Bernoulli-Carlitz and Cauchy-Carlitz numbers with Stirling-Carlitz numbers

By

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Abstract

Recently, the Cauchy-Carlitz number was defined as the counterpart of the Bernoulli-Carlitz number. Both numbers can be expressed explicitly in terms of so-called Stirling-Carlitz numbers. In this paper, we study the second analogue of Stirling-Carlitz numbers and give some general formulae, including Bernoulli and Cauchy numbers in formal power series with complex coefficients, and Bernoulli-Carlitz and Cauchy-Carlitz numbers in function fields. We also give some applications of Hasse-Teichmüller derivative to hypergeometric Bernoulli and Cauchy numbers in terms of associated Stirling numbers.

§ 1. The second analogue of Stirling-Carlitz numbers

The (unsigned) Stirling numbers of the first kind $\left[{n \atop k} \right]$ and the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ are defined by the generating functions

$$\frac{\left(-\log(1-x) \right)^k}{k!} = \sum_{n=0}^{\infty} \left[{n \atop k} \right] \frac{x^n}{n!} \quad \text{and} \quad \frac{(e^x - 1)^k}{k!} = \sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} ,$$

respectively. Based upon these generating functions, in [13] we introduced Stirling-Carlitz numbers $\left[{n \atop k} \right]_C$ and $\left\{ {n \atop k} \right\}_C$. Using these C-Stirling-Carlitz numbers, Bernoulli-Carlitz numbers BC_n (1.1) and Cauchy-Carlitz numbers CC_n (1.2) can be expressed

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explicitly. Notice that Bernoulli-Carlitz numbers BC_n ([1, 2, 3, 5]) are given by

$$(1.1) \qquad \frac{z}{e_C(z)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Pi(n)} z^n$$

and Cauchy-Carlitz numbers CC_n ([13]) are give by

$$(1.2) \qquad \frac{z}{\log_C(z)} = \sum_{n=0}^{\infty} \frac{CC_n}{\Pi(n)} z^n \,.$$

The (unsigned) Stirling numbers of the first kind $\left[{n \atop k} \right]$ appear in the falling factorial

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} (-1)^{n-k} \left[{n \atop k} \right] x^k$$

and the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ may be defined by

$$(1.3) \qquad x^n = \sum_{k=0}^{n} x(x-1)\cdots(x-k+1) \left\{ {n \atop k} \right\} \,.$$

Based upon such relations, we can introduce different type Stirling-Carlitz numbers $\left[{n \atop k} \right]_A$ and $\left\{ {n \atop k} \right\}_A$.

Throughout this paper, let $\mathbb{F}_r$ be the field with r elements and $\mathbb{A} = \mathbb{F}_r[T]$ (resp. $\mathbb{F}_r(T)$) the ring of polynomials (resp. the field of rational functions) in one variable over $\mathbb{F}_r$. According to the notations used in [5], set $[i] := T^{r^i} - T \in \mathbb{A}$ $(i \geq 1)$, $D_i := [i][i-1]^r \cdots [1]^{r^{i-1}}$ $(i \geq 1)$ with $D_0 = 1$, and $L_i := [i][i-1]\cdots[1]$ $(i \geq 1)$ with $L_0 = 1$. Then, The Carlitz exponential $e_C(x)$ is defined by

$$e_C(x) = \sum_{i=0}^{\infty} \frac{x^{r^i}}{D_i}$$

and the Carlitz logarithm $\log_C(x)$ is defined by

$$\log_C(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{r^i}}{L_i} \,.$$

The Carlitz factorial $\Pi(i)$ is defined by

$$\Pi(i) = \prod_{j=0}^{m} D_j^{c_j}$$

for a non-negative integer i with r-ary expansion:

$$i = \sum_{j=0}^{m} c_j r^j \quad (0 \leq c_j < r) \,.$$

Denote the d-dimensional $\mathbb{F}_r$-vector space of polynomials of degree$< d$ by $\mathbb{A}(d) := \{\alpha \in \mathbb{A}\,|\deg(\alpha) < d\}$. As

