sqrt{\gamma}\left[c_3\left(\gamma^{ij}\mathfrak{K} - \gamma^{il}\mathfrak{K}^j{}_l\right) + c_4\gamma^{ij}\right]\right\}. \tag{4.86}$$

Therefore one may solve $\frac{d}{dt}\tilde{\mathcal{R}}_0 \approx 0$ for one component of the shift N^i and $\frac{d}{dt}\mathcal{R}_i \approx 0$ for the lapse N. There are therefore only two new constraints generated from these equations, and except for the usual $Y_{[IJ]}$ no new constraints are generated. The total Hamiltonian for PMmG is therefore

$$\mathfrak{H}^{(\rm tot)}_{\rm PMmG} = \int d^3x\left(-N\tilde{\mathcal{R}}_0 - N^i\mathcal{R}_i + \lambda^\tau\mathcal{C}_\tau + \mathcal{H}_{\rm rest} + \alpha^{IJ}\mathcal{P}_{[IJ]} + \beta^{IJ}Y_{[IJ]}\right)\,, \tag{4.87}$$

with $\tau = 1, 2$. All the constraints are second class (one can show that the determinant of the matrix of Poisson brackets does not vanish), therefore one finds that the theory propagates three degrees of freedom.

We would like to make one comment. Suppose one could find $\mathcal{H}_{\rm rest} = 0$ on some background (since it depends on dynamical fields this is not possible in general, of course). Then one may not solve $\frac{d}{dt}\mathcal{R}_i \approx 0$ for the lapse N. On such backgrounds, which in fact correspond to the *self-accelerating branch* of dRGT, there are thus more constraints arising, and it turns out the the kinetic term for the third mode vanishes, which is problematic.

With the precursor theory we have discussed how *introducing* Lorentz violations may reduce the number of degrees of freedom, reaching a theory with three gravitational degrees of freedom. In the next subsection we thus continue on this route to review MTMG, a theory of massive gravity with two degrees of freedom. The Hamiltonians presented so-far, and that of MTMG are summarized in table 4.1.

Theory	Total Hamiltonian density	Degrees of freedom
GR	$-N\mathcal{R}_0 - N^i\mathcal{R}_i$	2
dRGT [47]	$-N\tilde{\mathcal{R}}_0 + \lambda\mathcal{C} + \mathcal{H}_{\rm rest}$	5
PMmG [51]	$-N\tilde{\mathcal{R}}_0 - N^i\mathcal{R}_i + \lambda^\tau\mathcal{C}_\tau + \mathcal{H}_{\rm rest}\,,\quad \tau \in \{1,2\}$	3
MTMG [51]	$-N\tilde{\mathcal{R}}_0 - N^i\mathcal{R}_i + \lambda^\alpha\mathcal{C}_\alpha + \mathcal{H}_{\rm rest}\,,\quad \alpha \in \{0,1,2,3\}$	2

Table 4.1: Short summary of total Hamiltonian and number of degrees of freedom in selected theories of massive gravity (in the four- or three-dimensional metric formalism). Red-colored quantities are first-class constraints while blue-colored quantities are second-class constraints. Only the structure is denoted, and quantities are not necessarily the same across the lines.

4.5.3 MTMG

In this section we finally review the construction of the *minimal theory of massive gravity*, as a theory of massive gravity with only two degrees of freedom [51, 311]. This theory is unique considering $SO(3)$ invariance and requiring that the cosmological background is the same as the precursor one, while keeping two degrees of freedom.

As we have seen, the symmetric condition is enough to ensure that a square root structure is possible. In the case of the precursor theory, the only relevant symmetry condition (4.76) is three-dimensional, and therefore it is possible to define a three-dimensional square-root structure out of the metrics. In what follows we thus use the metric formalism. The first step is to split the fiducial metric as

$$ds_f^2 = -M^2 dt^2 + \tilde{\gamma}_{ij}\left(x^i + M^i dt\right)\left(x^j + M^j dt\right) , \tag{4.88}$$

and we use (2.14) to define the physical lapse N, shift N^i, and three-dimensional metric γ_{ij}. We may then define the 3-dimensional matrix

$$\mathfrak{K}^i{}_k \mathfrak{K}^k{}_j = \gamma^{ik}\tilde{\gamma}_{kj} , \tag{4.89}$$

as well as the time derivative of the fiducial metric which will be useful later on

$$\tilde{\zeta}^i{}_j \equiv \frac{1}{2M}\tilde{\gamma}^{ik}\partial_t\tilde{\gamma}_{kj} , \tag{4.90}$$

where $\tilde{\gamma}^{ij}$ is the inverse of $\tilde{\gamma}_{ij}$. The potential term in the metric formalism is then given by

$$S_m = -\frac{M_{\rm P}^2 m^2}{2}\int d^4x\left(N\mathcal{H}_0 + M\mathcal{H}_1\right), \tag{4.91}$$

where

$$\mathcal{H}_0 \equiv \sqrt{\gamma}\sum_{i=1}^{4} c_i e_{4-i}(\mathfrak{K}) , \qquad \mathcal{H}_1 \equiv \sqrt{\gamma}\sum_{i=0}^{3} c_i e_{3-i}(\mathfrak{K}), \tag{4.92}$$

including the 3D symmetric polynomials $e_i(X)$, the structure needed to evade the Boulware-Deser ghost. The fact that this potential leads to the same theory as the potential (4.70) with the ADM form for the vielbein is justified by condition (4.76).

Hamiltonian structure of MTMG

Basing ourselves on (4.87), we can find the total Hamiltonian of MTMG, in its metric formulation, simply by (i) removing $\mathcal{P}_{[IJ]}$ and $Y_{[IJ]}$, which simply enforce the symmetry conditions for the physical metric and the composite vielbein, and (ii) replacing the constraints $\mathcal{C}_\tau$ by $\mathcal{C}_\alpha$, $\alpha \in \{0, \ldots, 3\}$, defined as

$$\mathcal{C}_0 \equiv -\{\tilde{\mathcal{R}}_0, \mathcal{H}_1\} + \frac{\partial}{\partial t}\mathcal{H}_0 , \qquad \frac{M_{\rm P}^2}{2}\mathcal{C}_i \equiv -\{\mathcal{R}_i, \mathcal{H}_1\} . \tag{4.93}$$

More explicitly, we can write the constraints as

$$\mathcal{C}_i \equiv \frac{1}{2}\sqrt{\gamma}M\mathcal{D}_j\left(\mathcal{F}^{jk}\gamma_{ki}\right) , \tag{4.94}$$

$$\mathcal{C}_0 \equiv \frac{1}{2}M\sqrt{\gamma}\left[\mathcal{F}^i{}_k\mathfrak{K}^k{}_j\tilde{\zeta}^j{}_i - \left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\mathcal{F}^{ij}\frac{\pi^{kl}}{\sqrt{\gamma}}\frac{2}{M_{\rm P}^2}\right] , \tag{4.95}$$

with the following definition (see appendix for the explicit expression)

$$\mathcal{F}^{ij} \equiv 2\gamma^{ik}\frac{\delta\left(\mathcal{H}_0/\sqrt{\gamma}\right)}{\delta\mathfrak{K}^k{}_j} = 2\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{H}_1}{\delta\gamma_{ij}} . \tag{4.96}$$

This last property derives from the symmetry of the potential term under exchange of fiducial and physical metrics. One can show that the $\mathcal{C}_\tau$ are included as linear combinations of these new constraints, and that no extra constraints are generated; the non-vanishing Poisson bracket

$$\{\mathcal{C}_0[\phi], \mathcal{C}_i[f^i]\} \not\approx 0 , \tag{4.97}$$

indeed indicates that the conservation equations can a priori be solved for Lagrange multipliers. One thus has

$$\mathfrak{H}_{\rm MTMG} = \int d^3x\left[-N\tilde{\mathcal{R}}_0 - N^i\mathcal{R}_i + \frac{M_{\rm P}^2 m^2}{2}\left(M\mathcal{H}_1 - \lambda\mathcal{C}_0 - \lambda^i\mathcal{C}_i\right)\right] , \tag{4.98}$$

where λ, λ^i, N and N^i are directly understood as Lagrange multipliers, and $M\mathcal{H}_1$ is the sole non-constraint part of the action. Note that λ and λ^i have dimensions of inverse energy. In MTMG, all constraints are again second class[6], for a total of 8 second class constraint on reducing the 12-dimensional phase space down to 4 dimensions. Back in field space, there are therefore 2 degrees of freedom: the two polarizations of the metric perturbations.

Action for MTMG

By making a Legendre transform, one easily finds the action for MTMG. From the Hamiltonian equations, the canonical momentum for the metric reads

$$\pi^{ij} = \sqrt{\gamma}\frac{M_{\rm P}^2}{2}\left[K^{ij} - \gamma^{ij}K - \lambda\frac{m^2}{4}\frac{M}{N}\mathcal{F}^{ij}\right] . \tag{4.99}$$

This expression can thus be replaced during the Legendre transformation. After conveniently redefining the Lagrange multipliers (in particular λ_i), the action of MTMG is thus given by

$$S = S_{\rm EH} + S_m + S_{\rm C} + S_{\rm mat} , \tag{4.100}$$

with $S_{\rm EH}$ the usual GR action derived from the Lagrangian (2.19), S_m the same graviton potential term as in the precursor theory (4.91), $S_{\rm C}$ a set of constraints explicited below and $S_{\rm mat}$ a generic minimally coupled matter action, which can be added a posteriori. The constraint part of the action reads

$$S_{\rm C} \equiv \frac{m^2 M_{\rm P}^2}{2}\int d^4x \left(\lambda^i\mathcal{C}_i + \lambda\mathcal{C}_\lambda + \lambda^2\mathcal{C}_{\lambda^2}\right) , \tag{4.101}$$

where λ_i has been redefined, $\mathcal{C}_i$ is given by (4.94), and where

$$\mathcal{C}_\lambda \equiv \frac{1}{2}M\sqrt{\gamma}\left(\mathcal{F}^i{}_k\mathfrak{K}^k{}_j\tilde{\zeta}^j{}_i - \mathcal{F}^{ij}K_{ij}\right) , \tag{4.102}$$

$$\mathcal{C}_{\lambda^2} \equiv \frac{m^2}{16}M\sqrt{\gamma}\frac{M}{N}\left(\mathcal{F}_{ij}\mathcal{F}^{ij} - \frac{1}{2}\mathcal{F}^2\right) , \tag{4.103}$$

Note that λ appears quadratically in the action since the constraints have mixed with the extrinsic curvature. Solving the equation λ will render explicit a modification to the kinetic part of the action.

Phenomenology of MTMG

In the previous discussion we have reviewed the construction of MTMG, following [51, 311], a theory of massive gravity member of the class of minimally modified gravity theories (see e.g. [50]). We now review some of its interesting phenomenology along the lines of [311, 312, 313].

One of the most interesting characteristics of MTMG is its cosmology. Indeed, while dRGT theory and several extensions do not allow for a (healthy) FLRW cosmology (see section 3.1.3), MTMG reduces the problem by getting rid of any potentially unstable modes, and allows for two healthy branches of cosmology.

Notably, it is possible to show [311] that on flat FLRW (2.34), jointly with a flat, FLRW fiducial metric ansatz (time dependent but not dynamical)

$$ds_f^2 = -M(t)^2dt^2 + \tilde{a}(t)^2\delta_{ij}dx^idx^j , \tag{4.104}$$

one may combine the equations of motion to find a unique solution $\lambda = 0$. Since $\lambda_i = 0$ already by symmetry, this considerably simplifies the study of cosmology. We then define for convenience

$$\mathcal{X} \equiv \frac{\tilde{a}}{a} , \qquad \tilde{H} \equiv \frac{\dot{\tilde{a}}}{M\tilde{a}} , \tag{4.105}$$

as well as

$$\Gamma = c_4 + 3c_3\mathcal{X} + 3c_2\mathcal{X}^2 + c_1\mathcal{X}^3 , \tag{4.106}$$

[6]Since the computation of the determinant of the matrix of Poisson brackets is cumbersome, we rely on the analysis of cosmological perturbations to show that *at least* two degrees of freedom propagate.

which is basically $\mathcal{H}_0$ evaluated on the Friedmann background.The combination Γ appears hence the Friedmann equations, and is directly responsible for the dark energy density contribution

$$3M_{\rm P}^2 H^2 = \rho_{\rm mat} + \rho_g \,, \tag{4.107}$$

$$2M_{\rm P}^2 \frac{\dot{H}}{N} = \rho_{\rm mat} + \rho_g + P_{\rm mat} + P_g \tag{4.108}$$

where $\rho_{\rm mat}$ and $P_{\rm mat}$ are the usual density and pressure of the matter fields and

$$\rho_g \equiv \frac{M_{\rm P}^2 m^2}{2}\Gamma \,, \qquad P_g \equiv -\frac{M_{\rm P}^2 m^2}{2}\left[\Gamma + \frac{1}{3}\Gamma_{,\mathcal{X}}\mathcal{X}\left(\frac{M}{N}\frac{1}{\mathcal{X}} - 1\right)\right] . \tag{4.109}$$

In the limit $\frac{M}{N}\frac{1}{\mathcal{X}} \to 1$ the background contribution of the graviton potential is an effective cosmological constant with $P_g = -\rho_g$. The equation for λ should still be considered and gives, once $\lambda = 0$ is replaced

$$\Gamma_{,\mathcal{X}}\left(\mathcal{X}\tilde{H} - H\right) = 0 \,. \tag{4.110}$$

This defines the two branches of MTMG, the *self-accelerating branch* with $\Gamma_{,\mathcal{X}} = 0$, and the *normal branch* with $\mathcal{X}\tilde{H} = H$. Without going into the details, we present the phenomenology general to both branches, and the one more specific to each of the two branches, following [311, 312].

We define the quantities to be used for the discussion. For the study of perturbations, we use the quantities defined in section 2.2.3. On top of these, the fields specific to MTMG are perturbed as

$$\lambda = \delta\lambda \,, \qquad \lambda^i = \frac{\delta^{ij}}{a^2}\partial_j \delta l + \delta l^i \,, \tag{4.111}$$

with $\partial_i \delta l^i = 0$, both vanish on the background and are therefore pure perturbations.

Self-accelerating branch of MTMG—The self-accelerating branch is characterized, on the background, by an exact agreement with ΛCDM cosmology. Indeed, one has

$$\Gamma_{,\mathcal{X}} = 0 \qquad \Rightarrow \qquad \mathcal{X} = cst. \tag{4.112}$$

and that the equation of state of dark energy becomes $w = -1$. For perturbations, the equations for $\delta\lambda$ and δl set $\psi = e = 0$. One then finds that the phenomenology for perturbations is the same as GR.

Normal branch of MTMG—In the normal branch, the equation of state of dark energy can differ from $w = -1$, unless the dynamics leads to $\mathcal{X} = cst.$ and $\frac{M}{N}\frac{1}{\mathcal{X}} \to 1$. The scalar perturbations have a modified behavior. There is a non-trivial no-ghost condition equivalent to

$$\frac{M_{\rm P}^2}{\rho_m}\left(m^2\Gamma_1 + H^2\right)^2\left(\frac{1}{3}\frac{k^2}{a^2} + m^2\Gamma_1(r-1)\right) + m^2\Gamma_1\left(m^2\Gamma_1 + H^2\right) > 0 \,, \tag{4.113}$$

with $\Gamma_1 \equiv -\frac{1}{12}\Gamma_{,\mathcal{X}}\mathcal{X}$. In the sub-Hubble limit, i.e. for $k \gg H \gtrsim m$, the no-ghost condition is automatically satisfied. Within the same limit, one finds for dust perturbations a non-trivial effective gravitational constant

$$\frac{G_{\rm eff}}{G_N} = \frac{2\rho^2 + 3m^2 M_{\rm P}^2\rho\left[\Gamma_1(2r+3) + \Gamma_2(1-r)\right] + m^2 M_{\rm P}^2 P_g - 18m^4 M_{\rm P}^4\Gamma_1^2(r-1)}{2\left(3\Gamma_1 m^2 M_{\rm P}^2 + \rho\right)^2} \,, \tag{4.114}$$

where $\rho = \rho_g + \rho_{\rm mat}$, and a non-trivial gravitational slip

$$\eta = \frac{3m^2 M_{\rm P}^2\Gamma_1 + \rho}{\rho}\frac{G_{\rm eff}}{G_N} \,. \tag{4.115}$$

In addition to deriving these results, it was shown in [311] that one may have $G_{\rm eff}/G_N < 1$ in a substantial region of the parameter space. Note that the phenomenology will tend to agree with GR whenever $|\Gamma_1 m^2| \ll H^2$, hence typically up to some time in our past.

The normal branch, since it provides non-trivial growth of the matter perturbations, has been tested against growth data [312], as well as ISW-galaxy cross-correlation data [313], in both cases assuming the same background as ΛCDM, with $\mathcal{X} = cst.$ and $r = 1$ for simplicity. In such a case the theory only has one extra parameter with respect to ΛCDM, which can be related to the graviton mass μ_T^2. Notwithstanding the extra parameter, MTMG still fits the observations better. In particular RSD data and weak lensing seems to allow, although not yet conclusively, for the range $G_{\rm eff}/G_N < 1$ (see section 3.2.1).

Non-branch specific phenomenology—In both branches of MTMG one finds that the vector and tensor modes are characterized by a similar behavior. Metric vector modes β_i are set to zero by the equation for l^i. There are therefore no dynamical vector modes. For tensor modes one finds the action [311]

$$\frac{M_{\rm P}^2}{8}\sum_{\epsilon=+,\times} Na^3\left[\frac{\dot{h}_\epsilon^2}{N^2}-\frac{1}{a^2}\left(\partial_i h_\epsilon\right)^2-\mu_T^2 h_\epsilon^2\right], \tag{4.116}$$

with the graviton mass

$$\mu_T^2 = m^2\frac{\mathcal{X}}{6}\left[\Gamma_{,\mathcal{X}}+(r-1)\frac{\Gamma_{,\mathcal{X}\mathcal{X}}\mathcal{X}}{2}\right]. \tag{4.117}$$

4.6 Summary

In this chapter, we have explored novel directions for constructing alternative theories of gravity. In particular, when relaxing Lorentz invariance, it is possible to construct theories which propagate as few degrees of freedom as general relativity, *minimally modified gravity theories (MMGs).*

In section 4.3 we segregate two types of MMGs, depending on whether they can be mapped by a redefinition into non-minimally coupled GR (type-I) or not (type-II). The separation of the cases allows to emphasize some possible systematic ways to construct new theories. In particular in section 4.4 we leverage the existence of an Einstein frame to build a class of novel and non-trivial MMGs. Just as scalar tensor theories are known to be linked by frame transformations (for example conformal or disformal transformations), which has helped find novel classes of theories, the study of classes of equivalence is essential for a systematic construction of MMGs and the discovery of new theories.

In addition to simply allowing to find new theories, adopting *minimalism* as a guiding principle can be interesting because (i) the theories are more tractable, (ii) the theories involve a smaller number stability conditions (since there are less potentially unhealthy modes), (iii) Lorentz-breaking theories allow for interesting phenomenology in view of recent and future data. For both types I and II, we have reviewed some interesting examples. These examples satisfy recent stringent bounds on modifications of gravity, in particular the bound on the speed of tensor modes.

The class of type-I theories we have realized during this thesis, constructed using canonical transformations, presents a time dependence of the effective gravitational constant at short (cosmological or astrophysical) scales. This dependence is mild and can be made to accommodate bounds such as the BBN bound. On the other hand, the cosmological gravitational constant is fixed. Furthermore, the equation of state for the dark energy (without extra mode) is non-trivial and can differ from $w = -1$. We argue that models of this type could in principle alleviate the H_0, tension if it becomes more significative with future data.

To summarize, we have found that MMGs hold interesting prospects, especially considering future data. However, as shown in image 4.1, and as briefly commented for each model (see also section 3.2.1), a lot of future development is still needed to both understand the theory space of MMGs and to connect theory and observations for all theories, along the lines of what was done for MTMG in [312, 313].

Chapter 5

Minimal quasidilaton

In this chapter present the construction of a quasidilaton theory which does pass the current observational constraints: the *minimal theory of quasidilaton massive gravity.* We first begin with a short review (section 5.1) of the origin and current status of quasidilaton theories. We then present the construction and study of the minimal quasidilaton, as realized during this thesis, in sections 5.2 to 5.4. We will see that the new construction has relative advantages compared to the other quasidilaton theories.

5.1 Quasidilaton theories

In the search for theories of massive gravity which could accommodate an exact FLRW cosmology one milestone was quasidilaton theories. The simple intuition is that one replaces the cumbersome (3.75) by

$$\dot{a} = 0 \qquad \to \qquad \partial_t(af(\sigma)) = 0\,, \tag{5.1}$$

where σ is a scalar field. It is therefore clear that by coupling a scalar field directly to the graviton potential, one would then have a chance to obtain a theory with a FLRW cosmology. This is the idea that has motivated several scalar field extensions of massive gravity.

As discussed previously, one possibility to couple a scalar to the graviton potential is to promote the constants of the graviton potential to general functions of the scalar field. This class of theories is called *mass-varying massive gravity.* In such a case one may have a heavier mass for the graviton at, say, early-times, with a lighter mass $m \sim 0$ in the late Universe (see [190, 201, 202, 203] and section 3.1.3 for a discussion of the cosmology of these models). It may however be necessary to tune the potential of the scalar mode to pass astrophysical tests. Here, we do not focus on this possibility.

Another situation of coupling between graviton potential may arise within the context of conformal transformations of the metric between a minimally coupled frame and an Einstein frame, as we have seen in section 3.1.1. An especially interesting class of theories are *dilaton theories*, which make the Einstein theory scale invariant, that may arise from brane scenarios, and which naturally allow for late-time attractors. Inspiration for building novel massive gravities was sought naturally within this class of theories, especially since the frame transformation allowed to build a frame in which the graviton potential was coupled to the scalar field. For the purposes of clarity we therefore review dilaton theories, before discussing quasi-dilaton theories.

5.1.1 Dilaton

The quasidilaton theory was inspired by theories with a dilaton, a scalar field appearing as the Goldstone boson of spontaneously broken scale invariance. Since scale invariance is generally a desired feature in the UV, and is broken in the standard model by the Planck mass, the cosmological constant, and the Higgs mass, the presence of such a scalar field is well motivated; it also appears in the low-energy effective action for string theory constructions [314].

Let us review shortly dilatation symmetry, the specific scale invariance we are interested in, as well as its relation to conformal symmetry. Dilatations are best introduced in the Jordan frame (we will discuss the coupling to matter a little ahead), where we require a transformation

$$x^\mu \to e^{\sigma_0} x^\mu\,, \qquad \tilde{g}_{\mu\nu}(x^\mu) \to \tilde{g}_{\mu\nu}(x^\mu)\,, \tag{5.2}$$

where σ_0 is a dimensionless constant, which can be understood as a particular conformal isometry followed by a conformal transformation to make the metric invariant. This definition maps well to Minkowski space-time, for which the metric is trivially constant. GR by itself, in this frame, is not scale invariant, since a dilatation maps

$$\frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-g}R \quad \to \quad \frac{\tilde{M}_{\rm P}^2}{2}\int d^4x\sqrt{-g}R \tag{5.3}$$

where $\tilde{M}_{\rm P} = e^{\sigma_0}M_{\rm P}$. For now still considering only the gravitational sector, one can build an example scalar-tensor theory invariant under these dilatations as

$$\frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-\tilde{g}}e^{2\sigma}\left(\tilde{R}-(\omega-6)\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right)\,, \tag{5.4}$$

where the constant ω is introduced for later convenience, as long as the dilaton σ maps as

$$\sigma \to \sigma - \sigma_0\,. \tag{5.5}$$

One may equivalently call this theory scale-invariant in the sense of (5.2), since the Planck scale can be arbitrarily changed by a dilatation. Then, we may focus on the matter fields, which in facts are the very ones that define the Jordan frame. Let us see how these transform under dilatations. Starting with an agnostic coupling

$$\Psi_i(x^\mu) \to e^{d_i\sigma_0}\Psi_i(e^{\sigma_0}x^\mu) \tag{5.6}$$

to the matter fields Ψ_i, we require that free, massless fields have an action invariant under dilatations, so as to be consistent with the usual notion of scale freedom (there can be other notions e.g. [315]). This leads for example to the following scaling (see e.g. [62]) for a scalar field in 4 dimensions

$$\phi(x^\mu) \to e^{-\sigma_0}\phi(e^{\sigma_0}x^\mu)\,. \tag{5.7}$$

Now as customary we couple matter minimally

$$\mathcal{L}_{\rm mat} = \mathcal{L}_m(\tilde{g}_{\mu\nu},\Psi_i)\,. \tag{5.8}$$

Due to the Higgs mass, the standard model of particle physics is not invariant under (5.2)—but it is approximately so at high enough energies. Therefore if, say, we want to include the standard model of particle physics in a scale-invariant fashion, in this Jordan frame, we should include for example a non-minimal coupling of the dilaton to the Higgs. Except for the Higgs mass, one may use the minimal coupling prescription (therefore still approximately relating to our notion of Jordan frame, the one in which matter fields couple minimally).

After having discussed the Jordan frame, we may study shortly what happens in the Einstein frame, which is defined via the conformal transformation[1]

$$g_{\mu\nu}(x) = e^{2\sigma(x)}\tilde{g}_{\mu\nu}(x)\,, \tag{5.9}$$

(here given in its passive form). The total action becomes

$$\frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-g}\left(R-\omega g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right)+\int d^4x\sqrt{-g}\mathcal{L}_{\rm mat}(e^{-2\sigma(x)}g_{\mu\nu},\Psi_i)\,, \tag{5.10}$$

where there exists a non-minimal coupling between matter and the gravitational sector. Of course, since we have just made a field redefinition, both frames should be equivalent, notwithstanding matter having acquired a non-minimal coupling. It is interesting to ask what happened to the dilatations: these have become

$$x^\mu \to e^{\sigma_0}x^\mu\,,\quad g_{\mu\nu}(x^\mu)\to e^{-2\sigma_0}g_{\mu\nu}(x^\mu)\,,\quad \sigma(x^\mu)\to\sigma(x^\mu)-\sigma_0\,,\quad \Psi_i(x^\mu)\to e^{d_i\sigma_0}\Psi_i(e^{\sigma_0}x^\mu)\,, \tag{5.11}$$

which, for the gravitational sector amounts to a constant shift in the dilaton and a global conformal isometry.

It is important to note (again) that it is only when choosing the coupling of gravity matter fields that we are choosing the physics. A theory of gravitational fields alone may be freely mapped into another via field redefinitions, but the real difference is made when including matter fields. What we have seen in the previous chapter is testimony of this fact.

[1]Concerning terminology, we employ the term *conformal transformation* in the sense of Wald's textbook [62].

5.1.2 Original quasidilaton

When massive gravity (or bigravity) are considered, one should review the previous results considering the newly introduced fields. In order to be able to retain a familiar notion of covariance, one may work with Stückelberg fields. The novel mass term may be written schematically as

$$\mathcal{L}_m = \mathcal{L}_m(g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b, f_{ab}(\phi^c))\,, \tag{5.12}$$

where indices $a, b, \ldots$ are internal, and are contracted with a frozen metric which may depend on the Stückelberg fields), where we allow for square-root structures, and we don't specify yet whether $g_{\mu\nu}$ is the Jordan frame or the Einstein frame metric. Given choices of f_{ab} may lead to different symmetries in the unitary gauge $\phi^a \to x^a$, e.g. $f_{ab} = \eta_{ab}$ obviously retaining its Lorentz invariance. In order to retain any properties of scale invariance in the unitary gauge, it is natural to demand that under dilatations, the Stückelberg transform as coordinates, i.e.

$$\phi^a(x^\mu) \to e_0^\sigma \phi^a(x^\mu)\,, \tag{5.13}$$

meaning that $Y_{ab} \equiv \tilde{g}^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b$ is invariant under dilatations and we may thus keep scale invariance by writing the gravitational action as

$$\frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-\tilde{g}}e^{2\sigma}\left[\tilde{R} - (\omega-6)\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + e^{2\sigma}\mathcal{L}_m(Y^{ab}, f_{ab})\right]\,. \tag{5.14}$$

In the conformally transformed frame, this will translate to

$$\frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-g}\left[R - \omega g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \mathcal{L}_m(e^{2\sigma}Y^{ab}, f_{ab})\right]\,. \tag{5.15}$$

Now, as we mentioned, a choice of matter coupling is crucial. Several choices are possible leading to different theories. Following [316], choosing to couple matter minimally in the frame (5.15), thus breaking the dilatation invariance in what we shall now call the *dilaton frame*, to avoid confusion with the other notions of frames. The action in the Jordan —minimally coupled—frame therefore reads

$$\frac{M_{\rm P}^2}{2}\int d^4x\sqrt{-g}\left[R - \omega g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \mathcal{L}_m(e^{2\sigma}Y^{ab}, f_{ab})\right] + \int d^4x\sqrt{-g}\mathcal{L}_{\rm mat}(g_{\mu\nu}, \Psi_i)\,. \tag{5.16}$$

It is invariant under a new symmetry which is called *quasidilatation*, i.e.

$$\sigma \to \sigma - \sigma_0\,, \qquad \phi^a(x^\mu) \to e^{\sigma_0}\phi^a(x^\mu)\,, \tag{5.17}$$

whereas scale invariance is generally broken by the matter fields. The scalar field is named *quasidilaton*. It is interesting to note that by coupling matter minimally with the metric $\tilde{g}_{\mu\nu}$ instead, we may have defined *dilatonic massive gravity*, a scale invariant theory (we will not explore this further here).

Let us now further describe the geometric meaning of the quasidilatation symmetry. In the unitary gauge $\phi^a \to x^a$, the transformation (5.17) can be transferred to the fiducial metric: every internal index being contracted with the fiducial metric one obtains the equivalent transformation

$$\sigma \to \sigma - \sigma_0\,, \qquad f_{\mu\nu} \to e^{2\sigma_0}f_{\mu\nu}\,. \tag{5.18}$$

We therefore have two equivalent pictures: i) rescaling of the internal space coordinates ϕ^a with respect to the space-time coordinates (in the covariant theory), and ii) global conformal transformation of the fiducial metric.

Finally let us point out that, in practice, the action (5.15) can be obtained by performing the following replacement on the dRGT Lorentz invariant action, in the covariant formulation

$$f_{\mu\nu} \equiv \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \to e^{2\sigma}f_{\mu\nu}\,. \tag{5.19}$$

5.1.3 Cosmology with a quasidilaton

The quasidilaton was mainly introduced for cosmological purposes, in particular to lead to a healthy cosmology and late-time acceleration. Since at late-times matter fields dilute away, the theory regains its scale invariance. Background cosmology was indeed enhanced as flat FLRW solutions can be found; here we reproduce some results, in order to show a recurring feature of quasidilaton models, the presence of late-time attractors.

On flat FLRW, it is customary to define

$$\mathcal{X} = \frac{e^{\sigma/M_\mathrm{P}}}{a}, \qquad r = \frac{a\, e^{\sigma/M_\mathrm{P}}}{N}, \qquad \Sigma = \frac{\dot{\sigma}}{N}. \tag{5.20}$$

Working in the covariant formulation, one may obtain the constraint equation from the ϕ^0 Stückelberg

$$\frac{d}{dt}\left[a^4 \mathcal{X}\, J(\mathcal{X})\right] = 0\,, \tag{5.21}$$

where $J \equiv \mathcal{X}^3 c_0 + 3\mathcal{X}^2 c_1 + 3\mathcal{X} c_2 + c_3 \propto \frac{\delta \mathcal{L}_m}{\delta M}$, implying that

$$\mathcal{X}\, J(\mathcal{X}) = \frac{cst}{a^4}. \tag{5.22}$$

Since necessarily $\dot{\mathcal{X}} \to 0$ at late times, equation (5.22) essentially amounts to having a late-time de Sitter attractor, as the Friedmann equation is, in vacuum,

$$H^2 = \frac{\omega}{6-\omega} \frac{\dot{\mathcal{X}}}{N\mathcal{X}} \left(\Sigma + H\right) + \tilde{\Gamma}\left(\mathcal{X}\right), \tag{5.23}$$

where we do not give $\tilde{\Gamma}\left(\mathcal{X}\right)$ for brevity. We have of course excluded the strongly coupled case $\mathcal{X} \to 0$.

Unfortunately, perturbations were quickly shown to be plagued with a ghost in the scalar sector (see [317, 203]). This motivated the development of extensions which we describe in the next section.

5.1.4 Extensions to the original formulation

Motivated by the absence of cosmology in the original formulation, several extensions to the quasidilaton theory have been proposed. Essentially three routes have been studied: an extended potential term which preserves the quasidilatation global symmetry [318, 319, 320, 321, 322], an extended coupling of the quasidilaton kinetic term [323], and Lorentz violations [324]. Although in some cases the extensions were unsuccessful to cure the original unhealthy features without introducing new problems [325, 320, 322], several promising approaches have emerged [324, 53, 321]. We will shortly present all these extensions in this subsection The last in date, the minimal theory of quasidilaton gravity (MQD), is one of the main results of this thesis and is studied from the next section on. A visual summary of the different quasidilaton theories can be found in table 5.1.

Theory	Lorentz	No BD-ghost?	Phenomenology
Quasidilaton	LI	✓	UV ghost [317, 203]
Extended quasidilaton $\omega \neq 0$		✗	[321, 322]
Extended quasidilaton $\omega = 0$		✓	-
New quasidilaton		✗	-
PC new quasidilaton	LV	✓	-
Precursor quasidilaton		✓	Strong coupling
Minimal quasidilaton		✓	✓

Table 5.1: Brief summary table of different formulations of the quasidilaton. "LI" stands for "Lorentz invariant", while "LV" stands for "Lorentz violating". Reintroducing the Boulware-Deser ghost may be softened as an issue if one ensures that it appears at high enough energies. This is the case in the new quasidilaton theory (and not in the partially constrained formalism, which removes the ghost non-linearly). "Phenomenology" stands for phenomenology with matter, or for distinctive problems when they appear.

The first extension of the quasidilaton theory has been called *generalized quasidilaton* and *extended quasidilaton* [318, 319]. The generalization is done in the way the quasidilaton couples to the metric, as an extra term is in principle allowed by the formulation. One may indeed generalize the replacement (5.19) to

$$f_{\mu\nu} \to e^{2\sigma} f_{\mu\nu} - \frac{\alpha_\sigma}{m^2} \partial_\mu\sigma\partial_\nu\sigma\,, \tag{5.24}$$

where α_σ is a new constant parameter, while retaining the symmetry under quasidilatations (5.17). From the perspective of extra-dimensions, this novel coupling can be naturally understood for a moving brane. There are thus at least 2 new parameters $\{\omega, \alpha_\sigma\}$ with respect to dRGT theory. Several studies have then explored the cosmology of this theory; it was however recognized that the extension (5.24) reintroduces the BD ghost, unless $\omega = 0$ or $\alpha_\sigma = 0$ [320]. It may be possible to generalize further the theory using Lovelock invariants of the curvature, while the tadpole extension proposed in [325] is readily taken into account by using the dRGT potentials (3.63) (see appendix A of [321]). Finally it should be noted that the addition of a Horndeski kinetic term for the quasidilaton, instead of the standard one, was proposed in [319].

The cosmology of the extended quasidilaton with $\omega = 0$ was studied in [321] in vacuum, and we report here the constraint equation

$$m^2 J\mathcal{X}\left(Hr - \Sigma\right) = 0\,, \tag{5.25}$$

where the definition of r is slightly modified, i.e.

$$r \equiv a\frac{n}{N}\,, \tag{5.26}$$

with n the effective lapse function for the fiducial metric. Condition (5.25) gives two branches, $J = 0$, for which the scalar and vector modes are strongly coupled, and $Hr = \Sigma$ which divides again into two further branches with different late-time attractors. One of these branches develops a late-time ghost, while the other can be healthy as far as vacuum cosmology goes. To our knowledge, cosmology with matter was not explored in details.

The second extension of the quasidilaton theory is commonly called the *new quasidilaton* [323]. This extension proposes a yet new class of terms that maintains the quasidilatation symmetry, by coupling the quasidilaton kinetic term to a composite metric,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + 2\beta e^{\sigma} g_{\mu\lambda}\mathcal{K}^{\lambda}{}_{\nu} + \beta^2 e^{2\sigma} f_{\mu\nu}\,, \tag{5.27}$$

i.e. defining the kinetic term as

$$S_{\sigma,\mathrm{NQD}} = -\omega\frac{M_{\mathrm{P}}^2}{2}\int d^4x\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\,. \tag{5.28}$$

It is expected that the BD ghost generically reappears in this theory. Indeed, viewing the quasidilaton as a matter field allows a straightforward analogy with *doubly-coupled matter* (see [43, 326] and references therein). The ghost is expected to have a mass (as explained in [323])

$$m_{\mathrm{BD}} \sim \frac{m^3 M_{\mathrm{P}}^2}{\sqrt{\beta}\dot{\sigma}\partial_i\sigma}\,. \tag{5.29}$$

With the scaling $\dot{\sigma} \sim H \sim m$ and for $\partial_i\sigma M_{\mathrm{P}} \lesssim \Lambda_3^2$, one expects the ghost to have the mass

$$m_{\mathrm{BD}} \sim \frac{\Lambda_3}{\sqrt{\beta}}\,, \tag{5.30}$$

i.e. much higher than Λ_3 for $\beta \ll 1$. In such a case the ghost may be integrated out, since the Lorentz invariant massive gravity theory can be seen as an effective theory for energies below Λ_3. Lastly, the cosmology of the new quasidilaton doesn't have two branches as the extended quasidilaton, but only a de Sitter attractor with $J = 0$.

The third avenue was to recognize that via the *partially constrained vielbein* formalism, i.e. a Lorentz violation introduced through a specific choice of vielbein, that the BD ghost that reappears due to the mixed coupling in the new quasidilaton theory could be removed non-linearly ([324], which we follow here). Recalling the arguments used in MTMG, one may use Lorentz invariance to put the fiducial vielbein in the ADM form via a boost

$$E^{\mathcal{A}}{}_{\mu} = \begin{pmatrix} M & 0 \\ M^k E^I{}_k & E^I{}_j \end{pmatrix}\,. \tag{5.31}$$

One imposes then the physical condition $Y_{IJ} = Y_{JI}$, with $Y_{IJ} \equiv E_I{}^i \delta_{JK} e^K{}_i$. In the case of a coupling of matter to $g_{\mu\nu}$ only, the theory is equivalent to dRGT theory; however, it becomes different (perturbatively on FLRW) when composite matter couplings are considered. The partially constrained vielbein formalism was used in the theory of massive gravity first [308], and then applied to the new quasidilaton theory [324].

Lorentz violations in fact allow for more constrained structures, even without composite matter coupling, and this was used to completely remove the scalar and vector modes from the theory of massive gravity, in the realization called *minimal theory of massive gravity* (presented in the previous chapter). Lorentz transformations are broken up to its $SO(3)$ subgroup, and are hence still in agreement with the phenomenology in cosmology. It turns out that this removal of modes can be applied to the quasidilaton as well, thereby defining the *minimal theory of quasidilaton massive gravity.* From the point of view of the extensions presented in this section, this has the advantage of making the treatment of perturbations relevantly simpler, as there will be less degrees of freedom involved.