$$e^{x\log(1+t)} = (1+t)^x = \sum_{n=0}^{\infty} \binom{x}{n} t^n$$

with

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}\,,$$

as an analogous version, set

$$e_C(z\log_C(x)) = \sum_{n=0}^{\infty} E_n(z) x^{r^n}\,,$$

where

$$E_n(z) = \frac{e_n(z)}{D_n}$$

([5, Corollary 3.5.3]). In addition, we have

$$(1.4) \qquad e_n(z) = \prod_{\alpha\in\mathbb{A}(n)} (z+\alpha) = \prod_{\alpha\in\mathbb{A}(n)} (z-\alpha) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_A z^{r^i}\,,$$

where

$$(1.5) \qquad \begin{bmatrix} n \\ i \end{bmatrix}_A = (-1)^{n-i} \frac{D_n}{D_i L_{n-i}^{r^i}}$$

([5, Theorem 3.1.5]).

As an analogue of the Stirling numbers of the second kind in (1.3), it is natural to define $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_A$ by

$$(1.6) \qquad \sum_{k=0}^{n} e_k(z) \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_A = z^{r^n}$$

Then, similarly to the C-Stirling-Carlitz numbers $\begin{bmatrix} n \\ k \end{bmatrix}_C$ and $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_C$ ([13, Theorem 5]), A-Stirling-Carlitz numbers $\begin{bmatrix} n \\ k \end{bmatrix}_A$ and $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_A$ satisfy the orthogonal identities.

Theorem 1.1. *For $i \leq n$,*

$$(1.7) \qquad \sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix}_A \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_A = \delta_{n,i}\,,$$

$$(1.8) \qquad \sum_{k=i}^{n} \left\{ \begin{matrix} k \\ i \end{matrix} \right\}_A \begin{bmatrix} n \\ k \end{bmatrix}_A = \delta_{n,i}\,.$$

Proof. Since

$$z^{r^n} = \sum_{k=0}^{n} e_k(z) \begin{Bmatrix} n \\ k \end{Bmatrix}_A$$

$$= \sum_{k=0}^{n} \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_A z^{r^i} \begin{Bmatrix} n \\ k \end{Bmatrix}_A$$

$$= \sum_{i=0}^{n} \sum_{k=i}^{n} \begin{bmatrix} k \\ i \end{bmatrix}_A \begin{Bmatrix} n \\ k \end{Bmatrix}_A z^{r^i},$$

by comparing the coefficients of z^{r^i} $(i = 0, 1, \ldots, n)$, we get (1.7). Since

$$e_n(z) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_A z^{r^i}$$

$$= \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_A \sum_{k=0}^{i} e_k(z) \begin{Bmatrix} i \\ k \end{Bmatrix}_A$$

$$= \sum_{k=0}^{n} \sum_{i=k}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_A \begin{Bmatrix} i \\ k \end{Bmatrix}_A e_k(z),$$

by comparing the coefficients of $e_k(z)$ $(k = n, n-1, \ldots, 1, 0)$, we have (1.8).   □

A-Stirling-Carlitz numbers of the second kind have an explicit expression as those of the first kind in (1.5).

Theorem 1.2.   *For $0 \le j \le n$, we have*

$$\begin{Bmatrix} n \\ j \end{Bmatrix}_A = \frac{D_n}{D_j D_{n-j}^{r^j}}.$$

Proof. We prove the theorem by induction on j. If $j = n$, then

$$\begin{Bmatrix} n \\ n \end{Bmatrix}_A = 1$$

by (1.7). Next, we consider the case of $j = n - i$ with $i > 0$. Using the inductive hypothesis and (1.7), we get

$$\begin{Bmatrix} n \\ n-i \end{Bmatrix}_A = -\sum_{d=0}^{i-1} \begin{bmatrix} n-d \\ n-i \end{bmatrix}_A \begin{Bmatrix} n \\ n-d \end{Bmatrix}_A$$

$$= -\sum_{d=0}^{i-1} \frac{(-1)^{i-d} D_{n-d}}{D_{n-i} L_{i-d}^{r^{n-i}}} \frac{D_n}{D_{n-d} D_d^{r^{n-d}}}$$

$$(1.9) \qquad\qquad = -\frac{D_n}{D_{n-i}} \sum_{d=0}^{i-1} \frac{(-1)^{i-d}}{L_{i-d}^{r^{n-i}} D_d^{r^{n-d}}}$$

Note for any $l \geq 0$ that
$$(-1)^{r^l} = -1 .$$
In fact, if r is even, then $-1 = 1$ because the characteristic of $\mathbb{F}_r$ is 2. Thus,
$$-\sum_{d=0}^{i-1} \frac{(-1)^{i-d}}{L_{i-d}^{r^{n-i}} D_d^{r^{n-d}}} = \left(-\sum_{d=0}^{i-1} \frac{(-1)^{i-d}}{L_{i-d}D_d^{r^{i-d}}} \right)^{r^{n-i}}$$
$$= \left(\frac{1}{D_i} - \sum_{d=0}^{i} \frac{(-1)^{i-d}}{L_{i-d}D_d^{r^{i-d}}} \right)^{r^{n-i}}$$
$$= \left(\frac{1}{D_i} - \sum_{a=0}^{i} \frac{(-1)^a}{L_a D_{i-a}^{r^a}} \right)^{r^{n-i}} .$$
It is well known for any $l \geq 0$ that
$$\sum_{a=0}^{l} \frac{(-1)^a}{L_a D_{l-a}^{r^a}} = \delta_{l,0}$$
(for instance, see equation (1.63) in [14]). Hence we obtain by $i \geq 1$ that
$$(1.10) \qquad -\sum_{d=0}^{i-1} \frac{(-1)^{i-d}}{L_{i-d}^{r^{n-i}} D_d^{r^{n-d}}} = \frac{1}{D_i^{r^{n-i}}} .$$
Combining (1.9) and (1.10), we deduce that
$$\left\{ \begin{matrix} n \\ n-i \end{matrix} \right\}_A = \frac{D_n}{D_{n-i} D_i^{r^{n-i}}} .$$

$\square$

Example 1.3. By using (1.4), we calculate $\begin{bmatrix} n \\ i \end{bmatrix}_A$ $(i = 0, 1, \ldots, n)$ in the case of $r = 3$ and $n = 1, 2, 3$. By
$$e_1(z) = z(z+1)(z-1) = \sum_{i=0}^{1} \begin{bmatrix} 1 \\ i \end{bmatrix}_A z^{3^i} ,$$
we have
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_A = -1 \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_A = 1 .$$
If $n = 2$, then by
$$e_2(z) = z(z+1)(z-1)(z+T)(z-T)$$
$$\times (z+T+1)(z+T-1)(z-T+1)(z-T-1)$$
$$= \sum_{i=0}^{2} \begin{bmatrix} 2 \\ i \end{bmatrix}_A z^{3^i} ,$$

we see

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix}_A = T^6 + T^4 + T^2, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_A = -(T^6 + T^4 + T^2 + 1), \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}_A = 1.$$

Moreover, by

$$\begin{aligned}
e_3(z) \\
&= z(z+1)(z-1)(z+T)(z-T) \\
&\quad \times (z+T+1)(z+T-1)(z-T+1)(z-T-1) \\
&\quad \times (z+T^2)(z+T^2+1)(z+T^2-1)(z+T^2+T)(z+T^2-T) \\
&\quad \times (z+T^2+T+1)(z+T^2+T-1)(z+T^2-T+1)(z+T^2-T-1) \\
&\quad \times (z-T^2)(z-T^2+1)(z-T^2-1)(z-T^2+T)(z-T^2-T) \\
&\quad \times (z-T^2+T+1)(z-T^2+T-1)(z-T^2-T+1)(z-T^2-T-1) \\
&= \sum_{i=0}^{3} \begin{bmatrix} 3 \\ i \end{bmatrix}_A z^{3^i},
\end{aligned}$$