5.2 Minimal quasidilaton

In the previous chapter, we have studied one particular massive gravity theory that followed from principles of minimalism: *MTMG*. This theory [51, 311, 312, 313] is healthy and viable for all known aspects of it. As we have seen, novel cosmological solutions can be obtained, provided that the background metric has a time-dependence. One may ask oneself whether this time dependence can be made dynamical, just as HR bigravity renders dynamical the fiducial metric of dRGT massive gravity.

The simplest realization of such a dynamical structure is to render the conformal mode of the fiducial metric dynamical. Indeed, as far as the cosmological solutions are concerned, only the presence of a time dependent scale factor is necessary to provide interesting solutions. Therefore, we just need one field: this is exactly what the quasidilaton can achieve.

In the following we present the construction of the minimal theory of quasidilaton massive gravity, which will be abridged here *minimal quasidilaton* (or MQD). Although the minimal quasidilaton is not a minimal theory in the sense discussed previously, it inherits at least the techniques used to build minimal theories. It therefore receives part of the advantages of these minimal theories, i.e. simplicity and healthiness.

We introduced the Horndeski extension of the quasidilaton kinetic term in order to implement a Vainshtein mechanism. Indeed, since the graviton scalar mode does not appear anymore, there are no higher order derivative interactions that can become important, and hence no Vainshtein radius. It is therefore important to add the derivative interactions in a different way, hence, through the kinetic term.

5.2.1 Properties of the minimal quasidilaton

The minimal quasidilaton is a theory satisfying the following properties:

- The tensor modes are massive in a way that lets the mass term contribute to the vacuum energy via the vacuum expectation value of the metric.
- The tensor modes travel at the speed of light.
- It is free of the Boulware-Deser ghost.
- It non-linearly propagates 2 tensor modes and 1 scalar mode.
- The extra scalar mode sustains a Vainshtein screening.
- A time-independent fiducial structure is enough to realize FLRW cosmology.

Just as in MTMG it is clear that in order to obtain massive tensor modes without having 5 associated degrees of freedom one needs to break Lorentz invariance. Again, in order to realize FLRW cosmology $SO(3)$ at least should be an unbroken subgroup of symmetries.

5.2.2 Action of the minimal quasidilaton

In what follows we detail the action for the minimal quasidilaton. Although the action itself may seem exceedingly complex, one must remember that in unitary gauge the minimal quasidilaton action reads

$$S = \tilde{S}_{\rm EH} + \tilde{S}_\sigma + S_m + S_{\rm C} + S_{\rm mat}\,, \tag{5.32}$$

We will develop on each of the terms one by one, but we first give an intuitive explanation. $\tilde{S}_{\rm EH}$ and $\tilde{S}_\sigma$ are related to the usual kinetic terms for the metric and the quasidilaton. S_m is a Lorentz-breaking graviton mass term (so as to satisfy the first property). $S_{\rm C}$ restricts the dynamical surface to remove consistently the graviton scalar mode. Finally, as usual, $\tilde{S}_{\rm mat}$ involves the minimally coupled matter fields. Starting from the Einstein-Hilbert and quasidilaton kinetic actions, we have

$$\tilde{S}_{\rm EH} = \frac{M_{\rm P}^2}{2}\int d^4x N\sqrt{\gamma}\left(R^{(3)} + \tilde{K}_{ij}\tilde{K}^{ij} - \tilde{K}^2\right)\,, \tag{5.33}$$

and the quasidilaton kinetic term as

$$\tilde{S}_\sigma = \int d^4x\sqrt{-g}\,\left[P(X) - G(X)\Box\sigma + \lambda_\chi\left(\tilde{\mathfrak{X}}_\sigma - X\right)\right]\,, \tag{5.34}$$

where $R^{(3)}$ is the three-dimensional Ricci curvature, λ_χ is a Lagrange multiplier, $P(X)$ and $G(X)$ are free functions of the auxiliary field X. The kinetic term of the quasidilaton corresponds to a cubic shift-symmetric Horndeski once all auxiliary fields are integrated out (the shift-symmetry is necessary to preserve the quasidilatation symmetry); this allows in principle to implement a Vainshtein screening. Note that the quasidilaton is not normalized as in the previous section but has dimensions of energy. We have also included, for compactness, other terms along the extrinsic curvature K_{ij} and the "normal" derivative of the quasidilaton $\partial_\perp\sigma \equiv \frac{1}{N}\left(\dot{\sigma} - N^i\partial_i\sigma\right)$, as

$$\begin{aligned}
\tilde{K}_{ij} &= K_{ij} + \frac{1}{M_{\rm P}^2}\frac{\lambda_T}{N}G_{,X}\left(X\gamma_{ij} + \sigma_{;i}\sigma_{;j}\right)\,,\\
\tilde{\mathfrak{X}}_\sigma &= \frac{1}{2}\left[\widetilde{\partial_\perp\sigma}^2 - \gamma^{ij}\partial_i\sigma\partial_j\sigma\right]\,,\quad \widetilde{\partial_\perp\sigma} = \partial_\perp\sigma + \frac{\lambda_T}{N} = \frac{1}{N}\left(\dot{\sigma} - N^i\partial_i\sigma\right) + \frac{\lambda_T}{N}\,.
\end{aligned} \tag{5.35}$$

In fact terms proportional to the Lagrange multiplier λ_T arise from the particular constraint structure of the theory. Note also that once the constraints are integrated out there will be novel contributions to the kinetic term coming from what we now defined as the constraint part of the action, $S_{\rm C}$. We move on to detail the potential term

$$S_m = -\frac{M_{\rm P}^2 m^2}{2}\int d^4x\left(N\mathcal{H}_0 + M\mathcal{H}_1\right)\,, \tag{5.36}$$

where the structure of $\mathcal{H}_0$ and $\mathcal{H}_1$ may remind one of what we have seen with MTMG in section 4.5.2, but with the additional presence of the quasidilaton,

$$\mathcal{H}_0 \equiv \sqrt{\gamma}\sum_{i=0}^{4} c_i e^{(4-i)\sigma/M_{\rm P}}e_{4-i}(\mathfrak{K})\,,\qquad \mathcal{H}_1 \equiv \sqrt{\gamma}\sum_{i=0}^{4} c_i e^{(\alpha+4-i)\sigma/M_{\rm P}}e_{3-i}(\mathfrak{K})\,, \tag{5.37}$$

including the 3D symmetric polynomials $e_i(X)$, just the structure needed to get rid of the Boulware-Deser ghost, and using the same definition for $\mathfrak{K}^i{}_j$ as in (4.89). We have introduced the extra constant parameter α, in order to account for the absence of Lorentz invariance at the level of the Stückelberg transformation; the time Stückelberg does not necessarily transform as the spatial ones

$$\sigma \to \sigma - \sigma_0\,,\qquad \phi^i(x^\mu) \to e^{\sigma_0}\phi^i(x^\mu)\,,\qquad \phi^0(x^\mu) \to e^{(1+\alpha)\sigma_0}\phi^0(x^\mu)\,, \tag{5.38}$$

The constraint side of the action can be decomposed as

$$S_{\rm C} = \int d^4x\left[\frac{M_{\rm P}^2 m^2}{2}\left(\lambda^i\mathcal{C}_i + \lambda\tilde{\mathcal{C}}_\lambda + \lambda^2\mathcal{C}_{\lambda^2}\right) + \lambda_T\mathcal{C}_{\lambda_T}\right]\,, \tag{5.39}$$

with

$$\mathcal{C}_i = \frac{1}{2}\sqrt{\gamma}\mathcal{D}_j\left(\mathcal{F}^{jk}\gamma_{ik}\right) - \mathcal{F}_\sigma\partial_i\sigma, \tag{5.40}$$

$$\tilde{\mathcal{C}}_\lambda = -\frac{1}{2}\left(2\mathcal{F}_\sigma\widetilde{\partial_\perp\sigma} + \sqrt{\gamma}\tilde{K}_{ij}\mathcal{F}^{ij}\right)\,, \tag{5.41}$$

$$\mathcal{C}_{\lambda^2} = \frac{m^2}{16N}\sqrt{\gamma}\left(\mathcal{F}_{ij}\mathcal{F}^{ij} - \frac{1}{2}\mathcal{F}^2\right)\,, \tag{5.42}$$

$$\mathcal{C}_{\lambda_T} = N\sqrt{\gamma}\frac{\lambda_T}{N}\left[G_{,X}\widetilde{\partial_\perp\sigma}\,\sigma^{;i}{}_{;i} - G_{,X}\left(\widetilde{\partial_\perp\sigma}\right)^{;i}\sigma_{;i} - G_{,X}\partial_\perp X - P_{,X}\widetilde{\partial_\perp\sigma}\right]\,, \tag{5.43}$$

where the semicolon indicates a covariant (with respect to γ_{ij}) derivative, and with the following definitions

$$\mathcal{F}^{ij} \equiv 2\gamma^{ik}e^{\alpha\sigma/M_{\rm P}}\frac{\partial\left(\mathcal{H}_0/\sqrt{\gamma}\right)}{\partial\mathfrak{K}^k{}_j} = 2\frac{1}{\sqrt{\gamma}}\frac{\delta\mathcal{H}_1}{\delta\gamma_{ij}}\,,\qquad \mathcal{F}_\sigma \equiv \frac{\partial\mathcal{H}_1}{\partial\sigma}\,. \tag{5.44}$$

The explicit expressions for these two last quantities are given in appendix D. Finally the matter action is, as usual, general but minimally coupled. For given calculations we will take a k-essence ansatz.

5.2.3 Total Hamiltonian of the minimal quasidilaton

In order to demonstrate that the minimal quasidilaton propagates 3 degrees of freedom non-linearly, we present the total Hamiltonian of the model (a few more details are given in appendix D). This will show that the theory propagates *at most* 3 degrees of freedom. Then, by the study of perturbations (see section 5.3.2), we will show that the theory propagates *at least* 3 degrees of freedom, hence completing our argument.

The total Hamiltonian of MQD, without matter, is

$$\begin{aligned}\mathfrak{H}^{(\mathrm{tot})}_{\mathrm{MQD,v}} = \int d^3x \Big[&-N\tilde{\mathcal{R}}_0 - N^i\mathcal{R}_i + \frac{M_{\mathrm{P}}^2 m^2}{2} M\mathcal{H}_1 + \xi_X P_X + \xi_\chi P_\chi + \xi_S P_S \\ &+ \sqrt{\gamma}\left(\lambda_X S_X + \lambda_\chi S_\chi + \lambda_S S_S + \lambda_T \tilde{T}\right) + \frac{M_{\mathrm{P}}^2 m^2}{2}\left(\lambda^i \mathcal{C}_i + \lambda \mathcal{C}_0\right)\Big] \,, \end{aligned} \tag{5.45}$$

where the subscript v stands for vacuum, and where N, N^i, ξ_*, λ_* ($*$ standing for any subscript), λ, and λ^i are all Lagrange multipliers. It is useful to define the analog of $\tilde{\mathfrak{X}}_\sigma$ in the Hamiltonian language, $\mathfrak{X}_H = \frac{1}{2}\left(\tilde{\pi}_\theta^2 - \gamma^{ij}\partial_i\sigma\partial_j\sigma\right)$. We then give the explicit expressions for the constraints, starting by the would-be Hamiltonian and momentum constraints which are given by

$$\tilde{\mathcal{R}}_0 = \frac{M_{\mathrm{P}}^2}{2}\sqrt{\gamma}\, R[\gamma] - \frac{2}{M_{\mathrm{P}}^2}\frac{1}{\sqrt{\gamma}}\left(\gamma_{il}\gamma_{jk} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\pi^{ij}\pi^{kl} - \mathcal{H}_\sigma - \frac{M_{\mathrm{P}}^2}{2}m^2\mathcal{H}_0 \,, \tag{5.46}$$

$$\mathcal{R}_i \equiv 2\sqrt{\gamma}\gamma_{ik}\mathcal{D}_j\tilde{\pi}^{kj} - \mathcal{H}_{\sigma i} \,, \tag{5.47}$$

with the following contributions from the Horndeski action

$$\frac{\mathcal{H}_\sigma}{\sqrt{\gamma}} = \chi\left(\mathfrak{X}_H - X\right) - \tilde{\pi}_\theta\tilde{\pi}_\sigma - F - \theta S - \gamma^{ij}\partial_i\sigma\partial_j\theta \,, \qquad \mathcal{H}_{\sigma i} \equiv \pi_\sigma\partial_i\sigma + \pi_\theta\partial_i\theta \,. \tag{5.48}$$

In order to keep expressions compact and easily accountable we have introduced the $\tilde{\pi}_* \equiv \pi_*/\sqrt{\gamma}$, where π_* stands for any of the canonical momenta. There are also the primary and secondary constraints associated with the auxiliary fields

$$P_X \equiv \pi_X \,, \qquad P_\chi \equiv \pi_\chi \,, \qquad P_S \equiv \pi_S \,, \tag{5.49}$$

$$S_X \equiv \chi + F_{,X} \,, \qquad S_\chi \equiv X - \mathfrak{X}_H \,, \qquad S_S \equiv \theta - G(X) \,. \tag{5.50}$$

A tertiary constraint is also generated, as

$$\tilde{T} \equiv -\tilde{\pi}_\sigma - P_{,X}\tilde{\pi}_\theta - G_{,X}\left[\frac{2}{M_{\mathrm{P}}^2}\tilde{\pi}^{ij}\left(\partial_i\sigma\partial_j\sigma + \gamma_{ij}X\right) - \tilde{\pi}_\theta\gamma^{ij}\mathcal{D}_i\mathcal{D}_j\sigma + \gamma^{ij}\mathcal{D}_i\sigma\mathcal{D}_j\tilde{\pi}_\theta\right] \,, \tag{5.51}$$

and its conservation can be solved for some Lagrange multiplier. Finally, the minimal theory is defined on the introduction of the four constraints

$$\begin{aligned}\mathcal{C}_0 &= M\left[\frac{1}{M_{\mathrm{P}}^2}\left(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\mathcal{F}^{ij}\pi^{kl} - \mathcal{F}_\sigma\tilde{\pi}_\theta\right] \,, \\ \mathcal{C}_i &= -\frac{1}{2}\sqrt{\gamma}\mathcal{D}_j\left(M\mathcal{F}^{jk}\gamma_{ki}\right) + M\mathcal{F}_\sigma\partial_i\sigma \,. \end{aligned} \tag{5.52}$$

These constraints are defined via the time evolution with the modified Hamiltonian H_1

$$\{\mathcal{R}_i, H_1\} \approx \frac{M_{\mathrm{P}}^2 m^2}{2}\mathcal{C}_i \,, \quad \{\tilde{\mathcal{R}}_0, H_1\} \approx \frac{M_{\mathrm{P}}^2 m^2}{2}\mathcal{C}_0 \,, \tag{5.53}$$

in which we defined

$$H_1 = \frac{M_{\mathrm{P}}^2 m^2}{2}\int d^3x M\mathcal{H}_1 \,. \tag{5.54}$$

Out of this set of constraints, it is possible to single out a first-class constraint $\tilde{P}_S \equiv P_S + G_{,X}P_\chi$ (whose associated symmetry has not yet been elucidated), also using suitable redefinitions of the other constraints. There are therefore 14 (at least) second-class and 1 first-class ($\tilde{P}_S$) constraints, which reduces the 22 phase space degrees of freedom (6 metric components, 1 quasidilaton, 4 scalar fields Horndeski-Lagrange multipliers) to 6 phase space degrees of freedom, i.e. *at most* 3 degrees of freedom. We give a few more details on the Hamiltonian analysis in appendix D. The Hamiltonian analysis of the cubic Horndeski theory has been detailed in appendix C, whereas the analysis of MTMG has been detailed in section 4.5.3. Roughly, the steps of the analysis of the minimal quasidilaton can be found in analogy to these. We refer the curious reader to the explicit construction in [54]. Among the differences between these approaches, one may note the additional non-vanishing Poisson brackets between S_χ, S_S and $\mathcal{C}_0$, as well as between $\tilde{T}$ and $\mathcal{C}_0$, $\mathcal{C}_i$. Since the conservation in time of these constraints is solved for Lagrange multipliers, this fortunately yields no difference in the counting of degrees of freedom.

The addition of matter does not complicate the previous discussion of the Hamiltonian analysis of MQD. It can be shown that with a suitable redefinition of the constraints $\tilde{\mathcal{R}}_0$ and $\tilde{\mathcal{R}}_i$, one may make the matter sector commute in the usual way with these new would-be Hamiltonian and momentum constraints, and that the conditions for the conservation in time of all constraints are unaffected (i.e. can still be solved for the same Lagrange multipliers). The same constraints are therefore generated, and the first-class constraint $\tilde{P}_S$ remains first-class. It is therefore possible to write formally

$$\mathfrak{H}^{(\mathrm{tot})}_{\mathrm{MQD}} = \mathfrak{H}^{(\mathrm{tot})}_{\mathrm{MQD,v}} + \mathfrak{H}^{(\mathrm{tot})}_{\mathrm{mat}} \,. \tag{5.55}$$

Since the minimal quasidilaton has been built from the Hamiltonian formulation of a different theory, the precursor theory (see section 4.5.3), this was a priori non-trivial and was checked in [55] for the k-essence ansatz.

Finally, note that the construction via the Hamiltonian language explains our use of auxiliary fields in (5.32). These are invoked in order to remove second time derivatives on the quasidilaton field σ, which appear from $\Box\sigma$. The use of auxiliary fields to perform the Hamiltonian analysis is standard [161, 158, 162] (see also section 3.1.1), but unfortunately renders the constraint algebra more opaque. After Legendre transformation, the auxiliary fields appear in a non-trivial way through the new constraints, and hence it is not trivial to integrate them out again.