we obtain

$$\begin{aligned}
\begin{bmatrix} 3 \\ 0 \end{bmatrix}_A &= -T^{42} - T^{40} - T^{38} - T^{36} + T^{34} + T^{32} + T^{30} + T^{28} \\
&\quad + T^{24} + T^{22} + T^{20} + T^{18} - T^{16} - T^{14} - T^{12} - T^{10} \\
\begin{bmatrix} 3 \\ 1 \end{bmatrix}_A &= T^{42} + T^{40} + T^{38} - T^{36} - T^{34} - T^{32} \\
&\quad - T^{16} - T^{14} - T^{12} + T^{10} + T^8 + T^6, \\
\begin{bmatrix} 3 \\ 2 \end{bmatrix}_A &= -T^{36} - T^{30} - T^{28} - T^{24} - T^{22} - T^{20} - T^{18} \\
&\quad - T^{16} - T^{14} - T^{12} - T^8 - T^6 - 1, \\
\begin{bmatrix} 3 \\ 3 \end{bmatrix}_A &= 1.
\end{aligned}$$

§2.   Applications to hypergeometric Bernoulli and Cauchy numbers

The Hasse-Teichmüller derivative $H^{(n)}$ of order n is defined by

$$H^{(n)} \left(\sum_{m=R}^{\infty} a_m z^m \right) = \sum_{m=R}^{\infty} a_m \binom{m}{n} z^{m-n}$$

for $\sum_{m=R}^{\infty} a_m z^m \in \mathbb{F}((z))$, where $\mathbb{F}$ is a field of any characteristic, R is an integer and $a_m \in \mathbb{F}$ for any $m \geq R$.

The Hasse-Teichmüller derivatives satisfy the product rule [20], the quotient rule [6] and the chain rule [8]. One of the product rules is described as follows:

Lemma 2.1. *For $f_i \in \mathbb{F}[[z]]$ ($i = 1, \dots, k$) with $k \geq 2$ and for $n \geq 1$, we have*

$$H^{(n)}(f_1 \cdots f_k) = \sum_{\substack{i_1, \dots, i_k \geq 0 \\ i_1 + \cdots + i_k = n}} H^{(i_1)}(f_1) \cdots H^{(i_k)}(f_k) \,.$$

The quotient rules are described as follows:

Lemma 2.2. *For $f \in \mathbb{F}[[z]] \backslash \{0\}$ and $n \geq 1$, we have*

$$(2.1) \qquad H^{(n)}\left(\frac{1}{f}\right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1, \dots, i_k \geq 1 \\ i_1 + \cdots + i_k = n}} H^{(i_1)}(f) \cdots H^{(i_k)}(f)$$

$$(2.2) \qquad = \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1, \dots, i_k \geq 0 \\ i_1 + \cdots + i_k = n}} H^{(i_1)}(f) \cdots H^{(i_k)}(f) \,.$$

In [11] Bernoulli numbers and Bernoulli-Carlitz numbers are expressed explicitly by using the Hasse-Teichmüller derivative. In [13], Cauchy numbers and Cauchy-Carlitz numbers are expressed explicitly as well.

In this section, by using the Hasse-Teichmüller derivative of order n, we shall obtain some explicit expressions of the hypergeometric Cauchy numbers $c_{N,n}$, defined by

$$\frac{1}{{}_2F_1(1, N; N+1; -x)} = \sum_{n=0}^{\infty} c_{N,n} \frac{x^n}{n!} \,,$$

where ${}_2F_1(a, b; c, z)$ is the hypergeometric function defined by

$${}_2F_1(a, b; c, z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}} \frac{z^n}{n!}$$

with $(a)^{(n)} = a(a+1)\cdots(a+n-1)$ ($n \geq 1$) and $(a)^{(0)} = 1$. We give a different proof for the following result shown in [15, Theorem 1].

Theorem 2.3. *For $n \geq 1$,*

$$c_{N,n} = (-1)^n n! \sum_{k=1}^{n} \sum_{\substack{i_1, \dots, i_k \geq 1 \\ i_1 + \cdots + i_k = n}} \frac{(-N)^k}{(N+i_1)\cdots(N+i_k)} \,.$$

Proof. Put

$$h := {}_2F_1(1, N; N+1; -x) = N \sum_{j=0}^{\infty} \frac{(-x)^j}{N+j} \, .$$

Note that

$$H^{(i)}(h)\Big|_{x=0} = \sum_{j=0}^{\infty} \frac{N(-1)^j}{N+j} \binom{j}{i} x^{j-i}\Big|_{x=0} = \frac{N(-1)^i}{N+i} \, .$$

Hence, by using Lemma 2.2 (2.1), we have

$$\begin{aligned}
\frac{c_{N,n}}{n!} &= H^{(n)}\left(\frac{1}{h}\right)\Big|_{x=0} \\
&= \sum_{k=1}^{n} \frac{(-1)^k}{h^{k+1}}\Big|_{x=0} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(h)\Big|_{x=0} \cdots H^{(i_k)}(h)\Big|_{x=0} \\
&= \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} \frac{N(-1)^{i_1}}{N+i_1} \cdots \frac{N(-1)^{i_k}}{N+i_k} \\
&= \sum_{k=1}^{n} \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} \frac{(-N)^k(-1)^n}{(N+i_1)\cdots(N+i_k)} \, .
\end{aligned}$$

$\square$

We express the hypergeometric Cauchy numbers in terms of the binomial coefficients, too. In fact, by using Lemma 2.2 (2.2) instead of Lemma 2.2 (2.1) in the proof of Theorem 2.3, we obtain the following:

Proposition 2.4. *For $n \geq 1$,*

$$c_{N,n} = (-1)^n n! \sum_{k=1}^{n} \binom{n+1}{k+1} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \frac{(-N)^k}{(N+i_1)\cdots(N+i_k)} \, .$$

Expressions of $c_{N,n}$ in Theorem 2.3 and Proposition 2.4 are explicit but not convenient to calculate them. Now, using associated Stirling numbers of the first kind, we introduce a more convenient expression of hypergeometric Cauchy numbers. Associated Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{\geq m}$ ([4, 16, 17, 18]) are given by

$$(2.3) \qquad \frac{\left(-\log(1-x) - F_{m-1}(x)\right)^k}{k!} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_{\geq m} \frac{x^n}{n!} \quad (m \geq 1),$$

where

$$F_m(x) = \begin{cases} 0 & (m = 0); \\ \sum_{n=1}^{m} \frac{x^n}{n} & (m \geq 1). \end{cases}$$

When $m = 1$, $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_{\geq 1}$ is the classical Stirling numbers of the first kind. Now, we obtain a simple expression for hypergeometric Cauchy numbers in terms of the binomial coefficients and incomplete Stirling numbers of the first kind.

Theorem 2.5.   *For $N \geq 1$ and $n \geq 1$, we have*

$$c_{N,n} = (-1)^n n! \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-N)^k k!}{(n+Nk)!} \begin{bmatrix} n+Nk \\ k \end{bmatrix}_{\geq N}.$$

Remark.   When $N = 1$, Theorem 2.5 is reduced to

$$c_n = \sum_{k=1}^{n} \frac{(-1)^{n-k} \binom{n+1}{k+1}}{\binom{n+k}{k}} \begin{bmatrix} n+k \\ k \end{bmatrix},$$

which is Proposition 2 in [13].