5.3 Cosmology with the minimal quasidilaton

Although the action for the minimal quasidilaton looks rather complex, its cosmology is relevantly simplified, just as in the case of MTMG, by

$$\lambda = \lambda_i = \lambda_T = 0 \,. \tag{5.56}$$

To study the cosmology at the background level, we use again the flat FLRW ansatz (2.34), keep a Minkowski fiducial metric ($\tilde{\gamma}_{ij} = \delta_{ij}$ and $M = 1$), and let $\sigma = \sigma(t)$ and $\phi_m = \phi_m(t)$. Since these combinations appear repeatedly, we collect

$$\mathcal{X} = \frac{e^{\sigma/M_{\mathrm{P}}}}{a} \,, \qquad r = \frac{a\, e^{\alpha\sigma/M_{\mathrm{P}}}}{N} \,, \qquad \Sigma = \frac{\dot{\sigma}}{N} \,. \tag{5.57}$$

We then reproduce here the cosmological background equations,

$$E_1 \equiv 3M_{\mathrm{P}}^2 H^2 - \rho_g - \rho_m = 0 \,, \tag{5.58}$$

$$E_2 \equiv M_{\mathrm{P}}^2 \frac{2\dot{H}}{N} + 3M_{\mathrm{P}}^2 H^2 + P_g + P_m = 0 \,, \tag{5.59}$$

$$E_m \equiv \dot{\rho}_m + 3H(\rho_m + P_m) = 0 \,, \tag{5.60}$$

$$\begin{aligned} E_\sigma \equiv\; & \frac{1}{2}\mathcal{X} M_{\mathrm{P}}^2 m^2 [(\alpha+1) r J + r J_{,\mathcal{X}}\mathcal{X} + \Gamma_{,\mathcal{X}}] + P_{,X}\Sigma \left(3H + \frac{\dot{\Sigma}}{\Sigma N}\right) \\ & + \Sigma^3 \frac{\dot{\Sigma}}{\Sigma N} \left(P_{,XX} + G_{,XX}\Sigma H\right) + 3G_{,X} H \Sigma^2 \left(3H + \frac{\dot{H}}{HN} + 2\frac{\dot{\Sigma}}{\Sigma N}\right) = 0 \,, \end{aligned} \tag{5.61}$$

$$E_X \equiv G_{,X}\left(\frac{\dot{\Sigma}}{N} + 3H\Sigma\right) + P_{,X} - \lambda_\chi = 0 \,, \tag{5.62}$$

$$E_{\lambda_T} \equiv E_X \Sigma = 0 \,, \tag{5.63}$$

$$E_{\lambda_\chi} \equiv X - \frac{1}{2}\Sigma^2 = 0 \,, \tag{5.64}$$

$$E_\lambda \equiv \Gamma_{,\mathcal{X}} H M_{\mathrm{P}} + \Sigma[(\alpha+1)J + J_{,\mathcal{X}}\mathcal{X}] = 0 \,, \tag{5.65}$$

corresponding respectively to the Friedmann equation, the second Einstein equation, the equation for matter, and the equations for the variables appearing as labels n of the E_n, and where several definitions are still needed. First of all we defined the new contributions from the gravity sector (including the quasidilaton) to the density and pressure balances

$$\rho_g \equiv 2XP_{,X} - P + 6HG_{,X}X\Sigma + \frac{M_{\rm P}^2 m^2}{2}\Gamma\,, \tag{5.66}$$

$$P_g \equiv P - 2G_{,X}X\frac{\dot{\Sigma}}{N} - \frac{M_{\rm P}^2 m^2}{2}\left[\Gamma + \frac{\Gamma_{,\mathcal{X}}\mathcal{X}}{3}(r-1)\right]. \tag{5.67}$$

These contributions can be seen to follow a conservation equation separately once the Bianchi identity and the conservation of matter are applied,

$$\dot{\rho}_g + 3H(\rho_g + P_g) = 0\,. \tag{5.68}$$

They play the role of dark energy at late-times, as we will show in the next subsection. We also have grouped

$$\Gamma = \mathcal{X}^3 c_1 + 3\mathcal{X}^2 c_2 + 3\mathcal{X} c_3 + c_4\,, \qquad J = \mathcal{X}^3 c_0 + 3\mathcal{X}^2 c_1 + 3\mathcal{X} c_2 + c_3\,, \tag{5.69}$$

combinations which can be identified as the contributions of $\mathcal{H}_0$ and $\mathcal{H}_1$ defined in (5.37). Contrary to MTMG, there is no tadpole term, and therefore we do not remove c_0.

5.3.1 Cosmological attractor

We first focus on the equation for λ, E_λ, which can be rewritten as a time-conservation equation

$$\frac{d}{dt}\left[a^{4+\alpha}\mathcal{X}^{1+\alpha}\, J(\mathcal{X})\right] = 0\,. \tag{5.70}$$

implying that

$$\mathcal{X}^{1+\alpha}\, J(\mathcal{X}) = \frac{cst}{a^{4+\alpha}}\,, \tag{5.71}$$

This result strongly motivates considering cosmology on the late-time attractor solutions given by

$$\begin{cases} \mathcal{X} = cst\,, & \text{for } \alpha = -4\,, \\ J = 0\,,\ \mathcal{X} = cst\,, & \text{for } \alpha \neq -4\,. \end{cases} \tag{5.72}$$

For $\alpha > -4$, the system will tend to the second attractor solution $J = 0$. In fact, for $\alpha \geq -1$, the attractor will settle faster than the end of the matter dominated era. When studying cosmology on the attractor, one may thus take

$$\Sigma|_{\rm attractor} = HM_{\rm P}\,, \qquad X|_{\rm attractor} = \frac{M_{\rm P}^2 H^2}{2}\,. \tag{5.73}$$

Whenever $\mathcal{X} = cst$, the graviton potential induces an effective cosmological constant even without c_4, which is the usual bare cosmological constant as found in GR. One therefore has in principle the possibility to sustain self-acceleration. In the remainder of the study, we will always assume $c_4 = 0$. Further requiring that an increase in the bare cosmological constant will always lead to an increase in the Hubble parameter gives the condition

$$g(H) \equiv P_{,X} - 6 + 12G_{,X}H^2M_{\rm P} + 3G_{,XX}H^4M_{\rm P}^3 + H^2M_{\rm P}^2P_{,XX} < 0\,, \quad \text{with } X = X|_{\rm attractor}\,. \tag{5.74}$$

This quantity appears in several expressions for the background and for its perturbations. One may label the complementary scenario as *background ghost*. In the remainder of the work, we also avoid the infinitely fine-tuned case $\alpha = -4$ and the strongly coupled solution $\mathcal{X} = 0$.

Assuming the attractor as a background, we can reach a few more relations. For example, r is given by

$$r - 1|_{\rm attractor} = \frac{6H^2(3G_{,X}H^2M_{\rm P} + P_{,X})}{m^2\Gamma_{,\mathcal{X}}\mathcal{X}} + \frac{6(P_m + \rho_m)}{m^2M_{\rm P}^2\Gamma_{,\mathcal{X}}\mathcal{X}}\frac{g - 3g_1}{g}\,, \tag{5.75}$$

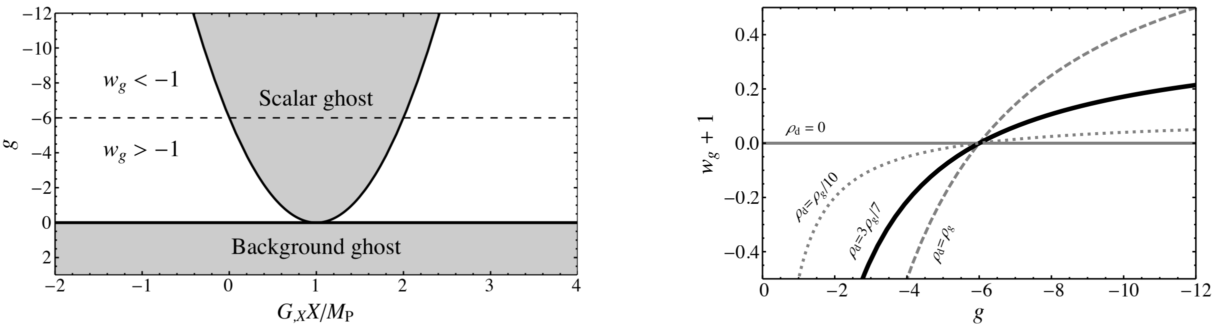

Figure 5.1: *(Figure reproduced from [55].) On the left-hand side*, one finds the unhealthy regions (shaded) of the minimal quasidilaton, in the $(g, G_{,X}X/M_{\rm P})$-parameter space, on the late-time attractor (5.72). There, the functions g and $G_{,X}X$ depend uniquely on Hubble rate H, as given by (5.74) and (5.73). *Background ghost* denotes a decrease in the Hubble rate when the bare cosmological constant is increased. This is avoided by fixing g to be negative. The scalar no-ghost condition 5.93 is obtained from studying the quadratic action for scalar perturbations. *On the right-hand side*, one finds the equation of state of dark energy w_g (5.76), on the attractor, as a function of g. Since the equation of state also depends on the matter density, we present it for different values of the ratio of dust and dark energy densities $\rho_{\rm d}/\rho_g$. Late-times (full gray line) are characterized by an effective cosmological constant. The thick line corresponds to a ratio close to the current ratio, and shows that any measure of the current or the past w_g will heavily constrain g, whereas the quantity $G_{,X}X$ is most constrained by the no-ghost condition. Bounds on $|w_g - 1|$ with e.g. Euclid may be smaller than $\mathcal{O}(10^{-1})$.

where g is given in (5.74) and $g_1 \equiv G_{,X}H^2M_{\rm P} - 2$ are functions of H only since $X = X|_{\rm attractor}$. One may further write the effective equation of state of dark energy $w_g = P_g/\rho_g$ as

$$w_g + 1|_{\rm attractor} = -\frac{(P_m + \rho_m)}{\rho_g}\frac{(g+6)}{g}\,, \tag{5.76}$$

Due to the presence of the attractor, and with the dilution of the matter fields, the Universe transitions to (and ends in) a de Sitter epoch, characterized by $w_g|_{\rm late} = -1$. Corrections close to the attractor appear as powers of $H/H_{\rm late}$.

As seen in equation (5.76), on the attractor, the equation of state depends on the matter fields, in a simple way. It is then possible to characterize its sign as a function of the parameters of the theory. For $-6 < g < 0$ the typical equation of state will be slightly $w_g > -1$ whereas in the complementary cases, the equation of state will be slightly $w_g < -1$. We reproduce Fig. 5.1 from [55] to illustrate these results.

Finally, we discuss the asymptotic Minkowski limit of our theory. This is given by $r \to 1$, since this choice cancels the effective cosmological constant $\rho_g|_{\rm late} = -P_g|_{\rm late}$ (see (5.75)). The contributions from $P(X)$ and $G(X)$ cannot be neglected in general, since this would correspond to a case of infinitely strong coupling.

5.3.2 Perturbed attractor

We now study the linear perturbations on the attractor solution detailed in the previous subsection, with $\alpha > -4$. One may further refine to $\alpha \geq -1$, if one wants to make sure that the attractor solution is reached before or during matter domination. Since a detailed study of the early Universe should include a study of the dynamics away from the attractor, we do not cover radiation fields here. The cosmological constant will also be set to zero by $c_4 = 0$ to study only self-accelerating cases.

Note that for the assumption of the attractor will not be needed in the case of tensor and vector perturbations. We start by these two cases, an then move on to scalar perturbations.

Tensor modes

The quadratic Lagrangian for the tensor perturbations, defined in (2.56) is

$$\mathcal{L} = \frac{M_{\rm P}^2}{8}\sum_{\epsilon=+,\times} Na^3\left[\frac{\dot{h}_\epsilon^2}{N^2} - \frac{1}{a^2}\left(\partial_i h_\epsilon\right)^2 - \mu_T^2\, h_\epsilon^2\right], \tag{5.77}$$

with, as usual, $+$ and $\times$ denoting the two different polarizations of tensor perturbations, and where

$$\mu_T^2 = \frac{1}{6}\mathcal{X}m^2\left[\Gamma_{,\mathcal{X}}\mathcal{X} + (r-1)\frac{\Gamma_{,\mathcal{X}\mathcal{X}}\mathcal{X}^2}{2}\right], \tag{5.78}$$

is the effective mass of the gravitational waves. The tensor mass doesn't vanish in the Minkowski limit $r \to 1$. Notably, one may then write the dispersion relation for modes $h_\epsilon \propto e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ as

$$\omega_T^2 = c_T^2\,\frac{k^2}{a^2} + \mu_T^2\,, \tag{5.79}$$

with $c_T^2 = 1$. Hence, the tensor modes satisfy all the present constraints (which we discussed in section 3.2.1). This result is valid both on and away from the attractor (5.72).

Vector modes

We consider the vector perturbations as defined in (2.55). In addition to these λ^i contributes by

$$\lambda^i = \delta\lambda^i\,, \tag{5.80}$$

since its background value is zero. All vector perturbations are considered to be transverse, i.e. $\partial^i e_i = \partial^i \beta_i = \partial^i \delta\lambda_i = 0$. Using the equations from $\delta\lambda_i$ one may integrate out all degrees of freedom, thus leaving no propagating vector mode. This is the result of the constraints $\mathcal{C}_i$. This result is valid both on and away from the attractor (5.72).

Scalar modes

Scalar modes are chosen as defined in (2.54). We also need to define perturbations for the quasidilaton, as well as the auxiliary fields and Lagrange multipliers that were not yet integrated out:

$$\sigma = \sigma(t) + \delta\sigma\,, \qquad X = X(t) + \delta X\,, \tag{5.81}$$

$$\lambda_\chi = \lambda_\chi(t) + \delta\lambda_\chi\,, \quad \lambda_T = \delta\lambda_T\,, \quad \lambda^i = \frac{1}{a^2}\,\partial_i\lambda_V\,, \quad \lambda = \delta\lambda\,. \tag{5.82}$$

Finally we use the density contrast δ, defined in (2.61) to represent matter perturbations. All the background quantities are chosen to satisfy the attractor equations (5.72). One finds that all but two modes can be integrated out, yielding

$$\mathcal{L} = k^2\,A_{11}\,\dot{\delta}_2^2 - A_{22}\,\dot{\delta\sigma}_2^2 + \mathcal{B}\,(\dot{\delta}_2\,\delta\sigma_2 - \delta_2\,\dot{\delta\sigma}_2) - \mathcal{C}_{11}\,\delta_2^2 - \mathcal{C}_{22}\,\delta\sigma_2^2 - 2\,\mathcal{C}_{12}\,\delta_2\,\delta\sigma_2\,, \tag{5.83}$$

in Fourier space, where we have used the field redefinitions

$$\delta\sigma = \delta\sigma_2 - Z\,\delta\,, \qquad \delta = k\,\delta_2\,, \tag{5.84}$$

with Z a function of k and the background variables (such that the kinetic matrix in (5.83) becomes diagonal), and where the order of each coefficient in the Lagrangian in the high-k limit is

$$A_{11} = \mathcal{O}(k^{-2})\,, \quad A_{22} = \mathcal{O}(k^0)\,, \quad \mathcal{B} = \mathcal{O}(k^{-1})\,, \quad \mathcal{C}_{11} = \mathcal{O}(k^0)\,, \quad \mathcal{C}_{12} = \mathcal{O}(k^1)\,, \quad \mathcal{C}_{22} = \mathcal{O}(k^2)\,. \tag{5.85}$$

We refer the reader to [55] for further details in the decomposition of the terms in (5.85). We have chosen not to present them in this thesis since they can be considered as intermediate results.

Subhorizon approximation—In order to say anything more about the scalar perturbations, we need to take the subhorizon approximation as given by (2.65). The quadratic Lagrangian then becomes

$$\mathcal{L} \approx \frac{1}{2}Na^3\left[Q_1\frac{\dot{\delta}_2^2}{N^2} + Q_2\frac{\dot{\delta\sigma}_2^2}{N^2} + \frac{a}{k}B\left(\frac{\dot{\delta}_2}{N}\,\delta\sigma_2 - \delta_2\,\frac{\dot{\delta\sigma}_2}{N}\right) - L_{11}\,\delta_2^2 - 2\,L_{12}\frac{k}{a}\,\delta_2\,\delta\sigma_2 - L_{22}\frac{k^2}{a^2}\,\delta\sigma_2^2\right], \tag{5.86}$$

where Q_1, Q_2, B, L_{lm} ($l, m \in \{1,2\}$), are all functions of time and of the parameters of the theory, with no more scale dependence. The explicit expression for the coefficients Q_1, Q_2, and L_{ij} are given in Appendix D.2.1, in order to keep a reasonable presentation.

The structure of the kinetic coefficients can be explored further, i.e.

$$Q_1 = a^2\,\rho_m\,, \tag{5.87}$$

$$Q_2 = \frac{\Gamma_1^2}{\Gamma_2^2}\frac{q^2}{4Qd^2}\,, \tag{5.88}$$

with

$$\Gamma_1 \equiv \mathcal{X}\left(c_1\mathcal{X}^2 + 2c_2\mathcal{X} + c_3\right) = \frac{1}{3}\Gamma_{,\mathcal{X}}\mathcal{X}\,, \tag{5.89}$$

$$\Gamma_2 \equiv \mathcal{X}^2\left(c_1\mathcal{X} + c_2\right) = \frac{1}{6}\Gamma_{,\mathcal{X}\mathcal{X}}\mathcal{X}^2\,, \tag{5.90}$$

$$d = m^2\Gamma_1 + 2\,H^2\,(4+\alpha)\left(M_{\rm P}H^2G_{,X} - 2\right), \tag{5.91}$$

$$Q \equiv g + \frac{3}{2}(G_{,X}H^2M_{\rm P} - 2)^2\,, \tag{5.92}$$

where g is the no-background-ghost condition (5.74) (which needs to be positive separately), and where we have chosen to explicit only those coefficients which are relevant to the discussion (see Appendix D.2.1). From these, it is possible to read the no ghost conditions in the high-k limit. The no-ghost conditions for the field δ_2 and $\delta\sigma_2$, read respectively

$$\rho_m > 0\,, \qquad Q > 0\,. \tag{5.93}$$

where the first is always satisfied for canonical matter and the second is a non-trivial condition. Now, from the L_{ij}, it also is possible to read the scalar sound speed in the high-k limit. In particular, for modes for which $\omega^2 = c_s^2\,\frac{k^2}{a^2}$, we have

$$\left(\omega^2\,Q_2 - L_{22}\,\frac{k^2}{a^2}\right)\delta\sigma_2 \approx 0\,, \tag{5.94}$$

so that

$$c_s^2 = \frac{L_{22}}{Q_2}\,, \tag{5.95}$$

which may be different from 1 (but generally $\sim \mathcal{O}(1)$) outside of the GR limit, whereas dust still has zero speed of propagation. The equations extracted from (5.86) are

$$-\frac{d}{dt}\left(a^3\,Q_1\,\frac{\dot\delta_2}{N}\right) - \frac{1}{2}\,\frac{d}{dt}\left(\frac{a^4}{k}\,B\,\delta\sigma_2\right) - \frac{a^4}{2k}\,B\dot{\delta\sigma_2} - N\,a^3\,L_{11}\,\delta_2 - N\,a^3\,L_{12}\,\frac{k}{a}\,\delta\sigma_2 = 0\,, \tag{5.96}$$

$$-\frac{d}{dt}\left(a^3\,Q_2\,\frac{\dot{\delta\sigma_2}}{N}\right) + \frac{a^4}{2k}\,B\,\dot\delta_2 + \frac{1}{2}\,\frac{d}{dt}\left(\frac{a^4}{k}\,B\,\delta_2\right) - N\,a^3\,L_{12}\,\frac{k}{a}\,\delta_2 - N\,a^3\,L_{22}\,\frac{k^2}{a^2}\,\delta\sigma_2 = 0\,, \tag{5.97}$$

a coupled system of equations, which cannot be diagonalized in general.

Quasi-static approximation—Although in the pure subhorizon approximation the equations for the quasidilaton and matter perturbations are coupled, it is possible to explore deeper the phenomenology by taking the quasi-static approximation, as described in equation (2.81), i.e. by assuming $\frac{\ddot{\delta\sigma_2}}{N^2} \simeq H\,\frac{\dot{\delta\sigma_2}}{N} \simeq H^2\,\delta\sigma_2 \ll \frac{k^2}{a^2}\,\delta\sigma_2$. Under the mild assumption, valid within the sound horizon of dark energy (2.79), the friction coefficient B is sub-leading, and it is possible to decouple the equations for δ_2 and $\delta\sigma_2$, the first of which yields

$$\frac{1}{N}\,\frac{d}{dt}\left(\frac{\dot\delta_2}{N}\right) + 2H\,\frac{\dot\delta_2}{N} + \frac{1}{\rho_m a^2}\left(L_{11} - \frac{L_{12}^2}{L_{22}}\right)\delta_2 \approx 0\,. \tag{5.98}$$

This allows to identify the time- and parameter-dependent effective gravitational constant

$$\frac{G_{\rm eff}}{G_N} = \frac{2M_{\rm P}^2}{\rho_m^2 a^2}\left(\frac{L_{12}^2}{L_{22}} - L_{11}\right), \tag{5.99}$$

where the coefficients L_{ij} are detailed in Appendix D.2.1. As it can be easily conceived, and as we will more explicitly show in the next subsection, the effective gravitational constant $G_{\rm eff}$ may be well different from G_N.