Proof.   From (2.3), we have

$$\left(\sum_{j=0}^{\infty} \frac{t^j}{j+N} \right)^k = \left(\frac{-\log(1-t) - F_{N-1}(t)}{t^N} \right)^k$$

$$= \sum_{n=k}^{\infty} k! \begin{bmatrix} n \\ k \end{bmatrix}_{\geq N} \frac{t^{n-Nk}}{n!}$$

$$= \sum_{n=-(N-1)k}^{\infty} \frac{k!}{(n+Nk)!} \begin{bmatrix} n+Nk \\ k \end{bmatrix}_{\geq N} t^n.$$

Notice that

$$H^{(i)} \left(\frac{-\log(1-t) - F_{N-1}(t)}{t^N} \right) \bigg|_{t=0} = \frac{1}{i+N}.$$

Applying Lemma 2.1 with

$$f_1(t) = \cdots = f_k(t) = \frac{-\log(1-t) - F_{N-1}(t)}{t^N},$$

we get

$$(2.4) \qquad \frac{k!}{(n+Nk)!} \begin{bmatrix} n+Nk \\ k \end{bmatrix}_{\geq N} = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \frac{1}{(i_1+N)\cdots(i_k+N)}.$$

Together with Proposition 2.4, we get the desired result.   □

Example 2.6. Let $N = 3$ and $n = 4$. By the definition in (2.3), we get

$$\frac{1}{7!}\begin{bmatrix} 7 \\ 1 \end{bmatrix}_{\geq 3} = \frac{1}{7}, \quad \frac{1}{10!}\begin{bmatrix} 10 \\ 2 \end{bmatrix}_{\geq 3} = \frac{153}{1400}, \quad \frac{1}{13!}\begin{bmatrix} 13 \\ 3 \end{bmatrix}_{\geq 3} = \frac{1751}{50400}$$

and $\quad \dfrac{1}{16!}\begin{bmatrix} 16 \\ 4 \end{bmatrix}_{\geq 3} = \dfrac{190261}{29030400}$.

Hence,

$$\begin{aligned}
c_{3,4} &= 4! \sum_{k=1}^{4} \binom{5}{k+1} \frac{(-3)^k k!}{(4+3k)!} \begin{bmatrix} 4+3k \\ k \end{bmatrix}_{\geq 3} \\
&= 4! \left(-\binom{5}{2} \frac{3}{7!} \begin{bmatrix} 7 \\ 1 \end{bmatrix}_{\geq 3} + \binom{5}{3} \frac{3^2 \cdot 2}{10!} \begin{bmatrix} 10 \\ 2 \end{bmatrix}_{\geq 3} \right. \\
&\qquad \left. -\binom{5}{4} \frac{3^3 \cdot 3!}{13!} \begin{bmatrix} 13 \\ 3 \end{bmatrix}_{\geq 3} + \binom{5}{5} \frac{3^4 \cdot 4!}{16!} \begin{bmatrix} 16 \\ 4 \end{bmatrix}_{\geq 3} \right) \\
&= -\frac{1971}{5600} .
\end{aligned}$$

Next, we shall obtain some explicit expressions of the hypergeometric Bernoulli numbers $B_{N,n}$ ([9, 10, 12]), defined by

$$\frac{1}{{}_1F_1(1; N+1; x)} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^n}{n!} ,$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function defined by

$$ {}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!} .$$

Then we have the following:

Theorem 2.7. *For $n \geq 1$,*

$$B_{N,n} = n! \sum_{k=1}^{n} (-N!)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+\cdots+i_k=n}} \frac{1}{(N+i_1)! \cdots (N+i_k)!} .$$

Proof. Put

$$h := {}_1F_1(1; N+1; x) = \sum_{j=0}^{\infty} \frac{N! x^j}{(N+j)!} .$$

Note that

$$H^{(i)}(h)\Big|_{x=0} = \sum_{j=0}^{\infty} \frac{N!}{(N+j)!}\binom{j}{i}x^{j-i}\Bigg|_{x=0} = \frac{N!}{(N+i)!}.$$