We evaluate the Bardeen potentials (2.59) and (2.60), which as usual satisfy Poisson equations, but with modified gravitational constants

$$-\frac{k^2}{a^2}\,\Psi = \frac{1}{2M_{\rm P}^2}\,\frac{G_{\rm eff}}{G_N}\,\rho_m\,\delta\,, \tag{5.100}$$

$$-\frac{k^2}{a^2}\,\Phi = \frac{1}{2M_{\rm P}^2}\,\frac{G_{\Phi}}{G_N}\,\rho_m\,\delta\,, \tag{5.101}$$

where the ratio $G_{\rm eff}/G_N$ is given in Eq. (5.99) and in the appendix D.2.1, whereas we detail the ratio G_Φ/G_N in the appendix D.2.2. Both are time- and parameter-dependent. Both equations can be combined to find the gravitational slip parameter η, given by

$$\eta \equiv \frac{\Psi}{\Phi} = \frac{G_{\rm eff}}{G_\Phi}\,, \tag{5.102}$$

which is also generally different from one, aside from the GR limit. In what follows, we verify the presence of this limit.

We expect that the GR limit is recovered at early times, when $\rho_m \to 3M_{\rm P}^2\,H^2$ and $m^2/H^2 \to 0$. Choosing a simple case for the illustration, $|\Gamma_2| \ll 1$, $G_{,X}H^2M_{\rm P} \to 0$, and $G_{,XX}H^4M_{\rm P}^3 \to 0$, we recover

$$G_{\rm eff} \to G_N\,, \tag{5.103}$$

$$\eta \to 1\,, \tag{5.104}$$

$$Q_2 \to \frac{(\alpha+4)^2\left(M_{\rm P}^2H^2P_{,XX}+P_{,X}\right)\Gamma_1^2}{4\Gamma_2^2}\,, \tag{5.105}$$

$$c_s^2 \to \frac{3}{6-M_{\rm P}^2H^2P_{,XX}-P_{,X}}\,. \tag{5.106}$$

There is therefore at least a region of the parameter space for which the early times limit is healthy and recovers the GR behavior for the growth of structure, provided that $0 < M_{\rm P}^2H^2P_{,XX}+P_{,X} < 6$. The simple limit $m \to 0$ corresponds to a limit towards GR plus the cubic shift-symmetric Horndeski sector.

5.4 Expected cosmological scenario and predictions

The construction of the minimal quasidilaton introduces a range of new parameters, yet it has some unique particularities. Thanks to these future tests may be able to tell apart MQD from other models. In this short section we summarize these particularities, but also more generally the expected cosmological scenario.

Summary of a conservative scenario

In our analysis of the phenomenology, we have chosen to remain conservative with respect to the standard cosmological evolution. If possible, the model should modify only late-times, since the early-time cosmological scenario—from reheating up to the end of the radiation dominated era—is relatively well understood. For this reason, we expect the scenario as summarized in the following table.

Early-time, matter domination	Due to the density of matter fields, and with a good choice of P and G, the modifications of gravity are expected to be negligible.
At or before matter domination	Settling of the attractor $J = 0$. The phenomenology discussed in this work becomes valid.
Attractor epoch	The equation of state can be temporarily $w_g \neq -1$ (see figure 5.1), but will eventually settle to $w_g = -1$ at late times during deep dark energy domination. There are modifications of gravity on unscreened scales.

It should be noted that although there are good reasons to think that early-time cosmology can remain close to GR (e.g. there exists a GR limit), a dedicated analysis would be necessary to be more affirmative.

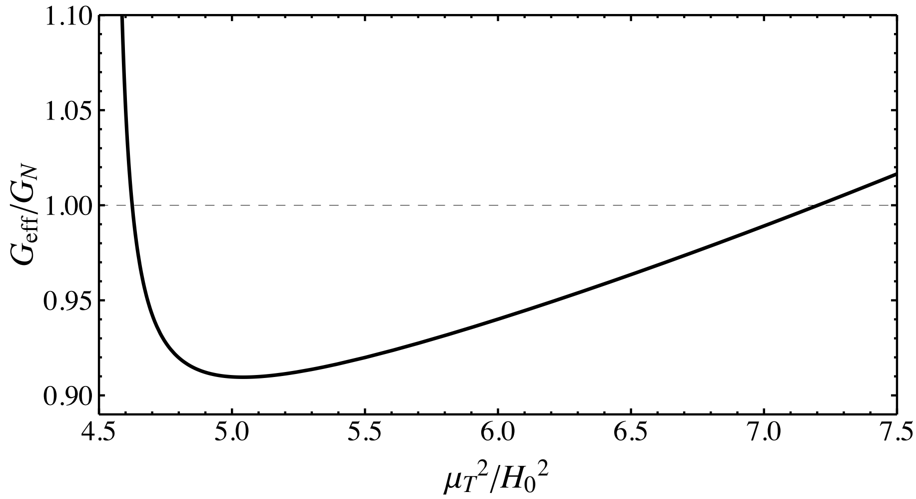

Figure 5.2: Gravitational constant for matter perturbations G_{eff}/G_N, evaluated in the present Universe, for a range of graviton masses μ_T^2/H_0^2, using an illustrative choice of parameters (5.107). In this example weak gravity ($G_{\mathrm{eff}}/G_N < 1$) for matter perturbations may be realized. *Figure reproduced from [55].*

Particular signatures

An essential feature for a model of modified gravity should be its distinguishability from other models. Although it is true that, at background level, the minimal quasidilaton on the attractor can be modelized by a dark energy fluid with a time dependent equation of state, it is possible to distinguish it from a pure fluid dark energy model at perturbative level, most notably due to the presence of non-trivial G_{eff} and η. We argue here that MQD can also be discriminated from a selection of modified gravity models that also modify G_{eff} and η; we discuss (i) scalar-tensor theories, (ii) Lorentz invariant massive gravity and extensions, (iii) Lorentz violating massive gravity such as MTMG.

A common model for dark energy physics is scalar-tensor gravity (see section 3.1.1). Here we would like to compare MQD to these models. Compared with scalar-tensor gravity, say for example Horndeski theories, MQD has an analog number of free functions (the scalar kinetic terms), $P(X)$ and $G(X)$, necessary for the screening, but also implements several additional real parameters that determine the graviton potential, 4 without counting the cosmological constant (say m, c_1, c_2, c_3, since for example c_0 can be absorbed by a redefinition of m). However, there are a positive trade-offs for adding a potential term: notably the model sustains weak gravity, i.e. $G_{\mathrm{eff}}/G_N\big|_{\mathrm{today}} < 1$, while keeping $c_T = 1$ on any background. This is not possible with Lorentz-invariant scalar-tensor theories, as we will discuss below. It can be also noted that late-time acceleration in MQD is due principally to the graviton potential, and not to the choice of free functions in the kinetic term. This means that there is more freedom in choosing $P(X)$ and $G(X)$ to produce a particular cosmology.

In what follows we discuss in more details the weak gravity feature of MQD. First of all, we show with an example choice of parameters that weak gravity can be obtained at late times. We choose

$$\alpha = 0\,, \qquad \Gamma_1 = 1\,, \qquad \Gamma_2 = 3\,, \qquad \Omega_{m,0} \equiv \frac{\rho}{3M_{\mathrm{P}}^2 H_0^2} = 0.3\,,$$
$$P(X) = X\,, \qquad G(X) = cst. \tag{5.107}$$

where all time-dependences are fixed to the present time t_0. As a result, as presented in Fig. 5.2, we obtain a one-parameter (say μ_T^2/H_0^2) family of models which has a wide allowed range of parameter space with the property $G_{\mathrm{eff}}/G_N\big|_{\mathrm{today}} < 1$. Illustratively, for $\mu_T^2/H_0^2 = 5$, one obtains $G_{\mathrm{eff}}/G_N\big|_{\mathrm{today}} \simeq 0.91$.

Second, we refer to section 3.1.1, in particular to equation (3.38) and related, where we followed [167, 168] to show that in the standard Lorentz-invariant scalar-tensor theory case, e.g. in the class of Horndeski theories and beyond, it is difficult to achieve weak gravity. The idea is that it is in fact technically possible to produce weak gravity through the time-dependent “squared Planck mass” G_4, if G_N is defined with the *present-day* value of the Newton constant (defined in a perfectly screened environment, or with the cosmological background). In other words, $G_4(z > 0) > G_4(z = 0)$ is a necessary condition. However other contributions (which can only contribute to increase the gravitational constant) do not favor this in practice. This holds in particular under the requirement $c_T = 1$, which is strongly favored by the recent observations (see section 3.2.4).

We note that although current observations do not yet have enough constraining power, future surveys will be relevantly more efficient to do so. Currently, deviations from $G_{\rm eff}/G_N = 1$ are constrained up to about 10% [85]. It is expected that future surveys will constrain much better $G_{\rm eff}/G_N$ and η (see section 3.2.1), and hence it may be possible to distinguish scalar-tensor from MQD (or similar theories such as MTMG). In fact, future surveys will not only probe the perturbative dynamics of MQD, but it will be also possible to constrain its background dynamics; the same future surveys will thus set strong constraints on g as defined in (5.74), and as can be seen from figure 5.1.

We now discuss possible comparisons between MQD, and other models of massive gravity. First of all the comparison with Lorentz-invariant massive gravity, i.e. dRGT theory, is difficult, as the exact necessary FLRW solutions do not exist, and hence the study of cosmology needs a different treatment. It would be interesting to compare MQD with future investigations of non-FLRW cosmology within dRGT theory. Extensions of dRGT theory that do have FLRW cosmology are for example the new quasidilaton theory, in particular its partially constrained ghost-free form, mass-varying massive gravity, or bigravity. In the case of the new quasidilaton theory, cosmology with matter has not yet been studied within that model. Note that due to the presence of more degrees of freedom the study may be more complex than in MQD. Finally healthy cosmology within bigravity (aside from less explored exotic branches) either has a degenerate phenomenology with GR in the case of a large hierarchy between Planck masses [204], or needs to rely on the Vainshtein mechanism to bypass instability issues [207].

Turning to models of Lorentz-violating massive gravity, we see that it is natural to compare with a version with the same number of degrees of freedom. In [327, 181], it is indeed pointed out that without modifying the kinetic term for gravity, as was instead done in this work, one may obtain theories with three or even two gravitational degrees of freedom. The theory with two degrees of freedom does not allow for self-accelerating solutions. See also [210] for a study of stability. On the other hand the theory with three degrees of freedom shows signs of strong coupling issue on Minkowski backgrounds. One may see the presence of free-functions for the quasidilaton field as the price to pay for a more interesting and varied phenomenology. We expect that observational constraints and the requirement of an efficient Vainshtein screening, may still efficiently restrict and constrain our model.

Finally, we discuss differences between MQD and MTMG. It should be noted that the presence of a dynamical scalar field should generically allow a more varied phenomenology than MTMG (see section 4.5.3), although a more complete investigation of cosmology (including the study of the Vainshtein screening) within MQD is still needed. We nevertheless expect that there may be a regime of sufficient self-acceleration, yet with sizable modifications of gravity at perturbative level, that can fit the observations and yet produce interesting signatures for future surveys to observe.

5.5 Summary

In this chapter, we have presented the *minimal theory of quasidilaton massive gravity* [53, 54, 55], a new model which propagates one scalar field in addition to endowing gravitational waves with a mass. Before introducing the model in section 5.2, we have reviewed, in section 5.1, other models of quasidilaton massive gravity, as well as discussed which of these had possibilities to achieve a viable cosmology. Although the original quasidilaton is unstable on cosmological backgrounds, some of its extensions may allow for viable cosmologies. These extensions have nonetheless not been explored thoroughly. In comparison with these models, the minimal quasidilaton is relevantly simpler (and stable by construction), and this has allowed us to explore the phenomenology in more details. The review of the minimal quasidilaton starts from the Lagrangian presented in a compact fashion in section 5.2.2. We have also discussed its properties via the total Hamiltonian in section 5.2.3. In the latter sections we explored in details the phenomenology, and in particular delineate the expected scenario, comparison with other models of modified gravity, as well as future prospects for constraining the theory in section 5.4.

Chapter 6

Conclusion

From front to end of this thesis, we have gone from depicting the present standard model of gravitation, general relativity (GR), to exploring different new theories of gravity. We are motivated to go beyond Einstein's theory by one of the biggest mysteries of cosmology and hence of gravitational physics: the nature of dark matter and dark energy remains elusive, and therefore we are still unable to converge towards a full picture of our cosmos. Exploring alternatives to GR will also allow to widen our perspective on future observations, as well as to understand GR better theoretically.

Research in modified gravity, i.e. alternatives to general relativity for which the laws of gravitation are modified at given scales, had already started several years ago, for example with the works of Jordan in the '50s. Despite this early start, and notably in light of many fundamental progresses of the last decade (e.g. on the theoretical front, holography, or more specifically to this work, scalar tensor theories, multi-metric theories, etc.), one may arguably say that there is still a lot to learn about gravitational theories, even only classically. As was done in this thesis with minimally modified gravity and the minimal quasidilaton, hopefully new theories of gravity are yet to be found and investigated. In fact, in these last years, on the front of alternative models of gravity, novel constructions have appeared more often than not. This has also gradually allowed to understand how they are related to each other; hopefully this "canvas" of relations may become even clearer with future research.

It is of course well possible that GR is the final theory at large scales, but we hope that Nature will have the last word on this question. At least as long as mysteries persist in our picture of the Universe, it is essential to probe the borders of our knowledge; one should work under the hypothesis that although GR has been favored by experiments so far, a subtle variation may in fact be more accurate at the end of the day. Epistemologically, we are forced to conflate, the understanding of *gravity*, as the force that our Universe exhibits, with the understanding of the *theories of gravity*, which may or may not describe our world. In addition to this, one may understand better the reason, if there is, as to why GR should be favored.

An element that gives reasonable boundaries to such an association is experimental research. Without experiments it would be difficult to give a direction to theoretical investigations. Recently, many constructions, such as a large subclass of scalar-tensor theories, have been severely constrained by gravitational waves, more precisely the observation of a neutron star binary merger via both gravitational waves and X-rays. This experiment has therefore reinforced GR as a solid construction. But this also had another implication: it has cast light on a smaller subset of theories which are still viable as far as it is presently known (theoretically and experimentally). This is a very strong motivation to revisit the ensemble of theories of modified gravity in light of these observations.

In this thesis, we have explored a novel set of models that are, as far as we understand, compatible with the recent observations, in particular those from gravitational wave observations. On the one hand, we have proposed a new class of type-I minimally modified gravity theories, which only propagate two degrees of freedom and have an Einstein frame. On the other hand, as an extension of the minimal theory of massive gravity, and a net simplification of previous works on the quasidilaton theories, we proposed the minimal theory of quasidilaton massive gravity. Both the minimal theory of quasidilaton massive gravity and a large subclass of the type-I minimally modified gravity theories we presented enjoy a unity speed for gravitational waves by construction, and are hence robust with respect to the binary neutron star bound. For type-I minimally modified gravity theories, the time-dependence of the effective gravitational constant at short scales sets a further bound on the theory, but it still allows for sizable deviations of gravity. On the other hand, in the case of the minimal quasidilaton, we made sure it allows for the possibility of a Vainshtein-type screening, which ensures that astrophysical tests are satisfied.

The construction of the minimal quasidilaton is noteworthy simply because it allows to propagate massive gravitons without instabilities (the theory is stable for a wide range of parameters), as recent years have seen several models of massive gravity be plagued by difficulties at the level of cosmology. The *minimization* as we have called it in this text, answers radically to this problematic: less degrees of freedom results in less potential unstable modes, which could invalidate the realization of a cosmological scenario. Thanks to it we obtained a model that can sustain self-acceleration, while having interesting phenomenology at perturbative level.

Considering minimal or minimized theories also allows for simplicity of treatment. For minimal theories, one does not need to rely on certain common approximations in the field of dark energy, such as the quasi-static approximation. Since the minimal quasidilaton propagates a third degree of freedom, we had to take this approximation. Within the context of Lorentz violating theories, *minimalism*, in the sense of having the least number of degrees of freedom, is a valid new take on the generally used principle: one should describe the world with the minimal number of extra assumptions, as has been argued by Occam's razor-type arguments.

Important observations will also be made in the coming years, with increased precision. The sensitivity of these new experiments is therefore another clear pointer as to which directions to explore in modified gravity: proposed new theories should have a chance to be constrained or reinforced by observations in a foreseeable future. The theories proposed in this thesis satisfy such a requirement. They will be indeed considerably constrained by the coming experiments, in particular those that will put constraints on weak lensing and structure growth. More interestingly, the theories we presented propose novel potentially detectable phenomenology, which is difficult to obtain in Lorentz-invariant scalar-tensor theories. An example is the fact that the minimal quasidilaton can sustain weak gravity in the context of cosmological matter perturbations. The situation is similar with type-I (and type-II) minimally modified gravity, which also propose interesting differences within perturbative dynamics. Among the future observations, the fate of current tensions within the standard model of cosmology will also be of special interest in the context of modified gravity. It was shown that modified gravity can relieve certain tensions [107], and if the mismatch is confirmed new elements will be called to explain the observations. The type-I minimally modified gravity may then offer a simple and minimal model to explain this mismatch.

The construction of interesting new theories of gravity might have seemed difficult from the point of view of Lovelock's theorem, which argues on the unicity of GR, under a set of assumptions. In fact, the theories presented in chapters 4 and 5 all rely on relaxing the symmetry under diffeomorphisms and under Lorentz transformations, both of which characterize GR. In the exploration of the wide range of theories that comes with the removal of these base assumptions, minimalism can be seen as a systematic guiding principle, obviously together with consistency of the theories. The results presented in this thesis all support this claim further. We have also shown that the prospects for finding new, interesting, and viable theories of modified gravity are good, and that the possibilities of finding novel theories in the future have a good chance not to be exhausted. The diagrammatic depiction of figure 4.1, for example, makes it clear that the space of minimally modified gravity theories can still be populated, and that the relationships between these minimal theories can still be investigated. We are certainly far from understanding the extent of minimal theories of gravity.

In the future, it will be essential to relate further these new constructions with observations. On the one hand, within current model-independent implementations of cosmological Einstein Boltzmann solvers, one often makes some assumptions to limit the complexity of the searches. It is therefore important to inquire model by model whether the usual simplified parametrizations are compatible, or whether model specific changes have to be implemented within the numerical schemes. Analytical advances may also be needed; for example, in the case of the minimal quasidilaton, further understanding of early-time dynamics and of the screening mechanism will be essential. On the other hand, with the advent of gravitational waves, it will be important to study more in details both astrophysical and strong field regimes, since these may be tested in the future. For this ultimately one will need a more detailed understanding of the solutions for compact objects (for example in the context of the minimal theory of massive gravity [52]), a study of their perturbations and their stability, as well as a study of modified gravitational wave production. Understanding gravitational wave production within GR has been technically (analytically and numerically) challenging (for some references see section 3.2.3), at it is fair to say that more effort will be needed within modified gravity theories.

The question of quantum corrections within modified gravity theories is important, and calls for a future in-depth study. On the one hand, one may show explicitly the suppression (by power counting of order $\mathcal{O}(m^2/M_{\rm P}^2)$) of Lorentz violating corrections appearing in the matter sector through graviton loops.

On the other hand, the constraint structures of some minimally modified gravity theories (this is of course also valid beyond this class) relies only on second class constraints, and hence are not protected against quantum corrections by some symmetries. UV cutoffs, for which the perturbative expansion becomes questionable at best, may also be found, for example within the minimal quasidilaton theory. Indeed, these cutoffs are found within several classes of scalar tensor, or massive gravity theories. Finally, still on the theoretical side, the search of novel models, and the delineation of the boundaries of minimally modified gravity theories will be crucial. Pioneering systematic studies have already been successful for type-I minimally modified gravity theories, whereas type-II may need a new approach to be studied more systematically. A generic Hamiltonian approach as in [57] may be a first step in this direction.

Appendix A

Tools for cosmology

A.1 Gauge transformation of the perturbations

Under diffeomorphisms $x^\mu \to x^\mu + \xi^\mu$, the first order perturbed metric transforms as

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi^\lambda + g^{(0)}_{\mu\lambda}\xi^\lambda_{,\nu} + g^{(0)}_{\nu\lambda}\xi^\lambda_{,\mu}\,. \tag{A.1}$$

By applying this to the 00, $0i$, and ij components, one can find

$$\tilde{\phi} = \phi + \frac{\dot{N}}{N}\xi^0 + \dot{\xi}^0\,, \tag{A.2}$$

$$\tilde{\beta} = \beta - \frac{N}{a}\xi^0 + \frac{a}{N}\dot{\xi}\,, \tag{A.3}$$

$$\tilde{B}_i = B_i + \frac{a}{N}\dot{\xi}^\perp_i\,, \tag{A.4}$$

$$\tilde{\psi} = \psi + HN\xi^0 + \frac{1}{3}\Delta\xi\,, \tag{A.5}$$

$$\tilde{e} = e + 2\xi\,, \tag{A.6}$$

$$\tilde{E}_i = E_i + 2\xi^\perp_i\,, \tag{A.7}$$

$$\tilde{h}_{ij} = h_{ij}\,, \tag{A.8}$$

where we decompose $\xi_i = \xi^\perp_i + \partial_i\xi$, with $\partial^i\xi^\perp_i = 0$. For a perfect fluid, one has the transformation of the density perturbation ($\rho \to \rho + \delta\rho$), at first order,

$$\tilde{\delta\rho} = \delta\rho + \dot{\rho}\xi^0\,. \tag{A.9}$$

By using these transformations it is simple to find gauge invariant variables, such as (2.59), (2.60), and (2.61).

A.2 Integrating by parts the friction matrix

Here we give, for reference, a reminder on the integration by parts of the friction matrix (multiplying terms with a single time-derivative over the perturbation variables). Assuming the terms in the quadratic action for perturbations

$$S^{(2)} \ni a\phi_1\dot{\phi}_2 + b\phi_2\dot{\phi}_1\,, \tag{A.10}$$

one can split the symmetric and antisymmetric part.

$$S^{(2)} \ni \frac{a+b}{2}\left(\phi_1\dot{\phi}_2 + \phi_2\dot{\phi}_1\right) + \frac{a-b}{2}\left(\phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1\right)\,. \tag{A.11}$$

The total derivative to be subtracted from the action is simply

$$S_{\rm totder} \equiv \frac{a+b}{2}\left(\phi_1\dot{\phi}_2 + \phi_2\dot{\phi}_1\right) + \frac{\dot{a}+\dot{b}}{2}\phi_1\phi_2\,, \tag{A.12}$$

which removes the symmetric part of the friction matrix.

A.3 Extracting the sound speeds

The equations of motion for the remaining perturbations (after all non-dynamical fields are integrated out) can be given in matrix form. One starts from the action at that stage, given by

$$S^{(2)} = \dot{\vec{\phi}}^\top \mathcal{K} \dot{\vec{\phi}} + \dot{\vec{\phi}}^\top \mathcal{F} \vec{\phi} + \vec{\phi} \mathcal{M} \vec{\phi} \,, \tag{A.13}$$

where $\vec{\phi}$ is a column vector containing the remaining perturbations, $\mathcal{K}$ is the kinetic matrix and it can be made diagonal, $\mathcal{F}$ is the friction matrix and it can be made antisymmetric by integration by parts, and $\mathcal{M}$ is the mass matrix and doesn't have any particular symmetries a priori. The equations of motion then are

$$-2\mathcal{K}\ddot{\vec{\phi}} - \frac{d\mathcal{F}}{dt}\vec{\phi} + 2\mathcal{M}\vec{\phi} = 0 \,, \tag{A.14}$$

which in the case of oscillating fields, $\ddot{\vec{\phi}} = -\Omega^2 \vec{\phi}$, reduces to

$$\left(\Omega^2 \mathcal{K} - \frac{1}{2}\frac{d\mathcal{F}}{dt} + \mathcal{M}\right)\vec{\phi} = 0 \,. \tag{A.15}$$

The solution is non-trivial only if

$$\det\left(\Omega^2 \mathcal{K} - \frac{1}{2}\frac{d\mathcal{F}}{dt} + \mathcal{M}\right) \equiv \det \mathcal{E} = 0 \tag{A.16}$$

One needs to consider the determinant of the equations of motion ($\det \mathcal{E}$), in the subhorizon limit, where $k, \Omega \gg 1$. In this case, it can be factorized as

$$A(\Omega^2 - c_{\mathrm{s},1}^2 k^2) \dots (\Omega^2 - c_{\mathrm{s},n}^2 k^2) \,, \tag{A.17}$$

where n is the number of degrees of freedom, A is some coefficient which—if the kinetic matrix is properly normalized—does not contain any power of k, and $c_{\mathrm{s},i}$ is a sound speed. By collecting the coefficients of different powers of k, one can obtain expressions for the partial sums and products of the sound speeds $c_{\mathrm{s},i}$. In the case in which one or several of these sound speeds are 0, then a corresponding power of Ω^2 will be factorisable.

A.3.1 Matter action for cosmology

Depending on the observables one wants to compute it is often convenient (and we will use this approach) to work directly in from the Lagrangian, as opposed to working with the equations of motion. In order to do so, one has to write not only the gravitational action but also the matter part of the theory. However, the matter action involves a lot of complex microphysics, which represents an unnecessary complication for understanding the physics on large scales. It is therefore customary to work with a simple proxy matter sector that is known to at least reproduce the physics of perfect fluids and their perturbations. Let us define one such proxies which will be used throughout the text.

One option is to use a so-called k-essence scalar field. Naming the field ϕ_m, its action reads

$$S_{\mathrm{mat}} = \int d^4x \sqrt{-g} P(X) \,, \qquad \mathfrak{X}_m \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi_m \partial_\nu \phi_m \,, \tag{A.18}$$

where, as indicated by the definition, X is the canonical kinetic term for a scalar field. Computing the energy-momentum tensor for the field yields

$$T_{\mu\nu} = P_{,X} \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} P \,, \tag{A.19}$$

and for a homogeneous and isotropic configuration one may identify

$$P \equiv P(X) \,, \quad \rho \equiv 2P_{,X} X - P(X) \,, \quad w \equiv \frac{P}{\rho} = \frac{P}{2P_{,X}X - P(X)} \,, \quad c_s^2 \equiv \frac{P_{,X}}{\rho_{,X}} = \frac{P_{,X}}{2P_{,XX}X + P_{,X}} \,. \tag{A.20}$$

where c_s is the typical speed of propagation of the perturbations in this k-essence component. After making the right replacements one can then adjust directly the equation of state, the pressure, and so forth.

In the treatment of perturbations it is then simple to change variables, from the k-essence perturbations, to the density perturbations $\delta\rho$ that we have defined in the previous sections. Perturbing linearly the expression for the density, and matching with $\rho' = \rho + \delta\rho$,

$$\delta\rho = (2P_{,XX}X + P_{,X})\,\delta X = \frac{1}{2Xc_s^2}\,(\rho + P)\,\delta X = \frac{\rho + P}{c_s^2}\left(\frac{\dot{\delta\phi}}{\dot{\phi}} - \Phi\right) \tag{A.21}$$

One can replace the k-essence perturbations with density perturbations by adding the adequate piece (constraint) to the action. The idea is to replace the kinetic term of the k-essence perturbation, by the corresponding value in terms of density perturbations. One adds, for example in Fourier space,

$$-\int d^3k\,dt\,a^3\frac{c_s^2}{2(P+\rho)}(\delta\rho - \bar{\delta\rho})^2\,, \tag{A.22}$$

up to some coefficient, such that the kinetic term of $\delta\phi$ vanishes, and the field can therefore be integrated out. Here $\delta\rho$ stands for its expression in terms of the k-essence and other perturbations.

Appendix B

Expressions for MTMG

B.1 Action and Hamiltonian

In the section 4.5.3 we have presented the construction of MTMG mostly schematically. However, the explicit details between each argument are rather tedious. Therefore, we give here, for reference, some important expressions and results.

In [311], the metric formulation, which we have implicitly used, was introduced. In addition to the square-root matrix $\mathfrak{K}^i{}_j \equiv (\sqrt{\gamma^{-1}\tilde{\gamma}})^i{}_j$, defined by (4.89), one may define its inverse $\mathcal{K}$ by

$$\mathcal{K}^a{}_c\mathfrak{K}^c{}_b = \mathfrak{K}^a{}_c\mathcal{K}^c{}_b = \delta^a_b\,, \qquad \mathcal{K}^a{}_c\mathcal{K}^c{}_b = \tilde{\gamma}^{ac}\gamma_{cb}\,. \tag{B.1}$$

Passing from the vielbein formalism to the metric formalism can only be achieved under the symmetric condition (4.76)

$$Y_{[IJ]} = 0\,. \tag{B.2}$$

In [51, 311] the theory was constructed from the perspective of the vielbein formalism. It is however possible to construct MTMG from a precursor theory entirely in the metric language. During the construction of the secondary Hamiltonian, one encounters the variations

$$\frac{\delta\mathcal{K}}{\delta\gamma_{ij}} = \frac{1}{2}\gamma^{k(i}\mathcal{K}^{j)}{}_k\,, \qquad \frac{\delta\mathfrak{K}}{\delta\gamma_{ij}} = -\frac{1}{2}\gamma^{k(i}\mathfrak{K}^{j)}{}_k\,, \tag{B.3}$$

which are easily obtained from the definition of $\mathfrak{K}^a{}_b$ and $\mathcal{K}^a{}_b$. We have noted $\mathfrak{K}$ and $\mathcal{K}$ the traces of the respective matrices. On the other hand, the variation $\frac{\delta\mathcal{K}^a{}_b}{\delta\gamma_{ij}}$ is not uniquely defined. This could seem to be a problem both for finding the equations of motion and for computing the secondary constraint algebra; indeed, the secondary Hamiltonian, in particular $\mathcal{C}_0$ and $\mathcal{C}_i$ contain such factors. Fortunately, the square root becomes (almost) uniquely defined in certain situations, for example if the expansion is done for a diagonal background (as in cosmology). In order to make contact with the choice of vielbein picture $\mathcal{K}^a{}_b = E_I{}^a e^I{}_b$ as in [311], we choose

$$\frac{\delta\mathcal{K}^a{}_b}{\delta\gamma_{ij}} = \frac{1}{2}\mathcal{K}^a{}_k\gamma^{k(j}\delta^{i)}_b\,, \qquad \frac{\delta\mathfrak{K}^a{}_b}{\delta\gamma_{ij}} = -\frac{1}{2}\gamma^{a(i}\mathfrak{K}^{j)}{}_b\,. \tag{B.4}$$

This allows to complete the analysis. We also give the following definition

$$\mathcal{F}^{ij} = \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}}\left[(c_1 + c_2\mathcal{K})\left(\mathcal{K}^i{}_k\gamma^{kj} + \gamma^{ik}\mathcal{K}^j{}_k\right) - 2c_2\tilde{\gamma}^{ij}\right] + 2c_3\gamma^{ij}\,, \tag{B.5}$$

which should not be confused with the analog expression for MQD (D.1).

Appendix C

Cubic Horndeski

We present here the steps leading to the complete Hamiltonian analysis of a cubic shift-symmetric Horndeski theory, using the method of auxiliary fields. This appendix is referred to in section 3.1.1, and in chapter 5, as the cubic Horndeski Lagrangian is invoked as the kinetic term for the quasidilaton scalar. This appendix is largely reproduced from [54].

C.1 Lagrangian of H3

The full Lagrangian density of the cubic Horndeski theory is given by

$$\mathcal{L}_{\mathrm{H3}} = \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_3 \,. \tag{C.1}$$

where $\mathcal{L}_{\mathrm{EH}} = M_{\mathrm{P}}^2\sqrt{-g}R[g]/2$ is the Einstein-Hilbert Lagrangian density, without cosmological constant. We define the scalar part of the (now restricting to shift invariant) cubic Horndeski Lagrangian by use of Lagrange multipliers

$$\mathcal{L}_3 = \sqrt{-g}\,[F(X,S) + \chi\,(X - \mathfrak{X}) + \theta S + g^{\mu\nu}\partial_\mu\theta\partial_\nu\sigma]\,, \tag{C.2}$$

where we write the canonical kinetic term for the scalar field σ as

$$\mathfrak{X} \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma, \tag{C.3}$$

and where

$$F(X,S) \equiv P(X) - G(X)S\,, \tag{C.4}$$

in which $P(X)$ and $G(X)$ are sufficiently well-behaved general functions. The Lagrangian density (C.1) is equivalent to the usual expression of the cubic Horndeski Lagrangian density once the e.o.m. of X, χ θ, and S are taken into account. The e.o.m. of X, χ θ, and S, calculated from (C.1), are respectively

$$\begin{cases} F_{,X} + \chi = 0\,, \\ X - \mathfrak{X} = 0\,, \\ S - \Box\sigma = 0\,, \\ \theta - G(X) = 0\,. \end{cases} \tag{C.5}$$

where we have used subscripts after a comma to denote derivatives, for instance, $F_{,X} \equiv \frac{\partial F}{\partial X}$. The system of equations is trivially solved by $\chi = -F_{,X}$, $X = \mathfrak{X}$, $\theta = G(\mathfrak{X})$, and $S = \Box\sigma$, and after replacing this solution in the Lagrangian density (C.1) one recovers its standard form (3.7). By using Lagrange multipliers one can evade all second or higher time derivatives. The equation of motion for the scalar field σ is

$$\chi\Box\sigma + \Box\theta + g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\sigma = 0\,, \tag{C.6}$$

which also reduces to the usual equation of motion once the auxiliary fields have been integrated out,

$$F_{,\mathfrak{X}}\,\Box\sigma - G_{,\mathfrak{X}}\,\Box(\mathfrak{X}) + g^{\mu\nu}F_{,\mathfrak{X}\mathfrak{X}}\,\nabla_\mu(\mathfrak{X})\nabla_\nu\sigma = 0\,. \tag{C.7}$$

We use here the (3+1) ADM decomposition of the 4-dimensional metric, which necessitates to define the lapse function N, the shift vector N^i as well as the spatial 3-dimensional metric γ_{ij}. These are defined via the line element (2.14). The indices $i, j, \dots \in \{1,2,3\}$ are used as spatial indices. The 4-dimensional metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are then given by

$$g_{00} = -N^2 + \gamma_{ij}N^iN^j \,, \quad g_{0i} = \gamma_{ij}N^i \,, \quad g_{ij} = \gamma_{ij} \,, \tag{C.8}$$

$$g^{00} = -\frac{1}{N^2} \,, \quad g^{0i} = \frac{N^i}{N^2} \,, \quad g^{ij} = \gamma^{ij} - \frac{N^iN^j}{N^2} \,, \tag{C.9}$$

where the spatial indices of the lapse function are raised and lowered using the spatial metric γ_{ij}, and its inverse γ^{ij}. In Eq. (C.3) we have used, and define from here on the "normal" derivative to the spatial hypersurfaces,

$$\partial_\perp * = \frac{1}{N}\left(\dot{*} - N^i\partial_i *\right) \,, \tag{C.10}$$

where $*$ stands for any field. Using these definitions we have for example that

$$\mathfrak{X} = \frac{1}{2}\left[(\partial_\perp\sigma)^2 - \gamma^{ij}\partial_i\sigma\partial_j\sigma\right] . \tag{C.11}$$

C.2 Variables, conjugated momenta, and primary constraints

We consider $\{\gamma_{(ij)}, \sigma, X, \chi, \theta, S\}$ and their conjugate momenta as 22 canonical variables, and $\{N, N^i\}$ as Lagrange multipliers, as these only appear linearly in the action. Upon calculating the conjugate momenta, we get

$$\pi^{ij} \equiv \frac{M_{\rm P}^2}{2}\sqrt{\gamma}\left(K^{ij} - K\gamma^{ij}\right) \,, \quad \pi_\sigma \equiv -\sqrt{\gamma}(\chi\partial_\perp\sigma + \partial_\perp\theta) \,, \quad \pi_\theta \equiv -\sqrt{\gamma}(\partial_\perp\sigma) \,, \tag{C.12}$$

$$\pi_\chi = 0 \,, \quad \pi_S = 0 \,, \quad \pi_X = 0 \,, \tag{C.13}$$

where

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - \mathcal{D}_iN_j - \mathcal{D}_jN_i\right) . \tag{C.14}$$

As an intermediate step before computing the Hamiltonian, we invert relations (C.12) as

$$\dot{\gamma}_{ij} = 2NK_{ij}(\pi^{kl}) + \mathcal{D}_iN_j + \mathcal{D}_jN_i \tag{C.15}$$

$$\dot{\sigma} = -N\tilde{\pi}_\theta + N^i\partial_i\sigma \tag{C.16}$$

$$\dot{\theta} = N(\chi\tilde{\pi}_\theta - \tilde{\pi}_\sigma) + N^i\partial_i\theta \,. \tag{C.17}$$

We have found useful to define the tilded momenta as three dimensional scalars, i.e., for instance, $\tilde{\pi}_\theta \equiv \frac{\pi_\theta}{\sqrt{\gamma}}$. In addition to previous relations, the primary constraints related to X, χ, and S are defined as

$$0 = P_X \equiv \pi_X \,, \quad 0 = P_\chi \equiv \pi_\chi \,, \quad 0 = P_S \equiv \pi_S \,. \tag{C.18}$$

C.3 Primary Hamiltonian, constraint algebra, and consistency conditions

The Hamiltonian with all primary constraints can be now written as

$$\mathfrak{H}^{(1)}_{\rm H3} = \int d^3x \left[-N\mathcal{R}_0 - N^i\mathcal{R}_i + \xi_XP_X + \xi_\chi P_\chi + \xi_SP_S\right] , \tag{C.19}$$

where the Lagrange multipliers λ_X, λ_χ and λ_S are scalars (density weight 0) of mass dimension 5, 3, and 4, respectively and where we define

$$\begin{aligned}
\mathcal{R}_0 &= \frac{M_{\rm P}^2}{2}\sqrt{\gamma}\,R[\gamma] - \frac{2}{M_{\rm P}^2}\sqrt{\gamma}\left(\gamma_{il}\gamma_{jk} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\right)\tilde{\pi}^{ij}\tilde{\pi}^{kl} - \mathcal{H}_\sigma \,, \\
\mathcal{H}_\sigma &= \sqrt{\gamma}\left[\frac{\chi}{2}\tilde{\pi}_\theta^2 - \tilde{\pi}_\theta\tilde{\pi}_\sigma - F - \chi\left(X + \frac{1}{2}\gamma^{ij}\partial_i\sigma\partial_j\sigma\right) - \theta S - \gamma^{ij}\partial_i\sigma\partial_j\theta\right] , \\
\mathcal{R}_i &= 2\sqrt{\gamma}\gamma_{ik}\mathcal{D}_j\tilde{\pi}^{kj} - \pi_\sigma\partial_i\sigma - \pi_\theta\partial_i\theta \,.
\end{aligned}$$

Just as in general relativity, the Hamiltonian is vanishing on the constraint surface. For further use we define the equivalent of the scalar canonical term $\mathfrak{X}$ in the Hamiltonian language,

$$\mathfrak{X}_H \equiv \frac{1}{2}\left(\tilde{\pi}_\theta^2 - \gamma^{ij}\partial_i\sigma\partial_j\sigma\right) . \tag{C.20}$$

From here one can compute the algebra of the primary constraints. The relations will be helpful for computing the evolution of the constraints. First, it is found that if we want to have the momentum constraint as a generator for translations, we need to modify it so as to include π_X, π_χ, and π_S. We thus define

$$\tilde{\mathcal{R}}_i \equiv 2\sqrt{\gamma}\gamma_{ik}\mathcal{D}_j\tilde{\pi}^{kj} - \pi_\sigma\partial_i\sigma - \pi_X\partial_i X - \pi_\chi\partial_i\chi - \pi_S\partial_i S - \pi_\theta\partial_i\theta\,. \tag{C.21}$$

Clearly, this momentum constraint is equivalent to the original one when restricted to the constraint surface. We turn to analyzing the constraint algebra, which is summarized in table C.1[1].