Hence, by using Lemma 2.2 (2.1), we have

$$\begin{aligned}
\frac{B_{N,n}}{n!} &= H^{(n)}\left(\frac{1}{h}\right)\Bigg|_{x=0}\\
&= \sum_{k=1}^{n}\frac{(-1)^k}{h^{k+1}}\Bigg|_{x=0} \sum_{\substack{i_1,\ldots,i_k\geq 1\\ i_1+\cdots+i_k=n}} H^{(i_1)}(h)\Big|_{x=0}\cdots H^{(i_k)}(h)\Big|_{x=0}\\
&= \sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1,\ldots,i_k\geq 1\\ i_1+\cdots+i_k=n}} \frac{N!}{(N+i_1)!}\cdots\frac{N!}{(N+i_k)!}\\
&= \sum_{k=1}^{n} \sum_{\substack{i_1,\ldots,i_k\geq 1\\ i_1+\cdots+i_k=n}} \frac{(-N!)^k}{(N+i_1)!\cdots(N+i_k)!}.
\end{aligned}$$

$\square$

We express the hypergeometric Bernoulli numbers in terms of the binomial coefficients, too. In fact, by using Lemma 2.2 (2.2) instead of Lemma 2.2 (2.1) in the proof of Theorem 2.7, we obtain the following:

Proposition 2.8. *For $n \geq 1$,*

$$B_{N,n} = n!\sum_{k=1}^{n}(-N!)^k\binom{n+1}{k+1}\sum_{\substack{i_1,\ldots,i_k\geq 0\\ i_1+\cdots+i_k=n}}\frac{1}{(N+i_1)!\cdots(N+i_k)!}.$$

In the same way as the proof of Theorem 2.5, using associated Stirling numbers of the second kind, we introduce a more convenient expression of hypergeometric Cauchy numbers. Associated Stirling numbers of the second kind $\left\{{n \atop k}\right\}_{\geq m}$ ([4, 16, 17, 18]) are given by

$$(2.5) \qquad \frac{\left(e^x - E_{m-1}(x)\right)^k}{k!} = \sum_{n=0}^{\infty}\left\{{n \atop k}\right\}_{\geq m}\frac{x^n}{n!} \quad (m\geq 1),$$

where

$$E_m(x) = \sum_{n=0}^{m}\frac{x^n}{n!}.$$

When $m = 1$, $\left\{{n \atop k}\right\} = \left\{{n \atop k}\right\}_{\geq 1}$ is the classical Stirling numbers of the second kind. Now, we obtain a simple expression for hypergeometric Bernoulli numbers in terms of the binomial coefficients and incomplete Stirling numbers of the second kind.

Theorem 2.9.   *For $N \geq 1$ and $n \geq 1$, we have*

$$B_{N,n} = n! \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-N!)^k k!}{(n+Nk)!} \left\{{n + Nk \atop k}\right\}_{\geq N}.$$

Remark.   When $N = 1$, Theorem 2.9 is reduced to

$$B_n = \sum_{k=1}^{n} \frac{(-1)^k \binom{n+1}{k+1}}{\binom{n+k}{k}} \left\{{n + k \atop k}\right\},$$

which is a simple formula of Bernoulli numbers, appeared in [7, 19].

Proof.   From (2.5), we have

$$\left(\sum_{j=0}^{\infty} \frac{t^j}{(j+N)!} \right)^k = \left(\frac{e^t - E_{N-1}(t)}{t^N} \right)^k$$

$$= \sum_{n=k}^{\infty} k! \left\{{n \atop k}\right\}_{\geq N} \frac{t^{n-Nk}}{n!}$$

$$= \sum_{n=-(N-1)k}^{\infty} \frac{k!}{(n+Nk)!} \left\{{n + Nk \atop k}\right\}_{\geq N} t^n.$$

Notice that

$$H^{(i)} \left(\frac{e^t - E_{N-1}(t)}{t^N} \right) \bigg|_{t=0} = \frac{1}{(i+N)!}.$$

Applying Lemma 2.1 with

$$f_1(t) = \cdots = f_k(t) = \frac{e^t - E_{N-1}(t)}{t^N},$$

we get

$$(2.6) \qquad \frac{k!}{(n+Nk)!} \left\{{n + Nk \atop k}\right\}_{\geq N} = \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ i_1+\cdots+i_k=n}} \frac{1}{(i_1+N)!\cdots(i_k+N)!}.$$

Together with Proposition 2.8, we get the desired result. $\square$

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