$\{\downarrow,\rightarrow\}$	$\mathcal{R}_0$	$\tilde{\mathcal{R}}_i$	P_X	P_χ	P_S
$\mathcal{R}_0$	0	0	$-(\chi + F_{,X})$	$-(X - \mathfrak{X}_H)$	$-(\theta - G(X))$
$\tilde{\mathcal{R}}_i$		0	0	0	0
P_X			0	0	0
P_χ				0	0
P_S					0

Table C.1: Primary constraint algebra in the cubic Horndeski theory. Dirac δ-functions were omitted in the entries.

We give some results in their integral form,

$$\{P_\star, \tilde{\mathcal{R}}_i[f^i]\} = -\int d^3x\sqrt{\gamma}\mathcal{D}_i\big(P_\star f^i\big) \approx 0\,, \tag{C.22}$$

$$\{\mathcal{R}_0[\phi_2], \mathcal{R}_0[\phi_2]\} = \int d^3x\mathcal{R}_i\left(\phi_1\mathcal{D}^i\phi_2 - \phi_2\mathcal{D}^i\phi_1\right) \approx 0\,, \tag{C.23}$$

$$\{\mathcal{R}_0[\phi], \tilde{\mathcal{R}}_i[f^i]\} = \int d^3x\mathcal{R}_0 f^i\mathcal{D}_i\phi \approx 0\,, \tag{C.24}$$

$$\{\tilde{\mathcal{R}}_i[f^i], \tilde{\mathcal{R}}_j[g^j]\} = \int d^3x\tilde{\mathcal{R}}_i\left(g^j\mathcal{D}_j f^i - f^j\mathcal{D}_j g^i\right) \approx 0\,. \tag{C.25}$$

One can observe that the Hamiltonian and momentum constraints obey the usual algebra. In equation (C.22) the symbol $\star$ stands for any of X, χ, and S.

Armed by the complete algebra of constraints one can move on to study the consistency conditions. Consistency of the primary constraints P_X, P_χ, and P_S with the time evolution of the system yields the following conditions which cannot be solved for Lagrange multipliers (unless one sets N to be zero, which is unphysical):

$$\dot{P}_X \equiv \sqrt{\gamma}\{\pi_X, \mathfrak{H}^{(1)}_{\mathrm{H3}}\} = N\sqrt{\gamma}(\chi + F_{,X}\,) \approx 0 \tag{C.26}$$

$$\dot{P}_\chi \equiv \sqrt{\gamma}\{\pi_\chi, \mathfrak{H}^{(1)}_{\mathrm{H3}}\} = N\sqrt{\gamma}(X - \mathfrak{X}_H) \approx 0 \tag{C.27}$$

$$\dot{P}_S \equiv \sqrt{\gamma}\{\pi_S, \mathfrak{H}^{(1)}_{\mathrm{H3}}\} = N\sqrt{\gamma}(\theta - G(X)) \approx 0\,. \tag{C.28}$$

We thus use these conditions to define the secondary constraints

$$S_X(X,\chi,S) \equiv \chi + F_{,X}\ , \tag{C.29}$$

$$S_\chi(\gamma,\sigma,X,\pi_\theta) \equiv X - \mathfrak{X}_H\,, \tag{C.30}$$

$$S_S(\theta,X) \equiv \theta - G(X)\,. \tag{C.31}$$

[1] In the entries of all tables we have omitted Dirac δ-functions, unless otherwise stated. In particular, we give more details whenever the result of the Poisson brackets formally includes derivatives of δ-functions – i.e. when these cannot be factorized out.

C.4 Secondary Hamiltonian, constraint algebra, and consistency conditions

We can now define the Hamiltonian with all primary and secondary constraints

$$\mathfrak{H}_{\mathrm{H3}}^{(2)} = \int d^3x \left[-N\mathcal{R}_0 - N^i\tilde{\mathcal{R}}_i + \xi_X P_X + \xi_\chi P_\chi + \xi_S P_S + \sqrt{\gamma}\left(\lambda_X S_X + \lambda_\chi S_\chi + \lambda_S S_S\right)\right], \tag{C.32}$$

where the Lagrange multipliers λ_X, λ_χ, and λ_S are spatial scalars, i.e. have a density weight of 0, of mass dimension 4, 0, and 3, respectively.

$\{\downarrow,\rightarrow\}$	$\mathcal{R}_0$	$\tilde{\mathcal{R}}_i$	P_X	P_χ	P_S	S_X	S_χ	S_S
$\mathcal{R}_0$	0	0	0	0	0	0	T_χ	T_S
$\tilde{\mathcal{R}}_i$		0	0	0	0	0	0	0
P_X			0	0	0	$-F_{,XX}$	-1	$G_{,X}$
P_χ				0	0	-1	0	0
P_S					0	$G_{,X}$	0	0
S_X						0	0	0
S_χ							0	$\pi_\theta/\sqrt{\gamma}$
S_S								0

Table C.2: Secondary constraint algebra in the cubic Horndeski theory. Dirac δ-functions were omitted in the entries.

The secondary constraint algebra is summarized in table C.2 where T_χ and T_S stand for

$$T_\chi = \frac{2}{M_{\mathrm{P}}^2}\tilde{\pi}^{ij}\left[\gamma_{ij}\mathfrak{X}_H - \mathcal{D}_i\sigma\mathcal{D}_j\sigma\right] - \tilde{\pi}_\theta\left[S - \gamma^{ij}\mathcal{D}_i\mathcal{D}_j\sigma\right] - \gamma^{ij}\mathcal{D}_j\sigma\mathcal{D}_i\tilde{\pi}_\theta\,, \tag{C.33}$$

$$T_S = \chi\tilde{\pi}_\theta - \tilde{\pi}_\sigma\,. \tag{C.34}$$

One can simplify a little the algebra by defining a revised version of the Hamiltonian constraint,

$$\tilde{\mathcal{R}}_0 = \mathcal{R}_0 - T_\chi(P_X - F_{,XX}P_\chi)\,, \tag{C.35}$$

which in turn yields Table C.3.

$\{\downarrow,\rightarrow\}$	$\tilde{\mathcal{R}}_0$	$\tilde{\mathcal{R}}_i$	P_X	P_χ	P_S	S_X	S_χ	S_S
$\tilde{\mathcal{R}}_0$	0	0	0	0	0	0	0	$T_S + G_{,X}T_\chi$
$\tilde{\mathcal{R}}_i$		0	0	0	0	0	0	0
P_X			0	0	0	$-F_{,XX}$	-1	$G_{,X}$
P_χ				0	0	-1	0	0
P_S					0	$G_{,X}$	0	0
S_X						0	0	0
S_χ							0	$\pi_\theta/\sqrt{\gamma}$
S_S								0

Table C.3: Secondary constraint algebra in the cubic Horndeski theory, with modified Hamiltonian constraint. Dirac δ-functions were omitted in the entries.

The consistency conditions yield the following equations

$$\dot{P}_\chi \approx 0 \approx \dot{P}_S : \lambda_X \approx 0 \tag{C.36}$$

$$\dot{P}_X \approx 0 : \lambda_\chi - \lambda_S G_{,X} \approx 0 \tag{C.37}$$

$$\dot{S}_\chi \approx 0 : \xi_X + \lambda_S\tilde{\pi}_\theta \approx 0 \tag{C.38}$$

$$\dot{S}_X \approx 0 : \xi_X F_{,XX} + \xi_\chi - \xi_S G_{,X} \approx 0 \tag{C.39}$$

$$\dot{S}_S \approx 0 : N(T_S + G_{,X}T_\chi) + \xi_X G_{,X} + \lambda_\chi\tilde{\pi}_\theta \approx 0 \tag{C.40}$$

$$\dot{\tilde{\mathcal{R}}}_0 \approx 0 : \lambda_S(T_S + G_{,X}T_\chi) \approx 0\,. \tag{C.41}$$

By plugging Eqs. (C.37) and (C.38) into Eq. (C.40), we obtain

$$N(T_S + G_{,X}T_\chi) = 0. \tag{C.42}$$

As a consequence, since setting the lapse to zero would be unphysical, we need to impose the tertiary constraint $T \approx 0$, where

$$T = T_S + G_{,X}T_\chi = \chi\tilde{\pi}_\theta - \tilde{\pi}_\sigma - G_{,X}\left[\frac{2}{M_{\mathrm{P}}^2}\tilde{\pi}^{ij}\left(\mathcal{D}_i\sigma\mathcal{D}_j\sigma + \gamma_{ij}\mathfrak{X}_H\right) + \tilde{\pi}_\theta\left(S - \gamma^{ij}\mathcal{D}_i\mathcal{D}_j\sigma\right) + \gamma^{ij}\mathcal{D}_j\sigma\mathcal{D}_i\tilde{\pi}_\theta\right] \tag{C.43}$$

In order to simplify further the constraint algebra, we can form the following combinations

$$\tilde{P}_X = P_X - F_{,XX}P_\chi\,, \tag{C.44}$$
$$\tilde{P}_S = \frac{1}{\sqrt{\gamma}}\left(P_S + G_{,X}P_\chi\right)\,, \tag{C.45}$$
$$\tilde{S}_S = S_S + G_{,X}S_\chi - \tilde{\pi}_\theta\tilde{P}_X\,, \tag{C.46}$$

where $\tilde{P}_S$ is a first-class constraint. The resulting algebra is summarized in table C.4, where

$$A = G_{,XX}T_\chi + F_{,XX}D\,, \tag{C.47}$$
$$D = \tilde{\pi}_\theta\,, \tag{C.48}$$
$$\begin{aligned} Q_\chi(x,y) = {} & \delta(x-y)\left[G_{,X}\frac{1}{M_{\mathrm{P}}^2}\left(\tilde{\pi}_\theta^2\gamma^{ij} - \gamma^{ki}\gamma^{lj}\mathcal{D}_k\sigma\mathcal{D}_l\sigma\right)\left(\mathcal{D}_i\sigma\mathcal{D}_j\sigma + \gamma_{ij}\mathfrak{X}_H\right) - \frac{1}{2}\gamma^{ij}\mathcal{D}_i\mathcal{D}_j\sigma\right] \\ & - \frac{1}{2}\left\{\left[\gamma^{ij}\mathcal{D}_i\sigma\right](y)\,\mathcal{D}_j^{(y)}\delta(x-y) - \left[\gamma^{ij}\mathcal{D}_i\sigma\right](x)\,\mathcal{D}_j^{(x)}\delta(x-y)\right\}\,, \end{aligned} \tag{C.49}$$
$$\begin{aligned} B = {} & P_{,X} + P_{,XX}\tilde{\pi}_\theta^2 - 2G_{,X}\gamma^{ij}\mathcal{D}_i\mathcal{D}_j\sigma - G_{,XX}\left[\tilde{\pi}_\theta^2\gamma^{ij}\mathcal{D}_i\mathcal{D}_j\sigma - \gamma^{ij}\gamma^{kl}\mathcal{D}_i\mathcal{D}_k\sigma\mathcal{D}_j\sigma\mathcal{D}_l\sigma\right] \\ & + \frac{1}{M_{\mathrm{P}}^2}\left[2G_{,X}\tilde{\pi}\tilde{\pi}_\theta + G_{3,X}^2\left(\frac{3}{2}\tilde{\pi}_\theta^4 - \frac{1}{2}\left(\gamma^{ij}\mathcal{D}_i\sigma\mathcal{D}_j\sigma\right)^2 - \tilde{\pi}_\theta^2\gamma^{ij}\mathcal{D}_i\sigma\mathcal{D}_j\sigma\right)\right] \\ & + \frac{G_{,XX}}{M_{\mathrm{P}}^2}\left[\tilde{\pi}_\theta^2\tilde{\pi} + \tilde{\pi}_\theta\mathcal{D}_i\sigma\mathcal{D}_j\sigma\left(2\tilde{\pi}^{ij} - \gamma^{ij}\tilde{\pi}\right)\right]\,, \end{aligned} \tag{C.50}$$

where A, D, and B are purely local expressions.

$\{\downarrow,\rightarrow\}$	$\tilde{\mathcal{R}}_0$	$\tilde{P}_X$	P_χ	S_X	S_χ	$\tilde{S}_S$	T
$\tilde{\mathcal{R}}_0$	0	0	0	0	0	0	$-\tilde{Q}$
$\tilde{P}_X$		0	0	0	-1	0	A
P_χ			0	-1	0	0	D
S_X				0	0	0	0
S_χ					0	0	$-Q_\chi$
$\tilde{S}_S$						0	B
T							Q_T

Table C.4: Tertiary constraint algebra, with some new combinations, in the cubic Horndeski theory. First-class constraints were omitted. Dirac δ-functions were omitted in the entries, excepting for $\tilde{Q}$, Q_χ, and Q_T, which yield derivatives of δ-functions.

At this point we define a new Hamiltonian constraint that will commute with all the constraints. Such a constraint becomes thus first-class in the tertiary constraint algebra.

$$\bar{\mathcal{R}}_0 = \tilde{\mathcal{R}}_0 + \frac{\tilde{Q}}{B}\tilde{S}_S\,, \tag{C.51}$$

This combination is well defined under the condition that $B \neq 0$. In the case $B = 0$, the constraint $\tilde{S}_S$ becomes first-class. As a result, we have the right number of constraints even though the Hamiltonian constraint is not first-class. In what follows we will assume that $B \neq 0$.

C.5 Closedness of the algebra and total Hamiltonian

To show that the algebra is closed, we need to show that

$$\det\left(\mathcal{M}(x,y)\right) \neq 0\,, \tag{C.52}$$

where $\mathcal{M}(x,y)$ is the constraint algebra matrix. This is equivalent to showing that, for all $\vec{u}$,

$$\int d^3x d^3y\, \vec{u}^{\intercal}(x)\mathcal{M}(x,y)\vec{v}(y) = 0\,, \tag{C.53}$$

implies that $\vec{v}=0$.

We make the constraint algebra explicit, and obtain

$$\begin{aligned} 0 = \int d^3x d^3y\, \{ & u_1\left[-v_5 + Av_6\right] + u_2\left[-v_3 - \tilde{\pi}_\theta v_6\right] + u_3\left[v_2\right] + u_4\left[Bv_6\right] + u_5\left[v_1 - Q^0_\chi v_6 - Q^i_\chi \partial_i v_6\right] \\ & u_6\left[-Av_1 + \tilde{\pi}_\theta v_2 - Bv_4 + Q^0_\chi v_5 + Q^i_\chi \partial_i v_5 - \partial_i\left(Q^i_T v_6\right) - Q^i_T \partial_i v_6\right]\}\,. \end{aligned} \tag{C.54}$$

Here we have defined Q^0_χ, Q^i_χ, Q^0_T, and Q^i_T by

$$\iint d^3x d^3y f_1(x) Q_\chi(x,y) f_2(y) = \int d^3x f_1\left(Q^0_\chi f_2 + Q^i_\chi \partial_i f_2\right)\,, \tag{C.55}$$

$$\iint d^3x d^3y f_1(x) Q_T(x,y) f_2(y) = \int d^3x\left[f_1 f_2 Q^0_T + Q^i_T\left(f_2 \partial_i f_1 - f_1 \partial_i f_2\right)\right]\,, \tag{C.56}$$

where f_1 and f_2 are any auxiliary well-behaved functions. As a consequence of setting the whole integral to zero we need to impose each expression between square brackets to vanish separately. As $B \neq 0$ we can use the first 5 square brackets to say $v_1 = v_2 = v_3 = v_5 = v_6 = 0$. By plugging these into the last squared bracket we obtain that also $v_4 = 0$ necessarily. The determinant of $\mathcal{M}(x,y)$ is thus different from zero.

The candidates for the second-class constraints are therefore indeed second-class if $B \neq 0$. In such a case the candidates for the first-class constraints are indeed first-class. The algebra is then closed. Furthermore, as the Hamiltonian is only composed of constraints, we have shown that the tertiary Hamiltonian is the total Hamiltonian and that there are no further constraints given by the consistency conditions.

C.6 Total Hamiltonian

The total Hamiltonian of the cubic Horndeski theory is given by

$$\mathfrak{H}^{(\mathrm{tot})}_{\mathrm{H3}} = \int d^3x\left[-N\tilde{\mathcal{R}}_0 - N^i\tilde{\mathcal{R}}_i + \xi_X \tilde{P}_X + \xi_\chi P_\chi + \sqrt{\gamma}\left(\xi_S \tilde{P}_S + \lambda_X S_X + \lambda_\chi S_\chi + \lambda_S \tilde{S}_S + \lambda_T T\right)\right]. \tag{C.57}$$

where the total number of degrees of freedom is 3. This is due to the presence of the 5 first-class constraints $(\tilde{\mathcal{R}}_0, \tilde{\mathcal{R}}_i, \tilde{P}_S)$ and 6 second-class constraints (the remaining ones), which kill the degrees of freedom introduced by the auxiliary fields.

Appendix D

Expressions for MQD

In this appendix we describe some further details of the action and the study of perturbations within the minimal quasidilaton model (studied in chapter 5), since several expressions are rather large. This appendix has been in part reproduced from [55].

D.1 Details for the action and Hamiltonian of MQD

In sections 5.2.2 and 5.2.3, we have skipped some large explicit expressions and manipulations when they would not have significantly affected the discussion. In this section, we reproduce the most useful of these relations. We start by giving the explicit expression of $\mathcal{F}^{ij}$ and $\mathcal{F}_\sigma$ defined compactly in (5.44), as

$$\mathcal{F}^{ij} \equiv e^{\alpha\sigma/M_{\rm P}} \left\{ \frac{\sqrt{\tilde{\gamma}}}{\sqrt{\gamma}} \left[\left(c_1 e^{3\sigma/M_{\rm P}} + c_2 e^{2\sigma/M_{\rm P}} \mathcal{K} \right) \left(\mathcal{K}^i{}_k \gamma^{kj} + \gamma^{ik} \mathcal{K}^j{}_k \right) - 2 c_2 e^{2\sigma/M_{\rm P}} \tilde{\gamma}^{ij} \right] + 2 c_3 e^{\sigma/M_{\rm P}} \gamma^{ij} \right\}, \tag{D.1}$$

$$\mathcal{F}_\sigma \equiv \frac{e^{(\alpha+1)\sigma/M_{\rm P}}}{M_{\rm P}} \left\{ \sqrt{\tilde{\gamma}} \left[(4+\alpha) c_0 e^{3\sigma/M_{\rm P}} + (3+\alpha) c_1 e^{2\sigma/M_{\rm P}} \mathcal{K} + \frac{2+\alpha}{2} c_2 e^{\sigma/M_{\rm P}} \left(\mathcal{K}^2 - \mathcal{K}^i{}_j \mathcal{K}^j{}_i \right) \right] + (1+\alpha) c_3 \sqrt{\gamma} \right\}. \tag{D.2}$$

The first expression should not be confused with the analog expression for MTMG (B.5). These first derivatives (with respect to the metric, and the quasidilaton field) of part of the graviton potential are central in MQD, as they appear in the additional constraints one introduces to "minimize" the theory and hence, at the end of the day, in the equations of motion.

We now give more details regarding the Hamiltonian analysis: we discuss the would-be Hamiltonian and momentum constraints. These should yield Poisson brackets corresponding to the diffeomorphism algebra (2.29) in the limit of zero graviton mass. For the would-be Hamiltonian constraint, as in (C.35) and (C.51), one may define (note that we are using a slightly different notation w.r.t. [54])

$$\bar{\mathcal{R}}_0 = \tilde{\mathcal{R}}_0 - T_\chi \tilde{P}_X + \frac{\tilde{Q}}{B} \tilde{S}_S \,, \tag{D.3}$$

where we have used the following constraints and quantities to simplify the constraint algebra at primary and secondary level

$$\tilde{P}_X = P_X - F_{,XX} P_\chi \,, \tag{D.4}$$

$$\tilde{S}_S = S_S + G_{,X} S_\chi - \tilde{\pi}_\theta \tilde{P}_X \,, \tag{D.5}$$

$$T_\chi \equiv \{\tilde{\mathcal{R}}_0, S_\chi\} = \frac{2}{M_{\rm P}^2} \tilde{\pi}^{ij} \left[\gamma_{ij} \mathfrak{X}_H - \mathcal{D}_i \sigma \mathcal{D}_j \sigma \right] - \tilde{\pi}_\theta \left[S - \gamma^{ij} \mathcal{D}_i \mathcal{D}_j \sigma \right] - \gamma^{ij} \mathcal{D}_j \sigma \mathcal{D}_i \tilde{\pi}_\theta \,, \tag{D.6}$$

whereas the following other quantities were used to simplify the tertiary constraint algebra

$$\tilde{Q} \equiv \{T, \tilde{\mathcal{R}}_0 - T_\chi \tilde{P}_X\} \,, \qquad B \equiv \{\tilde{S}_S, T\} \,, \tag{D.7}$$

$$T - \tilde{T} = G_{,X} \frac{2}{M_{\rm P}^2} \tilde{\pi} S_\chi + \tilde{\pi}_\theta S_X \,. \tag{D.8}$$

The would-be momentum constraint is given as in (C.21) by

$$\tilde{\mathcal{R}}_i \equiv 2\sqrt{\gamma} \gamma_{ik} \mathcal{D}_j \tilde{\pi}^{kj} - \pi_\sigma \partial_i \sigma - \pi_X \partial_i X - \pi_\chi \partial_i \chi - \pi_S \partial_i S - \pi_\theta \partial_i \theta \,. \tag{D.9}$$

Note again that whereas the constraints $\tilde{\mathcal{R}}_i$ and $\bar{\mathcal{R}}_0$ were actually first-class in the cubic shift-symmetric Horndeski case, in MQD their Poisson bracket does not vanish due to the graviton potential.

We now discuss the main differences of with the cases of Horndeski (appendix C), and MTMG (section 4.5.3). At the level of the structure of the constraint algebra, one may see by choosing appropriate constraints, that the MTMG structure and Horndeski structure couple only through the additional non-vanishing Poisson brackets between S_χ, S_S and $\mathcal{C}_0$, as well as between $\tilde{T}$ and $\mathcal{C}_0$, $\mathcal{C}_i$. On the one hand, since we expect the conservation in time of $\tilde{T}$, $\mathcal{C}_0$, and $\mathcal{C}_i$ to be solved for Lagrange multipliers (in the cubic shift-symmetric Horndeski case, and in MTMG, respectively), the presence of the extra non-vanishing brackets between them will be harmless. On the other hand, one may worry that the Poisson brackets D_χ and D_S defined by

$$D_\chi \equiv \{S_\chi, \mathcal{C}_0\}\,, \qquad D_S \equiv \{S_S, \mathcal{C}_0\}\,, \tag{D.10}$$

may produce some changes in the structure of the algebra[1]. To see that it is not the case, we inspect the conservation in time of S_χ and $\tilde{S}_S$ (the equivalents of (C.38) and (C.40)) with the total Hamiltonian MQD. These conditions become respectively

$$\{S_\chi, \mathfrak{H}^{(\mathrm{tot})}_{\mathrm{MQD,v}}\} \approx \xi_X - \lambda_T Q_\chi + \frac{M_{\mathrm{P}}^2 m^2}{2}\lambda D_\chi \approx 0\,, \tag{D.11}$$

$$\{\tilde{S}_S, \mathfrak{H}^{(\mathrm{tot})}_{\mathrm{MQD,v}}\} \approx \lambda_T B + \frac{M_{\mathrm{P}}^2 m^2}{2}\lambda\left(D_S + G_{,X} D_\chi\right) \approx 0\,, \tag{D.12}$$

where for a simple calculation, we have used the combinations $\bar{\mathcal{R}}_0$ and $\tilde{P}_X$. Whether in the pure Horndeski case, or in the minimal quasidilaton case, these two relations have to be solved for λ_T and ξ_X. Therefore the presence of the extra terms does not change the structure of these consistency conditions. This of course is valid only if $B \neq 0$. Finally note that $\tilde{P}_S$ is still a primary constraint.

The constraint algebra, for the constraints appearing in the Hamiltonian (5.45) can be found summarized in table D.1. Notably, there is a single first-class constraint $\tilde{P}_S$, not presented in the table, of which the associated symmetry has not yet been elucidated.

$\{\downarrow,\rightarrow\}$	$\mathcal{H}_1$	$\bar{\mathcal{R}}_0$	$\tilde{\mathcal{R}}_i$	$\mathcal{C}_0$	$\mathcal{C}_i$	$\tilde{P}_X$	P_χ	S_X	S_χ	$\tilde{S}_S$	T
$\mathcal{H}_1$	0	$\not\approx 0$	$\not\approx 0$	$\not\approx 0$	0	0	0	0	0	0	0
$\bar{\mathcal{R}}_0$		0	$\not\approx 0$	$\not\approx 0$	$\not\approx 0$	0	0	0	0	0	0
$\tilde{\mathcal{R}}_i$			0	$\not\approx 0$	$\not\approx 0$	0	0	0	0	0	0
$\mathcal{C}_0$				0	$\not\approx 0$	0	0	0	$-D_\chi$	$-D_S - G_{,X}D_\chi$	$-Q_0$
$\mathcal{C}_i$					0	0	0	0	0	0	$-Q_i$
$\tilde{P}_X$						0	0	0	-1	0	A
P_χ							0	-1	0	0	D
S_X								0	0	0	0
S_χ									0	0	$-Q_\chi$
$\tilde{S}_S$										0	B
T											Q_T

Table D.1: Constraint algebra of the minimal quasidilaton. A first-class constraint $\tilde{P}_S$ was omitted. Dirac δ-functions were omitted in the entries, excepted for Q_0, Q_i, Q_χ, and Q_T, which formally include derivatives of δ-functions. When $\not\approx 0$ is indicated, the entry may formally include not only Dirac δ-functions but also derivatives of Dirac δ-functions.

D.2 Explicit expressions in the subhorizon limit

In this appended section, we give the explicit expressions for the important phenomenological quantities of the subhorizon and quasi-static approximations. For compactness, we start with some useful

[1]However, the existence of T is not put into question: the minimal theory is defined by the addition of $\mathcal{C}_0$ and $\mathcal{C}_i$ a posteriori (not at the secondary step), hence T should be kept as a constraint.

definitions

$$\Gamma_1 = \mathcal{X}\left(c_1\mathcal{X}^2 + 2c_2\mathcal{X} + c_3\right) = \frac{1}{3}\Gamma_{,\mathcal{X}}\mathcal{X}\,, \tag{D.13}$$

$$\Gamma_2 = \mathcal{X}^2\left(c_1\mathcal{X} + c_2\right) = \frac{1}{6}\Gamma_{,\mathcal{X}\mathcal{X}}\mathcal{X}^2\,, \tag{D.14}$$

$$q = 2Q\left[2H^2\left(4+\alpha\right)^2 + m^2\Gamma_2\right] - 3m^2\left[\left(2 - M_{\rm P}H^2G_{,X}\right)^2\Gamma_2 + \left(4+\alpha\right)\left(2 - M_{\rm P}H^2G_{,X}\right)\Gamma_1\right], \tag{D.15}$$

$$d = m^2\Gamma_1 + 2\,H^2\left(4+\alpha\right)\left(M_{\rm P}H^2G_{,X} - 2\right), \tag{D.16}$$

$$Q = 3\,M_{\rm P}^3\,G_{,XX}H^4 + \frac{3}{2}\,M_{\rm P}^2\,H^4\left(G_{,X}\right)^2 + M_{\rm P}^2P_{,XX}H^2 + 6\,M_{\rm P}\,G_{,X}H^2 + P_{,X} \equiv g + \frac{3}{2}(G_{,X}H^2M_{\rm P} - 2)^2\,. \tag{D.17}$$

as well as $\iota_n \equiv (4+\alpha)\Gamma_1 + n\Gamma_2$, for $n \in \mathbb{R}$, and

$$g_1 = G_{,X}H^2M_{\rm P} - 2\,, \tag{D.18}$$

$$g_2 = 2G_{,X}H^2M_{\rm P} + G_{,XX}H^4M_{\rm P}^3\,, \tag{D.19}$$

$$\epsilon = 2(\alpha+4)\Gamma_1 - (3+2\alpha)\Gamma_2 \tag{D.20}$$

$$\xi_1 = (\alpha+4)^2\Gamma_1^2 - 4(\alpha+4)\Gamma_1\Gamma_2 - 2(P_{,X}+6)\Gamma_2^2 \tag{D.21}$$

$$\xi_2 = (\alpha+4)^2\Gamma_1^2 + 6(\alpha+4)\Gamma_1\Gamma_2 + 2(P_{,X}+6)\Gamma_2^2 \tag{D.22}$$

$$\xi_3 = 3(\alpha+4)^2\Gamma_1^2 - 4(\alpha+4)(\alpha+g_1+2)\Gamma_1\Gamma_2 + 2(2(\alpha+3)g_1 + P_{,X}+6)\Gamma_2^2 \tag{D.23}$$

$$g = P_{,X} - 6 + 12G_{,X}H^2M_{\rm P} + 3G_{,XX}H^4M_{\rm P}^3 + H^2M_{\rm P}^2P_{,XX} = Q - \frac{3}{2}g_1^2\,. \tag{D.24}$$

We remind the reader that Γ (see equation (5.69)) is the contribution of the graviton interaction term to the Friedmann equation. The free parameters characterizing the graviton potential are c_1, c_2, c_3, α, and m, while the free functions P and G characterize the quasidilaton kinetic term.

D.2.1 Mass terms of scalar perturbations and G_{eff}/G_N

For the ease of reading, we first decompose the mass coefficients in powers of m as

$$L_{11} = -\frac{a^2\rho_m^2}{2q^2M_{\rm P}^2\Gamma_1^2}\left(L_{11,0} + m^2L_{11,2} + m^4L_{11,4}\right), \tag{D.25}$$

$$L_{12} = -\frac{a\rho_m}{2dqM_{\rm P}^3\Gamma_1\Gamma_2}\left(L_{12,0} + m^2L_{12,2} + m^4L_{12,4}\right), \tag{D.26}$$

$$L_{22} = \frac{1}{2d^2M_{\rm P}^2\Gamma_2^2}\left(L_{22,0} + m^2L_{22,2} + m^4L_{22,4}\right). \tag{D.27}$$

The coefficients appearing in the mass terms for scalar perturbations, as defined in equations (D.25)-(D.27), are given by

$$L_{11,0} = 16(\alpha+4)^2H^2Q^2\left[H^2(\xi_2 - 6\Gamma_2\iota_{-g_1}) + \frac{2\Gamma_2^2(g-3g_1)\rho_m}{M_{\rm P}^2g}\right], \tag{D.28}$$

$$L_{11,2} = 8(\alpha+4)\Gamma_1Q\left[H^2(3g_1\iota_0\iota_2 + 2\Gamma_2Q\epsilon) + \frac{6\Gamma_2^2Q\rho_m}{M_{\rm P}^2g}\right], \tag{D.29}$$

$$L_{11,4} = 3\Gamma_1^2g_1\left[3g_1(8\Gamma_2\iota_{-g_1} + \xi_1) + 8\Gamma_2Q\iota_{-(1+\alpha)} - \frac{18\Gamma_2^2g_1g_2\rho_m}{H^2M_{\rm P}^2g}\right], \tag{D.30}$$

$$L_{12,0} = 8(\alpha+4)^2H^2Q\left\{H^2M_{\rm P}^2\Big[(g_1+2)\left(\iota_0^2 - 6\Gamma_2\iota_{-g_1}\right) - 2\Gamma_2P_{,X}\iota_{-g_1}\Big] + \frac{2\Gamma_2\iota_{-g_1}(3g_1-g)\rho_m}{g}\right\}, \tag{D.31}$$

$$\begin{aligned} L_{12,2} = {}& 2(\alpha+4)\Gamma_1M_{\rm P}^2\Bigg(H^2\Big\{3g_1^2\iota_{-6}\iota_0 + 2g_1\left[3(\alpha+4)\Gamma_1\iota_{-(6+P_{,X})} + 2\Gamma_2Q\epsilon\right] - 2Q\iota_0\iota_{-2(3+\alpha)}\Big\} \\ & - \frac{6\Gamma_2\rho_m(3g_1g_2\iota_0 + 2Q\iota_{-g_1})}{M_{\rm P}^2g}\Bigg), \end{aligned} \tag{D.32}$$

$$\begin{aligned} L_{12,4} = {}& \Gamma_1^2M_{\rm P}^2\Big\{12\Gamma_2g_1^2\iota_{-(3+\alpha)} - 3g_1\left(\iota_{-(3+\alpha)}^2 + \Gamma_2^2\left(-\alpha^2 - 6\alpha + 2P_{,X} + 3\right)\right) + 4\Gamma_2g\iota_{-(1+\alpha)} \\ & - \frac{18\Gamma_2^2g_1g_2\rho_m}{H^2M_{\rm P}^2g}\Big\}, \end{aligned} \tag{D.33}$$

$$\begin{aligned}
L_{22,0} &= 4(\alpha+4)^2 H^2 \Big(g_1 H^2 M_{\rm P}^2 \Big\{ 2\Gamma_2 P_{,X}(\iota_{-g_1}+\iota_0) - (g_1+2)\left[\iota_0^2 - 6\Gamma_2(\iota_{-g_1}+\iota_0)\right] \Big\} \\
&\quad + 2\rho_m \left[\frac{3}{g}(g_1 \iota_{-g_1}^2 + g_2 \iota_0^2) - \iota_{-g_1}^2 \right] \Big) , \qquad \text{(D.34)} \\
L_{22,2} &= 4(\alpha+4)\Gamma_1 \Big(H^2 M_{\rm P}^2 \Big\{ g_1(\iota_{2-2\alpha}\iota_{-g_1} - \Gamma_2 g_1 \iota_{-5}) - (\alpha+4)\Gamma_1\left[\iota_0 - 2\Gamma_2(P_{,X}+6)\right] \Big\} \\
&\quad + \frac{\rho_m}{g}\left[6(\alpha+4)\Gamma_1\Gamma_2 g_2 - 3\iota_{-g_1}^2 \right] \Big) , \qquad \text{(D.35)} \\
L_{22,4} &= \frac{\Gamma_1^2}{H^2} \left(H^2 M_{\rm P}^2 \xi_3 + 6\Gamma_2^2 g_2 \frac{\rho_m}{g} \right) , \qquad \text{(D.36)}
\end{aligned}$$

D.2.2 G_Φ/G_N

The Bardeen potential Φ (see (2.59)) satisfies a Poisson equation (5.101) with a modified gravitational constant, which we detail in this appendix. In order to maintain relatively contained expressions, we first collect powers of the graviton mass coefficient m, as

$$\frac{G_\Phi}{G_N} = \frac{g_{\Phi,n,4} m^4 + g_{\Phi,n,2} m^2 + g_{\Phi,n,0}}{g_{\Phi,d,4} m^4 + g_{\Phi,d,2} m^2 + g_{\Phi,d,0}} . \qquad \text{(D.37)}$$

The coefficients appearing in Eq. (D.37), are then explicitly given by

$$\begin{aligned}
g_{\Phi,n,4} &= -2\Gamma_1^2 H^2 \iota_2 M_{\rm P}^2 g \iota_{-(1+\alpha)} , \qquad \text{(D.38)} \\
g_{\Phi,n,2} &= -4(\alpha+4)\Gamma_1 H^2 \big(3\rho_m \left[\Gamma_2 g_1 \iota_2 + \Gamma_2(g_2-2)\iota_0 - \iota_0^2 \right] \\
&\quad + H^2 M_{\rm P}^2 g \left\{ g_1 \left[\iota_0 \iota_{6-\alpha} - 2(2\alpha+3)\Gamma_2^2 \right] + \iota_0 \left[\Gamma_2(2\alpha + P_{,X} + 12) - \iota_0 \right] \right\} \big) , \qquad \text{(D.39)} \\
g_{\Phi,n,0} &= -8(\alpha+4)^2 H^4 \left\{ \rho_m \left[\iota_2 \iota_{-g_1}(3g_1 - g) + 3g_2 \iota_0^2 \right] \right. \\
&\quad \left. + H^2 M_{\rm P}^2 g \left[\Gamma_2 g_1 \iota_2 (3g_1 + P_{,X} + 6) + \iota_0 (g_1 \iota_{-6} + 2\iota_{-(6+P_{,X})}) \right] \right\} , \qquad \text{(D.40)} \\
g_{\Phi,d,4} &= -\Gamma_1^2 \left[H^2 M_{\rm P}^2 g (4\iota_{-(3+\alpha)}\iota_{-g_1} - \xi_1) + 6\Gamma_2^2 g_2 \rho_m \right] , \qquad \text{(D.41)} \\
g_{\Phi,d,2} &= 12(\alpha+4)\Gamma_1 H^2 \rho_m \left[\iota_{-g_1}^2 - 2(\alpha+4)\Gamma_1\Gamma_2 g_2 \right] \qquad \text{(D.42)} \\
&\quad - 4(\alpha+4)\Gamma_1 H^4 M_{\rm P}^2 g \left\{ (g_1+2)\left[-2\alpha\Gamma_2 \iota_{-g_1} + 3\Gamma_2(\iota_{g_1} + \iota_2) + \iota_0 \iota_{-2(2+g_1)} \right] - \xi_3 + 2\Gamma_2 \iota_1 P_{,X} \right\} , \\
g_{\Phi,d,0} &= -24(\alpha+4)^2 H^4 \rho_m \left[g_1 \iota_{-g_1}^2 + (\alpha+4)^2 \Gamma_1^2 g_2 \right] + 8(\alpha+4)^2 H^4 \rho_m g \iota_{-g_1}^2 \\
&\quad + 4(\alpha+4)^2 g_1 H^6 M_{\rm P}^2 g \left\{ (g_1+2)\left[\iota_0^2 - 6\Gamma_2(\iota_{-g_1}+\iota_0) \right] - 2\Gamma_2 P_{,X}(\iota_{-g_1}+\iota_0) \right\} . \qquad \text{(D.43)}
\end{aligned}$$

where we have used definitions (D.18)-(D.24) and (D.13)-(D.17).

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